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NON-AFFINE HOPF ALGEBRA DOMAINS OF GELFAND–KIRILLOV DIMENSION TWO

K. R. GOODEARL
Department of Mathematics University of California at Santa Barbara, Santa Barbara, CA 93106, USA
e-mail: goodearl@math.ucsb.edu

and J. J. ZHANG
Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195, USA
e-mail: zhang@math.washington.edu

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Abstract. We classify all non-affine Hopf algebras H over an algebraically closed field k of characteristic zero that are integral domains of Gelfand–Kirillov dimension two and satisfy the condition $\mathrm{Ext}^1_H(k, k) \neq 0$. The affine ones were classified by the authors in 2010 (Goodearl and Zhang, *J. Algebra* **324** (2010), 3131–3168).

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1. Introduction. Throughout let k be a base field that is algebraically closed of characteristic zero. All algebras and Hopf algebras are assumed to be k-algebras. The main result of [**6**] is the classification of the affine Hopf k-algebras H that are integral domains of Gelfand–Kirillov dimension two and satisfy the extra homological condition:

$$\mathrm{Ext}^1_H(k, k) \neq 0, \tag{$\natural$}$$

where k also denotes the trivial module $H/\ker\epsilon$. We say that H is *affine* if it is finitely generated over k as an algebra. Geometrically, the condition ($\natural$) means that the tangent space of the corresponding quantum group is non-trivial. By [**6**, Theorem 3.9], the condition ($\natural$) is equivalent to the condition that the corresponding quantum group contains a classical algebraic subgroup of dimension one. The authors asked whether the condition ($\natural$) is automatic when H is an affine domain of Gelfand–Kirillov dimension (or GK-dimension, for short) two [**6**, Question 0.3]. This question was answered negatively in [**21**]. Some affine Hopf algebra domains of GK-dimension two that do not satisfy ($\natural$) were given and studied in [**21**].

All Hopf domains of GK-dimension one are listed in [**6**, Proposition 2.1]. All affine Hopf domains of GK-dimension two satisfying ($\natural$) are listed in [**6**, Theorem 0.1]. The main goal of the present paper is to classify *non-affine* Hopf domains of GK-dimension two satisfying ($\natural$). Together with [**6**, Theorem 0.1], this provides a complete list of all Hopf domains of GK-dimension two satisfying ($\natural$).

Theorem 1.1. *Let H be a Hopf domain of GK-dimension two satisfying* ($\natural$). *Then, it is isomorphic, as a Hopf algebra, to one of the following.*

(1) *kG where G is a subgroup of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$.*
(2) *kG where $G = L \rtimes R$ for some subgroup L of $\mathbb{Q}$ containing $\mathbb{Z}$ and some subgroup R of $\mathbb{Z}_{(2)}$ containing $\mathbb{Z}$.*
(3) *$U(\mathfrak{g})$ where $\mathfrak{g}$ is a two-dimensional Lie algebra over k.*
(4) *$A_G(e, \chi)$ where G is a non-zero subgroup of $\mathbb{Q}$* [Example 2.2].
(5) *$C_G(e, \tau)$ where G is a non-zero subgroup of $\mathbb{Q}$* [Example 2.3].
(6) *$B_G(\{p_i\}, \chi)$ where G is a non-zero subgroup of $\mathbb{Q}$* [Construction 3.1].

In part (2) of the above theorem, $\mathbb{Z}_{(2)}$ denotes the localization of the ring $\mathbb{Z}$ at the maximal ideal (2), that is, the ring of rational numbers with odd denominators.

There are more Hopf domains of GKdim two if the hypothesis ($\natural$) is removed from Theorem 1.1, see [**21**].

We also study some algebraic properties of the algebras in Theorem 1.1. Parts (1)–(5) of the following are easy consequences of Theorem 1.1, whereas part (6) is given in [**6**, Theorem 0.1].

Corollary 1.2. *Let H be as in Theorem 1.1. Then, the following hold.*

(1) *H is pointed and generated by grouplike and skew primitive elements.*
(2) *H is countable dimensional over k.*
(3) *The antipode of H is bijective.*
(4) *Let K be a Hopf subalgebra of H. Then ${}_KH$ and ${}_KH$ are free.*
(5) *If H is as in parts* (1–5) *of Theorem 1.1, then* $2 \leq \operatorname{gldim} H \leq 3$*, whereas if H is as in part* (6)*, then* $\operatorname{gldim} H = \infty$.
(6) *H is affine if and only if it is noetherian.*

By [**6**, Proposition 0.2(b)], if H in Theorem 1.1 is noetherian, then $\operatorname{injdim} H = 2$. So, we conjecture that $\operatorname{injdim} H = 3$ if H in Theorem 1.1 is non-noetherian.

There have been extensive research activities concerning infinite dimensional Hopf algebras (or quantum groups) in recent years. The current interests are mostly on noetherian and/or affine Hopf algebras. One appealing research direction is to understand some global structure of noetherian and/or affine and/or finite GK-dimensional Hopf algebras.

A classical result of Gromov states that a finitely generated group G has polynomial growth, or equivalently, the associated group algebra has finite GK-dimension, if and only if G has a nilpotent subgroup of finite index [**7**]. So, group algebras of finite GK-dimension are understood. It is natural to look for a Hopf analogue of this result, see [**22**, Question 0.1]. Another vague question is "what can we say about a Hopf algebra of finite GK-dimension?". Let us mention a very nice result in this direction. Zhuang proved that every connected Hopf algebra of finite GK-dimension is a noetherian and affine domain with finite global dimension [**25**]. Here, the term "connected" means that the coradical is one-dimensional. In general, the noetherian and affine properties are not consequences of the finite GK-dimension property. To have any sensible solution, we might restrict our attention to the domain case. A secondary goal of this paper is to promote research on Hopf domains of finite GK-dimension which are not necessarily noetherian nor affine.

Let us start with some definitions.

Definition 1.3. Let H be a Hopf algebra with antipode S.

(1) H is called *locally affine* if every finite subset of H is contained in an affine Hopf subalgebra of H.
(2) H is said to have *S-finite type* if there is a finite dimensional subspace $V \subseteq H$ such that H is generated by $\bigcup_{i=0}^{\infty} S^i(V)$ as an algebra.
(3) H is said to satisfy (*FF*) if for every Hopf subalgebra $K \subseteq H$, the K-modules H_K and ${}_KH$ are faithfully flat.

It is clear that H is affine if and only if H is both locally affine and of S-finite type. The question of whether H satisfies (FF) has several positive answers [**4, 12, 15, 16, 19, 20**]. In 1993, Montgomery asked if every Hopf algebra satisfies (FF) [**11**, Question 3.5.4]. A counterexample was given in [**17**]. Hence, Montgomery's question was modified to encompass only the Hopf algebras with bijective antipode. By a result of Skryabin [**18**, Theorem A], every Hopf domain of finite GK-dimension has bijective antipode. Prompted by Zhuang's result and Corollary 1.2, we ask the following questions.

QUESTIONS 1.4. Let H be a Hopf domain of finite GK-dimension.

(1) Is the k-dimension of H countable, or equivalently, is H countably generated as an algebra?
(2) Is H locally affine?
(3) Is H equal to the union of an ascending chain of affine Hopf subalgebras?
(4) Is "affine" equivalent to "noetherian"? See also [**24**, Question 5.1], [**2**, Questions D and E] and [**5**, Question 2.4].
(5) Does H satisfy (FF)?
(6) Is injdim H bounded by a function of GKdim H?
(7) If gldim H is finite, is gldim H bounded by a function of GKdim H?

If any of Questions 1.4(1–3) has a positive answer, it would indicate that H is somewhat close to being affine. We also have the following result connecting some of these concepts. Note that pointed Hopf algebras satisfy (FF) by [**16**].

THEOREM 1.5. *Let H be a Hopf algebra that is left noetherian.*

(1) *Suppose H satisfies* (FF). *Then, H is of S-finite type. As a consequence,* $\dim_k H$ *is countable.*
(2) *Suppose H satisfies* (FF) *and H is locally affine. Then, H is affine.*
(3) *If H is pointed, then it is affine.*

Theorem 1.5(3) partially answers [**24**, Question 5.1] in the pointed case, see also [**5**, Question 2.4] and [**2**, Question D].

1.1. Notation. Fix an algebraically closed base field k of characteristic zero throughout.

Elements u and v of a k-algebra are said to *quasi-commute* if $uv = qvu$ for some $q \in k^{\times}$, in which case they are also said to *q-commute*.

We shall reserve the term *skew primitive* for $(1, g)$-skew primitive elements z, meaning that g is grouplike and $\Delta(z) = z \otimes 1 + g \otimes z$. In this situation, g is called the *weight* of z, denoted wt(z). General skew primitive elements can be normalized to the kind above, since if w is (a, b)-skew primitive, then $a^{-1}w$ is $(1, a^{-1}b)$-skew primitive.

Let G be an additive subgroup of $(\mathbb{Q}, +)$. For convenience we sometimes identify it with the multiplicative x-power group, namely,

$$G = \{x^g \mid g \in G\},$$

where $x^g x^h = x^{g+h}$ for $g, h \in G$ and $x^0 = 1$. If $1 \in G$, then we also write x for x^1. Similarly, the group algebra kG is identified with $\bigoplus_{g\in G} k\, x^g$. In Section 3, we will also use an additive submonoid $M \subseteq \mathbb{Q}$. Then, in this case, we identify M with the multiplicative y-power monoid, namely,

$$M = \{y^m \mid m \in M\}.$$

By using these different notations, one sees the different roles played by G and M in Section 3.

2. Non-affine construction of types A and C. We start by recalling a result of [**6**] that classifies all (not necessarily affine) Hopf domains of GK-dimension one. Note that a domain of GK-dimension one is automatically commutative (e.g., [**6**, Lemma 4.5]).

Lemma 2.1 ([**6**, Proposition 2.1]). *Assume that a Hopf algebra H is a domain of GK-dimension one. Then, H is isomorphic to one of the following:*

(1) *an enveloping algebra $U(\mathfrak{g})$, where* $\dim \mathfrak{g} = 1$*;*
(2) *a group algebra kG, where G is infinite cyclic;*
(3) *a group algebra kG, where G is a non-cyclic torsionfree abelian group of rank* 1*, i.e., a non-cyclic subgroup of* $\mathbb{Q}$.

As a consequence, H satisfies ($\natural$).

Proof. The main assertion is [**6**, Proposition 2.1] and the consequence follows by an easy computation. □

Note that every Hopf algebra H in Lemma 2.1 is countable dimensional and is completely determined by its coradical.

In [**6**, Constructions 1.2–1.4], we constructed some affine Hopf domains of GK-dimension two, labelled as types A, B and C. Non-affine versions of types A and C can be constructed similarly and appeared also in other papers. One way of defining these is to use the Hopf Ore extensions introduced in [**3, 14**]. We review the definition briefly, and refer to [**3, 14**] for more details.

Given a Hopf algebra K, an automorphism σ and a σ-derivation δ of K, a *Hopf Ore extension* (or *HOE*, for short) of K, denoted by $K[z; \sigma, \delta]$, is a Hopf algebra H that is isomorphic to the usual Ore extension $K[z; \sigma, \delta]$ as an algebra and contains K as a Hopf subalgebra. HOEs have been studied in several papers including [**3, 14, 23**]. When $\delta = 0$, the HOE H is abbreviated to $K[z; \sigma]$, and when $\sigma = Id_K$, it is abbreviated to $K[z; \delta]$. If K is a domain, then H is also a domain.

Example 2.2 ([**23**, Example 5.4]). Let $K = kG$ where G is a group and let $\chi : G \to k^\times$ be a character of G. Define an algebra automorphism $\sigma_\chi : K \to K$ by

$$\sigma_\chi(g) = \chi(g)g, \quad \forall\, g \in G.$$

Let $\delta = 0$. By [**23**, Example 5.4], $H := K[z; \sigma_\chi]$ is a HOE of K with $\Delta(z) = z \otimes 1 + e \otimes z$ for any choice of e in the center of G. This Hopf algebra is denoted by $A_G(e, \chi)$.

We are mostly interested in non-trivial subgroups $G \subseteq \mathbb{Q}$. In this case,

$$\operatorname{GKdim} kG = 1, \quad \text{and} \quad \operatorname{GKdim} H = 2,$$

using, e.g., [**8**, Lemma 2.2] for the second equality. If $k = \mathbb{C}$, then there are many characters of G. For example, let λ be a real number, then $\exp_\lambda : r \to \exp(2\pi i r\lambda)$ is a character from $\mathbb{Q}$ to $\mathbb{C}^\times$.

A special case is when $G = \mathbb{Z}$ (identified with $\{x^i\}_{i\in\mathbb{Z}}$). Suppose the character $\chi : G \to k^\times$ is trivial (in this case $\sigma = \mathrm{Id}_{kG}$) and $\Delta(z) = z \otimes 1 + x \otimes z$. This special HOE, denoted by $A_{\mathbb{Z}}(1, 0)$, is the commutative Hopf algebra $A(1, 1)$ given in [**6**, Construction 1.2] (by taking $(n, q) = (1, 1)$). More generally, if $n \in \mathbb{Z}$, $q \in k^\times$, and $\chi : \mathbb{Z} \to k^\times$ is the character given by $\chi(i) = q^{-i}$, then $A_{\mathbb{Z}}(n, \chi)$ is the Hopf algebra $A(n, q)$ of [**6**, Construction 1.2].

Example 2.3 ([**23**, A special case of Example 5.5]). Let $K = kG$, where G is a group and let e be an element in the center of G. Let $\tau : G \to (k, +)$ be an additive character of G. Define a k-linear derivation $\delta : K \to K$ by

$$\delta(g) = \tau(g)g(e - 1), \quad \forall\, g \in G.$$

Then, $H := K[z; \delta]$ is a HOE of K with $\Delta(z) = z \otimes 1 + e \otimes z$. This Hopf algebra is denoted by $C_G(e, \tau)$.

Later, we will take G to be a subgroup of $\mathbb{Q}$. Since we assume k has characteristic zero, there are many additive characters from G to $(k, +)$. For example, let λ be a rational number, then $i_\lambda : r \to r\lambda$ is an additive character from $G \to (k, +)$.

A special case is when $G = \mathbb{Z}$ (identified with $\{x^i\}_{i\in\mathbb{Z}}$), $e = 1 - n$ for some $n \in \mathbb{N}$, and $\tau = i_1$ (the inclusion map $\mathbb{Z} \to k$). Then $C_{\mathbb{Z}}(1 - n, i_1)$ is the Hopf algebra $C(n)$ of [**6**, Construction 1.4].

The next result of [**23**] says that $A_G(e, \chi)$ and $C_G(e, \tau)$ are natural classes of Hopf algebras. Let $G(H)$ denote the group of all grouplike elements in a Hopf algebra H.

Theorem 2.4 ([**23**, Theorem 7.1]). *Let H be a pointed Hopf domain. Suppose that $G := G(H)$ is abelian and that*

$$\operatorname{GKdim} kG < \operatorname{GKdim} H < \operatorname{GKdim} kG + 2 < \infty.$$

If H does not contain $A(1, 1)$ as a Hopf subalgebra, then H is isomorphic to either $A_G(e, \chi)$ or $C_G(e, \tau)$ as given in Examples 2.2 and 2.3.

As a consequence, the following corollary is obtained.

Corollary 2.5. *Let H be a pointed Hopf domain of GK-dimension two. Suppose that the coradical of H has GK-dimension one and that H does not contain $A(1, 1)$ as a Hopf subalgebra. Then, H is isomorphic to either $A_G(e, \chi)$ or $C_G(e, \tau)$, where G is a non-zero subgroup of $\mathbb{Q}$.*

Proof. Since H is pointed, the coradical of H is kG, where $G = G(H)$. Since $\operatorname{GKdim} kG = 1$, by Lemma 2.1, G is a non-zero subgroup of $\mathbb{Q}$. So, $G(H) = G$ is abelian. Now,

$$1 = \operatorname{GKdim} kG < 2 = \operatorname{GKdim} H < \operatorname{GKdim} kG + 2 = 3.$$

Hence, the hypothesis of Theorem 2.4 holds, and the assertion follows from the theorem. □

Hopf algebras of GK-dimension two that contain $A(1,1)$ are more complicated. We will construct a family of them in the next section.

3. Non-affine construction of type B.

3.1. Construction. We construct a Hopf algebra $B_G(\{p_i\}, \chi)$ based on the following data.

Data. Let G be a subgroup of $(\mathbb{Q}, +)$ that contains $\mathbb{Z}$, and write its group algebra in the form

$$kG = \bigoplus_{a \in G} k\, x^a,$$

as in Notation 1.1.

Let I be an index set with $|I| \geq 2$. Let $\{p_i \mid i \in I\}$ be a set of pairwise relatively prime integers such that $p_i \geq 2$ and $1/p_i \in G$ for all $i \in I$, and let M be the additive submonoid of $\mathbb{Q}$ generated by $\{1/p_i \mid i \in I\}$. Due to the relative primeness assumption, $1/p_ip_j \in G$ for all distinct $i, j \in I$. Obviously $M \subseteq G$, but we want to keep the algebras of M and G separate, as these are playing different roles. Following Notation 1.1 we write the monoid algebra of M in the following form:

$$kM = \bigoplus_{b \in M} k\, y^b.$$

Set

$$GM = \{a_1 m_1 + \cdots + a_t m_t \mid t \in \mathbb{Z}_{\geq 0},\ a_l \in G,\ m_l \in M\} = \sum_{i \in I} G(1/p_i),$$

an additive subgroup of $\mathbb{Q}$, and let $\chi : GM \to k^\times$ be a character (i.e., a group homomorphism) such that

(1) $\chi(1/p_i^2)$ is a primitive p_i-th root of unity for all $i \in I$.

Note that (1) implies that

(2) $\chi(1/p_i) = \chi(1/p_i^2)^{p_i} = 1$ for $i \in I$, and

(3) $\chi(1) = \chi(1/p_i)^{p_i} = 1$.

Observation. For any distinct $i, j \in I$, we have $1/p_ip_j \in GM$ and there are $c_i, c_j \in \mathbb{Z}$ such that $c_ip_i + c_jp_j = 1$, whence

$$\chi(1/p_ip_j) = \chi((c_i/p_j) + (c_j/p_i)) = \chi(1/p_j)^{c_i} \chi(1/p_i)^{c_j} = 1. \tag{E3.1.1}$$

Algebra structure. Let G act on kM by k-algebra automorphisms such that

$$a \cdot y^b = \chi(ab) y^b \qquad \forall\, a \in G, \quad b \in M.$$

Use this action to turn kM into a left kG-module algebra, form the smash product

$$B = B_G(\{p_i\}, \chi) := kM \# kG,$$

and omit #s from expressions in B. If we write $y_i := y^{1/p_i}$ for $i \in I$, then we can present B by the generators $\{x^a \mid a \in G\} \sqcup \{y_i \mid i \in I\}$ and the relations

$$\begin{aligned} x^0 &= 1, && \\ x^a x^{a'} &= x^{a+a'} && (a, a' \in G), \\ x^a y_i &= \chi(a/p_i) y_i x^a && (a \in G, \quad i \in I), \\ y_i y_j &= y_j y_i && (i, j \in I), \\ y_i^{p_i} &= y_j^{p_j} && (i, j \in I). \end{aligned} \tag{E3.1.2}$$

These relations are very similar to the relations in [**6**, (E1.2.1) in Construction 1.2].

Consider a non-empty finite subset $J \subseteq I$. If $c := \prod_{j \in J} p_j$, the submonoid of M generated by $\{1/p_j \mid j \in J\}$ is a submonoid of the additive monoid $M_c := \mathbb{Z}_{\geq 0}(1/c)$. The subalgebra of B generated by $\{x^a \mid a \in G\} \sqcup \{y_j \mid j \in J\}$ is a subalgebra of a skew polynomial ring:

$$kM_c \# kG = kG[y^{1/c}; \sigma_c],$$

where σ_c is the automorphism of kG such that $\sigma_c(x^a) = \chi(-a/c)x^a$ for all $a \in G$. It follows that $kM_c \# kG$ is a domain of GK-dimension two.

Since B is a directed union of subalgebras of the form $kM_c \# kG$, we conclude that B is a domain of GK-dimension two.

Hopf structure. It is clear from the presentation in (E3.1.2) that there is an algebra homomorphism $\varepsilon : B \to k$ such that $\varepsilon(x^a) = 1$ for all $a \in G$ and $\varepsilon(y_i) = 0$ for all $i \in I$.

Obviously $(x^a \otimes x^a)(x^{a'} \otimes x^{a'}) = x^{a+a'} \otimes x^{a+a'}$ for all $a, a' \in G$. Set $x_i := x^{1/p_i}$ and $\delta_i := y_i \otimes 1 + x_i \otimes y_i$ for $i \in I$. It is clear that $(x^a \otimes x^a)\delta_i = \chi(a/p_i)\delta_i(x^a \otimes x^a)$ for all $a \in G$ and $i \in I$. For any distinct $i, j \in I$, we have

$$x_i y_j = \chi(1/p_i p_j) y_j x_i = y_j x_i \tag{E3.1.3}$$

because of (E3.1.1), and likewise $x_j y_i = y_i x_j$. It follows that $\delta_i \delta_j = \delta_j \delta_i$. Moreover, since

$$(x_i \otimes y_i)(y_i \otimes 1) = \chi(1/p_i^2)(y_i \otimes 1)(x_i \otimes y_i)$$

with $\chi(1/p_i^2)$ a primitive p_i-th root of unity, it follows from the q-binomial formula that $\delta_i^{p_i} = y^1 \otimes 1 + x^1 \otimes y^1$. Likewise, $\delta_j^{p_j} = y^1 \otimes 1 + x^1 \otimes y^1$, so that $\delta_i^{p_i} = \delta_j^{p_j}$. Therefore, there is an algebra homomorphism $\Delta : B \to B \otimes B$ such that $\Delta(x^a) = x^a \otimes x^a$ for all $a \in G$ and $\Delta(y_i) = \delta_i$ for all $i \in I$.

Observe that $(\varepsilon \otimes \mathrm{id})\Delta(x^a) = x^a$ for all $a \in G$ and $(\varepsilon \otimes \mathrm{id})\Delta(y_i) = y_i$ for all $i \in I$. Consequently, $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$, and similarly $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$. We also observe that $(\Delta \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes \Delta)\Delta$ agree on x^a and y_i for all $a \in G$ and $i \in I$, and consequently $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Therefore, (B, Δ, ε) is a bialgebra.

Next, observe that $(x^{-a'})(x^{-a}) = x^{-(a+a')}$ for all $a, a' \in G$ and that

$$(-x_i^{-1} y_i)x^{-a} = \chi(a/p_i)x^{-a}(-x_i^{-1} y_i)$$

for all $a \in G$ and $i \in I$. For any distinct $i, j \in I$, we see using (E3.1.3) that

$$(-x_j^{-1} y_j)(-x_i^{-1} y_i) = (-x_i^{-1} y_i)(-x_j^{-1} y_j).$$

We also have

$$(-x_i^{-1}y_i)^{p_i} = (-1)^{p_i} x_i^{-1} y_i x_i^{-1} y_i \cdots x_i^{-1} y_i = (-1)^{p_i} \chi(-1/p_i^2)^{p_i(p_i+1)/2} y^1 x^{-1}.$$

If p_i is odd, then p_i divides $p_i(p_i+1)/2$ and so $\chi(-1/p_i^2)^{p_i(p_i+1)/2} = 1$. On the other hand, if p_i is even, then $\chi(-1/p_i^2)^{p_i(p_i+1)/2} = (-1)^{p_i+1} = -1$ due to the primitivity of $\chi(-1/p_i^2)$. In both cases, we end up with $(-x_i^{-1}y_i)^{p_i} = -y^1x^{-1}$. Likewise, $(-x_j^{-1}y_j)^{p_j} = -y^1x^{-1}$, so that $(-x_i^{-1}y_i)^{p_i} = (-x_j^{-1}y_j)^{p_j}$. Therefore, there is an algebra homomorphism $S : B \to B^{\mathrm{op}}$ such that $S(x^a) = x^{-a}$ for all $a \in G$ and $S(y_i) = -x_i^{-1}y_i$ for all $i \in I$.

Finally, observe that $m(S \otimes \mathrm{id})\Delta(x^a) = 1 = \varepsilon(x^a)1$ for all $a \in G$, where $m : B \otimes B \to B$ is the multiplication map, and $m(S \otimes \mathrm{id})\Delta(y_i) = 0 = \varepsilon(y_i)1$ for all $i \in I$, from which we conclude that $m(S \otimes \mathrm{id})\Delta = u \circ \varepsilon$, where $u : k \to B$ is the unit map. Similarly, $m(\mathrm{id} \otimes S)\Delta = u \circ \varepsilon$. Therefore, $(B, \Delta, \varepsilon, S)$ is a Hopf algebra. We shall denote it $B_G(\{p_i\}, \chi)$, as indicated above. The Hopf algebra structure is uniquely determined by the conditions

$$\begin{aligned} &x^a \ \text{ is grouplike for all } \ a \in G, \\ &y_i \ \text{ is } \ (1, x_i)\text{-skew primitive for all } \ i \in I. \end{aligned} \tag{E3.1.4}$$

The construction above can also be carried out when the index set I is a singleton, but then the resulting Hopf algebra is isomorphic to $A_G(e, \chi)$ for suitable e and χ. We leave the details to the reader.

3.2. Examples. The data $(G, \{p_i\}, \chi)$ can be chosen so that $B_G(\{p_i\}, \chi)$ contains infinitely many skew primitive elements which are linearly independent modulo its coradical, as follows.

Example 3.1. Let $p_1, p_2, \ldots$ be any infinite sequence of pairwise relatively prime integers ≥ 2, and let G be the subgroup of $(\mathbb{Q}, +)$ generated by $1/p_i$ for all $i \in \mathbb{N}$. Set $M := \sum_{i=1}^{\infty} \mathbb{Z}_{\geq 0}(1/p_i)$ and $G^2 := \sum_{i=1}^{\infty} \mathbb{Z}(1/p_i^2)$. Note that $GM = G^2$, which is a subgroup of $(\mathbb{Q}, +)$. For each $i \in \mathbb{N}$, let $\beta_i \in k^\times$ be a primitive p_i-th root of unity. The cosets $\overline{1/p_i^2}$ in $\mathbb{Q}/\mathbb{Z}$ generate finite cyclic subgroups of pairwise relatively prime orders, whence the sum of these subgroups is a direct sum. Hence, there is a homomorphism

$$\overline{\chi}_0 : \sum_{i=1}^{\infty} \mathbb{Z}\big(\overline{1/p_i^2}\big) = \bigoplus_{i=1}^{\infty} \mathbb{Z}\big(\overline{1/p_i^2}\big) \to k^\times$$

such that $\overline{\chi}_0\big(\overline{1/p_i^2}\big) = \beta_i$ for all i. Since $k^\times$ is a divisible abelian group, it is injective in the category of all abelian groups, and so $\overline{\chi}_0$ extends to a homomorphism $\overline{\chi} : G^2/\mathbb{Z} \to k^\times$. Compose $\overline{\chi}$ with the quotient map $G^2 \to G^2/\mathbb{Z}$ to obtain a character $\chi : G^2 \to k^\times$. By the choice of χ, we have $\chi(1/p_i^2) = \beta_i$ for all $i \in \mathbb{N}$. As a consequence, $\chi(a) = 1$ for all $a \in G$.

Thus, by the construction in the previous subsection, we obtain a Hopf algebra $B = B_G(\{p_i\}, \chi)$ that contains distinct skew primitive elements y_i, for $i \in \mathbb{N}$, which are linearly independent modulo the coradical kG of B.

Since there are uncountably many different choices of $\overline{\chi}_0$, there are uncountably many non-isomorphic Hopf domains of GK-dimension two by Proposition 3.6 below.

Certain natural finitely generated subalgebras of $B_G(\{p_i\}, \chi)$ are Hopf algebras isomorphic to some of the Hopf algebras $A(n,q)$ and $B(n,p_0,\dots,p_s,q)$ of [**6**, Constructions 1.1, 1.2], as follows.

LEMMA 3.2. *Let $B := B_G(\{p_i\}, \chi)$ as in the previous subsection. Let $\widetilde{G}$ be a finitely generated subgroup of G, $\widetilde{I}$ a non-empty finite subset of I such that $1/p_i \in \widetilde{G}$ for all $i \in \widetilde{I}$, and $\widetilde{B}$ the subalgebra of B generated by $\{x^a \mid a \in \widetilde{G}\} \sqcup \{y_i \mid i \in \widetilde{I}\}$. Then, $\widetilde{B}$ is a Hopf subalgebra of B.*

Assume that $\widetilde{I} = \{1,\dots,s\}$ for some positive integer s and $p_1 < \dots < p_s$. Set $m := p_1p_2\cdots p_s$ and $m_i := m/p_i$ for $i \in \widetilde{I}$.

(1) *There are positive integers n and p_0 such that $\widetilde{G} = \mathbb{Z}(1/mn)$ and $1/m^2n \in GM$, while $q := \chi(1/m^2n)$ is a primitive ℓ-th root of unity, where $\ell := mn/p_0$. Moreover, $p_0 \mid n$ and p_0 is relatively prime to each of $p_1,\dots,p_s$.*

(2) *If $s = 1$, then $\widetilde{B} \cong A(n,q)$.*

(3) *If $s \geq 2$, then $\widetilde{B} \cong B(n,p_0,\dots,p_s,q)$.*

Proof.

(1) Since $\widetilde{G}$ is a finitely generated subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, it has the form $\widetilde{G} = \mathbb{Z}(1/t)$ for some positive integer t. For $i \in \widetilde{I}$, we have $1/p_i \in \widetilde{G}$, whence $p_i \mid t$. Then, since the p_i are pairwise relatively prime, $m \mid t$. Thus, $t = mn$ for some positive integer n. Let $\widetilde{M}$ be the submonoid $\sum_{i\in\widetilde{I}} \mathbb{Z}_{\geq 0}(1/p_i)$.

The pairwise relative primeness of the p_i implies that $\gcd(m_1,\dots,m_s) = 1$, and so there exist integers c_i such that $c_1m_1 + \dots + c_sm_s = 1$, whence

$$(c_1/p_1) + \dots + (c_s/p_s) = 1/m. \tag{E3.2.1}$$

This does not imply that $1/m \in \widetilde{M}$, since some of the c_i may be negative, but we do get

$$\frac{1}{m^2n} = \frac{c_1}{p_1mn} + \dots + \frac{c_s}{p_smn} = \left(\frac{c_1}{t}\right)\left(\frac{1}{p_1}\right) + \dots + \left(\frac{c_s}{t}\right)\left(\frac{1}{p_s}\right) \in \widetilde{G}\widetilde{M} \subseteq GM.$$

Therefore $q := \chi(1/m^2n) \in k^\times$ is defined. Since $q^{m^2n} = \chi(1) = 1$, the order of q in the group $k^\times$ is finite, say $|q| = \ell$. Thus, q is a primitive ℓ-th root of unity. For $i \in \widetilde{I}$, the power

$$q^{m_i^2n} = \chi(1/m^2n)^{m_i^2n} = \chi(1/p_i^2)$$

is a primitive p_i-th root of unity, which implies that $p_i \mid \ell$. Consequently, $m \mid \ell$. On the other hand, $q^{p_im_i^2n} = 1$, whence ℓ divides $p_im_i^2n = m_imn$ for all $i \in \widetilde{I}$, and so $\ell \mid mn$. Thus, $mn = \ell p_0$ for some positive integer p_0. Since $m \mid \ell$, it follows that $p_0 \mid n$.

Since $\chi(1/p_i) = 1$ for all $i \in \widetilde{I}$, we can invoke (E3.2.1) to obtain $\chi(1/m) = 1$, from which it follows that

$$\chi(1/m_i) = 1, \quad \forall\, i \in \widetilde{I}. \tag{E3.2.2}$$

Set $d_i := \gcd(p_0, p_i)$, and write $p_0 = d_iu_i$ and $p_i = d_iv_i$ for some positive integers u_i, v_i. Since $\ell d_iu_i = \ell p_0 = mn = d_iv_im_in$, we find that $\ell \mid v_im_in$, and consequently

$$1 = q^{v_im_in} = \chi(1/m^2n)^{v_im_in} = \chi(1/p_i^2m_i)^{v_i}.$$

Now p_i^2 and m_i are relatively prime, whence $a_i p_i^2 + b_i m_i = 1$ for some integers a_i, b_i, and so $1/p_i^2 m_i = (a_i/m_i) + (b_i/p_i^2)$. Thus, taking account of equation (E3.2.2),

$$1 = \chi(1/p_i^2 m_i)^{v_i} = \chi(a_i/m_i)^{v_i}\chi(b_i/p_i^2)^{v_i} = \chi(1/p_i^2)^{b_i v_i}.$$

Since $\chi(1/p_i^2)$ is a primitive p_i-th root of unity, $p_i \mid b_i v_i$, from which it follows that p_i divides $a_i p_i^2 v_i + b_i m_i v_i = v_i$, and so $d_i = 1$. Thus, p_0 and p_i are relatively prime, for each $i \in \widetilde{I}$. By now, we have checked all assertions in part (1).

(2) In this case, $m = p_1$. Set $\widetilde{x} := x^{1/mn}$, so that $\widetilde{B}$ is generated by $\{\widetilde{x}^{\pm 1}, y_1\}$. We have $\widetilde{x}y_1 = \chi(1/mnp_1)y_1\widetilde{x} = qy_1\widetilde{x}$, so there is an algebra isomorphism $\phi : \widetilde{B} \to A(n, q)$ with $\phi(\widetilde{x}) = x$ and $\phi(y_1) = y$. Since $x_1 = x^{1/p_1} = \widetilde{x}^n$ and $\Delta(y_1) = y_1 \otimes 1 + x_1 \otimes y_1$, we see that ϕ preserves comultiplication. Observe also that ϕ preserves counit and antipode. Therefore, ϕ is an isomorphism of Hopf algebras.

(3) Again, set $\widetilde{x} := x^{1/mn}$, and observe that $\widetilde{B}$ can be presented by the generators $\widetilde{x}^{\pm 1}, y_1, \dots, y_s$ and the relations

$$\begin{aligned} \widetilde{x}\widetilde{x}^{-1} &= \widetilde{x}^{-1}\widetilde{x} = 1 && \\ \widetilde{x}y_i &= q^{m_i} y_i \widetilde{x} && (1 \le i \le s) \\ y_i y_j &= y_j y_i && (1 \le i < j \le s) \\ y_i^{p_i} &= y_j^{p_j} && (1 \le i < j \le s). \end{aligned} \tag{E3.2.3}$$

Comparing (E3.2.3) with [**6**, (E1.2.1)], we see that there is an algebra isomorphism $\phi : \widetilde{B} \to B(n, p_0, \dots, p_s, q)$ such that $\phi(\widetilde{x}) = x$ and $\phi(y_i) = y_i$ for $i = 1, \dots, s$. Since ϕ also preserves the Hopf algebra structures, we conclude that ϕ is an isomorphism of Hopf algebras.

□

PROPOSITION 3.3. *Let $B := B_G(\{p_i\}, \chi)$ as in Section 3.1. There is an ascending chain of Hopf subalgebras*

$$B\langle 1\rangle \subseteq B\langle 2\rangle \subseteq \cdots \subseteq B\langle n\rangle \cdots \subseteq B$$

*such that $B = \bigcup_{n=1}^{\infty} B\langle n\rangle$ and each $B\langle n\rangle$ is a finitely generated Hopf algebra of type B as in [**6**, Construction 1.2].*

Proof. Since G is countable, we list its elements as $\{g_1, g_2, \dots, g_n, \dots\}$. Write I as either $\{1, 2, \dots\}$ or $\{1, \dots, t\}$, and in the latter case set $p_i = p_t$ for all $i > t$. For $n \in \mathbb{N}$, let $\widetilde{G}\langle n\rangle$ be the subgroup of G generated by $\{g_1, \dots, g_n, 1/p_1, \dots, 1/p_{n+1}\}$. Let $B\langle n\rangle$ be the subalgebra of B generated by $\widetilde{G}\langle n\rangle$ and $\{y_1, \dots, y_{n+1}\}$. By Lemma 3.2(3), $B\langle n\rangle$ is a finitely generated Hopf algebra of type B as in [**6**, Construction 1.2]. It is clear that $B = \bigcup_{n=1}^{\infty} B\langle n\rangle$. □

3.3. Basic properties.

We next derive some basic properties of the Hopf algebras $B_G(\{p_i\}, \chi)$.

LEMMA 3.4. *Let $B := B_G(\{p_i\}, \chi)$ as in Section 3.1.*

(1) *There is a unique Hopf algebra map $\pi : B \to kG$ such that π is the identity on kG and $\pi(y_i) = 0$ for all $i \in I$.*

(2) *The maps* $\lambda := (\pi \otimes \mathrm{id})\Delta$ *and* $\rho := (\mathrm{id} \otimes \pi)\Delta$ *make* B *into a left and a right* kG*-comodule algebra, respectively.*
(3) B *is a* G*-graded algebra via* λ*, with*

$$B_a = \{z \in B \mid \lambda(z) = x^a \otimes z\}, \quad \forall\, a \in G.$$

(4) $x^a \in B_a$ *for all* $a \in G$*, and* $y^b \in B_b$ *for all* $b \in M$*.*
(5) *The subalgebra* $B^{\mathrm{co}\,\rho}$ *of* ρ*-coinvariants in* B *equals* kM*.*

Proof. (1) It is clear from the presentation in (E3.1.2) that the identity map on kG extends to an algebra map $\pi : B \to kG$ such that $\pi(y_i) = 0$ for all $i \in I$, and we observe that π is a Hopf algebra map.

(2, 3) These are standard consequences of (1).

(4) The first statement is clear since $\lambda(x^a) = x^a \otimes x^a$ for all $a \in G$.

Given $b \in M$, write $b = \sum_{i \in I} n_i/p_i$ for some $n_i \in \mathbb{Z}_{\geq 0}$, where at most finitely many n_i are non-zero. Then, $y^b = \prod_{i \in I} y_i^{n_i}$, whence

$$\lambda(y^b) = (\pi \otimes \mathrm{id}) \left(\prod_{i \in I} (y_i \otimes 1 + x_i \otimes y_i)^{n_i} \right) = \prod_{i \in I} (x_i^{n_i} \otimes y_i^{n_i}) = x^b \otimes y^b.$$

Thus, $y^b \in B_b$, as claimed.

(5) Since $\rho(y_i) = (\mathrm{id} \otimes \pi)(y_i \otimes 1 + x_i \otimes y_i) = y_i \otimes 1$ for $i \in I$, we see that each y_i is a ρ-coinvariant, and consequently $kM \subseteq B^{\mathrm{co}\,\rho}$.

Consider a non-zero element $z \in B$, and write $z = \sum_{l=1}^{m} x^{a_l} z_l$ for some distinct elements $a_l \in G$ and some non-zero elements $z_l \in kM$. Then,

$$\rho(z) = \sum_{l=1}^{m} x^{a_l} z_l \otimes x^{a_l}.$$

If z is a ρ-coinvariant, we must have $m = 1$ and $a_1 = 0$, whence $z = z_1 \in kM$. Therefore, $B^{\mathrm{co}\,\rho} = kM$. □

A skew primitive element of the Hopf algebra $B_G(\{p_i\}, \chi)$ is called *non-trivial* if it is not in kG. Recall the notation $x := x^1$ and $y := y^1$.

LEMMA 3.5. *Let* $B := B_G(\{p_i\}, \chi)$ *as in Section 3.1.*

(1) *The only grouplike elements of* B *are the* x^a *for* $a \in G$*.*
(2) *For* $a \in G$*, all* $(1, x^a)$*-skew primitive elements of* B *are in* $k(1 - x^a) + kM$*.*
(3) *Every non-trivial skew primitive element of* B *is of the form* $by_i + c(1 - x_i)$ *or* $by + c(1 - x)$ *for some scalars* $b, c \in k$*.*

Proof.

(1) Recall from Section 3.1 that B is a directed union of subalgebras of skew polynomial rings of the form $kM_c \# kG = kG[y^{1/c}; \sigma_c]$ (see the end of **Algebra structure**). In such a skew polynomial ring, the only units are the units of kG, so the only units in B are those in kG, that is, the elements αx^a for $\alpha \in k^\times$ and $a \in G$. Since grouplike elements are units, statement (1) follows.

(2) We first show that any non-zero $(1, x^a)$-skew primitive element $w \in kG$ must be a scalar multiple of $1 - x^a$. Write $w = \sum_{l=1}^{m} \alpha_l x^{a_l}$ for some $\alpha_l \in k^\times$ and some distinct $a_l \in G$. Then,

$$\sum_{l=1}^{m} \alpha_l x^{a_l} \otimes x^{a_l} = \Delta(w) = \sum_{l=1}^{m} \alpha_l \big(x^{a_l} \otimes 1 + x^a \otimes x^{a_l}\big). \tag{E3.5.1}$$

It follows that any non-zero a_l must equal a, whence $m \le 2$. If $m = 1$, then, after multiplying by α_1^{-1}, (E3.5.1) reduces to

$$x^{a_1} \otimes x^{a_1} = x^{a_1} \otimes 1 + x^a \otimes x^{a_1},$$

which is impossible. Thus, after a possible renumbering, we must have $a \neq 0$ and $w = \alpha_1 + \alpha_2 x^a$. In this case, (E3.5.1) says that

$$\alpha_1 \otimes 1 + \alpha_2 x^a \otimes x^a = \alpha_1(1 \otimes 1 + x^a \otimes 1) + \alpha_2(x^a \otimes 1 + x^a \otimes x^a).$$

It follows that $\alpha_1 + \alpha_2 = 0$ and so $w = \alpha_1(1 - x^a)$, as desired.
Now suppose that z is a $(1, x^a)$-skew primitive element of B, for some $a \in G$. Then, $\pi(z)$ must be a $(1, x^a)$-skew primitive element of kG, and so the claim above shows that $\pi(z) = \alpha(1 - x^a)$ for some $\alpha \in k$. Consequently, $z' := z - \alpha(1 - x^a)$ is a $(1, x^a)$-skew primitive element of B with $\pi(z') = 0$. Then,

$$\rho(z') = (\mathrm{id} \otimes \pi)(z' \otimes 1 + x^a \otimes z') = z' \otimes 1,$$

whence $z' \in B^{\mathrm{co}\,\rho}$. By Lemma 3.4(4), $z' \in kM$, and therefore $z \in k(1 - x^a) + kM$.

(3) By Proposition 3.3, we may assume that B is finitely generated and isomorphic to a Hopf algebra of type B as in [**6**, Construction 1.2]. By [**21**], these type B Hopf algebras form a special class of the $K(\{p_s\}, \{q_s\}, \{\alpha_s\}, M)$ defined in [**21**, Section 2]. By [**21**, Lemma 2.9(a)], any non-trivial skew primitive element f in B is a linear combination of $\{y_i\}_{i\in I}$ and y modulo kG. Write a $(1, g)$-skew primitive element f as $f = ay + \sum_{i\in I} a_i y_i + f_0$ where $f_0 \in kG$. By [**21**, Lemma 2.9(a)], $g = x_i$ or x. Since all x_i and x are distinct, we have that only one of $\{a_i\}_{i\in I} \cup \{a\}$ is non-zero. The assertion follows by combining this with part (2). □

The next proposition is similar to [**6**, Lemma 1.3].

Proposition 3.6. *Let $B := B_G(\{p_i\}, \chi)$ and $\widetilde{B} := B(\widetilde{G}, \{\widetilde{p}_i\}, \widetilde{\chi})$ as in Section 3.1. Then, $B \cong \widetilde{B}$ if and only if $G = \widetilde{G}$, $\{p_i \mid i \in I\} = \{\widetilde{p}_i \mid i \in \widetilde{I}\}$, and $\chi = \widetilde{\chi}$.*

Proof. Let $\phi : B \to \widetilde{B}$ be an isomorphism of Hopf algebras.
Label the canonical generators of B as above, namely as x^a for $a \in G$ and y_i for $i \in I$, and label those of $\widetilde{B}$ as $\widetilde{x}^a$ for $a \in \widetilde{G}$ and $\widetilde{y}_i$ for $i \in \widetilde{I}$. Write M and $\widetilde{M}$ for the additive submonoids of $\mathbb{Q}$ generated by $\{1/p_i \mid i \in I\}$ and $\{1/\widetilde{p}_i \mid i \in \widetilde{I}\}$, respectively.
The group of grouplike elements of B is isomorphic to G, and that of $\widetilde{B}$ to $\widetilde{G}$, so it follows from Lemma 3.5(1) that there is an isomorphism $\gamma : G \to \widetilde{G}$ such that $\phi(x^a) = \widetilde{x}^{\gamma(a)}$ for all $a \in G$. Since G and $\widetilde{G}$ are additive subgroups of $\mathbb{Q}$, γ is given by multiplication by some $r \in \mathbb{Q}^\times$. Thus, $\widetilde{G} = rG$ and $\phi(x^a) = \widetilde{x}^{ra}$ for all $a \in G$.

We next show that ϕ maps kM onto $k\widetilde{M}$. For $i \in I$, the element y_i is $(1, x^{1/p_i})$-skew primitive, and $x^{1/p_i} y_i = q_i y_i x^{1/p_i}$ where $q_i := \chi(1/p_i^2)$ is a primitive p_i-th root of unity. Then, $\phi(y_i)$ is a $(1, \widetilde{x}^{r/p_i})$-skew primitive element of $\widetilde{B}$ such that $\widetilde{x}^{r/p_i}\phi(y_i) = q_i\phi(y_i)\widetilde{x}^{r/p_i}$. By Lemma 3.5(2), $\phi(y_i) = \alpha(1 - \widetilde{x}^{r/p_i}) + z$ for some $\alpha \in k$ and $z \in k\widetilde{M}$. Hence,

$$q_i\alpha(1 - \widetilde{x}^{r/p_i}) + q_i z = q_i\phi(y_i) = \widetilde{x}^{r/p_i}\phi(y_i)\widetilde{x}^{-r/p_i} = \alpha(1 - \widetilde{x}^{r/p_i}) + \widetilde{x}^{r/p_i} z \widetilde{x}^{-r/p_i},$$

and so $(q_i - 1)\alpha(1 - \widetilde{x}^{r/p_i}) \in k\widetilde{M}$. Since $q_i \neq 1$ and $r \neq 0$, this forces $\alpha = 0$, whence $\phi(y_i) = z \in k\widetilde{M}$. Thus, $\phi(kM) \subseteq k\widetilde{M}$. By symmetry, $\phi^{-1}(k\widetilde{M}) \subseteq kM$, and therefore $\phi(kM) = k\widetilde{M}$.

By Lemma 3.4(1), the identity map on kG extends to a Hopf algebra map $\pi : B \to kG$ such that $\pi(y_i) = 0$ for all $i \in I$, and there is a corresponding Hopf algebra map $\widetilde{\pi} : \widetilde{B} \to k\widetilde{G}$. Since ϕ maps $kM \cap \ker\varepsilon$ to $k\widetilde{M} \cap \ker\varepsilon \subseteq \ker\widetilde{\pi}$, we conclude that $\phi|_{kG}\pi = \widetilde{\pi}\phi$. The lemma shows that the maps

$$\lambda := (\pi \otimes \mathrm{id})\Delta : B \to kG \otimes B \qquad \text{and} \qquad \widetilde{\lambda} := (\widetilde{\pi} \otimes \mathrm{id})\Delta : \widetilde{B} \to k\widetilde{G} \otimes \widetilde{B}$$

make B and $\widetilde{B}$ into left comodule algebras over kG and $k\widetilde{G}$, respectively, whence B is G-graded and $\widetilde{B}$ is $\widetilde{G}$-graded. Since $\phi|_{kG}\pi = \widetilde{\pi}\phi$, we see that $\big(\phi|_{kG} \otimes \phi\big)\lambda = \widetilde{\lambda}\phi$, and thus ϕ transports the grading on B to the grading on $\widetilde{B}$, namely, $\phi(B_a) = \widetilde{B}_{ra}$ for all $a \in G$.

We claim that $kM \cap B_a = 0$ for all $a \in G \setminus M$. Any non-zero element $v \in kM$ can be written $v = \sum_{l=1}^{n} \alpha_l y^{b_l}$ for some distinct $b_l \in M$ and some $\alpha_l \in k^\times$. In view of Lemma 3.4(4),

$$\lambda(v) = \sum_{l=1}^{n} \alpha_l (x^{b_l} \otimes y^{b_l}).$$

Hence, $\lambda(v) = x^a \otimes v$ for some $a \in G$ only if $n = 1$ and $b_1 = a$. This forces $a \in M$ and verifies the claim. Similarly, $k\widetilde{M} \cap \widetilde{B}_{a'} = 0$ for all $a' \in \widetilde{G} \setminus \widetilde{M}$. Since ϕ maps $kM \cap B_a$ isomorphically onto $k\widetilde{M} \cap \widetilde{B}_{ra}$ for all $a \in G$, and $rG = \widetilde{G}$, it follows that $rM = \widetilde{M}$. Note that this forces $r > 0$.

The atoms of the monoid M (i.e., the additively indecomposable elements) are exactly the $1/p_i$ for $i \in I$, as one sees from the pairwise relative primeness of the p_i. Similarly, the atoms of $\widetilde{M}$ are exactly the $1/\widetilde{p}_i$. Since we have an isomorphism $b \mapsto rb$ from M onto $\widetilde{M}$, it follows that $\{1/\widetilde{p}_i \mid i \in \widetilde{I}\} = \{r/p_i \mid i \in I\}$. Consequently, we may assume that $\widetilde{I} = I$ and $1/\widetilde{p}_i = r/p_i$ for all $i \in I$.

Finally, write $r = s/t$ for some relatively prime positive integers s, t, and reduce the final equation of the previous paragraph to $tp_i = s\widetilde{p}_i$. Thus, s divides tp_i for all $i \in I$. Since there are distinct indices $i, j \in I$, and p_i, p_j are relatively prime, it follows that $s \mid t$. By symmetry, $t \mid s$, whence $r = 1$. Therefore, $\widetilde{G} = G$ and $\widetilde{p}_i = p_i$ for all $i \in I$. Moreover, $\phi(kM \cap B_{1/p_i}) = k\widetilde{M} \cap \widetilde{B}_{1/p_i}$, from which we see that $\widetilde{y}_i$ is a non-zero scalar multiple of $\phi(y_i)$ (due to the fact that $k\widetilde{M} \cap \widetilde{B}_a$ is one-dimensional for all $a \in \widetilde{G}$). It thus follows from the relations $x^a y_i = \chi(a/p_i) y_i x^a$ that $\widetilde{x}^a\widetilde{y}_i = \chi(a/p_i)\widetilde{y}_i\widetilde{x}^a$, whence $\widetilde{\chi}(a/\widetilde{p}_i) = \chi(a/p_i)$ for all $a \in G$ and $i \in I$. Therefore, $\widetilde{\chi} = \chi$. $\square$

4. Initial analysis.

In this section, we will finish most of the analysis of the pointed case.

4.1. Classification by GK-dimension of the coradical. Throughout this subsection we assume that Hopf algebras are pointed. We will review some results from other papers.

Suppose H is a Hopf domain of GK-dimension two. Since H is pointed, the coradical $C_0(H)$ of H is a group algebra kG for the group $G = G(H)$ of all grouplikes in H. Hence, $\operatorname{GKdim} kG \le \operatorname{GKdim} H = 2$. Since $\operatorname{GKdim} kG$ is an integer, $\operatorname{GKdim} kG$ is either 0, or 1, or 2, see [**21**, Section 2.1]. We shall refer to $\operatorname{GKdim} kG$ as the *GK-dimension of* G, for short.

If $\operatorname{GKdim} kG = 0$, then $C_0(H) = k$ (as H is a domain). This means that H is connected. By [**21**, Theorem 1.9], H is isomorphic to $U(\mathfrak{g})$ for a two-dimensional Lie algebra $\mathfrak{g}$. This is part (1) of the following proposition.

For the statement of part (3) in the next proposition, recall that the localization $\mathbb{Z}_{(2)}$ is the subring of $\mathbb{Q}$ consisting of all rational numbers with odd denominators. There is a non-trivial group homomorphism $\varphi : \mathbb{Z}_{(2)} \to \operatorname{Aut} \mathbb{Q}$ such that $\ker \varphi = 2\mathbb{Z}_{(2)}$ and the remaining elements of $\mathbb{Z}_{(2)}$ are sent to the automorphism $(-1) \cdot (-)$. We shall also use φ to denote the corresponding homomorphism from a subgroup R of $\mathbb{Z}_{(2)}$ to the automorphism group of a subgroup L of $\mathbb{Q}$.

PROPOSITION 4.1. *Let H be a pointed Hopf domain of GK-dimension two.*

(1) *If* $\operatorname{GKdim} C_0(H) = 0$*, then* $H \cong U(\mathfrak{g})$ *for a two-dimensional Lie algebra* $\mathfrak{g}$.

(2) *If* $\operatorname{GKdim} C_0(H) = 2$ *and* $C_0(H)$ *is commutative, then* $H \cong kG$ *where* G *is a subgroup of* $\mathbb{Q}^2$ *containing* $\mathbb{Z}^2$.

(3) *If* $\operatorname{GKdim} C_0(H) = 2$ *and* $C_0(H)$ *is not commutative, then* $H \cong kG$ *where* $G = L \rtimes_\varphi R$ *for some subgroup* L *of* $\mathbb{Q}$ *containing* $\mathbb{Z}$ *and some subgroup* R *of* $\mathbb{Z}_{(2)}$ *containing* $\mathbb{Z}$.

Proof.

(1) This is [**21**, Theorem 1.9].

Let $C_0(H) = kG$. In both (2) and (3), we have $\operatorname{GKdim} kG = 2$. By [**21**, Lemma 1.6], $H = C_0(H) = kG$. Then,

$$G = \bigcup_{N \in \mathcal{N}} N, \tag{E4.1.1}$$

a directed union, where $\mathcal{N}$ is the set of all finitely generated subgroups of G of GK-dimension two.

(2) If $H = C_0(H)$ is commutative, meaning G is abelian, then every $N \in \mathcal{N}$ is isomorphic to $\mathbb{Z}^2$ by [**21**, Theorem 1.7]. If $M \subseteq N \subseteq G$, where M has GK-dimension two, then $M \cong N \cong \mathbb{Z}^2$ and N/M is finite. This is true for all such N, which implies that G/M is torsion. Therefore, G is isomorphic to a subgroup of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$. Conversely, every subgroup of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$ has GK-dimension two. The assertion follows.

(3) This is the case when G is non-abelian. By [**21**, Theorem 1.7], we may assume that every N in $\mathcal{N}$ is isomorphic to the non-trivial semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$. Then, N is generated by elements x_N and y_N satisfying $x_N y_N x_N^{-1} = y_N^{-1}$. It is clear that x_N and y_N have infinite order. Moreover, the following properties are easily checked:

(a) $Z(N) = \langle x_N^2 \rangle$.

(b) $Y_N := \langle y_N \rangle$ is a normal subgroup of N.

(c) $C_N := C_N(Y_N) = Y_N Z(N)$ and $[N : C_N] = 2$.

(d) For any $a \in N \setminus C_N$, we have $a^2 \in Z(N)$ and $a y_N a^{-1} = y_N^{-1}$. Moreover, $\langle a \rangle Z(N)$ is infinite cyclic.

A short calculation reveals that, for a fixed a in (d),

$$\{b \in N \mid aba^{-1} = b^{-1} \text{ for some } a \in N\} = Y_N$$

for any $N \in \mathcal{N}$. It follows that $Y_N \subseteq Y_M$ whenever $N \subseteq M$ in $\mathcal{N}$, and so

$$Y_G := \bigcup_{N \in \mathcal{N}} Y_N$$

is a normal subgroup of G. Since each Y_N is infinite cyclic, Y_G is isomorphic to a subgroup L of $\mathbb{Q}$ containing $\mathbb{Z}$.

If $N \subseteq M$ in $\mathcal{N}$, then $N \cap C_M \subseteq C_N$, because $Y_N \subseteq Y_M$. Since $[M : C_M] = 2$, it follows that $N \cap C_M = C_N$. In particular, $C_N \subseteq C_M$. Now

$$C_G := \bigcup_{N \in \mathcal{N}} C_N$$

is a normal subgroup of G containing Y_G. Pick some $N_0 \in \mathcal{N}$, and set

$$x_G := x_{N_0} .$$

For any $N \in \mathcal{N}$ containing N_0, the equation $N_0 \cap C_N = C_{N_0}$ implies $x_G \notin C_N$. Consequently, in view of (c), we have

(e) $x_G \notin C_G$ and $[G : C_G] = 2$.

When $N \subseteq M$ in $\mathcal{N}$, we have $x_N \notin C_M$ because $N \cap C_M = C_N$, and so it follows from (d) and (a) that $Z(N) \subseteq Z(M)$. Thus,

$$Z(G) = \bigcup_{N \in \mathcal{N}} Z(N).$$

If we now set

$$X_G := \langle x_G \rangle Z(G) = \bigcup_{N_0 \subseteq N \in \mathcal{N}} \langle x_G \rangle Z(N),$$

then (d) tells us that X_G is a directed union of infinite cyclic groups. Hence, X_G is isomorphic to a subgroup R of $\mathbb{Q}$ containing $\mathbb{Z}$, with $x_G \mapsto 1$. Moreover,

$$X_G Y_G = \langle x_G \rangle C_G = G,$$

because of (e). Note also that $Y_G Z(G) \subseteq C_G$, whence $X_G \cap Y_G \subseteq \langle x_G^2 \rangle Z(G) = Z(G)$, and so $X_G \cap Y_G \subseteq Y_G \cap Z(G) = 1$. Therefore

$$G \cong Y_G \rtimes X_G \cong L \rtimes_\alpha R$$

for some homomorphism $\alpha : R \to \operatorname{Aut} L$.

Because of (e) and (d), we have $x_G y x_G^{-1} = y^{-1}$ for all $y \in Y_G$, so $\alpha(1)$ must be the automorphism $\nu := (-1) \cdot (-)$ of L. Since the automorphisms of L are given by multiplication by certain elements of $\mathbb{Q}^\times$, the automorphism ν has no square root in $\operatorname{Aut} L$, whence $1 \notin 2R$. It follows that $a/b \notin R$ for any odd integer a and any non-zero even integer b, and therefore $R \subseteq \mathbb{Z}_{(2)}$. We similarly conclude that $\alpha(R) = \{\mathrm{id}_L, \nu\}$, whence $\alpha(R \cap 2\mathbb{Z}_{(2)}) = \{\mathrm{id}_L\}$ and $\alpha(R \setminus 2\mathbb{Z}_{(2)}) = \{\nu\}$. Therefore, $\alpha = \varphi$, completing the proof. □

The only case left is when $C_0(H) = kG$ and G has rank one. By Lemma 2.1(3), G is isomorphic to a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. Further analysis assuming this is given in the next subsection.

4.2. Analysis of skew primitives. In the first result of this subsection, we assume that H is a pointed Hopf domain of GK-dimension two and that $C_0(H) = kG$ where G is a subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. We identify elements $a \in G \subseteq \mathbb{Q}$ with elements labelled $x^a \in G(H)$. Since type A and type C Hopf algebras are easy to understand (Examples 2.2 and 2.3), we are focusing on algebras that are not types A and C. By Corollary 2.5, we may assume that H contains $A(1, 1)$ as a Hopf subalgebra. This means that H contains a grouplike element $x \in G$ and a skew primitive element $y \notin kG$ such that

$$xy = yx, \quad \text{and} \quad \Delta(y) = y \otimes 1 + x \otimes y. \tag{E4.1.2}$$

By replacing G with an isomorphic subgroup of $\mathbb{Q}$, we may assume that $x = x^1$.

Lemma 4.2. *If y is, up to a scalar, the only non-trivial skew primitive element* (*modulo* $kG(H)$) *in H, then H is either type A or type C.*

Proof. This follows from the proof of [**23**, Theorem 7.1].

Let K be the subalgebra of H generated by y and $C_0(H)$. It is clear that K is a Hopf subalgebra. Applying [**22**, Lemma 2.2(c)] to $V := ky + k(x - 1)$, there is an element $z \in V \setminus k(x - 1)$ such that either

(i) there is a character $\chi : G \to k^\times$ such that $h^{-1}zh = \chi(h)z$ for all $h \in G$, or
(ii) there is an additive character $\tau : G \to k$ such that $h^{-1}zh = z + \tau(h)(x - 1)$ for all $h \in G$.

In the first case K is a quotient of $A_G(x, \chi)$ and in the second case K is a quotient of $C_G(x, \tau)$. We claim that $K \cong A_G(x, \chi)$ in the first case and that $K \cong C_G(x, \tau)$ in the second case. We only prove the claim for the second case (the first case was given in the proof of [**23**, Theorem 7.1]). Consider the natural Hopf map $f : C_G(x, \tau) \to H$ which is injective on $C_0 + C_0z = C_1(H)$ by definition. By [**11**, Theorem 5.3.1], f is injective. Consequently, $K \cong C_G(x, \tau)$. By definition, K is generated by all the grouplikes and skew primitive elements of H. By [**23**, Corollary 6.9(2)], the primitive cohomological dimension PCdim K defined in [**23**, Definition 1.2] equals 1. Finally, by [**23**, Proposition 2.4(2)], $H = K$ as desired. □

Notation 4.3. We shall also need information about the skew primitive elements of the affine Hopf domains of types A, B, C from [**6**, Section 1]. To make the notation compatible with the present paper, we express these Hopf algebras as follows.

(1) $A(n, q)$, for $n \in \mathbb{Z}_{\geq 0}$ and $q \in k^\times$, is presented by generators $x^{\pm 1}, z$ with $xz = qzx$, where x is grouplike and z is $(1, x^n)$-skew primitive. We restrict to $n \geq 0$ because $A(m, q) \cong A(-m, q^{-1})$.
(2) $B(n, p_0, \dots, p_s, q)$, for $s \in \mathbb{Z}_{\geq 2}$, $n, p_0, \dots, p_s \in \mathbb{Z}_{>0}$, and $q \in k^\times$ satisfying certain conditions, is presented by generators $x^{\pm 1}, y_1, \dots, y_s$ with relations described in [**6**, Eq. (E1.2.1)], where x is grouplike and each y_i is $(1, x^{m_i n})$-skew

primitive. (Here $m_i = m/p_i$ with $m = p_1p_2\cdots p_s$.) The restriction $s \geq 2$ rules out the situation $B(n, p_0, p_1, q) \cong A(n, q)$.

(3) $C(n)$, for $n \in \mathbb{Z}_{\geq 2}$, is presented by generators $x^{\pm 1}$, z with the relation $zx = xz + (x^{2-n} - x)$, where x is grouplike and z is $(1, x^{1-n})$-skew primitive. We restrict to $n \geq 2$ because $C(1) \cong A(0, 1)$ and $C(m) \cong C(2 - m)$. In particular, this means $zx \neq xz$.

PROPOSITION 4.4.

(1) *Let $A = A(n, q)$ where q^n is either 1 or a non-root of unity. Then all non-trivial skew primitive elements in A have weight x^n, and they are linear combinations of z and $1 - x^n$.*

(2) *Let $A = A(n, q)$ where q^n is a primitive d-th root of unity for some $d > 1$. Then any non-trivial skew primitive element of A has weight either x^n or x^{dn}, and it is a linear combination of z and $1 - x^n$ or of z^d and $1 - x^{dn}$, respectively.*

(3) *Let $B = B(n, p_0, \dots, p_s, q)$, with m, m_i as above. Any non-trivial skew primitive element of B has weight either x^{mn} or x^{m_in} for some $i = 1, \dots, s$, and it is a linear combination of $y_1^{p_1}$ and $1 - x^{mn}$ or of y_i and $1 - x^{m_in}$, respectively.*

(4) *Let $C = C(n)$. All non-trivial skew primitive elements in C have weight x^{1-n}, and they are linear combinations of z and $1 - x^{1-n}$.*

Proof. In each case, the elements of the stated forms are skew primitive with the given weights, as proved in [**6**, Constructions 1.1, 1.2, 1.4].

(1,2) Suppose $f \in A$ is a non-trivial skew primitive with weight x^m, for some $m \in \mathbb{Z}$. Write $f = \sum_{i=0}^{d} c_iz^i$ for some $c_i \in k[x^{\pm 1}]$ with $c_d \neq 0$. Then,

$$\sum_{i=0}^{d} \Delta(c_i)(z \otimes 1 + x^n \otimes z)^i = \Delta(f) = \sum_{i=0}^{d} c_iz^i \otimes 1 + \sum_{i=0}^{d} x^m \otimes c_iz^i. \tag{E4.4.1}$$

Comparing terms from $A \otimes k[x^{\pm 1}]$ in this equation, we see that

$$\sum_{i=0}^{d} \Delta(c_i)(z^i \otimes 1) = x^m \otimes c_0 + \sum_{i=0}^{d} c_iz^i \otimes 1.$$

A comparison of terms from $k[x^{\pm 1}]z^i \otimes k[x^{\pm 1}]$ then yields $\Delta(c_i) = c_i \otimes 1$ for $i > 0$ and $\Delta(c_0) = x^m \otimes c_0 + c_0 \otimes 1$. It follows that $c_0 \in k(1 - x^m)$ and $c_i \in k$ for $i > 0$. The non-triviality of f forces $d > 0$.

If $d = 1$, (E4.4.1) implies that $c_1x^n \otimes z = x^m \otimes c_1z$, whence $m = n$. In this case, f is a linear combination of z and $1 - x^n$, and we are done. Assume now that $d > 1$.

Since $c_0 \in k(1 - x^m)$ and $c_1, \dots, c_d$ are scalars, (E4.4.1) reduces to

$$\sum_{i=1}^{d} c_i(z \otimes 1 + x^n \otimes z)^i = \sum_{i=1}^{d} c_iz^i \otimes 1 + \sum_{i=1}^{d} c_ix^m \otimes z^i.$$

Comparing terms from $A \otimes kz^i$ yields

$$\sum_{i=j}^{d} \binom{i}{j}_{q^n} c_iz^{i-j}x^{jn} = c_jx^m, \qquad \text{for } 1 \leq j \leq d. \tag{E4.4.2}$$

From the case $j = d$, we get $m = dn$.

For $1 \leq j < d$, (E4.4.2) implies that $\binom{d}{j}_{q^n} = 0$. Thus, by [**6**, Lemma 7.5], q^n must be a primitive d-th root of unity. As in [**6**, Construction 1.1], it follows that z^d is skew primitive with weight x^m, hence so is $f - c_d z^d$. If $f - c_d z^d$ is non-zero, let e be its z-degree. Applying the above analysis to $f - c_d z^d$, we find that $e = 0$ or $m = en$, the latter case being impossible. Therefore $f - c_d z^d$ is a scalar multiple of $1 - x^{dn}$, and f has the required form.

(3) This follows from Lemma 3.5.

(4) Suppose $f \in C$ is a non-trivial skew primitive with weight x^m, for some $m \in \mathbb{Z}$, and write $f = \sum_{i=0}^{d} c_i z^i$ for some $c_i \in k[x^{\pm 1}]$ with $c_d \neq 0$. As in cases (1)(2), we get equation (E4.4.1), but with x^n replaced by x^{1-n}. Moreover, it follows that $c_0 \in k(1 - x^m)$ and $c_i \in k$ for $i > 0$, and then that $d > 0$. In case $d = 1$, we obtain $m = 1 - n$ and f is a linear combination of z and $1 - x^{1-n}$.

Now suppose that $d > 1$. After canceling common terms, (E4.4.1) reduces to

$$\sum_{i=1}^{d} c_i (z \otimes 1 + x^{1-n} \otimes z)^i = \sum_{i=1}^{d} c_i z^i \otimes 1 + \sum_{i=1}^{d} c_i x^m \otimes z^i.$$

Comparing terms in $C \otimes k[x^{\pm 1}]z^d$ in this equation, we find that $m = d(1 - n)$. Turning to $C \otimes k[x^{\pm 1}]z^{d-1}$, we obtain

$$c_{d-1} x^{(d-1)(1-n)} + d c_d x^{(d-1)(1-n)} z + c_d h = c_{d-1} x^{1-n}$$

for some $h \in k[x^{\pm 1}]$. Since $d c_d \neq 0$, this is impossible, and the proof is complete. □

Corollary 4.5. *Suppose H is one of the affine Hopf domains of types A, B, C. For any grouplike g in H, the space of skew primitive elements in H with weight g has k-dimension at most* 2.

Corollary 4.6. *Let $H_1 \subsetneq H_2$ be affine Hopf domains of types A, B, C.*
(1) *If H_2 is of type A, so is H_1.*
(2) *If H_1 is of type B, so is H_2.*
(3) *H_1 is of type C if and only if H_2 is of type C.*

Proof. We take account of the behaviour of the non-trivial skew primitives described in Proposition 4.4. In all three types, non-trivial skew primitives exist. In types A and B, each skew primitive quasi-commutes with its weight, whereas in type C, no non-trivial skew primitive quasi-commutes with its weight. Statement (3) follows.

In type A, at most two grouplikes are weights of non-trivial skew primitives, while in type B, at least three grouplikes are weights of skew primitives. Statements (1) and (2) now follow. □

4.3. Locally affine Hopf algebras. We recall the definition of the local affine property.

Definition 4.7. Let H be a Hopf algebra.
(1) An element $f \in H$ is called *locally affine* if it is contained in a Hopf subalgebra that is affine.
(2) Let V be a subset of H. We say that V is *locally affine* if every element in V is locally affine.

LEMMA 4.8. *Let H be a Hopf algebra.*

(1) *Every finite set of locally affine elements of H is contained in an affine Hopf subalgebra of H.*
(2) *The locally affine elements in H form a Hopf subalgebra of H, and this Hopf subalgebra is a directed union of affine Hopf subalgebras.*
(3) *If a subset $V \subseteq H$ is locally affine, and if H is generated by V as an algebra, then H is locally affine.*
(4) *[**25**, Corollary 3.4] If H is pointed, then it is locally affine.*
(5) *Let V be a subset of H such that $\sum_{i\geq 0} kS^i(v)$ is finite dimensional for each $v \in V$. If H is the k-span of V, then H is locally affine.*
(6) *If S has finite order, then H is locally affine. As a consequence, if H is commutative or cocommutative, then it is locally affine.*

Proof.

(1) This follows from the observation that if $\Omega_1, \ldots, \Omega_n$ are affine Hopf subalgebras of H, then the subalgebra of H generated by $\bigcup_{i=1}^n \Omega_i$ is an affine Hopf subalgebra.
(2) If $f, g \in H$ are locally affine, then in view of part (1), $f \pm g$ and fg are locally affine. Of course, the identity $1 \in H$ is locally affine, because $1 \in k$. Therefore the set L of locally affine elements of H is a subalgebra of H. Any finite subset of L is contained in an affine Hopf subalgebra Ω of H by (1), and $\Omega \subseteq L$ by definition of L. Hence, L is a directed union of affine Hopf subalgebras of H. In particular, L is a Hopf subalgebra.
(3) This is clear from part (2).
(4) This was proved by Zhuang [**25**], and we give a different proof below.
We show, by induction, that $f \in C_n(H)$ is locally affine where $\{C_n(H)\}_{n\geq 0}$ is the coradical filtration of H. Suppose $n = 0$. Since H is pointed, $C_0(H) = kG$ for a group G. It is clear that kG is locally affine. Now suppose that $C_{n-1}(H)$ is locally affine and let $f \in C_n(H)$ for some $n \geq 1$. By [**11**, Theorem 5.4.1], $f = \sum_{g,h\in G(H)} f_{g,h}$ where $\Delta(f_{g,h}) = f_{g,h} \otimes g + h \otimes f_{g,h} + w_{g,h}$ for some $w_{g,h} \in C_{n-1} \otimes C_{n-1}$. It suffices to show that each $f_{g,h}$ is locally affine, so assume that $f = f_{g,h}$. It is clear that f is locally affine if and only if xf is locally affine for some (or any) grouplike element x. By replacing f by xf for some grouplike x, we can assume that $\Delta(f) = f \otimes 1 + g \otimes f + w$, where g is grouplike and $w \in C_{n-1} \otimes C_{n-1}$. By the antipode axiom, $\epsilon(f) = S(f) + g^{-1}f + w_0$, where $w_0 \in C_{n-1}^2$, or $S(f) = -g^{-1}f + v$ for some $v \in C_{n-1}^2$. By part (2) and the induction hypothesis, all tensor components of w are contained in an affine Hopf subalgebra of H, say Ω. In particular, $v \in \Omega$. Let B be the subalgebra of H generated by $g^{\pm 1}$, f and Ω. Then, B is an affine Hopf subalgebra of H. Since $f \in B$, f is locally affine. The assertion follows by induction.
(5) Let $f \in H$ and let W be a finite dimensional subcoalgebra of H containing f. By hypothesis, $X := \sum_{i\geq 0} S^i(W)$ is finite dimensional. Then f is contained in the affine subalgebra $k\langle X\rangle$ which is a Hopf subalgebra as X is a subcoalgebra with $S(X) \subseteq X$.
(6) This is a consequence of part (5). (Recall from [**11**, Corollary 1.5.12] that $S^2 = \mathrm{Id}$ if H is commutative or cocommutative.)

□

PROPOSITION 4.9. *Let H be a Hopf algebra and K a locally affine Hopf subalgebra of H. If H is generated by K and $x \in H$ as an algebra, then H is locally affine.*

Proof. Let $f \in H$. Since H is generated by K and x, there is a finite dimensional subspace $V \subseteq K$ such that

(a) $f, S(x) \in k\langle x, V\rangle =: A$, and

(b) $\Delta(x) \in A \otimes A \subseteq H \otimes H$.

Since $V \subseteq K$ is finite dimensional, there is an affine Hopf subalgebra $K_0 \subseteq K$ that contains V. Let H_0 be the subalgebra of H generated by K_0 and x. By definition, $A \subseteq H_0$ and H_0 is affine. Moreover, $f \in H_0$. By (a) and (b), $S(x) \in H_0$ and $\Delta(x) \in H_0 \otimes H_0$. Since K_0 is a Hopf subalgebra of K, it is easy to see that H_0 is a Hopf subalgebra of H. Therefore, H is locally affine. $\square$

Given a Hopf algebra H, note that $\operatorname{Ext}^1_H(k,k) \cong (\mathfrak{m}/\mathfrak{m}^2)^*$ where $\mathfrak{m} = \ker \epsilon$ [**6**, Lemma 3.1(a)]. So, $\operatorname{Ext}^1_H(k,k) \neq 0$ if and only if $\mathfrak{m} \neq \mathfrak{m}^2$.

PROPOSITION 4.10. *Let H be a Hopf algebra domain of GK-dimension two satisfying ($\natural$). If H is locally affine, then it is pointed.*

Proof. Let V be a simple subcoalgebra of H, and let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. By Lemma 4.8(1), there is an affine Hopf subalgebra $K \subseteq H$ that contains V and f. Since $f \in \mathfrak{m}_K \setminus \mathfrak{m}_K^2$ where $\mathfrak{m}_K := \ker \epsilon_K$, K satisfies ($\natural$). By [**6**, Theorem 0.1], K is pointed, whence $\dim_k V = 1$. Therefore, H is pointed. $\square$

We finish this section with a well-known lemma.

LEMMA 4.11. *Let H be a Hopf algebra with countable dimensional $C_1(H)$. If H is generated* (*as an algebra*) *by $C_1(H)$, then H is a union of an ascending chain of affine Hopf subalgebras, each of which is finitely generated by its grouplikes and skew primitives.*

Proof. Let $G = \{g_i\}_{i\in I}$ be the group of grouplikes in H and $C = \{y_j\}_{j\in J}$ a set of non-trivial skew primitive elements with weights in G. Then $\bigcup_{i\in I} g_i C$ spans the space $C_1(H)$. Since $\dim_k C_1(H)$ is countable, so are I and J. We list elements in G and C

$$G = \{g_1, \ldots, g_n, \ldots\} \qquad \text{and} \qquad C = \{y_1, \ldots, y_n, \ldots\}.$$

Let $B\langle n\rangle$ be the Hopf subalgebra of H generated by $y_1, \ldots, y_n$ and a finite set of grouplike elements containing $g_1^{\pm 1}, \ldots, g_n^{\pm 1}$ and all $x^{\pm 1}$ where x appears in the expression $\Delta(y_i)$ for some $i = 1, \ldots, n$. Then $B\langle n\rangle$ is an affine Hopf subalgebra of H and $H = \bigcup_n B\langle n\rangle$. We may choose the $B\langle n\rangle$ so that $B\langle n\rangle \subseteq B\langle n+1\rangle$ for all n. $\square$

5. Classification results.

In this section, we prove a couple of classification theorems for Hopf domains of GK-dimension two.

5.1. Classification in the pointed case with ($\natural$).

We start with a classification of pointed Hopf domains of GK-dimension two satisfying ($\natural$).

LEMMA 5.1. *Let H be a locally affine Hopf domain of GK-dimension two satisfying ($\natural$), and assume that* $\operatorname{GKdim} G(H) = 1$. *Then,* $\dim_k H = \aleph_0$.

Proof. Obviously H is infinite dimensional.

By Lemma 4.8(2), H is a directed union of affine Hopf subalgebras K_α, and we may assume that all of them have GK-dimension two. We may also assume that $G(K_\alpha)$

is non-trivial, whence $\operatorname{GKdim} G(K_\alpha) = 1$. Because of ($\natural$), there is some $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, where $\mathfrak{m} = \ker \epsilon$, and we may assume that $f \in K_\alpha$ for all α. Then, $f \in \mathfrak{m}_{K_\alpha} \setminus \mathfrak{m}^2_{K_\alpha}$, where $\mathfrak{m}_K = \ker \epsilon_K$, so that K_α satisfies ($\natural$). Now by [**6**, Theorem 0.1], each K_α is of type A, B, or C. In particular, K_α is generated by its grouplikes and skew primitives, so the same holds for H.

In view of Corollary 4.5, we see that, for any grouplike $g \in H$, the space of $(1, g)$-skew primitive elements in H is at most two-dimensional. Since H has only countably many grouplikes (Lemma 2.1), there is a countable dimensional subspace V of H that contains $G(H)$ and all skew primitives. Therefore $H = k\langle V\rangle$ has countable dimension. □

THEOREM 5.2. *Let H be a pointed Hopf domain of GK-dimension two satisfying* ($\natural$). *Suppose H is not affine* (*or equivalently, not noetherian*). *Then, it is isomorphic to one of the following.*

(1) *kG where G is a subgroup of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$ that is not finitely generated.*
(2) *kG where $G = L \rtimes_\varphi R$ for some subgroup L of $\mathbb{Q}$ containing $\mathbb{Z}$ and some subgroup R of $\mathbb{Z}_{(2)}$ containing $\mathbb{Z}$, and at least one of L or R is not finitely generated.*
(3) *$A_G(e, \chi)$ where G is a non-cyclic subgroup of $\mathbb{Q}$.*
(4) *$C_G(e, \tau)$ where G is a non-cyclic subgroup of $\mathbb{Q}$.*
(5) *$B_G(\{p_i\}, \chi)$ where G is a non-cyclic subgroup of $\mathbb{Q}$.*

Proof. The cases when $kG(H)$ has GK-dimension zero or two are done by Proposition 4.1. Now assume $\operatorname{GKdim} G(H) = 1$. We also assume that H is not isomorphic to any Hopf algebra in parts (1)–(4), and we will prove that it is isomorphic to one of those in part (5). By Corollary 2.5, it remains to consider the case when H contains a Hopf subalgebra isomorphic to $A(1, 1)$.

In view of Lemma 2.1, $G(H)$ is isomorphic to a non-zero subgroup G of $\mathbb{Q}$. Write $G(H)$ in the form $\{x^a \mid a \in G\}$ as in Notation 1.1. Since H contains a copy of $A(1, 1)$, there are a grouplike x and a non-trivial skew primitive y in H such that

$$xy = yx, \quad \text{and} \quad \Delta(y) = y \otimes 1 + x \otimes y.$$

After replacing G by an isomorphic subgroup of $\mathbb{Q}$ if necessary, we may assume that $1 \in G$ and $x = x^1$.

Since H is pointed, it is locally affine by Lemma 4.8(4). Thus, by Lemma 5.1 and its proof, H is the union of an increasing sequence of affine Hopf subalgebras

$$K_1 \subsetneqq K_2 \subsetneqq \dots,$$

each being one of type A, B or C from Notation 4.3. Since the Hopf subalgebra $k\langle x^{\pm 1}, y\rangle \cong A(1, 1)$ is contained in some K_j, Corollary 4.6 implies that none of the K_i is of type C. From the same corollary, we find that either all the K_i are of type A or all but finitely many K_i are of type B. Since we may delete any K_i that does not properly contain $k\langle x^{\pm 1}, y\rangle$, there is no loss of generality in assuming that

$$k\langle x^{\pm 1}, y\rangle \subsetneqq K_1 .$$

From Proposition 4.4 and the details of [**6**, Constructions 1.1, 1.2, 1.4], we see that in each K_i, there is at most one grouplike g_i which is the weight of a non-trivial skew primitive that commutes with g_i. Consequently, taking also Corollary 4.5 into account the following.

(i) the unique grouplike $g \in G(H)$ which is the weight of a non-trivial skew primitive that commutes with g is $g = x$. The space of $(1, x)$-skew primitives in H is $ky + k(1 - x)$.

Suppose first that all the K_i are of type A. If $K_i \cong A(n_i, q_i)$ with $q_i = 1$ or q_i not a root of unity, it follows from Proposition 4.4(1) that y is, up to a scalar, the only non-trivial skew primitive element modulo $kG(K_i)$ in K_i. As a consequence, y is the only non-trivial skew primitive element modulo $kG(H)$ in H. By Lemma 4.2, H is either type A or type C. This yields a contradiction.

Thus, after deleting some of the K_i, we may assume that $K_1 \cong A(n_1, q_1)$ where q_1 is a primitive d_1-th root of unity for some $d_1 > 1$. By Proposition 4.4(2), $K_1 = k\langle x_1^{\pm 1}, z_1\rangle$ for some grouplike x_1 and some non-trivial skew primitive z_1 with weight $x_1^{n_1}$ such that $x_1^{d_1 n_1} = x$ and $z_1^{d_1} \in ky + k(1 - x)$. Since K_1 then contains two non-trivial skew primitives with different weights, so do all the K_i, and another application of the proposition yields $K_i \cong A(n_i, q_i)$ where q_i is a primitive d_i-th root of unity for some $d_i > 1$. Moreover, $K_i = k\langle x_i^{\pm 1}, z_i\rangle$ for some grouplike x_i and some non-trivial skew primitive z_i with weight $x_i^{n_i}$ such that $x_i^{d_i n_i} = x$ and $z_i^{d_i} \in ky + k(1 - x)$. Further, $x_{i+1}^{n_{i+1}} = x_i^{n_i}$ and $z_{i+1} \in kz_i + k(1 - x_i^{n_i})$, from which we see that $K_{i+1} = k\langle x_{i+1}^{\pm 1}, z_i\rangle$.

At this point, H is generated by $G(H) \cup \{z_1\}$. Let χ be the character of $G(H)$ determined by

$$g^{-1} z_1 g = \chi(g) z_1 + \tau(g)(1 - x_1^{n_1}), \qquad \forall\, g \in G := G(H).$$

Since χ is non-trivial, one can choose $\tau(g) = 0$ for all g by [**22**, Lemma 2.2(c)]. Then, there is a surjective Hopf algebra map $\phi : A_G(x_1^{n_1}, \chi) \to H$. But $A_G(x_1^{n_1}, \chi)$ is a domain of GK-dimension two, so ϕ is an isomorphism, contradicting one of our assumptions.

Therefore, all but finitely many K_i are of type B. After deleting the exceptions, we may assume that all K_i are of type B.

Each K_i is now generated by grouplikes $x_i^{\pm 1}$ and finitely many non-trivial skew primitives, say y_{ij} for $j \in J_i$. From the details of [**6**, Construction 1.2], we have positive integers n_i, p_{0i}, and p_{ij} for $j \in J_i$ and some $q_i \in k^\times$ such that for all $j, l \in J_i$,

(ii) $|J_i| \geq 2$ and $p_{ij} \geq 2$.

(iii) $p_{0i} \mid n_i$ and p_{0i} together with the p_{ij} are pairwise relatively prime.

(iv) q_i is a primitive l_i-th root of unity, where $l_i = m_i n_i / p_{0i}$ and $m_i = \prod_{j \in J_i} p_{ij}$, and $q_i^{m_{ij}^2 n_i}$ is a primitive p_{ij}-th root of unity.

(v) $x_i y_{ij} x_i^{-1} = q_i^{m_{ij}} y_{ij}$, where $m_{ij} = m_i / p_{ij}$.

(vi) $y_{ij} y_{il} = y_{il} y_{ij}$ and $y_{ij}^{p_{ij}} = y_{il}^{p_{il}}$.

(vii) y_{ij} has weight $x_i^{m_{ij} n_i}$, and these elements do not commute.

(viii) $y_{ij}^{p_{ij}}$ is a non-trivial skew primitive element with weight $x_i^{m_i n_i}$, and these elements commute.

In view of (i), it follows that $x_i^{m_i n_i} = x$ and $y_{ij}^{p_{ij}} \in ky + k(1 - x)$. In particular,

$$1/m_i n_i \in G \qquad \text{and} \qquad x_i = x^{1/m_i n_i}.$$

Set $G_i := \mathbb{Z}(1/m_i n_i)$, so that $G(K_i) = \{x^c \mid c \in G_i\}$.

Define a character χ_i on the group $G_i(1/m_i) = \mathbb{Z}(1/m_i^2 n_i)$ so that $\chi_i(1/m_i^2 n_i) = q_i$, and observe that

$$x^{c/m_i n_i} y_{ij} x^{-c/m_i n_i} = q_i^{c m_{ij}} y_{ij} = \chi_i(c/m_i n_i p_{ij}) y_{ij} \qquad \forall\, c \in \mathbb{Z},\ j \in J_i\,.$$

Temporarily set $t := i+1$, and consider $j \in J_i$. Since y_{ij} is a non-trivial skew primitive element of K_t with weight $x_i^{m_{ij}n_i}$ and y_{ij} does not commute with its weight, Proposition 4.4(3) implies that there is some $s \in J_t$ such that $x_i^{m_{ij}n_i} = x_t^{m_{ts}n_t}$ and $y_{ij} \in ky_{ts} + k(1 - x_t^{m_{ts}n_t})$. Now, $x_i^{m_{ij}n_i}$ quasi-commutes with y_{ij} and y_{ts} but does not commute with these elements, whereas it does commute with $1 - x_t^{m_{ts}n_t}$. Since $y_{ts} \notin kG(K_t)$, it follows that $y_{ij} \in ky_{ts}$. After rearranging indices, we may thus assume that $J_i \subseteq J_t$ and

$$x_i^{m_{ij}n_i} = x_t^{m_{tj}n_t} \qquad \text{and} \qquad y_{ij} = \alpha_j y_{tj} \ \text{ with } \ \alpha_j \in k^\times, \qquad \forall\, i \in J_i\,.$$

Note that $x^{1/p_{ij}} = x_i^{m_{ij}n_i} = x_t^{m_{tj}n_t} = x^{1/p_{tj}}$ implies $p_{ij} = p_{tj}$. For $j, l \in J_i$, we have

$$(\alpha_j y_{tj})^{p_{tj}} = y_{ij}^{p_{ij}} = y_{il}^{p_{il}} = (\alpha_l y_{tl})^{p_{tl}},$$

and for $r \in J_t \setminus J_i$ we may choose $\alpha_r \in k^\times$ such that $\alpha_r^{p_{tr}} = \alpha_j^{p_{tj}}$, so that $(\alpha_r y_{tr})^{p_{tr}} = (\alpha_j y_{tj})^{p_{tj}}$. Hence, we may replace all the generators y_{tu} of K_t by the elements $\alpha_u y_{tu}$. This means there is no loss of generality in assuming that

$$x_i^{m_{ij}n_i} = x_{i+1}^{m_{i+1,j}n_{i+1}}, \qquad p_{ij} = p_{i+1,j}\,, \qquad \text{and} \qquad y_{ij} = y_{i+1,j} \qquad \forall\, j \in J_i\,.$$

If d_i denotes the product of the $p_{i+1,r}$ for $r \in J_t \setminus J_i$ (where an empty product equals 1), then $m_{i+1} = d_i m_i$ and $m_{i+1,j} = d_i m_{ij}$ for $j \in J_i$. Since $x_i \in G(K_{i+1}) = \langle x_{i+1} \rangle$, we have $x_i = x_{i+1}^{e_i}$ for some non-zero integer e_i, whence $m_{i+1}n_{i+1} = e_i m_i n_i$. For any $j \in J_i$,

$$q_i^{m_{ij}} y_{ij} = x_i y_{ij} x_i^{-1} = x_{i+1}^{e_i} y_{i+1,j} x_{i+1}^{-e_i} = q_{i+1}^{e_i m_{i+1,j}} y_{i+1,j} = q_{i+1}^{d_i e_i m_{ij}} y_{ij},$$

whence $q_i^{m_{ij}} = q_{i+1}^{d_i e_i m_{ij}}$. Since the GCD of $\{m_{ij} \mid j \in J_i\}$ is 1, it follows that $q_i = q_{i+1}^{d_i e_i}$. Consequently,

$$\chi_{i+1}(1/m_i^2 n_i) = \chi_{i+1}(d_i e_i / m_{i+1}^2 n_{i+1}) = q_{i+1}^{d_i e_i} = q_i\,,$$

and therefore χ_{i+1} restricted to $G_i(1/m_i)$ equals χ_i.

Now $G = \bigcup_{i=1}^\infty G_i = \bigcup_{i=1}^\infty \mathbb{Z}(1/m_i n_i)$, and

$$M := \sum_{i=1}^{\infty} \sum_{j \in J_i} \mathbb{Z}_{\geq 0}(1/p_{ij}) = \bigcup_{i=1}^{\infty} \mathbb{Z}_{\geq 0}(1/m_i),$$

whence

$$GM = \sum_{i=1}^{\infty} \sum_{j \in J_i} G(1/p_{ij}) = \bigcup_{i=1}^{\infty} \mathbb{Z}_{\geq 0}(1/m_i^2 n_i) = \bigcup_{i=1}^{\infty} G_i(1/m_i).$$

Consequently, there is a well-defined character χ on GM which restricts to χ_i on $G_i(1/m_i)$ for all i. The set

$$P := \{p_{ij} \mid i \in \mathbb{Z}_{\geq 0},\ j \in J_i\}$$

is a set of pairwise relatively prime integers ≥ 2, and each $1/p_{ij} = m_{ij}n_i/m_i n_i \in G$.

Moreover, the scalar

$$\chi(1/p_{ij}^2) = \chi_i(m_{ij}^2 n_i / m_i^2 n_i) = q_i^{m_{ij}^2 n_i}$$

is a primitive p_{ij}-th root of unity.

Our data now satisfy all the conditions required to define the Hopf algebra $B_G(P, \chi)$ as in Section 3.1, and there is a surjective Hopf algebra map $\pi : B_G(P, \chi) \to H$ sending the generators $x^{\pm a}$, y_{ij} of $B_G(P, \chi)$ to the elements with the same names in H. Since both $B_G(P, \chi)$ and H are domains of GK-dimension two, π is an isomorphism.

By construction, H is generated by $\{x^a \mid a \in G\} \cup \{y_{ij} \mid i \in \mathbb{Z}_{>0},\ j \in J_i\}$. If G were cyclic, the non-affine hypothesis on H would imply that there are infinitely many y_{ij}, whence P would be infinite. However, P consists of pairwise relatively prime integers $p \geq 2$ with $1/p \in G$. This is not possible with G cyclic. Therefore, G is non-cyclic and H is isomorphic to a Hopf algebra in part (5). □

5.2. Removing the "pointed" hypothesis. Our next goal is to prove Theorem 5.2 without assuming that H is pointed. By Proposition 4.10, it suffices to show H is locally affine.

LEMMA 5.3. *Let H be a Hopf domain of GK-dimension two satisfying ($\natural$). If H is not commutative, then there is a quotient Hopf algebra $K := H/I$ that is a commutative domain of GK-dimension one. Furthermore, K is one of the Hopf algebras listed in Lemma 2.1.*

Proof. Let $\mathfrak{m} = \ker \epsilon$ and let $I = \bigcap_{i \geq 1} \mathfrak{m}^i$. By [**9**, Lemma 4.7], I is a Hopf ideal.

Let $e(H)$ be the dimension of $\mathrm{Ext}^1_H(k, k)$ and let $\mathrm{gr}\, H = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. By [**6**, Proposition 3.4(b)], $e(H) \leq \mathrm{GKdim}\, H = 2$. Hence, $e(H) = 1$ or 2. By [**6**, Proposition 3.4(a)], $\mathrm{gr}\, H \cong U(\mathfrak{g})$, where $\mathfrak{g}$ is a graded Lie algebra generated in degree 1 and $\dim_k \mathfrak{g}_1 = e(H)$. If $e(H) = 1$, then $\mathrm{gr}\, H = k[x]$, which is commutative. If $e(H) = 2$, then $\mathfrak{g}_1 = \mathfrak{g}$ by [**6**, Proposition 3.4(b)]. So, $\mathfrak{g}$ is abelian and again $\mathrm{gr}\, H \cong U(\mathfrak{g})$ is commutative.

By [**6**, Lemma 3.5], H/I is commutative. Since H is a domain of GK-dimension two and H is not commutative, $I \neq 0$ and H/I has GK-dimension at most one. On the other hand, $\mathrm{Ext}^1_H(k, k) \neq 0$ implies $\mathfrak{m} > \mathfrak{m}^2$, and consequently $\mathrm{Ext}^1_{H/I}(k, k) \neq 0$. Another application of [**6**, Proposition 3.4(b)] yields $\mathrm{GKdim}\, H/I \geq e(H/I) > 0$, and thus $\mathrm{GKdim}\, H/I = 1$. Moreover, $\mathrm{gr}(H/I)$ is an enveloping algebra and thus a domain. Therefore H/I is a commutative domain of GK-dimension one. The assertion follows. □

THEOREM 5.4. *Let H be a Hopf domain of GK-dimension two satisfying ($\natural$). Then, H is locally affine.*

Proof. If H is commutative, the assertion follows by Lemma 4.8(6). From now on, assume that H is not commutative. By Lemma 5.3, there is a Hopf ideal I such that $K := H/I$ is a Hopf domain of GK-dimension one. By Lemma 2.1, we are in one of the following two cases:

Case 1: $K = kG$ where G is a non-zero subgroup of $\mathbb{Q}$.

Case 2: $K = k[t]$ is a polynomial ring, with $\Delta(t) = t \otimes 1 + 1 \otimes t$.

The following analysis is similar to the one in [**6**]. In fact, the ideas and arguments are copied from [**6**]. Let $\pi : H \to K$ be the quotient map, set

$$\rho := (\mathrm{id} \otimes \pi)\Delta : H \to H \otimes K \qquad \text{and} \qquad \lambda := (\pi \otimes \mathrm{id})\Delta : H \to K \otimes H,$$

and note that H becomes a right (respectively, left) comodule algebra over K via ρ (respectively, λ), see [**6**, Section 4.1].

Case 1. Write $K = kG$ in the form $\bigoplus_{a\in G} kx^a$. For $a \in G$, let

$$H_a := \{h \in H \mid \rho(h) = h \otimes x^a\} \qquad \text{and} \qquad {}_aH := \{h \in H \mid \lambda(h) = x^a \otimes h\}.$$

Then, H is a G-graded algebra in two ways:

$$H = \bigoplus_{a\in G} H_a = \bigoplus_{a\in G} {}_aH,$$

where the first decomposition is called the ρ-grading and the second is called the λ-grading. Let π_a^r and π_a^l be the respective projections from H onto H_a and ${}_aH$ in the above decompositions. Then, by G-graded versions of [**6**, (E5.0.1) and (E5.0.2)], we have

$$\pi_a^r \pi_b^l = \pi_b^l \pi_a^r, \quad \forall\, a, b \in G \tag{E5.4.1}$$

and, writing ${}_aH_b = {}_aH \cap H_b$ for all $a, b \in G$, we have

$$H_b = \bigoplus_{a\in G} {}_aH_b, \quad \text{and} \quad {}_aH = \bigoplus_{b\in G} {}_aH_b. \tag{E5.4.2}$$

In particular, these give G-gradings for the algebras H_0 and ${}_0H$, and a $(G \times G)$-grading $\bigoplus_{a,b\in G} {}_aH_b$ for H.

By the proof of [**6**, Lemmas 5.2 and 5.3], $H_a \cap {}_aH \neq 0$ for each $a \in G$, and H is strongly G-graded with respect to both the ρ-grading and the λ-grading. Then, G-graded versions of [**6**, Lemma 5.4(a)(b)] imply that $\dim_k H_0 = \infty$ and $\operatorname{GKdim} H \geq \operatorname{GKdim} H_0 + 1$. Since H_0 is a domain, it cannot be algebraic over k, and therefore $\operatorname{GKdim} H_0 = 1$.

Case 1a. Suppose that $H_0 = {}_0H$. By [**6**, Lemma 4.3(c)], H_0 is a Hopf subalgebra of H. By Lemma 2.1, H_0 is either $k[t]$ or kG' where G' is a torsionfree abelian group of rank one. Hence, H_0 is a Bezout domain (see the proof of [**6**, Lemma 6.2]). Thus each H_a is a free H_0-module of rank one, say $H_a = h_a H_0$. Since H is strongly G-graded, h_a must be invertible. In particular, $\varepsilon(h_a) \neq 0$, so we may replace h_a by $\varepsilon(h_a)^{-1}h_a$ and thus assume that $\varepsilon(h_a) = 1$.

We claim that each h_a is grouplike. By a G-graded version of [**6**, Lemma 5.1(b)], $\Delta(h_a) \in H_a \otimes H_a$, and so $\Delta(h_a) = (h_a \otimes h_a)w$ for some $w \in H_0 \otimes H_0$. Since h_a is invertible, so is w, and $(\varepsilon \otimes \mathrm{id})(w) = (\mathrm{id} \otimes \varepsilon)(w) = 1$ by the counit axiom applied to h_a. By [**6**, Lemma 4.4(a)], w is a homogeneous invertible element of $H_0 \otimes H_0$. If $H_0 = k[t]$, then $w = c1 \otimes 1$ for some $c \in k^\times$, while if $H_0 = kG'$, then $w = cg \otimes g'$ for some $c \in k^\times$ and $g, g' \in G'$. In either case, it follows from the equations $(\varepsilon \otimes \mathrm{id})(w) = (\mathrm{id} \otimes \varepsilon)(w) = 1$ that $w = 1 \otimes 1$. Therefore, h_a is grouplike, as claimed.

Now, $S^2(h_a) = h_a$ for all $a \in G$. Since H_0 is commutative, S^2 is the identity on H_0. As H is generated by H_0 and the h_a, we find that S^2 is the identity on H. Therefore, by Lemma 4.8(6), H is locally affine.

Case 1b. Suppose that $H_0 \neq {}_0H$. Either $H_0 \not\subseteq {}_0H$ or ${}_0H \not\subseteq H_0$, say $H_0 \not\subseteq {}_0H$. Then, ${}_aH_0 \neq 0$ for at least one non-zero $a \in G$. By a G-graded version of [**6**, Lemma 5.4(c)], $\dim_k {}_aH_0 \leq 1$ for all $a \in G$. For any $a, b \in G$, multiplication by a non-zero element of ${}_{-b}H_{-b}$ embeds ${}_aH_b$ in ${}_{a-b}H_0$, whence $\dim_k {}_aH_b \leq 1$. By a G-graded version of [**6**, Lemma 5.1(d)], $S^2({}_aH_b) \subseteq {}_aH_b$ for all $a, b \in G$. This implies that $\sum_{i\geq 0} S^i({}_aH_b)$ has dimension at most two. Since H is spanned by the ${}_aH_b$, we conclude by Lemma 4.8(5) that H is locally affine.

Case 2. By [**6**, Lemma 8.1], there are two commuting locally nilpotent derivations δ_r and δ_l on H such that

$$\rho(h) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta_r^n(h) \otimes t^n \qquad \text{and} \qquad \lambda(h) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \otimes \delta_l^n(h), \qquad \forall\, h \in H.$$

In particular, $H_0 = \ker \delta_r$ and ${}_0H = \ker \delta_l$. Following the proofs of [**6**, Lemmas 9.2, 9.3], we find that H_0 and ${}_0H$ have GK-dimension one.

We claim that $H_0 = {}_0H$. Suppose not, say, $H_0 \not\subseteq {}_0H = \ker \delta_l$. Since δ_l commutes with δ_r, H_0 is δ_l-invariant. Thus, δ_l restricts to a non-zero locally nilpotent derivation on H_0, denoted by δ. Let $H_{00} = \ker \delta$, and choose $u \in H_0 \setminus H_{00}$. Then, $H_{00}[u]$ is a polynomial subalgebra of H_0 (by the argument of [**6**, Lemma 9.2]), which implies that $\operatorname{GKdim} H_{00} = 0$. Since k is algebraically closed, $H_{00} = k$. Since $\ker \delta = k$, there is an element $u \in H_0 \setminus H_{00}$ such that $\delta(u) = 1$. This implies $\lambda(u) = 1 \otimes u + t \otimes 1$, whereas $\rho(u) = u \otimes 1$ as $u \in H_0$. Set $y = \pi(u) \in K$ and compute $\Delta(y)$ in the following two ways:

$$\Delta(y) = (\pi \otimes \pi)\Delta(u) = (\pi \otimes \mathrm{id})\rho(u) = y \otimes 1,$$
$$\Delta(y) = (\pi \otimes \pi)\Delta(u) = (\mathrm{id} \otimes \pi)\lambda(u) = 1 \otimes y + t \otimes 1.$$

The counit axioms then yield $y = \epsilon(y)$ and $y = \epsilon(y) + t$, giving a contradiction. Therefore, we have proved that $H_0 = {}_0H$.

Since $H_0 = {}_0H$, by [**6**, Lemma 4.3(c)], H_0 is a Hopf subalgebra of H. By [**6**, Theorem 8.3(b)], H has the form $H = H_0[x; \partial]$, which is generated by H_0 and x. Since H_0 is commutative, H_0 is locally affine by Lemma 4.8(6). So H is locally affine by Proposition 4.9.

Combining Cases 1 and 2, we have that H is locally affine. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 By Theorem 5.4, H is locally affine. By Proposition 4.10, H is pointed. Therefore, the assertion follows from Theorem 5.2. □

We also have a slight modification of Theorem 1.1.

PROPOSITION 5.5. *Let H be a Hopf domain of GK-dimension two. Suppose H is a union of an ascending chain of Hopf subalgebras $\{K_i\}_{i=1}^{\infty}$ such that all K_i satisfy* (♮). *Then, H satisfies* (♮). *As a consequence, H is isomorphic to one of Hopf algebras in Theorem* 1.1.

Proof. By Theorem 1.1, each K_i is pointed and generated by grouplikes and skew primitives. Hence, H is pointed and generated by grouplikes and skew primitives.

It is well-known that all algebras in Proposition 4.1 satisfy (♮). Therefore, we can assume that $\operatorname{GKdim} C_0(H) = \operatorname{GKdim} G(H) = 1$.

Now, repeating the proof of Theorem 5.2 gives the result. The hard part is concerning type B, where we give an alternative proof as below.

If all K_i are of type B, let I_i be the Hopf ideal of K_i generated by all non-trivial skew primitives that quasi-commute with their weight. Then, $K_i/I_i \cong kG(K_i)$, which induces an injection $K_i/I_i \subseteq K_{i+1}/I_{i+1}$ for all i. It is easy to see that $I_i = I_{i+1} \cap K_i \subseteq I_{i+1}$. Let $I = \bigcup_i I_i$. Then, I is a Hopf ideal of H and H/I is a union of Hopf subalgebras isomorphic to K_i/I_i. As a consequence, $H/I \cong kG(H)$, where $G(H)$ is a non-trivial subgroup of $\mathbb{Q}$. Since $kG(H)$ satisfies (♮) (Lemma 2.1), so does H. □

6. Other properties. In this section, we will prove Corollary 1.2 and Theorem 1.5. We use some ideas of Takeuchi [**19**]. Some parts of the proofs were also suggested by Q.-S. Wu (personal communication). We would like to thank Wu for sharing his comments and proofs with us.

6.1. Takeuchi's idea. In this subsection, we review some ideas of Takeuchi [**19**]. The following lemma was proved in [**19**] in the commutative case and the proof works for a general Hopf algebra. Let H^+ be the kernel of the counit of H.

LEMMA 6.1 ([**19**, Lemma 3.9]). *Let H be a Hopf algebra. Suppose that K and K' are Hopf subalgebras of H such that $K' \subseteq K$. Then, there is a right H-module isomorphism:*

$$K \otimes_{K'} H \xrightarrow{\tau} (K/K(K'^+)) \otimes H, \quad \tau : x \otimes y \mapsto \sum \bar{x}_1 \otimes x_2 y,$$

with the inverse map

$$(K/K(K'^+)) \otimes H \xrightarrow{\mu} K \otimes_{K'} H, \quad \mu : \bar{u} \otimes v \mapsto \sum u_1 \otimes S(u_2)v.$$

Proof. Since the proof is very nice, we include it here.

For any $x \in K$, $z \in K'$, and $y \in H$, we have

$$\begin{aligned}\sum x_1 z_1 \otimes x_2 z_2 y - \sum x_1 \otimes x_2 z y &= \sum x_1 z_1 \otimes x_2 z_2 y - \sum x_1 \epsilon(z_1) \otimes x_2 z_2 y \\ &= \sum x_1 (z_1 - \epsilon(z_1)) \otimes x_2 z_2 y \in KK'^+ \otimes H.\end{aligned}$$

This implies that the map τ is well defined by the definition of the tensor product. On the other hand, for any $u = xz \in K(K'^+)$, where $x \in K$, $z \in K'^+$, and $v \in H$, we have

$$\begin{aligned}\sum u_1 \otimes_{K'} S(u_2)v = \sum x_1 z_1 \otimes_{K'} S(x_2 z_2)v &= \sum x_1 z_1 S(z_2) \otimes_{K'} S(x_2)v \\ &= \sum x_1 \epsilon(z) \otimes_{K'} S(x_2)v = 0.\end{aligned}$$

Hence, μ is well defined.

It is easy to see that

$$\begin{aligned}\mu\tau(x \otimes_{K'} y) = \mu\left(\sum \bar{x}_1 \otimes x_2 y\right) &= \sum x_1 \otimes_{K'} S(x_2) x_3 y \\ &= \sum x_1 \otimes_{K'} \epsilon(x_2) y = x \otimes_{K'} y,\end{aligned}$$

and

$$\tau\mu(\bar{u} \otimes v) = \tau\left(\sum u_1 \otimes_{K'} S(u_2)y\right) = \sum \bar{u}_1 \otimes u_2 S(u_3)v = \bar{u} \otimes v.$$

Therefore, τ is invertible with the inverse μ. □

There is also a left H-module version of the above lemma. The next lemma is also well known (even before [**19**]) and is used in the study of "faithfully flat descent" when A is commutative. Since the proof is short, it is included here.

LEMMA 6.2. *Suppose that $B \subseteq A$ is a ring extension such that A_B (or ${}_BA$) is faithfully flat. Then, the sequence $0 \longrightarrow B \longrightarrow A \xrightarrow{f} A \otimes_B A$, where f is the map sending $x \mapsto x \otimes 1 - 1 \otimes x$, is exact.*

Proof. Let $C = \ker(f) = \{x \in A \mid x \otimes 1 = 1 \otimes x\}$. It is clear that $B \subseteq C$. Since A_B is flat, there are embeddings

$$A = A \otimes_B B \hookrightarrow A \otimes_B C \hookrightarrow A \otimes_B A.$$

For any element $a \otimes c \in A \otimes_B C$, we obtain that

$$a \otimes c = a(1 \otimes c) = a(c \otimes 1) = ac \otimes 1,$$

which implies that the map $(A =)A \otimes_B B \to A \otimes_B C$ is subjective. Consequently, $A \otimes_B C/B = 0$. Since A_B is faithfully flat, $C/B = 0$ and $C = B$ as desired. □

PROPOSITION 6.3. *Let H be a Hopf algebra and $K \subsetneq H$ be a Hopf subalgebra of H. Suppose that H_K is faithfully flat. Then, $HK^+ \neq H^+$.*

Proof. We modify Takeuchi's proof [**19**]. By Lemma 6.2, the sequence

$$0 \to K \longrightarrow H \xrightarrow{f} H \otimes_K H$$

is exact, where f is the map $x \mapsto x \otimes 1 - 1 \otimes x$. By Lemma 6.1, the right H-module map

$$\tau : H \otimes_K H \to H/HK^+ \otimes H, \quad x \otimes y \mapsto \sum \bar{x}_1 \otimes x_2 y$$

is an isomorphism. Hence,

$$0 \to K \longrightarrow H \xrightarrow{\tau \circ f} H/HK^+ \otimes H,$$

where $\tau \circ f : x \mapsto \sum \bar{x}_1 \otimes x_2 - 1 \otimes x$, is exact. Since $K \neq H$, the map $\tau \circ f \neq 0$, which implies that $H/HK^+ \neq 0$. This completes the proof. □

6.2. Some consequences. If H is pointed, then it satisfies (FF) [**16**]. We now prove Theorem 1.5.

THEOREM 6.4. *Let H be a left noetherian Hopf algebra.*

(1) *If H satisfies* (FF)*, then H is of S-finite type. As a consequence,* $\dim_k H$ *is countable.*

(2) *If H is locally affine and satisfies* (FF)*, then H is affine.*
(3) *If H is pointed, then H is affine.*

Proof.

(1) Suppose H is not of S-finite type. Let $K_0 = k$ and define a sequence of S-finite type Hopf subalgebras K_n inductively. Suppose K_n is generated by $\bigcup_{i=0}^{\infty} S^i(V_n)$ where V_n is a finite dimensional subcoalgebra of H. Let W be a finite dimensional subcoalgebra of H such that $W \nsubseteq K_n$. Let $V_{n+1} = V_n + W$ and K_{n+1} be the Hopf subalgebra generated by $\bigcup_{i=0}^{\infty} S^i(V_{n+1})$. By definition, $K_n \neq K_{n+1}$ and K_{n+1} is of S-finite type. Let $K = \bigcup_n K_n$. Then, K is a Hopf subalgebra that is not of S-finite type. Since H satisfies (FF), H_K is faithfully flat. Then, K is left noetherian. We may replace H by K and assume that $H = \bigcup_n K_n$ without loss of generality.
Since H is left noetherian, there is an N such that $HK_n^+ = HK_N^+$ for $n \geq N$. Since $H = \bigcup_n K_n$,

$$HK_N^+ = \bigcup_{n \geq N} HK_n^+ = H \bigcup_{n \geq N} K_n^+ = HH^+ = H^+.$$

By Proposition 6.3, $H^+ \neq HK_N^+$, yielding a contradiction. Therefore, H is of S-finite type.
Any Hopf algebra of S-finite type is countably generated and so has countable k-dimension.
(2) This follows from part (1) and the fact that S-finite type plus local affineness implies that H is affine.
(3) If H is pointed, then it is (FF) by [**16**]. By Lemma 4.8(4), H is locally affine. The assertion follows from part (2).

□

6.3. Global dimension. In this subsection, we will prove Corollary 1.2. The following lemma is known.

LEMMA 6.5 ([**1**, Corollary 1], [**13**, Proposition 2.1]). *Let A be an algebra and $\{A(n)\}_{n=1}^{\infty}$ an ascending chain of subalgebras of A such that $A = \bigcup_n A(n)$. Then,*

$$\operatorname{gldim} A \leq \max\{\operatorname{gldim} A(n) \mid \forall\, n\} + 1.$$

Proof of Corollary 1.2

(1) and (2) follow by construction.
(3) This is a consequence of [**18**, Theorem A(ii)].
(4) This follows by [**16**].
(5) If H is in Theorem 1.1(1–5), then every affine Hopf subalgebra K of H of GK-dimension two has global dimension two by the proof of [**6**, Proposition 0.2(1)]. Since H_K and ${}_K H$ are free (see part (4)), by [**10**, Theorem 7.2.6], $\operatorname{gldim} H \geq \operatorname{gldim} K = 2$. By Lemmas 4.11 and 6.5, $\operatorname{gldim} H \leq 3$.
If H is in Theorem 1.1(6), then there is an affine Hopf subalgebra K of H of GK-dimension two that has global dimension ∞ by the proof of [**6**,

Proposition 0.2(1)]. Since H_K and ${}_KH$ are free (see part (4)), by [**10**, Theorem 7.2.8(i)], gldim $H \geq$ gldim $K = \infty$. □

Questions 6.6. Let H be as in Theorem 1.1.

(1) In cases (1)–(5), is H affine if and only if gldim $H = 2$?

(2) In case (6), what is the injective dimension of H?

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