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Tensor products of classifiable C^*-algebras

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Let $\mathcal{A}_1$ be the class of all unital separable simple C^*-algebras A such that $A \otimes U$ has tracial rank no more than one for all UHF-algebra U of infinite type. It has been shown that all amenable $\mathcal{Z}$-stable C^*-algebras in $\mathcal{A}_1$ which satisfy the Universal Coefficient Theorem can be classified up to isomorphism by the Elliott invariant. In this note, we show that $A \in \mathcal{A}_1$ if and only if $A \otimes B$ has tracial rank no more than one for some unital simple infinite dimensional AF-algebra B. In fact, we show that $A \in \mathcal{A}_1$ if and only if $A \otimes B \in \mathcal{A}_1$ for some unital simple AH-algebra B. We actually prove a more general result. Other results regarding the tensor products of C^*-algebras in $\mathcal{A}_1$ are also obtained.

Keywords: Classification; tensor products; tracial rank; rational tracial rank; TAC; tracially AF; AH-algebra.

1. Introduction

The Elliott program of classification of amenable C^*-algebras is to classify separable amenable C^*-algebras up to isomorphism by their K-theoretic data known as the Elliott invariant. It is a very successful program. Two important classes of unital separable simple C^*-algebras, the class of amenable separable purely infinite simple C^*-algebras satisfying the Universal Coefficient Theorem (UCT) and the class of unital simple AH-algebras with no dimension growth, are classified by their Elliott invariant (see [5, 7–10, 13, 16, 24, 31] among many articles in the literature). There has been other significant progress in the Elliott program. Related to this note, it has been shown that unital separable amenable simple C^*-algebras with tracial

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rank at most one and satisfying the UCT are classifiable by the Elliott invariant. In fact, they are isomorphic to unital simple AH-algebras with no dimension growth. More recently, with a remarkable method developed by Winter ([36]), the notion of rational tracial rank at most one was introduced (a unital separable simple C^*-algebra A is said to have rational tracial rank at most one if $A \otimes U$ has tracial rank at most one for every UHF-algebra U of infinite type), and it was shown in [23] that unital separable amenable simple $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT and have rational tracial rank at most one are also classifiable by the Elliott invariant (see also [25, 26, 36]). This class is significantly larger than the class of all unital simple AH-algebras with no dimension growth. Denote by $\mathcal{A}_1$ the class of all unital separable simple C^*-algebras which have rational tracial rank at most one. A special unital separable simple C^*-algebra in $\mathcal{A}_1$ which does not have finite tracial rank is the Jiang–Su algebra $\mathcal{Z}$. The range of the Elliott invariant for C^*-algebras of rational tracial rank at most one has been characterized and computed (see [27]). This class of C^*-algebras includes C^*-algebras whose ordered K_0-groups may not have the Riesz interpolation property. The verification that a particular unital simple C^*-algebra is in the class $\mathcal{A}_1$ was slightly eased when it was proved in [27] that, $A \in \mathcal{A}_1$ if and only if $A \otimes U$ has tracial rank at most one for some UHF-algebra U of infinite type (instead of for all UHF-algebras of infinite type). Suppose that A is a unital separable simple C^*-algebra such that $A \otimes B$ has tracial rank at most one for some unital simple infinite dimensional AF-algebra B. Does it follow that $A \in \mathcal{A}_1$? We will answer this question affirmatively in this short note. In fact, we will show that if $A \otimes B$ has tracial rank at most one for some unital infinite dimensional separable simple C^*-algebra B with tracial rank at most one then $A \in \mathcal{A}_1$. This may provide a better way to determine which C^*-algebras are in $\mathcal{A}_1$. In a more recent development, the class of all finite unital separable simple C^*-algebras which satisfy the UCT has been classified ([12, 14, 33]). These C^*-algebras have rational generalized tracial rank at most one (see the end of Definition 3.1 below). As defined in Definition 3.1, a unital separable simple C^*-algebra A has rational generalized tracial rank at most one if $gTR(A \otimes U) \leq 1$ for all UHF-algebras U of infinite type. It is much more convenient to deal with $A \otimes Q$ as demonstrated in [12]. This short note also provides such a convenient passage.

Denote by $\mathcal{N}$ the class of all unital separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem. For the purpose of classification, we also consider $\mathcal{A}_1 \cap \mathcal{N}$, the class of all unital separable simple amenable C^*-algebras which have rational tracial rank at most one and satisfy the UCT. We will show that if A and B are both in $\mathcal{A}_1 \cap \mathcal{N}$, then $A \otimes B$ is also in $\mathcal{A}_1 \cap \mathcal{N}$. Assume that $A \in \mathcal{A}_1 \cap \mathcal{N}$ and B is a simple amenable infinite dimensional C^*-algebra with tracial rank at most one and satisfies the UCT. From the fact above, $A \otimes B$ is also in $\mathcal{A}_1 \cap \mathcal{N}$. One might ask whether $A \otimes B$ has tracial rank at most one. We will also give an affirmative answer to this question.

Most of the results are in a more general setting which may provide an opportunity for the future applications. In fact, with a much more recent classification result in [14], we expect some of the results presented in this note can be used to ease some technical constrains. In fact, for example, for a unital simple separable C^*-algebra A, it is much more delightful to work with $A \otimes Q$ than $A \otimes U$, since $K_i(A \otimes Q)$ $(i = 0, 1)$ is torsion free and divisible, while $K_i(A \otimes U)$ could have torsion in general. Some applications of results in this short note can be found in [14].

2. Preliminaries

Definition 2.1. Let A be a C^*-algebra, $\mathcal{F}$ and $\mathcal{G}$ be two subsets of A and $\epsilon > 0$. We say that $\mathcal{F} \subset_\epsilon \mathcal{G}$ if for each $x \in \mathcal{F}$, there exists $y \in \mathcal{G}$, such that $\|x - y\| < \epsilon$.

By A_+, we mean the positive cone of all positive elements in A.

If $a, b \in A_+$, we write $a \lesssim b$ if there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} \|x_n^* b x_n - a\| = 0$. We say two positive elements x and y are Cuntz equivalent and write it as $x \sim y$, if $x \lesssim y$ and $y \lesssim x$.

Let A be a unital stably finite simple C^*-algebra. Denote by $T(A)$ the tracial state space of A. Define $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$ for all $a \in A_+$ and $\tau \in T(A)$. A is said to have strict comparison property for positive elements if for any pair $a, b \in A_+ \backslash \{0\}$, $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$ implies that $a \lesssim b$.

Let $\mathcal{F} \subset A$ be a finite subset and let $p \in A$ be a projection. We use $p\mathcal{F}p$ to denote $\{pxp \colon x \in \mathcal{F}\}$.

Definition 2.2. Let $\mathcal{B}$ be a family of unital C^*-algebras. We say a unital simple separable C^*-algebra A is tracially approximated by C^*-subalgebras in $\mathcal{B}$ and write it as $A \in \mathrm{TA}\mathcal{B}$, if the following holds: For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $a \in A_+ \backslash \{0\}$, there exist a projection $p \in A$ and a C^*-subalgebra $B \subset A$ with $B \in \mathcal{B}$ and $1_B = p$ such that

$$\|px - xp\| < \epsilon \quad \text{for all } x \in \mathcal{F}, \tag{2.1}$$

$$p\mathcal{F}p \subset_\epsilon B \quad \text{and} \tag{2.2}$$

$$1 - p \lesssim a. \tag{2.3}$$

Let $\mathcal{B} = \mathcal{I}_1$ be the family of C^*-algebras of the form $C([0, 1], F)$, where F is a unital finite dimensional C^*-algebra. Then we write $TR(A) \le 1$ if $A \in \mathrm{TA}\mathcal{I}_1$.

Note that, in the original Definition 3.1 of [20], $\mathcal{I}_1$ is replaced by the class of all finite direct sums of C^*-algebras of the form $M_n(C(X_n))$, where each X_n is a finite CW complex with dimension one. But those definitions are equivalent. Please see Theorems 6.13 and 7.1 of [20] for more details on such equivalence. In the definition above, if we replace $\mathcal{B}$ by $\mathcal{I}_0$, the class of finite dimensional C^*-algebras, then we say that A has tracial rank zero (see Theorem 7.1 of [20]). If A has tracial rank at most one, we denote it by $TR(A) \le 1$. If A has tracial rank zero, we denote it by $TR(A) = 0$. For more details, see [17, 18, 20].

Notations. Let A be a unital C^*-algebra. For each $n \in \mathbb{N}$, there is an embedding of $M_n(A)$ into $M_{n+1}(A)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Denote by $M_\infty(A)$ the algebraic inductive limit of $M_1(A) \to M_2(A) \to \cdots$, whose connecting maps are just the embeddings above. Suppose that $T(A) \neq 0$. For any $p \in M_\infty(A)$ and for any $\tau \in T(A)$, we may assume that $p \in M_n(A)$ for certain n. By identifying $M_n(A)$ with $A \otimes M_n(\mathbb{C})$, we define $\tau(p)$ to be $(\tau \otimes \mathrm{Tr})(p)$, where Tr is the standard matrix trace (not normalized) on $M_n(\mathbb{C})$. Note that the value $(\tau \otimes \mathrm{Tr})(p)$ is independent of the choice of n.

Denote by $\mathcal{N}$ the class of all unital separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem.

Denote by Q the UHF-algebra with $(K_0(Q), K_0(Q)_+, [1_Q]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$.

Use $\mathcal{A}_0$ to denote the class of all unital separable simple C^*-algebras A for which $TR(A \otimes M_{\mathfrak{p}}) = 0$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

Use $\mathcal{A}_1$ to denote the class of all unital separable simple C^*-algebras A for which $TR(A \otimes M_{\mathfrak{p}}) \leq 1$ for all supernatural numbers $\mathfrak{p}$ of infinite type.

By the above defined notations, $\mathcal{A}_0 \cap \mathcal{N}$ is the class of all C^*-algebras which are separable, amenable, satisfies the UCT, and are in $\mathcal{A}_0$, and $\mathcal{A}_1 \cap \mathcal{N}$ is the class of all C^*-algebras which are separable, amenable, satisfies the UCT, and are in $\mathcal{A}_1$.

Definition 2.3. Let $\epsilon > 0$. Define

$$f_\epsilon(t) = \begin{cases} 1 & t \geq 2\epsilon, \\ (1/\epsilon)t - 1 & \epsilon < t < 2\epsilon, \\ 0 & 0 \leq t \leq \epsilon. \end{cases}$$

It is easy to check that such f_ϵ is a continuous function on $[0, \infty)$.

3. Tensor with AF-Algebras

Definition 3.1. Throughout this section and the next, let $\mathcal{C}$ be a class of unital separable amenable C^*-algebras which satisfy the following properties: (1) Every finite dimensional C^*-algebras is in $\mathcal{C}$; (2) If $A \in \mathcal{C}$, then $A \otimes F \in \mathcal{C}$, for every finite dimensional C^*-algebra F; (3) Every C^*-algebra in $\mathcal{C}$ is weakly semiprojective (see [28, Chap. 4] for the definition and some basic properties of weak semiprojectivity); (4) Every unital hereditary C^*-subalgebra of C^*-algebras in $\mathcal{C}$ is in $\mathcal{C}$; (5) Suppose that $A \in \mathcal{C}$ and $I \subset A$ is a closed ideal. Then, for any finite subset $\mathcal{F} \subset A/I$ and any $\epsilon > 0$, there exists a C^*-subalgebra $B \subset A/I$ such that $B \in \mathcal{C}$ and $\mathrm{dist}(x, B) < \epsilon$ for all $x \in \mathcal{F}$.

It is easy to verify that the class $\mathcal{I}_1$ defined in Sec. 2 satisfies (1)–(5).

Let F_1 and F_2 be two finite dimensional C^*-algebras, and let $\varphi_1, \varphi_2 : F_1 \to F_2$ be two homomorphisms. Define the mapping torus

$$\begin{aligned} A &= A(F_1, F_2, \varphi_1, \varphi_2) \\ &= \{(f, a) \in C([0,1], F_2) \oplus F_1 : f(0) = \varphi_1(a) \text{ and } f(1) = \varphi_2(a)\}. \end{aligned}$$

Let $\mathcal{C}'$ be the class consisting of all such mapping tori and all finite dimensional C^*-algebras. It is obvious that $\mathcal{C}'$ satisfy properties (1) and (2) above. It is proved in [6] that all C^*-algebras in $\mathcal{C}'$ are semiprojective (property (3)). It is proved in [14] that the class also satisfies properties (4) and (5).

Unital separable simple C^*-algebras which are in TA$\mathcal{C}'$ are also called C^*-algebras with generalized tracial rank at most one. If A is in TA$\mathcal{C}'$, then we write $gTR(A) \le 1$. We say a unital separable simple C^*-algebra A has rational generalized tracial rank at most one, if $gTR(A \otimes U) \le 1$ for all UHF-algebras U of infinite type. Via Theorem 3.4 below, we will show that if $gTR(A \otimes U) \le 1$ for one UHF-algebra U of infinite type (preferably $U = Q$), then A has rational generalized tracial rank at most one.

We begin with the following:

Proposition 3.2. Let A be a unital separable simple infinite dimensional C^*-algebra in TA$\mathcal{C}$. Then, for any simple AF-algebra B (B could be finite dimensional), $A \otimes B \in$ TA$\mathcal{C}$.

Proof. The case that B is finite dimensional follows from properties (1) and (2) of C^*-algebras in $\mathcal{C}$.

Now we assume B is infinite dimensional. It is easy to see that $A \otimes B$ is a unital simple C^*-algebra. Note that B is approximately divisible (see [1] for the definition). By Theorem 1.4 of [1], $A \otimes B$ has the strict comparison property for positive elements. Let $\mathcal{F} \subset A \otimes B$ be a finite subset, $\epsilon > 0$ and $c \in (A \otimes B)_+ \setminus \{0\}$. Since A is a unital infinite dimensional simple C^*-algebra, it is non-elementary. It is easy to find, for any integer $n \ge 1$, n nonzero mutually orthogonal and Cuntz equivalent positive elements in A. By the strict comparison, one obtains a nonzero element $a_0 \in A_+$ such that $a_0 \otimes 1_B \lesssim c$.

To prove that $A \otimes B$ is in TA$\mathcal{C}$, we may assume, without loss of generality, that $\mathcal{F} = \{a \otimes b : a \in \mathcal{F}_1 \text{ and } b \in \mathcal{F}_2\}$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite subsets in A and B, respectively. Since B is AF, we may further assume that $\mathcal{F}_2 \subset F$, where F is a unital finite dimensional C^*-subalgebra of B. Moreover, to simplify notation further, without loss of generality, we may also assume that $\|a\|, \|b\| \le 1$ for all $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$.

Since $A \in$ TA$\mathcal{C}$, there exists a projection $p_1 \in A$ and a C^*-subalgebra $C_0 \in \mathcal{C}$ of A with $1_{C_0} = p_1$ such that

$$\|ap_1 - p_1 a\| < \epsilon/2 \quad \text{for all } a \in \mathcal{F}_1, \tag{3.1}$$

$$\mathrm{dist}(p_1 a p_1, C_0) < \epsilon/2 \quad \text{for all } a \in \mathcal{F}_1 \quad \text{and} \tag{3.2}$$

$$1 - p_1 \lesssim a_0. \tag{3.3}$$

Define $C_1 = C_0 \otimes F$ and $p = p_1 \otimes 1_B$. Then $C_1 \in \mathcal{C}$ and $1_{C_1} = p$. It follows that

$$\|xp - px\| < \epsilon \quad \text{for all } x \in \mathcal{F} \tag{3.4}$$

$$\mathrm{dist}(pxp, C_1) < \epsilon \quad \text{for all } x \in \mathcal{F} \quad \text{and} \tag{3.5}$$

$$1 - p \lesssim a_0 \otimes 1_B \lesssim c. \tag{3.6}$$

□

Remark 3.3. If A is finite dimensional, Proposition 3.2 still holds.

Theorem 3.4. Let A be a unital separable simple C^*-algebra. Suppose that $A \otimes U \in$ TA$\mathcal{C}$ for some infinite dimensional UHF-algebra U. Then $A \otimes B \in$ TA$\mathcal{C}$ for any unital infinite dimensional simple AF-algebra B.

Proof. Suppose that $A \otimes U \in$ TA$\mathcal{C}$. Let B be a unital infinite dimensional simple AF-algebra.

Fix $\epsilon > 0$, and also fix a finite subset $\mathcal{F} \subset A \otimes B$ and $a \in (A \otimes B)_+ \setminus \{0\}$.

As B is approximately divisible, so is $A \otimes B$. It follows from Theorem 1.4(a) of [1] that $A \otimes B$ is either purely infinite or has the strict comparison property for positive elements. In either case, there is a nonzero element $a_0 \in 1_A \otimes B$ such that $a_0 \lesssim a$ in $A \otimes B$. As $A \otimes B$ is simple, there is an integer $N_0 \geq 1$ such that

$$1_{A \otimes B} \lesssim N_0[a_0]. \tag{3.7}$$

We write $B = \lim_{n \to \infty}(B_n, \psi_n)$, where each B_n is a finite dimensional C^*-algebra and $\psi_n : B_n \to B_{n+1}$ is a unital embedding. If $n > m$, put $\psi_{m,n} = \psi_{n-1} \circ \cdots \circ \psi_m : B_m \to B_n$. We will also use $\psi_{n,\infty} : B_n \to B$ for the unital embedding induced by the inductive limit. Write

$$B_n = M_{R(n,1)} \oplus M_{R(n,2)} \oplus \cdots \oplus M_{R(n,k(n))}.$$

According to Proposition 2.2 and Lemma 2.3(b) of [30], to simplify notation, without loss of generality, by replacing a_0 with a smaller (in sense of the Cuntz relation) element, we may assume that $a_0 \in B_n$ for some large n. Moreover, we may assume that $a_0 = a_{1,n} \oplus a_{2,n} \oplus \cdots \oplus a_{k(n),n}$, where $a_{i,n} \in B_{R(n,i)}, i = 1, 2, \ldots, k(n)$. Since B is simple, we may assume that $R(n,j) > 4N_0$ for all j and all n. It follows from (3.7) that we may assume that the range projection of $a_{j,n}$ has rank at least two. Then we may write $a_{j,n} \geq a_{j,n}^{(0)} + a_{j,n}^{(1)}$, where $a_{j,n}^{(i)}$ has exactly rank one range projection, and $a_{j,n}^{(0)}$ and $a_{j,n}^{(1)}$ are mutually orthogonal. Thus

$$a_0 \geq a_0^{(i)} = a_{1,n}^{(i)} \oplus a_{2,n}^{(i)} \oplus \cdots \oplus a_{k(n),n}^{(i)}, \quad i = 0, 1.$$

By choosing possibly smaller a_0 (in the sense of the Cuntz relation), we may assume that $a_{j,n}^{(i)}$ is a rank one projection for each j and n, $i = 0, 1$.

By changing notation, without loss of generality, we may further assume that $\mathcal{F} \subset A \otimes B_1$ and $a_0, a_0^{(0)}, a_0^{(1)} \in B_1$. Define $\pi_j : B_1 \to M_{R(1,j)}$ to be the canonical projection to the jth summand, $j = 1, 2, \ldots, k(1), n = 2, 3, \ldots$. Put $\mathcal{F}_j = \pi_j(\mathcal{F}), j = 1, 2, \ldots, k(1)$.

For each $A \otimes M_{R(1,i)} \otimes U \in$ TA$\mathcal{C}$, there exists a projection $p_i \in A \otimes M_{R(1,i)} \otimes U$ and a C^*-subalgebra $D_{0,i} \in \mathcal{C}$ with $1_{D_{0,i}} = p_i$ such that

$$\|[p_i, x]\| < \epsilon/8 \quad \text{for all } x \in \mathcal{F}_i, \tag{3.8}$$

$$\mathrm{dist}(p_i x p_i, D_{0,i}) < \epsilon/8 \quad \text{for all } x \in \mathcal{F}_i \quad \text{and} \tag{3.9}$$

$$1_{A \otimes M_{R(1,i)} \otimes U} - p_i \lesssim a_{i,1}^{(0)}. \tag{3.10}$$

Let $\mathcal{G}_i \subset B_{0,i}$ be a finite subset such that, for every $x \in \mathcal{F}_i$, there exists $x' \in \mathcal{G}_i$ such that $\|p_i x p_i - x'\| < \epsilon/16$. We may also assume that $1_{D_{0,i}} \in \mathcal{G}_i$.

Write $U = \overline{\bigcup_{n=1}^{\infty} M_{r(n)}}$, where $\lim_{n\to\infty} r(n) = \infty$ and $M_{r(n)} \subset M_{r(n+1)}$ unitally. Since each $D_{0,i}$ is weakly semiprojective, we can choose n_0 large enough, such that for each $i = 1, 2, \ldots, k(1)$, there exists a unital homomorphism $\varphi_i : D_{0,i} \to A \otimes M_{R(1,i)} \otimes M_{r(n_0)}$, satisfying

$$\|\varphi_i(x') - x'\| < \epsilon/8 \quad \text{for all } x' \in \mathcal{G}_i. \tag{3.11}$$

Without loss of generality (by replacing φ_i with $\mathrm{Ad}\, u_i \circ \varphi_i$ for some unitary u_i in $A \otimes M_{R(1,i)} \otimes M_{r(n_0)}$ if necessary), we may assume that $\varphi_i(1_{B_{0,i}}) = p_i$. It follows from property (5) of C^*-algebras in $\mathcal{C}$ that there exists a unital C^*-algebra $D_i \subset \varphi_i(D_{0,i})$ such that $D_i \in \mathcal{C}$, $1_{D_i} = \varphi_i(D_{0,i})$ and

$$\mathrm{dist}(\varphi_i(x'), D_i) < \epsilon/16 \quad \text{for all } x' \in \mathcal{G}_i. \tag{3.12}$$

Note that $1_{D_i} = \varphi_i(1_{D_{0,i}}) = p_i$. Thus

$$\mathrm{dist}(p_i x p_i, D_i) < \epsilon/4 \quad \text{for all } x \in \mathcal{F}_i. \tag{3.13}$$

Denote by $\imath_{0,i} : M_{R(1,i)} \to M_{R(1,i)r(n_0)}$ the map defined as $\imath_{0,i}(x) = x \otimes 1_{M_{r(n_0)}}$, $i = 1, 2, \ldots, k(1)$, and define $\imath_0 : A \otimes B_1 \to A \otimes B_1 \otimes M_{r(n_0)}$ by $\imath_0(x) = x \otimes 1_{M_{r(n_0)}}$ for all $x \in A \otimes B_1$.

Since B is a unital simple AF-algbera, we may assume that $\psi_{1,n_1} : B_1 \to B_{n_1}$ has multiplicities at least $N \geq 1$ for each simple summand of B_1, such that

$$\frac{2r(n_0)\left(\sum_{j=1}^{k(1)} R(1,j)\right)^2}{N} < 1. \tag{3.14}$$

Put $\Psi_{i,j} = \pi_{n_1,j} \circ (\psi_{1,n_1}|_{M_{R(1,i)}}) : M_{R(1,i)} \to M_{R(n_1,j)}$, where $\pi_{n_1,j}$ is the canonical projection to the jth summand of B_{n_1}. The assumption on the multiplicity implies that $\Psi_{i,j}(1_{M_{R(1,i)}}) = 1_{M_{m(i,j)}} \in M_{R(n_1,j)}$ with $m(i,j) \geq N, i = 1, 2, \ldots, k(1)$ and $j = 1, 2, \ldots, k(n_1)$. It follows that $R(1,i) \,|\, m(i,j), i = 1, 2, \ldots, k(1)$ and $j = 1, 2, \ldots, k(n_1)$. Note that $1_{M_{R(n_1,j)}} = \sum_{i=1}^{k(1)} \Psi_{i,j}(1_{M_{R(1,i)}}), j = 1, 2, \ldots, k(n_1)$. Write

$$m(i,j) = l(i,j) r(n_0) R(1,i) + s_{i,j}, \tag{3.15}$$

where $l(i,j) \geq 1$ and $r(n_0)R(1,i) > s_{i,j} \geq 0$ are integers. It follows that

$$\sum_{i=1}^{k(1)} \frac{s_{i,j}}{m(i,j)} < \sum_{i=1}^{k(1)} \frac{r(n_0)R(1,i)}{N} < \sum_{i=1}^{k(1)} \frac{r(n_0)R(1,i)}{2r(n_0)(\sum_{i=1}^{k(1)} R(1,i))^2} < \frac{1}{2\sum_{i=1}^{k(1)} R(1,i))}. \tag{3.16}$$

Since $R(1,i) \,|\, m(i,j)$, we may write $s_{i,j} = s_{i,j}^{(r)} R(1,i), i = 1, 2, \ldots, k(1)$ and $j = 1, 2, \ldots, k(n_1)$. Define $\rho_{i,j} : M_{R(1,i)} \to M_{s_{i,j}}$ by $x \to x \otimes 1_{M_{s_{i,j}^{(r)}}}$. Note also that $\sum_{i=1}^{k(1)} m(i,j) = R(n_1,j), j = 1, 2, \ldots, k(n_1)$.

It follows from (3.16) that

$$\sum_{i=1}^{k(1)} \rho_{i,j}(1_{M_{R(1,i)}}) \lesssim \psi_{1,n_1}(a_{j,1}^{(1)}). \tag{3.17}$$

Let $\imath_{1,i,j} : M_{r(n_0)R(1,i)} \to M_{l(i,j)r(n_0)R(1,i)}$ be the embedding defined by $a \mapsto a \otimes 1_{M_{l(i,j)}}$. Let $\imath_{2,i,j} : M_{l(i,j)r(n_0)R(1,i)} \to \Psi_{i,j}(M_{R(1,i)})$ be defined by the embedding which sends rank one projection to rank one projection. Put $\imath_{3,i,j} = \imath_{2,i,j} \circ \imath_{1,i,j}$. Define $\imath_{4,i,j} : A \otimes M_{R(1,i)} \otimes M_{r(n_0)} \to A \otimes M_{R(n_1,j)}$ by $\imath_{4,i,j}(a \otimes b) = a \otimes \imath_{3,i,j}(b)$ for all $a \in A$ and $b \in M_{r(n_0)R(1,i)}$. Note that

$$\left(\sum_{j=1}^{k(n_1)} \imath_{3,i,j} \circ \imath_{0,i}\right) \oplus \left(\sum_{j=1}^{k(n_1)} \rho_{i,j}\right) = \sum_{j=1}^{k(n_1)} \Psi_{i,j} = \psi_{1,n_1}|_{M_{R(1,i)}}$$

and

$$\sum_{i=1}^{k(1)} \left(\left(\sum_{j=1}^{k(n_1)} \imath_{3,i,j} \circ \imath_{0,i}\right) \oplus \left(\sum_{j=1}^{k(n_1)} \rho_{i,j}\right)\right) = \psi_{1,n_1}.$$

Define $\imath : A \otimes B_{n_1} \to A \otimes B$ to be the map given by $a \otimes b \mapsto a \otimes \psi_{1,\infty}(b)$.

Put $C_1 = \bigoplus_{i=1}^{k(1)} \imath\left(\bigoplus_{j=1}^{k(n_1)} \imath_{4,i,j}(D_i)\right)$. Then $C_1 \in \mathcal{C}$ and $p = 1_{C_1}$ has the form $\imath\left(\sum_{i=1}^{k(1)} p_i \otimes p_i'\right)$, where $p_i \in A \otimes M_{R(1,i)} \otimes M_{r(n_0)}$ and $p_i' = \sum_{j=1}^{k(n_1)} 1_{M_{l(i,j)}}$. A fact we use here is

$$\|xp - px\| = \|x(p_i \otimes p_i') - (p_i \otimes p_i')x\| = \|xp_i - p_i x\| < \epsilon/4 \tag{3.18}$$

for all $x \in \mathcal{F}$, since $x = a \otimes b$, where $a \in A$ and $b \in B_1$. We also have

$$\operatorname{dist}(pxp, C_1) < \epsilon/4 \quad \text{for all } x \in \mathcal{F}. \tag{3.19}$$

By (3.10) and (3.17),

$$1 - p \le \sum_{i=1}^{k(1)} (1 - p_i) + \sum_{j=1}^{k(n_1)} \rho_{i,j}(1_{M_{R(1,i)}}) \lesssim a_0^{(0)} + a_0^{(1)} \lesssim a, \tag{3.20}$$

where we identify $a_0^{(i)}$ with $\psi_{1,\infty}(a_0^{(i)})$. Therefore $1 - p \lesssim a$. This implies that $A \otimes B \in \mathrm{TA}\mathcal{C}$. □

Proposition 3.5. Let A be a unital separable simple C^*-algebra in $\mathrm{TA}\mathcal{C}$ and let $p \in A$ be a nonzero projection. Then $pAp \in \mathrm{TA}\mathcal{C}$.

Proof. Let $1/4 > \epsilon > 0$. Let $\mathcal{F} \subset pAp$ be a finite subset and let $a \in (pAp)_+ \setminus \{0\}$. Without loss of generality, we may assume that $p \in \mathcal{F}$ and $\|x\| \le 1$ for all $x \in \mathcal{F}$. Since A is in $\mathrm{TA}\mathcal{C}$, there is a projection $e \in A$ and a C^*-subalgebra $C_0 \in \mathcal{C}$ of A

with $1_{C_0} = e$ such that

$$\|ex - xe\| < \epsilon/64 \quad \text{for all } x \in \mathcal{F}, \tag{3.21}$$

$$\text{dist}(exe, C_0) < \epsilon/64 \quad \text{for all } x \in \mathcal{F} \quad \text{and} \tag{3.22}$$

$$1 - e \lesssim a. \tag{3.23}$$

We have $\|ep - pe\| < \epsilon/64$. One computes that there is a projection $e_0 \le p$ and $e_0' \in C_0$ such that $\|e_0 - pep\| < \epsilon/32$, $\|e_0' - epe\| < \epsilon/32$ and $\|e_0 - e_0'\| < \epsilon/8$. Then there is a unitary $u \in A$ such that $\|u - 1\| < \epsilon/4$ and $u^* e_0' u = e_0$. Define $C_1 = u^*(e_0' C_0 e_0')u$. By property (4) as in the definition of $\mathcal{C}$, $e_0' C_0 e_0' \in \mathcal{C}$. Therefore $C_1 \in \mathcal{C}$. Note that $px = xp = x$ for all $x \in \mathcal{F}$. We then estimate that (with $\|x\| \le 1$ for $x \in \mathcal{F}$ as assumed), for all $x \in \mathcal{F}$,

$$\begin{aligned}\|e_0 x - x e_0\| &\le \|e_0 x - pepx\| + \|pepx - xe_0\| \\ &< \epsilon/16 + \|pepx - pxep\| + \|pxep - xe_0\| \\ &< \epsilon/16 + \epsilon/64 + \epsilon/16 < \epsilon. \end{aligned} \tag{3.24}$$

By (3.23),

$$p - pep \lesssim a. \tag{3.25}$$

Since $\|(p - pep) - (p - e_0)\| < \epsilon/32$, by Proposition 2.2 and Lemma 2.3(b) of [30],

$$p - e_0 = f_{\epsilon/16}(p - e_0) \lesssim p - pep \lesssim a. \tag{3.26}$$

We also estimate that

$$\text{dist}(e_0 x e_0, C_1) < \epsilon \quad \text{for all } x \in \mathcal{F}. \tag{3.27}$$

It follows that $pAp \in \text{TA}\mathcal{C}$. □

Theorem 3.6. Let A be a unital simple separable C^*-algebra. Then $A \otimes C \in \text{TA}\mathcal{C}$ for all unital simple AF-algebra C if and only if $A \otimes C \in \text{TA}\mathcal{C}$ for some infinite simple AF-algebra C.

Proof. Following Theorem 3.4, it suffices to show the following: Suppose that $A \otimes C \in \text{TA}\mathcal{C}$ for some unital simple AF-algebra C. Then $A \otimes Q \in \text{TA}\mathcal{C}$.

Since every finite dimensional C^*-algebra is in $\mathcal{C}$, it is easy to see that we only need to consider the case that A is infinite dimensional.

Let $B = A \otimes Q$, $1/4 > \epsilon > 0$, $a \in B_+ \backslash \{0\}$ and let $\mathcal{F} \subset B$ be a finite subset. To simplify the notation, without loss of generality, we may assume that $\|x\| \le 1$ for all $x \in \mathcal{F}$ and $\|a\| = 1$.

We will write $A \otimes Q$ as $\lim_{k \to \infty}(A \otimes M_{k!}, j_k)$, where $j_k : A \otimes M_{k!} \to A \otimes M_{(k+1)!}$ is given by $j_k(a) = a \otimes 1_{M_{(k+1)}}$ for all $a \in A \otimes M_{k!}, k = 1, 2, \ldots$. Without loss of generality, we may assume that $\mathcal{F} \subset A \otimes M_{k!}$ for some $k \ge 1$.

Without loss of generality again, we may assume that there exists a positive element $a' \in A \otimes M_{k!}$ such that $\|a - a'\| < \epsilon$. By Proposition 2.2 and Lemma 2.3(b) of [30], $f_{\delta}(a') \lesssim a$. Put $a_0 = f_{\delta}(a')$. As $\|a\| = 1$ and $\epsilon < 1/4$, it is clear that $a_0 \in (A \otimes M_{k!})_+ \setminus \{0\}$.

For C in the statement, we write it as $\lim_{m\to\infty}(C_m, \imath_m)$, where each C_m is a finite dimensional C^*-algebra and $\imath_m$ is a unital embedding of C_m into C_{m+1}. Since C is an infinite dimensional unital simple AF-algebra, for k above, we can assume that for m large enough, each C_m satisfies

$$C_m = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{s(m)}}, \tag{3.28}$$

where $n_j \geq k!, j = 1, 2, \ldots, s(m)$. Fix one such m. Then one obtains a projection $q \in C_m$ such that $M_{k!}$ is a unital C^*-subalgebra of $qC_m q$ (with unit q). Put $e = 1_A \otimes q$ in $A \otimes C$ and let $\varphi_1' : M_{k!} \to qC_m q$ be a unital embedding. Define $\varphi_1 : A \otimes M_{k!} \to A \otimes qC_m q$ by $\varphi_1(x \otimes y) = x \otimes \varphi_1'(y)$ for all $x \in A$ and $y \in M_{k!}$.

By Proposition 3.5, $e(A \otimes C)e \in \mathrm{TA}\mathcal{C}$. Therefore there exists a projection $p \in e(A \otimes C)e$ and a C^*-subalgebra $I_0 \in \mathcal{C}$ of $e(A \otimes C)e$ with $1_{I_0} = p$, satisfying

$$\|px - xp\| < \epsilon/16 \quad \text{for all } x \in \varphi_1(\mathcal{F}), \tag{3.29}$$

$$\mathrm{dist}(pxp, I_0) < \epsilon/16 \quad \text{for all } x \in \varphi_1(\mathcal{F}) \quad \text{and} \tag{3.30}$$

$$1 - p \lesssim \varphi_1(a_0). \tag{3.31}$$

Choose a finite set $\mathcal{G}_0$ in I_0 such that $p\varphi_1(\mathcal{F})p \subset_{\epsilon/16} \mathcal{G}_0$. Since I_0 is weakly semi-projective, for n large enough, there exists a homomorphism $h\colon I_0 \to A \otimes (qC_n q)$ such that $\|h(y) - y\| < \epsilon/32$ for all $y \in \mathcal{G}_0$. Without loss of generality, replacing h by $\mathrm{Ad}\, u \circ h$ for some unitary u if necessary, we may assume that $h(p) = p$. Using property (5), we obtains a unital C^*-subalgebra $I_{00} \subset h(I_0)$ with $1_{I_{00}} = 1_{h(I_0)} = p$ such that $I_{00} \in \mathcal{C}$,

$$\mathrm{dist}(g, I_{00}) < \epsilon/16 \quad \text{for all } g \in \mathcal{G}_0. \tag{3.32}$$

Therefore

$$\mathrm{dist}(pxp, I_{00}) < \epsilon/4 \quad \text{for all } x \in \varphi_1(\mathcal{F}). \tag{3.33}$$

Write $qC_n q$ as $M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_r}$. Note that $k! \mid m_j$ for $j = 1, 2, \ldots, r$, as φ_1' is unital. Put $N = \sum_{j=1}^{r} m_j$. Then there is a unital embedding $\varphi_2' : qC_n q \to M_{N!}$. Consider the canonical embedding $j_k : M_{k!} \to M_{N!}$ and $\varphi_2' \circ \varphi_1' : M_{k!} \to M_{N!}$. Since they are both unital, there is a unitary $u \in M_{N!}$ such that

$$\mathrm{Ad}u \circ \varphi_2' \circ \varphi_1' = j_k.$$

Define $\varphi_2 : A \otimes qC_n q \to A \otimes M_{N!}$ by

$$\varphi_2(x \otimes y) = x \otimes (\mathrm{Ad}\, u \circ \varphi_2'(y))$$

for all $x \in A$ and $y \in qC_n q$.

Then

$$(\varphi_2 \circ \varphi_1)(c) = c \quad \text{for all } c \in A \otimes M_{k!}. \tag{3.34}$$

Put $p_1 = \varphi_2(p) \in A \otimes M_{N!} \subset A \otimes Q$ and $D = \varphi_2(I_{00}) \subset A \otimes M_{N!} \subset A \otimes Q$ with $1_D = p_1$. Note also $D \in \mathcal{C}$. Moreover, by (3.29), (3.30) and (3.33), we have

$$\begin{aligned}\|p_1 x - x p_1\| &= \|\varphi_2(p\varphi_1(x) - \varphi_1(x)p)\| \\ &= \|p\varphi_1(x) - \varphi_1(x)p\| < \epsilon/2 \quad \text{for all } x \in \mathcal{F};\end{aligned} \tag{3.35}$$

$$\mathrm{dist}(p_1 x p_1, D) \le \mathrm{dist}(p\varphi_1(x)p, I_{00}) < \epsilon/2 \quad \text{for all } x \in \mathcal{F}. \tag{3.36}$$

Then, by (3.31),

$$1 - p_1 = \varphi_2(1 - p) \lesssim \varphi_2(\varphi_1(a)) = a. \tag{3.37}$$

This implies that $A \otimes Q \in \mathrm{TA}\mathcal{C}$. □

The following corollary is a special case of Theorem 3.6.

Corollary 3.7. Let A be a unital simple separable C^*-algebra, and let C be a unital infinite dimensional simple AF-algebra. Suppose that $A \otimes C$ has tracial rank at most one. Then $A \in \mathcal{A}_1$.

4. Criterions for C^*-algebras to be in $\mathcal{A}_1$

Lemma 4.1. Let A be a unital separable simple C^*-algebra. Let C be a unital simple AH-algebra with no dimension growth and with $\mathrm{Tor}(K_0(C)) = \{0\}$. Suppose that $A \otimes C$ is in $\mathrm{TA}\mathcal{C}$. Then for any simple unital infinite dimensional AF algebra F, $A \otimes F$ is also in $\mathrm{TA}\mathcal{C}$.

Proof. According to Theorem 3.6, we just need to show that $A \otimes F$ is in $\mathrm{TA}\mathcal{C}$ for some simple unital AF-algebra F.

By Theorem 3.7 of [29], we know that $K_0(C)$ is weakly unperforated. By Theorem 2.7 of [15], $K_0(C)$ has the Riesz interpolation property. As $\mathrm{Tor}(K_0(C)) = 0$, we have that $K_0(C)$ is an unperforated Riesz group. It follows from the Effros–Handelman–Shen theorem (Theorem 2.2 of [5]) that there exists a unital separable simple AF-algebra B with

$$(K_0(B), K_0(B)_+, [1_B]) = (K_0(C), K_0(C)_+, [1_C]). \tag{4.1}$$

We will show that $A \otimes B$ is in $\mathrm{TA}\mathcal{C}$. For that, let $1/4 > \epsilon > 0$. Let $\mathcal{F} \subset A \otimes B$ be a finite subset and let $a \in (A \otimes B)_+ \backslash \{0\}$. Without loss of generality, we may assume that $1/2 > \epsilon, \mathcal{F}$ is a subset of the unit ball and $\|a\| = 1$.

For any $f \in \mathcal{F}$, we may assume that there are $a_{f,1}, a_{f,2}, \ldots, a_{f,n(f)} \in A$ and $b_{f,1}, b_{f,2}, \ldots, b_{f,n(f)} \in B$ such that

$$\left\| f - \sum_{i=1}^{n(f)} a_{f,i} \otimes b_{f,i} \right\| < \epsilon/32. \tag{4.2}$$

We may also assume that there exist $x_1, x_2, \ldots, x_{n(a)} \in A$ and $y_1, y_2, \ldots, y_{n(a)} \in B$ such that

$$\left\| f_{1/4}(a) - \sum_{i=1}^{n(a)} x_i \otimes y_i \right\| < \epsilon/32. \tag{4.3}$$

Let

$$K_1 = n(a) + \max\{n(f)\colon f \in \mathcal{F}\}, \tag{4.4}$$

$$K_2 = \max\{\|x_i\| + \|y_i\|\colon 1 \le i \le n(a)\} \quad \text{and} \tag{4.5}$$

$$K_3 = \max\{\|a_{f,i}\| + \|b_{f,i}\|\colon 1 \le i \le n(f) \text{ and } f \in \mathcal{F}\}. \tag{4.6}$$

Put $a_1 = f_{1/2}(a)$.

As B is an AF-algebra and C has stable rank one (see [4]), it is known that there exists a unital homomorphism $\varphi'\colon B \to C$ such that $(\varphi')_*$ gives the identification (4.1). Define $\varphi\colon A \otimes B \to A \otimes C$ as $\varphi = \mathrm{id}_A \otimes \varphi'$. Now since $A \otimes C$ is in TA$\mathcal{C}$, there exists a C^*-subalgebra D of $A \otimes C$ such that $D \in \mathcal{C}$ and (using p to denote 1_D)

$$\|px - xp\| < \epsilon/32 \quad \text{for all } x \in \varphi(\mathcal{F}), \tag{4.7}$$

$$\mathrm{dist}(pxp, D) < \epsilon/32 \quad \text{for all } x \in \varphi(\mathcal{F}) \quad \text{and} \tag{4.8}$$

$$1 - p \lesssim \varphi(a_1). \tag{4.9}$$

Thus there exists $w \in A \otimes C$ such that $w^*w = 1 - p$ and $ww^*\varphi(f_{1/4}(a)) = ww^*$. Let $\mathcal{G}_0 \subset D$ be a finite subset such that, for each $x \in \mathcal{F}$, there exists $y \in \mathcal{G}_0$ such that $\|x - y\| < \epsilon/32$.

By the UCT (see [3]), we obtain $\kappa \in KL(C, B)$ such that $\kappa|_{K_1(C)} = 0$ and $\kappa|_{K_0(C)} = (\varphi')_{*0}^{-1}$. Choose a unital AH-algebra C_0 with no dimension growth whose Elliott invariant is

$$\begin{aligned}(K_0(C_0), (K_0(C_0))_+, [1_{C_0}], K_1(C_0), T(C_0), r_{C_0}) \\ = (K_0(C), (K_0(C)_+, [1_C], \{0\}, T(C), r_C).\end{aligned}$$

With the identification above, it is known (by Theorem 6.10 of [22], for example) that there exists a unital homomorphism $H : C \to C_0$ such that $H_{*0} = \mathrm{id}_{K_0(C)}$, $H_{*1} = 0$ and H induces the identity map on $T(C)$. By the UCT, we obtain $\kappa \in KL(C_0, B)$ such that $\kappa|_{K_0(C_0)} = \mathrm{id}_{K_0(C_0)}$ (with the above identification). Note that $K_1(C_0) = \{0\} = K_1(B)$. It follows from Theorem 9.12 of [21] (see also Theorem 4.7 of [19]) that there exists a sequence of unital contractive completely positive linear maps $\Psi'_n : C_0 \to B$ such that

$$[\{\Psi'_n\}] = \kappa \quad \text{and} \quad \lim_{n\to\infty} \|\Psi'_n(x)\Psi'_n(y) - \Psi'_n(xy)\| = 0 \tag{4.10}$$

for all $x, y \in C_0$. Define $\Psi_n = \Psi'_n \circ H : C \to B, n = 1, 2, \ldots$. By Theorem 3.5.3 of [2], there exists a sequence of unital contractive completely positive linear maps

$\Phi_n : A \otimes C \to A \otimes B$ such that

$$\Phi_n(x \otimes y) = x \otimes \Psi_n(y) \tag{4.11}$$

for all $x \in A$ and $y \in C$. Since D is weakly semiprojective, we may assume that, there exists $n_0 \geq 1$ such that, for all $n \geq n_0$, there exists a unital homomorphism $h_n : D \to A \otimes B$ such that

$$\|h_n(g) - \Phi_n(g)\| < \epsilon/32 \quad \text{for all } g \in \mathcal{G}_0. \tag{4.12}$$

Let

$$\mathcal{G}_1 = \{1_B\} \cup \{y_i : 1 \leq i \leq n(a)\} \cup \{b_{f,i} : 1 \leq i \leq n(f) \text{ and } f \in \mathcal{F}\} \quad \text{and} \tag{4.13}$$

$$\mathcal{G}_2 = \{a_{f,i} : 1 \leq i \leq n(f) \text{ and } f \in \mathcal{F}\} \cup \{x_i : 1 \leq i \leq n(a)\}. \tag{4.14}$$

As B is an AF algebra, without loss of generality, we may assume that there exists a finite dimensional C^*-subalgebra $E \subset B$ such that $\mathcal{G}_1 \subset E$.

Put

$$\delta = \frac{\epsilon}{32K_1K_2K_3}. \tag{4.15}$$

As E is weakly semiprojective, so is $\varphi'(E)$ (note that E is simple). There then exists a unital homomorphism $h'_n : \varphi'(E) \to B$ such that, when n is large enough,

$$\|h'_n(g) - \Psi_n(g)\| < \delta/2 \quad \text{for all } g \in \varphi'(\mathcal{G}_1). \tag{4.16}$$

We may also assume, without loss of generality, that $(h'_n \circ \varphi')_{*0} = ((\mathrm{id}_B)|_E)_{*0}$. Then we can choose sufficiently large n_1, such that for each $n > n_1$, there exists a unitary $v_n \in C$ satisfying

$$\|(\mathrm{Ad}\, v_n \circ h'_n \circ \varphi')(y) - y\| < \delta/2 \quad \text{for all } y \in \mathcal{G}_1. \tag{4.17}$$

For $n \geq n_1$, define $\Phi'_n : A \otimes C \to A \otimes B$ by $\Phi'_n = \mathrm{Ad}\,(1_A \otimes v) \circ \Phi_n$. Put $p_1 = h'_n(p)$ and $D_1 = h_n(D)$. By choosing even larger n_1, we may assume, without loss of generality

$$\|\Phi'_n(w)^*\Phi'_n(w) - (1 - p_1)\| < \epsilon/16 \quad \text{and} \tag{4.18}$$

$$\|\Phi'_n(w)\Phi'_n(w)^*(\Phi'_n \circ \varphi(f_{1/4}(a))) - \Phi'_n(w)\Phi'_n(w)^*\| < \epsilon/16. \tag{4.19}$$

Then, one estimates, by (4.17) and (4.2), that

$$\|\Phi'_n \circ \varphi(f) - f\| < \epsilon/32 + \epsilon/32 + K_1K_3\delta < 3\epsilon/32 \quad \text{for all } f \in \mathcal{F}. \tag{4.20}$$

Similarly,

$$\|\Phi'_n \circ \varphi(f_{1/4}(a)) - f_{1/4}(a)\| < \epsilon/32 + \epsilon/32 + K_1K_2\delta < 3\epsilon/32. \tag{4.21}$$

By applying (4.7), (4.12) and (4.20), we then have that

$$\begin{aligned} \|p_1x - xp_1\| &\leq \|p_1x - \Phi'_n(p)\Phi'_n \circ \varphi(x)\| + \|\Phi'_n(p)\Phi'_n \circ \varphi(x) - \Phi'_n \circ \varphi(x)\Phi'_n(p)\| \\ &\quad + \|\Phi'_n \circ \varphi(x)\Phi'_n(p) - xp_1\| < 3\epsilon/16 \\ &\quad + \|p\varphi(x) - \varphi(x)p\| + 3\epsilon/16 < 7\epsilon/16 \end{aligned} \tag{4.22}$$

for all $x \in \mathcal{F}$. Similarly,

$$\mathrm{dist}(p_1 x p_1, D_1) < \epsilon/2 \quad \text{for all } x \in \mathcal{F}. \tag{4.23}$$

By property (5) of C^*-algebras in $\mathcal{C}$, there is a C^*-subalgebra $D_2 \subset D_1$ with $1_{D_2} = 1_{D_1} = p_1$ such that $D_2 \in \mathcal{C}$ and

$$\mathrm{dist}(p_1 x p_1, D_2) < \epsilon \quad \text{for all } x \in \mathcal{F}. \tag{4.24}$$

Now, by (4.18) and (4.19), there are projections $e_1 \in \overline{\Phi'_n(w)^*\Phi'_n(w)(A \otimes B)\Phi'_n(w)^*\Phi'_n(w)}$ and $e_2 \in \overline{\Phi'_n(w)\Phi'_n(w)^*(A \otimes B)\Phi'_n(w)\Phi'_n(w)^*}$ such that $e_1 \sim e_2$ and

$$\|e_1 - \Phi'_n(w)^*\Phi'_n(w)\| < \epsilon/8 \quad \text{and} \quad \|e_2 - \Phi'_n(w)\Phi'_n(w)^*\| < \epsilon/8. \tag{4.25}$$

Moreover,

$$\|(1 - p_1) - e_1\| < \epsilon/8 \quad \text{and} \quad \|e_2 \Phi'_n \circ \varphi(f_{1/4}(a)) - e_2\| < \epsilon/4. \tag{4.26}$$

It follows from (4.21) that

$$\|e_2 f_{1/4}(a) - e_2\| < \epsilon/2 + 3\epsilon/32. \tag{4.27}$$

Thus

$$\|f_{1/4}(a) e_2 f_{1/4}(a) - e_2\| < \epsilon + 6\epsilon/32 < 1/2. \tag{4.28}$$

Then we can find a projection in $\mathrm{Her}(f_{1/4}(a))$ which is unitarily equivalent to e_2. It follows that $e_2 \lesssim f_{1/16}(a)$. Therefore, by (4.26),

$$1 - p_1 \sim e_1 \sim e_2 \lesssim f_{1/16}(a) \lesssim a. \tag{4.29}$$

From (4.22), (4.23), and (4.29), we conclude that $A \otimes B$ is in TA$\mathcal{C}$. By Theorem 3.6, for any unital simple infinite dimensional AF algebra F, $A \otimes F \in$ TA$\mathcal{C}$.

□

Theorem 4.2. *Let A be a unital separable simple C^*-algebra. Suppose that $A \otimes C$ is in TA$\mathcal{C}$ for some unital amenable separable simple C^*-algebra C such that $TR(C) \leq 1$ and C satisfies the UCT. Then $A \otimes F$ is in TA$\mathcal{C}$ for any simple unital infinite dimensional AF algebra F.*

Proof. We may assume that C has infinite dimension. Otherwise, as C is simple, $C \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$. As $M_n(A)$ is in TA$\mathcal{C}$, by applying Proposition 3.5, we conclude that A is also in TA$\mathcal{C}$. It follows from Proposition 3.2 that $A \otimes F$ is in TA$\mathcal{C}$ for any unital simple infinite dimensional AF algebra F.

Now assume that C is infinite dimensional. It follows from the assumption that $A \otimes C$ is in TA$\mathcal{C}$ and from Proposition 3.2 that $(A \otimes C) \otimes Q$ is in TA$\mathcal{C}$. Note that $(A \otimes C) \otimes Q \cong A \otimes (C \otimes Q)$. Since $TR(C) \leq 1$, it follows that $TR(C \otimes Q) \leq 1$. Since C is amenable and satisfies the UCT, $C \otimes Q$ is also a unital separable amenable simple C^*-algebra which satisfies the UCT. It follows from Theorem 10.4 of [21] that $C \otimes Q$ is a unital simple AH-algebra with no dimension growth. One computes that $K_0(C\otimes Q)$ is torsion-free. Applying Lemma 4.1, we have that $A\otimes F$ is in TA$\mathcal{C}$ for any unital simple infinite dimensional AF algebra F. □

As a special case to Theorem 4.2, we have the following corollary.

Corollary 4.3. Let A be a unital separable simple C^*-algebra. Then $A\in\mathcal{A}_0$ if and only if $TR(A\otimes C)=0$ for some unital amenable separable simple C^*-algebra C with $TR(C)\leq 1$ which satisfies the UCT.

Corollary 4.4. Let A be a unital separable simple C^*-algebra. Then $A\in\mathcal{A}_1$ if and only if $TR(A\otimes C)\leq 1$ for some unital simple AH-algebra C.

Proof. For the "only if" part, we only need to choose C to be Q. The corollary then follows from Corollary 3.7.

For the "if" part, note that by Theorem 10.4 of [21], $C\otimes Q$ is a unital simple AH-algebra with no dimension growth. Since $TR(A\otimes C)\leq 1$, we have $TR(A\otimes C\otimes Q)\leq 1$. Theorem 4.2 then applies. □

5. Tensor Products

In this section we are ready to answer the following three questions:

(1) Let A and B be both in $\mathcal{A}_1\cap\mathcal{N}$. Is $A\otimes B$ in $\mathcal{A}_1\cap\mathcal{N}$?
(2) Let A be a unital separable simple C^*-algebra and let $B\in\mathcal{A}_1\cap\mathcal{N}$. Suppose that $A\otimes B\in\mathcal{A}_1$. Is it true that $A\in\mathcal{A}_1$?
(3) Let $A\in\mathcal{A}_1\cap\mathcal{N}$ and $B\in\mathcal{N}$ with $TR(B)\leq 1$. Is it true that $TR(A\otimes B)\leq 1$?

Proposition 5.1. Let A and B be two unital separable simple C^*-algebras in $\mathcal{A}_1\cap\mathcal{N}$. Then $A\otimes B\in\mathcal{A}_1\cap\mathcal{N}$.

Proof. Let $A,B\in\mathcal{A}_1\cap\mathcal{N}$. Then

$$(A\otimes B)\otimes Q\cong(A\otimes B)\otimes(Q\otimes Q)\cong(A\otimes Q)\otimes(B\otimes Q).$$

Since both A and B are in $\mathcal{A}_1\cap\mathcal{N}$, $A\otimes Q$ and $B\otimes Q$ have tracial rank no more than one and satisfy the UCT. Therefore, by Lemma 10.9 and Theorem 10.10 of [21], each of them is isomorphic to some unital simple AH-algebra with no dimension growth. It is then easy to see that $(A\otimes Q)\otimes(B\otimes Q)$ can be written as a unital simple AH-algebra with no dimension growth, which implies that $TR(A\otimes B\otimes Q)\leq 1$. □

Theorem 5.2. Let A be a unital separable simple C^*-algebra. Suppose that there exists a unital separable simple C^*-algebra $B\in\mathcal{A}_1\cap\mathcal{N}$ such that $A\otimes B\in\mathcal{A}_1$, then $A\in\mathcal{A}_1$.

Proof. Since $A \otimes B \in \mathcal{A}_1$, we have that $TR(A \otimes B \otimes Q) \leq 1$. As $B \in \mathcal{A}_1 \cap \mathcal{N}$, we have that $B \otimes Q$ satisfies the UCT and $TR(B \otimes Q) \leq 1$. By Lemma 10.9 and Theorem 10.10 of [21], $B \otimes Q$ is a unital simple AH-algebra with no dimension growth. Note that $\mathrm{Tor}(K_0(B \otimes Q)) = 0$. It follows from Lemma 4.1 (by setting TA$\mathcal{C}$ to TAI algebras) that $A \in \mathcal{A}_1$. □

We now consider the converse of a special case of Theorem 4.2 (when TA$\mathcal{C}$ are just TAI algebras) in the following sense. Let $A \in \mathcal{A}_1 \cap \mathcal{N}$. Is it true that $TR(A \otimes C) \leq 1$ if C is a unital separable amenable infinite dimensional simple C^*-algebra with $TR(C) \leq 1$ and satisfies the UCT? An affirmative answer is given in Theorem 5.6.

Lemma 5.3. Let G be a countable weakly unperforated simple ordered group which is rationally Riesz. Suppose that G also has the following property: for any $x, y \in G$ with $x < y$ and for any integer $N \geq 1$, there exists $z \in G$ such that

$$x < Nz < y. \tag{5.1}$$

Then G has the Riesz interpolation property.

Proof. Let $u \in G_+$ be an order unit. Denote by $S_u(G)$ the state space of G, i.e. the set of order and unit preserving homomorphisms from G to the additive group $\mathbb{R}$. First, we claim the following: For any $a_1, a_2 \in G_+ \backslash \{0\}$, there is $b \in G_+ \backslash \{0\}$ such that

$$0 < b < a_i, \quad i = 1, 2. \tag{5.2}$$

In fact, as G is simple, there exists an integer $n_1 > 0$ such that

$$n_1 a_i > u, \quad i = 1, 2. \tag{5.3}$$

By the assumption, there exists $b_0 \in G$ such that

$$0 < n_1 b_0 < u. \tag{5.4}$$

As G is weakly unperforated, we get

$$0 < b_0 < a_i, \quad i = 1, 2, \tag{5.5}$$

which proves the claim.

Suppose that $x_i \leq y_j$ for $i, j = 1, 2$. We will show that there exists $z \in G$ such that

$$x_i \leq z \leq y_j, \quad i, j = 1, 2. \tag{5.6}$$

If $x_{i'} = y_{j'}$ for some pair of i' and j', choose $z = y_{j'}$. Then $x_i \leq y_{j'} = z = x_{i'} \leq y_i$, $i = 1, 2$. Now assume that $x_i < y_j$ for all i and j.

Since G is rationally Riesz, there are $m, n \in \mathbb{N} \backslash \{0\}$ and $w \in G$ such that

$$nw \leq my_j \ \text{ and } \ mx_i \leq nw, \quad i, j = 1, 2. \tag{5.7}$$

If $nw = mx_{i'} = my_{j'}$ for certain i' and j', then $m(y_{j'} - x_{i'}) = 0$. Since $y_{j'} - x_{i'} > 0$ and G is an ordered group, this is impossible.

If $nw < my_j$ for all j, by the claim above, there exists $b_0 \in G_+$ such that

$$0 < b_0 < my_j - nw, \quad j = 1, 2. \tag{5.8}$$

By the assumption, there exists $z \in G$ such that

$$mx_i \leq nw < mz < nw + b_0 < my_j, \quad j = 1, 2. \tag{5.9}$$

By the weak unperforation,

$$x_i < z < y_j, \quad i, j = 1, 2. \tag{5.10}$$

If $nw > mx_i$, $i = 1, 2$, by the claim above again, there exists $b_0 \in G_+$ such that

$$0 < b_0 < nw - mx_i, \quad i = 1, 2. \tag{5.11}$$

Then, as above, we obtain $z \in G$ such that

$$mx_i < nw - b_0 < mz < nw \leq my_j, \quad i, j = 1, 2. \tag{5.12}$$

We then conclude, as above,

$$x_i < z < y_j, \quad i, j = 1, 2. \tag{5.13}$$

Thus G has the Riesz interpolation property. □

Lemma 5.4. Let $A \in \mathcal{A}_1 \cap \mathcal{N}$. Suppose that B is a unital separable amenable simple C^*-algebra with $TR(B) \leq 1$ which satisfies the UCT. Then $K_0(A \otimes B)$ has the Riesz interpolation property.

Proof. Since $A \in \mathcal{A}_1 \cap \mathcal{N}$ and $TR(B) \leq 1$ where B is amenable and satisfies the UCT, by Proposition 5.1, $A \otimes B \in \mathcal{A}_1 \cap \mathcal{N}$. It follows from [27] that $K_0(A \otimes B)$ is rationally Riesz.

By Lemma 10.9 and Theorem 10.10 of [21], B is isomorphic to a unital simple AH-algebra with no dimension growth. It follows from Theorem 2.1 of [11] that B is approximately divisible. Therefore $A \otimes B$ is approximately divisible. It follows that, for any pair $x, y \in K_0(A \otimes B)$ and any integer $N \geq 1$ with $x < y$, there exists $z \in K_0(A \otimes B)$ such that

$$x < Nz < y. \tag{5.14}$$

Moreover, from the approximate divisibility, by Theorem 1.4 of [1], $A \otimes B$ has the strict comparison for positive elements. In particular, it follows that $K_0(A \otimes B)$ is weakly unperforated. The lemma then follows by applying Lemma 5.3. □

Theorem 5.5. Let $A \in \mathcal{A}_1 \cap \mathcal{N}$. Then, for any unital infinite dimensional simple AH-algebra B with slow dimension growth, $A \otimes B$ is a unital simple AH-algebra with no dimension growth.

Proof. Since $A \in \mathcal{A}_1 \cap \mathcal{N}$, it follows from Proposition 5.1 that $A \otimes B \in \mathcal{A}_1 \cap \mathcal{N}$. By Lemma 5.4, $K_0(A \otimes B)$ has the Riesz interpolation property. Since B is an infinite dimensional simple AH-algebra of no dimension growth, from Theorem 2.1 of [11], B is approximately divisible. So $A \otimes B$ is approximately divisible. It follows that $K_0(A \otimes B) \neq \mathsf{Z}$. Since $A \otimes B \otimes Q$ is a unital simple AH-algebra of no dimension growth, the canonical map $r_{\mathsf{A} \otimes \mathsf{B} \otimes \mathsf{Q}} : T(A \otimes B \otimes Q) \to S_{[1]}(K_0(A \otimes B \otimes Q))$ maps extreme points to extreme points. Therefore the canonical map $r_{\mathsf{A} \otimes \mathsf{B}} : T(A \otimes B) \to S_{[1]}(K_0(A \otimes B))$ maps the extreme points to extreme points (see Lemma 5.6 of [27]). It follows from [35] (see also [32, 34] for more background knowledge) that there is a unital simple AH-algebra C with no dimension growth such that its Elliott invariant is exactly the same as that of $A \otimes B$. According to Theorem 10.4 of [21], we have that $A \otimes B \cong C$. □

We end this note with the following summary:

Theorem 5.6. Let $A \in \mathcal{N}$ be a unital separable simple amenable C^*-algebra that satisfies the UCT. Then the following are equivalent.

(1) $A \in \mathcal{A}_1$;
(2) $TR(A \otimes Q) \leq 1$;
(3) $A \otimes Q \in \mathcal{A}_1$;
(4) $TR(A \otimes B) \leq 1$ for some unital simple infinite dimensional AF-algebra B;
(5) $TR(A \otimes B) \leq 1$ for all unital simple infinite dimensional AF-algebras B;
(6) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional AF-algebra B;
(7) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional AF-algebras B;
(8) $TR(A \otimes B) \leq 1$ for some unital simple infinite dimensional AH-algebra B with no dimension growth;
(9) $TR(A \otimes B) \leq 1$ for all unital simple infinite dimensional AH-algebras B with no dimension growth;
(10) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional AH-algebra B with no dimension growth;
(11) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional AH-algebras B with no dimension growth;
(12) $A \otimes B \in \mathcal{A}_1$ for some unital simple infinite dimensional C^*-algebra B in $\mathcal{A}_1 \cap \mathcal{N}$;
(13) $A \otimes B \in \mathcal{A}_1$ for all unital simple infinite dimensional C^*-algebras B in $\mathcal{A}_1 \cap \mathcal{N}$.

Proof. Note that "(1) $\Rightarrow$ (2)", "(2) $\Rightarrow$ (3)", "(5) $\Rightarrow$ (4)", "(4) $\Rightarrow$ (6)", "(7) $\Rightarrow$ (6)", "(9) $\Rightarrow$ (8)", "(9) $\Rightarrow$ (10)", "(11) $\Rightarrow$ (10)", "(11) $\Rightarrow$ (7)", "(13) $\Rightarrow$ (11)", "(13) $\Rightarrow$ (7)" and "(13) $\Rightarrow$ (12)" are straightforward.

Note that "(1) $\Rightarrow$ (5)" and "(1) $\Rightarrow$ (9)" follow from Theorem 5.5. To see that "(1) $\Rightarrow$ (13)", let $A \in \mathcal{A}_1 \cap \mathcal{N}$ and $B \in \mathcal{A}_1 \cap \mathcal{N}$. Then $TR(B \otimes Q) \leq 1$. So $B \otimes Q$ is a unital simple infinite dimensional AH-algebra with no dimension growth. Since "(1) $\Rightarrow$ (9)", this implies that $TR(A \otimes (B \otimes Q)) \leq 1$. It then follows that $A \otimes B \in \mathcal{A}_1$.

For "(12) $\Rightarrow$ (1)", assume that $TR(A \otimes B \otimes Q) \leq 1$. It follows that $TR(A \otimes (B \otimes Q)) \leq 1$. Since $TR(B \otimes Q) \leq 1$, again, $B \otimes Q$ is a unital simple infinite dimensional AH-algebra with no dimension growth. It follows from Corollary 4.4 that $A \in \mathcal{A}_1$.

That "(3) $\Rightarrow$ (1)" follows from [27] and "(4) $\Rightarrow$ (1)" follows from Corollary 3.7.

For "(6) $\Rightarrow$ (4)", one considers $A \otimes B \otimes Q$ and notes that $B \otimes Q$ is a unital simple infinite dimensional AF-algebra.

That "(8) $\Rightarrow$ (4)" follows from Corollary 4.4.

The rest of the implications follow similarly as established above. □

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