Convex projective structures on nonhyperbolic three-manifolds

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Y Benoist proved that if a closed three-manifold M admits an indecomposable convex real projective structure, then M is topologically the union along tori and Klein bottles of finitely many submanifolds each of which admits a complete finite volume hyperbolic structure on its interior. We describe some initial results in the direction of a potential converse to Benoist's theorem. We show that a cusped hyperbolic three-manifold may, under certain assumptions, be deformed to convex projective structures with totally geodesic torus boundary. Such structures may be convexly glued together whenever the geometry at the boundary matches up. In particular, we prove that many doubles of cusped hyperbolic three-manifolds admit convex projective structures.

1 Introduction

The previous decade has seen tremendous progress in the study of three-dimensional manifolds. Much of that progress stems from Perelman's proof of Thurston's geometrization conjecture, which states that any closed orientable prime three-manifold admits a decomposition into geometric pieces modeled on the eight homogeneous Thurston geometries. However, because these geometric pieces do not glue together in any sensible geometric way, there are some questions about three-manifolds for which a Thurston geometric decomposition of the manifold may not be useful. One example is the question of linearity of three-manifold groups, ie whether a three-manifold fundamental group admits a faithful linear representation and in which dimensions. While, in most cases, the Thurston geometric structure on each piece of a geometric decomposition determines a faithful linear representation of its fundamental group, these representations cannot be directly synthesized into a representation of the fundamental group of the entire manifold. In order to make progress on this and other problems, it

is natural, given a manifold of interest, to search for a homogeneous geometry capable of describing the entire manifold all at once.

This article studies properly convex real projective structures on three-manifolds. A domain Ω in the real projective space $\mathbb{RP}^n$ is called properly convex if there is an affine chart containing Ω in which Ω is convex and bounded. A properly convex projective n–manifold is the quotient $\Gamma\backslash\Omega$ of a convex domain Ω by a discrete group Γ of projective transformations preserving Ω. Given a manifold N, a properly convex projective structure on N is a diffeomorphism of N with some properly convex projective manifold $\Gamma\backslash\Omega$, considered up to a certain equivalence. A convex projective structure therefore induces a representation, called the holonomy representation, identifying $\pi_1 N$ with the discrete subgroup $\Gamma \subset \mathrm{PGL}_{n+1}\mathbb{R}$. Hyperbolic structures are special examples of convex real projective structures, but there are many nonhyperbolic manifolds that admit such structures as well. See Benoist [2; 3] or Kapovich [30] for some examples. See Benoist [4] for a survey of the subject of convex projective structures on closed manifolds.

We mention that there are simple examples of convex projective structures on three-manifolds coming from a convex hull construction applied to lower-dimensional domains; such structures are called *decomposable* and are not of interest to us in the present article. By work of Benoist [2], the Thurston geometric decomposition of any closed three-manifold that admits an indecomposable properly convex projective structure contains only hyperbolic pieces glued together along tori and Klein bottles. We are concerned with the converse problem: if a closed three-manifold N has geometric decomposition containing only hyperbolic pieces, must N admit a properly convex projective structure? Our main theorem gives a positive answer to this question in a special case.

Theorem 1.1 *Let M be a compact, connected, orientable three-manifold with a union of tori as boundary such that the interior of M admits a finite-volume hyperbolic structure which is infinitesimally projectively rigid rel boundary. Then the double $N = 2M$ of M admits a properly convex projective structure.*

Using cube complex techniques, Przytycki and Wise [34] showed that any mixed three-manifold, and therefore any manifold N as in Theorem 1.1, has linear fundamental group. However, their methods give no control on the dimension of the linear representation. On the other hand, it was shown by Button [7] that there exist three-manifold groups which admit no linear representation in dimension 4 or lower. To determine

the smallest possible dimension of a linear representation of a three-manifold group remains an interesting open question. Since the holonomy representation of a convex projective structure lifts to the special linear group (see Section 2.1), we obtain:

Corollary 1.2 *Let $N = 2M$ be as in Theorem 1.1. Then $\pi_1 N$ admits a discrete faithful representation into* $\mathrm{SL}_4\mathbb{R}$.

The corollary says that the property of linearity of the two hyperbolic pieces in N may be extended to all of N. Indeed, the proof of Theorem 1.1 will show that the representation of $\pi_1 N$ in the corollary may be chosen such that the restriction to each copy of $\pi_1 M$ is arbitrarily close to the holonomy representation of the finite-volume hyperbolic structure on M.

The assumption of infinitesimal projective rigidity rel boundary (Definition 3.1), studied by Heusener and Porti [25], is satisfied for many hyperbolic manifolds, for example for infinitely many fillings of one component of the Whitehead link. On the other hand, this assumption does fail in certain cases, for example when the hyperbolic structure on M contains a totally geodesic surface. A related rigidity condition in closed hyperbolic three-manifolds was studied in Cooper, Long and Thistlethwaite [13; 14] and shown experimentally to hold very often in small examples. However, in the setting of cusped hyperbolic manifolds, it is not yet known in what degree of generality infinitesimal projective rigidity rel boundary will hold. Nonetheless, the theorem gives a large new source of examples of convex projective structures on nonhyperbolic three-manifolds. The only other known examples come from taking covers of convex projective reflection orbifolds. Benoist [2] constructed the first example of such an orbifold by realizing a truncation polyhedron, ie a polyhedron obtained from a three-dimensional tetrahedron by successively truncating vertices, as a reflection polyhedron in projective space. Later, Marquis [32] completely classified the three-dimensional convex projective orbifolds obtained from projective truncation polyhedra. Generalizing Benoist's examples, Choi, Lee and Marquis [12] are currently classifying convex projective reflection polyhedra and studying their deformation theory.

1.1 Convex projective structures with totally geodesic boundary

The proof of Theorem 1.1 is motivated by Benoist's beautiful theory [2] describing the geometry of convex projective structures on three-manifolds, which we briefly review here. If $N = \Gamma \backslash \Omega$ is a properly convex projective closed three-manifold which is indecomposable, then either

(i) Ω is strictly convex and N admits a hyperbolic structure, or

(ii) Ω is not strictly convex and the points on $\partial\Omega$ at which strict convexity fails form a dense set in $\partial\Omega$ each component of which bounds a *properly embedded triangle*, which is the intersection of Ω with a hyperplane.

We will refer to these triangles as *totally geodesic triangles*. Each totally geodesic triangle descends to a totally geodesic embedded torus or Klein bottle in N and, after cutting along these tori and Klein bottles, N is decomposed into a union of properly convex submanifolds M_i with *totally geodesic boundary*. Each piece in this decomposition (which topologically is exactly the JSJ or geometric decomposition of N) admits a hyperbolic structure.

In light of the Benoist theory, in order to construct convex projective structures on nonhyperbolic three-manifolds, we first need a source of convex projective building blocks, ie convex projective manifolds with totally geodesic boundary. Under suitable assumptions, we are able to find such structures by deforming the hyperbolic structure.

Theorem 1.3 Let M be a connected, orientable, finite-volume, noncompact hyperbolic three-manifold which is infinitesimally projectively rigid rel boundary. Then M admits nearby properly convex projective structures where each cusp becomes a principal totally geodesic boundary component.

The term *principal totally geodesic boundary* (see Definition 5.2, following Goldman [22] in the setting of convex projective surfaces) refers to a totally geodesic boundary component which admits a convex thickening. That all totally geodesic boundary components are principal is a necessary condition for a convex projective manifold to appear as a submanifold in the Benoist decomposition of a closed convex projective manifold described above.

The deformations described in the previous theorem may be understood by analogy with the related phenomenon of the deformation of a two-dimensional finite-volume hyperbolic surface whose cusp "opens up" to a very small geodesic circle coming in from infinity. Hyperbolic surfaces with geodesic boundary are indeed convex projective structures; the associated convex domain, a subset of the hyperbolic plane, has in its boundary a dense collection of segments, each of which covers the geodesic boundary circle. Although cusp opening is not possible in three-dimensional hyperbolic geometry by Mostow–Prasad rigidity, Theorem 1.3 shows that it is possible in the category of convex projective manifolds. Indeed, as the convex projective structures in the

conclusion of the theorem approach the hyperbolic structure, the totally geodesic boundary tori become very small (with respect to the Hilbert metric) and escape to infinity as the totally geodesic triangles in the boundary of the associated convex domain converge to points; see Figure 1.
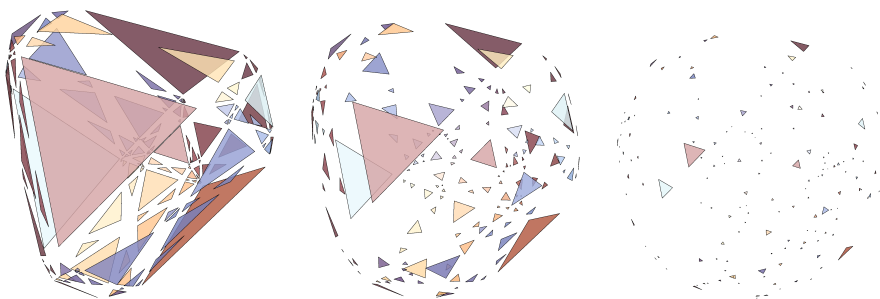

Figure 1: The principal totally geodesic triangles in the boundaries of the convex domains from Theorem 1.3 collapse to points as the convex domains converge to the round ball.

The proof of Theorem 1.3 boils down to a transversality argument in the space $\mathrm{Hom}(\pi_1 \partial M, \mathrm{PGL}_4 \mathbb{R})$ of representations of the peripheral fundamental groups $\pi_1 \partial M$. The assumption of infinitesimal projective rigidity rel boundary, also appearing in Theorem 1.1 above, guarantees that $\mathrm{Hom}(\pi_1 M, \mathrm{PGL}_4 \mathbb{R})$ is smooth at the holonomy representation ρ_{hyp} of the complete hyperbolic structure and that (an augmented version of) the restriction map $\mathrm{res} \colon \mathrm{Hom}(\pi_1 M, \mathrm{PGL}_4 \mathbb{R}) \to \mathrm{Hom}(\pi_1 \partial M, \mathrm{PGL}_4 \mathbb{R})$ submerses a neighborhood of ρ_{hyp} onto a submanifold of $\mathrm{Hom}(\pi_1 \partial M, \mathrm{PGL}_4 \mathbb{R})$. We prove that this submanifold transversely intersects a certain family of diagonalizable representations constructed explicitly in Lemma 4.3. This family of diagonalizable representations is a partial slice (never tangent to conjugation orbits) whose construction is geometric in nature (see Section 4.1 for details). For dimensional reasons the intersection is positive-dimensional and thus we can find representations of $\pi_1 M$ into $\mathrm{PGL}_4 \mathbb{R}$ whose restriction to the fundamental group of each boundary component is diagonalizable over the reals. Finally, we are able to conclude that the resulting representations are the holonomy representations of convex projective structures as in the theorem using a "holonomy principle" that follows from recent work of Cooper, Long and Tillmann [15] or Choi [11].

We note that the proof of Theorem 1.3 encounters immediate problems upon removing the assumption that M is orientable. Indeed, Poincaré duality, applied to both M

and its torus boundary components, is used to determine the dimensions of relevant representation spaces.

1.2 Convex gluing

Consider a finite disjoint union $\mathcal{M}$ of properly convex projective three-manifolds and two disjoint closed submanifolds ∂ and ∂' of its boundary, each of which is a finite disjoint union of principal totally geodesic tori. Given a homeomorphism $f\colon \partial \to \partial'$, let $\mathcal{M}_f$ denote the (topological) manifold obtained by identifying ∂ with ∂' via f; the topology of $\mathcal{M}_f$ depends only on the isotopy class of f. By straightforward general arguments, the projective structure on $\mathcal{M}$ defines a projective structure (in fact many different structures) on $\mathcal{M}_f$ provided that the *holonomy matching condition* is satisfied: for each component ∂_j of ∂, there exists $g_j \in \mathrm{PGL}_4\mathbb{R}$ such that

$$(1\text{-}1) \qquad\qquad \rho'_j(f_*\gamma) = g_j \rho_j(\gamma) g_j^{-1}$$

for all $\gamma \in \pi_1 \partial_j$, where ρ_j and ρ'_j denote the holonomy representations for the projective structures on neighborhoods of ∂_j and $\partial'_j = f(\partial_j)$, respectively. Indeed, if this condition is satisfied then f may be isotoped to a projective map on each component of ∂ and the pieces of $\mathcal{M}$ may be glued together projectively. We prove that the resulting projective structure on $\mathcal{M}_f$ may in fact be taken to be properly convex. The following theorem is the three-dimensional analogue of a result of Goldman [22] in the setting of convex projective surfaces.

Theorem 1.4 Let $\mathcal{M}$ be a finite disjoint union of properly convex projective three-manifolds, let ∂ and ∂' be disjoint closed submanifolds of its boundary, each of which is a disjoint union of principal totally geodesic tori, and let $f\colon \partial \to \partial'$ be a homeomorphism. Assume that the holonomy matching condition (1-1) is satisfied along all components of ∂. Then $\mathcal{M}_f$ admits a projective structure in which all connected components are properly convex. The natural map $\mathcal{M} \to \mathcal{M}_f$ is isotopic to a projective map.

When $\mathcal{M} = M \sqcup M$ is the disjoint union of two copies of the same properly convex projective manifold M with principal totally geodesic boundary and $f\colon \partial M \to \partial M$ is the identity map, the holonomy matching condition (1-1) is trivially satisfied. In this case, $\mathcal{M}_f = 2M$ is the double of M. Hence, Theorems 1.3 and 1.4 imply Theorem 1.1.

In general, any given projective manifold $\mathcal{M}$ and homeomorphism $f\colon \partial \to \partial'$ as in the hypotheses of Theorem 1.4 are unlikely to satisfy the holonomy matching condition (1-1). To find a properly convex projective structure on the glued manifold $\mathcal{M}_f$,

one may attempt to deform the properly convex projective structures on the connected components of $\mathscr{M}$ with the goal of aligning the geometry of the principal totally geodesic boundary tori of ∂ and ∂' so that (1-1) is satisfied. However, global deformation theory problems such as this are in general very difficult. Consider for example the case that $\mathscr{M} = M_1 \sqcup M_2$ is a disjoint union of two three-manifolds, each with boundary homeomorphic to a torus $\partial = \partial M_1$ and $\partial' = \partial M_2$, and that $f\colon \partial \to \partial'$ is any fixed gluing homeomorphism. The space of representations $\mathrm{Hom}(\pi_1 \partial_j, A)$ into the diagonal subgroup $A \subset \mathrm{PGL}_4 \mathbb{R}$ is six-dimensional. Furthermore, the subset of representations which extend to $\pi_1 M_1$ is a half-dimensional (Lagrangian) subvariety, as is the subset of representations which extend via f_* to $\pi_1 M_2$. Therefore, the expected intersection between these two sets is zero-dimensional. We do not know any reason in general to expect this intersection to be nonempty. On the other hand, no example seems to be known in which the intersection turns out to be empty. An experimental study of some basic cases could prove enlightening; the authors hope to conduct such experiments in future work. Of course, even if the matching problem (1-1) is solved at the level of representations, one needs to find convex projective structures realizing those representations as their holonomy representations in order to apply Theorem 1.4. The following theorem shows that any deformation of the holonomy representation of a properly convex projective manifold through representations with diagonalizable peripheral holonomy is indeed the holonomy representation of a properly convex projective structure with totally geodesic boundary.

Theorem 1.5 *Let M be a complete finite-volume hyperbolic three-manifold. Then the set of holonomy representations of properly convex projective structures on M with principal totally geodesic boundary is closed in the subspace of representations in $\mathrm{Hom}(\pi_1 M, \mathrm{PGL}_4 \mathbb{R})$ whose restriction to $\pi_1 \partial M$ is diagonalizable.*

Indeed, this theorem is not difficult using our techniques. Any convex projective structure on M with totally geodesic boundary yields a convex projective structure on the double $2M$ by Theorem 1.4. By a theorem of Benoist [1], the space of holonomy representations of convex projective structures on this closed manifold $2M$ is closed in $\mathrm{Hom}(\pi_1 2M, \mathrm{PGL}_4 \mathbb{R})$ and any deformation of the original structure on $2M$ may be cut back into pieces with principal totally geodesic boundary.

1.3 Gluing together covers of reflection orbifolds

In Section 7, we apply the work of Benoist [2] and Marquis [32] and Theorem 1.4 to produce many new examples of convex projective structures on nonhyperbolic three-

manifolds N which, in contrast to Theorem 1.1, are not doubles. In each of these examples, the pieces $\{M_i\}$ of the JSJ decomposition of N come from covers of a reflection orbifold $\mathcal{O}$ with a cusp ∂ isomorphic to the Euclidean $(3, 3, 3)$–triangle orbifold. Any convex projective structure on $\mathcal{O}$ induces a convex projective structure at infinity on ∂, the space of which is well known to be one-dimensional. Considering only projective structures on the pieces M_i which cover a convex projective structure on such an orbifold $\mathcal{O}$ greatly simplifies the holonomy matching problem of Theorem 1.4. This technique allows us, for example, to identify manifolds in the census of tetrahedral cusped hyperbolic manifolds and gluing maps so that the resulting glued-up manifold admits properly convex projective structures. See Section 7 for precise results.

Acknowledgements

2 Background

2.1 Properly convex domains

The n–dimensional real projective space $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1}\setminus\{0\}$ by the action of $\mathbb{R}^\times$ by scaling. A point in $\mathbb{RP}^n$ is an equivalence class $[v]$ of vectors

$v \in \mathbb{R}^{n+1} \setminus \{0\}$. The projective general linear group $\mathrm{PGL}_{n+1}\mathbb{R}$, the quotient of $\mathrm{GL}_{n+1}\mathbb{R}$ by its center, acts faithfully on $\mathbb{RP}^n$.

The image, or projectivization, of a two-dimensional vector subspace under the quotient map is called a *projective line* and the image of an n–plane in $\mathbb{RP}^n$ is called a *projective hyperplane*. Each hyperplane is determined by a unique projective class of linear functionals on $\mathbb{R}^{n+1}$, so the space of hyperplanes identifies with the dual projective space $\mathbb{RP}^{n*}$, which is the projective space associated to $(\mathbb{R}^{n+1})^*$. This correspondence is known as *projective duality*. The natural action of $\mathrm{GL}_{n+1}\mathbb{R}$ on $(\mathbb{R}^{n+1})^*$ descends to a faithful action of $\mathrm{PGL}_{n+1}\mathbb{R}$ on $\mathbb{RP}^{n*}$.

Let H be a projective hyperplane. The complement of H in $\mathbb{RP}^n$ is called an *affine patch* and is denoted by $\mathbb{A}_H$. Up to change of coordinates by a projective transformation, ie an element of $\mathrm{PGL}_{n+1}\mathbb{R}$, any affine patch $\mathbb{A}_H$ may be identified with the standard affine patch

$$\{[x_1, \ldots, x_n, 1] \in \mathbb{RP}^n \mid (x_1, \ldots, x_n) \in \mathbb{R}^n\}.$$

A subset Ω of $\mathbb{RP}^n$ is called *convex* if it is contained in some affine patch (ie is disjoint from a projective hyperplane) and its intersection with every projective line is connected. If in addition its closure $\overline{\Omega}$ is contained in an affine patch then Ω is *properly convex*. Equivalently, Ω is properly convex if Ω does not contain any complete affine line. The boundary $\partial\Omega := \overline{\Omega} \setminus \mathrm{Int}(\Omega)$ is said to be *strictly convex* at $p \in \partial\Omega$ if p is not contained in the interior of any affine line segment in $\partial\Omega$. If Ω is properly convex and $\partial\Omega$ is strictly convex at every point $p \in \partial\Omega$, then we say that Ω is *strictly convex*.

Every open properly convex domain Ω gives rise to a dual domain

$$\Omega^* = \{[\phi] \in \mathbb{RP}^{n*} \mid \phi(v) \neq 0 \text{ for all } [v] \in \overline{\Omega}\}.$$

It is easily verified that Ω^* is also open, nonempty and properly convex and that $(\Omega^*)^* = \Omega$. For each $p = [v] \in \partial\Omega$ there is a (possibly nonunique) $q = [\phi] \in \partial\Omega^*$ such that $\phi(v) = 0$. The projective hyperplane dual to q is called a *supporting hyperplane* at p. A point $p \in \partial\Omega$ has a unique supporting hyperplane if and only if p is a C^1 point of the boundary.

Let $\mathbb{S}^n$ denote the n–sphere, realized as the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by positive scaling, and let $\pi \colon \mathbb{S}^n \to \mathbb{RP}^n$ denote the two-to-one covering map. The automorphisms of $\mathbb{S}^n$ are given by the linear transformations $\mathrm{SL}_{n+1}^{\pm}\mathbb{R}$ with determinant ± 1. Let $[T] \in \mathrm{PGL}_{n+1}\mathbb{R}$ be an equivalence class of linear transformations. By scaling we may arrange that $T \in \mathrm{SL}_{n+1}^{\pm}\mathbb{R}$. Additionally, since $T \in \mathrm{SL}_{n+1}^{\pm}\mathbb{R}$ if and only if

$-T \in \mathrm{SL}^{\pm}_{n+1}\mathbb{R}$, there is a two-to-one covering $\mathrm{SL}^{\pm}_{n+1}\mathbb{R} \to \mathrm{PGL}_{n+1}\mathbb{R}$, which by abuse we also call π, given by $\pi(T) = [T]$. If Ω is a properly convex domain then we let $\mathrm{SL}^{\pm}(\Omega)$ and $\mathrm{PGL}(\Omega)$ denote the subgroups of $\mathrm{SL}^{\pm}_{n+1}\mathbb{R}$ and $\mathrm{PGL}_{n+1}\mathbb{R}$ preserving $\pi^{-1}(\Omega)$ and Ω, respectively. Then π restricts to a two-to-one covering homomorphism from $\mathrm{SL}^{\pm}(\Omega)$ onto $\mathrm{PGL}(\Omega)$. When Ω is properly convex, a homomorphic section of π is constructed as follows. Since $\Omega \subset \mathbb{RP}^n$ is properly convex, the preimage of Ω under π will consist of two connected components. An element $T \in \mathrm{SL}^{\pm}(\Omega)$ either preserves both of these components individually or it interchanges them. Furthermore, T preserves both components if and only if $-T$ interchanges them. The desired section of π is defined by mapping $[T] \in \mathrm{PGL}(\Omega)$ to the unique lift of $[T]$ to $\mathrm{SL}^{\pm}(\Omega)$ that preserves both components of $\pi^{-1}(\Omega)$. Using this section, we are able to identify $\mathrm{PGL}(\Omega)$ with a subgroup of $\mathrm{SL}^{\pm}(\Omega)$. We may therefore regard elements of $\mathrm{PGL}(\Omega)$ as linear transformations when convenient.

2.2 Projective structures on manifolds

Let M be an n–manifold. A *projective atlas* on M is a collection $\phi_\alpha\colon U_\alpha \to \mathbb{RP}^n$ of charts that cover M with the property that if U_α and U_β are charts with nonempty intersection then $\phi_\alpha \circ \phi_\beta^{-1}$ is locally the restriction of an element of $\mathrm{PGL}_{n+1}\mathbb{R}$. Every projective atlas determines a unique maximal projective atlas and we call a maximal projective atlas on M a *projective structure* on M. In other words, a projective structure on M is a (G, X) structure on M (see Ratcliffe [35] for an introduction to (G, X) structures) where (G, X) is real projective geometry: $G = \mathrm{PGL}_{n+1}\mathbb{R}$ and $X = \mathbb{RP}^n$. A manifold equipped with a projective structure is called a *projective manifold*. Note that a projective manifold is also a smooth manifold.

If M and M' are projective manifolds of the same dimension, then a continuous map $f\colon M \to M'$ is *projective* if, for each pair of charts $\phi\colon U \to \mathbb{RP}^n$ of M and $\psi\colon V \to \mathbb{RP}^n$ of M' such that $U \cap f^{-1}(V) \neq \varnothing$, the map

$$\psi \circ f \circ \phi^{-1}\colon \phi(U \cap f^{-1}(V)) \to \psi(f(U) \cap V)$$

agrees with an element of $\mathrm{PGL}_{n+1}\mathbb{R}$ on each connected component. Such a map is necessarily smooth. If in addition f is a diffeomorphism, we say that f is a *projective equivalence*.

After fixing a universal covering $\widetilde{M} \to M$, the local data of a projective structure may be replaced, via analytic continuation of the charts, by global data (dev, ρ),

where dev: $\widetilde{M} \to \mathbb{RP}^n$ is a local diffeomorphism, called a *developing map*, which is equivariant with respect to a representation $\rho\colon \pi_1 M \to \mathrm{PGL}_{n+1}\mathbb{R}$ called the *holonomy representation* in the sense that

$$\mathrm{dev}(\gamma x) = \rho(\gamma)\mathrm{dev}(x)$$

for all $\gamma \in \pi_1 M$ and all $x \in \widetilde{M}$. Any two developing maps dev and dev$'$ for the same structure satisfy that dev$' = g \circ \mathrm{dev}$ for some $g \in \mathrm{PGL}_{n+1}\mathbb{R}$; the respective holonomy representations are related by conjugation: $\rho' = g\rho g^{-1}$. More generally, two projective structures on M are considered equivalent if developing maps dev and dev$'$ for the respective structures are related by the equation dev$' = g \circ \mathrm{dev} \circ \widetilde{\varphi}$, where $\widetilde{\varphi}$ is the lift to $\widetilde{M}$ of a diffeomorphism φ of M that is isotopic to the identity.

Suppose that we are given a projective structure on M with development pair (dev, ρ). If dev is a diffeomorphism onto a convex (resp. properly convex or strictly convex) domain Ω of $\mathbb{RP}^n$ then we say that the projective structure is *convex* (resp. *properly convex* or *strictly convex*). In this case the holonomy representation ρ is faithful, with image a discrete subgroup of $\mathrm{PGL}(\Omega)$. Here are some useful equivalent characterizations of convexity.

Theorem 2.1 (Goldman [22, Proposition 3.1]) *Let M be a projective n–manifold. Then the following are equivalent:*

(1) *M is convex.*

(2) *Every path in M is homotopic (rel endpoints) to a* unique *geodesic segment, ie a segment that develops into a projective line.*

(3) *M is projectively equivalent to the natural projective structure on $\Gamma\backslash\Omega$ where $\Omega = \mathrm{dev}(\widetilde{M}) \subset \mathbb{RP}^n$ is convex and $\Gamma = \rho(\pi_1 M) \subset \mathrm{PGL}(\Omega)$ is a discrete group acting properly discontinuously and freely on Ω.*

Remark 2.2 If $M = \Gamma\backslash\Omega$ is properly convex then M can be equipped with the Hilbert metric (see de la Harpe [23]). Segments of projective lines, also known as projective geodesics, are always geodesics in the Hilbert metric. If Ω is strictly convex, then Hilbert geodesics are always projective geodesics.

Via the Klein model, the hyperbolic n–space $\mathbb{H}^n$ is realized as a ball in $\mathbb{RP}^n$ and its group of isometries is realized as $\mathrm{PO}(n,1) \subset \mathrm{PGL}_{n+1}\mathbb{R}$ (see Ratcliffe [35] for details). Thus a hyperbolic structure on a manifold is a projective structure and a *complete*

hyperbolic structure is a strictly convex projective structure. Incomplete hyperbolic structures could be convex, eg the interior of the convex core of a convex cocompact hyperbolic structure, or could fail to be convex. For example, given a cusped hyperbolic manifold M, Thurston's generalized Dehn surgery space [37] consists of one complete structure, which is strictly convex, and many incomplete structures, none of which are convex. Some of those incomplete structures can be completed to give hyperbolic structures on Dehn fillings of M, but even those incomplete structures are not convex: the developing map is an infinite-sheeted covering onto the complement of a discrete countable collection of lines.

2.3 Deformation theory and projective manifolds with generalized cusps

There is a natural correspondence between (G, X) structures and their holonomy representations, often called the *holonomy principle* or *Ehresmann–Thurston principle*. The correspondence, originally discovered by Ehresmann [18] and popularized by Thurston [37], is a crucial tool in the study of deformation spaces of geometric structures. We describe this holonomy principle in the projective setting as well as some relevant generalizations in the setting of convex projective structures. For this discussion we let $G = \mathrm{PGL}_{n+1}\mathbb{R}$. Let M be the interior of a compact n–manifold, possibly with or without boundary, let $\Gamma = \pi_1 M$, and let $\mathfrak{D}(M)$ be the space of equivalence classes of marked projective structures on M. Let $\mathrm{Hom}(\Gamma, G)$ denote the space of representations of Γ into G and let $\chi(\Gamma, G)$ denote its quotient by the G action by conjugation. Let $\mathrm{hol}\colon \mathfrak{D}(M) \to \chi(\Gamma, G)$ denote the map that associates to each equivalence class of projective structure the conjugacy class of its holonomy representation. Each space is equipped with a natural topology; we refer the reader to Goldman [21] for details. When M is closed, the holonomy principle states simply that the map $\mathrm{hol}\colon \mathfrak{D}(M) \to \chi(\Gamma, G)$ is a local homeomorphism; in other words, *the small deformations of a projective structure are, up to equivalence, parametrized by small deformations of the conjugacy class of its holonomy representation*. In the case that ∂M is nonempty, the principle holds as stated only once the definition of equivalence of projective structures is relaxed so that two projective structures are considered equivalent if there is a diffeomorphism of M which is projective, with respect to the one structure in the domain and the other in the target, away from some collar neighborhood of ∂M. In general, the holonomy principle does not guarantee any control of the geometry at the boundary; this is an important issue in many studies of deformations of geometric structures (eg in the context of cone-manifold structures; see Hodgson and Kerckhoff [27] or Danciger [17]),

including this one. Proofs of the holonomy principle are found in Canary, Epstein and Green [9] or Goldman [21].

In the context of this paper, we need a more powerful holonomy principle that allows for control of more refined geometric properties, specifically that of convexity. Koszul [31] proved that when M is closed, proper convexity is an open condition: a small deformation of the holonomy representation of a properly convex projective structure is the holonomy representation of a nearby properly convex projective structure. However, when $\partial M \neq \varnothing$, the same statement fails. A simple example, that of incomplete hyperbolic structures near a complete hyperbolic structure, was already given in the previous subsection. We now describe recent work of Cooper, Long and Tillmann [15], which, given a properly convex projective structure satisfying certain assumptions, determines which deformations of the holonomy representation are the holonomy representation of a nearby properly convex projective structure. Further, the result allows for some control over the geometry at the boundary. We note that related results of Choi [10; 11] on projective structures with radial ends also imply the holonomy principle needed in our context.

A *generalized cusp* is a properly convex manifold B (with boundary) such that $B \cong \partial B \times [0, \infty)$, ∂B is compact and strictly convex (ie locally the graph of a strictly convex function), and $\pi_1 B$ is virtually nilpotent. The manifold ∂B is called the *cusp cross-section*. We now discuss some motivating examples for this definition. The first is an end of a finite-volume hyperbolic manifold. In this case the strictly convex boundary ∂B of the generalized cusp is the quotient of a horosphere by a virtually abelian group. The second example, which is more relevant for our purposes, is a regular neighborhood of a principal totally geodesic torus boundary component. The following holonomy principle follows immediately from [15, Theorem 0.1].

Theorem 2.3 Let $M = M^c \cup \mathcal{V}$ be a connected n–manifold such that M^c is compact and connected, $\mathcal{V} \cong \partial \mathcal{V} \times [0, \infty)$ and $\partial M^c = \partial \mathcal{V}$. Suppose that $V_1, \ldots, V_k$ are the components of $\mathcal{V}$ and that ρ is the holonomy of a properly convex projective structure on M in which each V_i is a generalized cusp. If ρ' is sufficiently close to ρ in $\mathrm{Hom}(\pi_1 M, G)$ and $\rho'|_{\pi_1 V_i}$ is the holonomy of a generalized cusp structure on V_i for each i, then ρ' is the holonomy of a properly convex structure on M in which each V_i is a generalized cusp.

We will use this theorem in Section 5 to prove Theorem 1.3.

3 Smoothness

We begin by setting some notation that will be used for the remainder of the article. Henceforth, we let M denote an orientable finite-volume noncompact hyperbolic three-manifold with k torus cusps, which we denote by ∂_i for $1 \le i \le k$. Let $\Gamma = \pi_1 M$ denote the fundamental group of M, and for $1 \le i \le k$ let Δ_i denote a representative of the conjugacy class of peripheral subgroups of the ith cusp. Each Δ_i is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. We denote the groups $\mathrm{PGL}_4\mathbb{R}$, $\mathrm{SL}_4^{\pm}\mathbb{R}$ and $\mathrm{SL}_4\mathbb{R}$ by G, $\widetilde{G}$ and $\widetilde{G}_0$, respectively. All three of these Lie groups are locally isomorphic and thus have the same Lie algebra, which we will denote by $\mathfrak{g}$.

Let ρ_{hyp} be a representative of the unique conjugacy class of discrete faithful representations of Γ into the isometry group of hyperbolic three-space. Via the Klein model, the isometry group of hyperbolic three-space is realized as the subgroup $\mathrm{PO}(3, 1)$ of G that preserves the standard round ball $\mathbb{H}^3$ in $\mathbb{RP}^3$. Hence, we may regard ρ_{hyp} as a point in the representation space $\mathrm{Hom}(\Gamma, G)$. We also, by abuse, regard ρ_{hyp} as a point in the quotient $\chi(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/G$ of the representation space by the action of G by conjugation. The sense in which the quotient is taken may be left ambiguous; if one desires $\chi(\Gamma, G)$ to have the structure of an algebraic variety, then the Mumford geometric invariant theory quotient is needed rather than the naive topological quotient. It is a standard fact that the two quotients agree (as topological spaces) locally near any irreducible representation such as ρ_{hyp}.

We will also need to study the representation space $\mathrm{Hom}(\Delta_i, G)$ of a peripheral subgroup Δ_i, which has no irreducible representations. In this case the quotient by conjugation is not well behaved. In the main sections of the paper, we will avoid such issues by working exclusively in Hom rather than in the quotient χ.

This section is dedicated to some basic results about the local structure of $\mathrm{Hom}(\Gamma, G)$ and $\chi(\Gamma, G)$ near ρ_{hyp}. These results are straightforward and many of them seem to be well known to experts, although we are not aware of their existence anywhere in the current literature. Some analogous results from the context of deformations in $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ can be found in Boden and Friedl [5], Heusener and Medjerab [24] and Heusener, Porti and Suárez Peiró [26]. The main result of this section shows that if a certain cohomological condition is satisfied (Definition 3.1), then ρ_{hyp} is a smooth point of $\mathrm{Hom}(\Gamma, G)$ and $\chi(\Gamma, G)$. Since we are only interested in local behavior, it is equivalent and will be marginally less cumbersome to work in the representation space $\mathrm{Hom}(\Gamma, \widetilde{G})$ for the matrix group $\widetilde{G}$. Note that, since ρ_{hyp} is the holonomy

representation of a convex projective structure, it admits a unique lift, denoted by abuse again by ρ_{hyp}, to $\widetilde{G}$ and so does every nearby representation (see Section 2.1).

3.1 Infinitesimal deformations and cohomology

Let $\rho_t\colon \Gamma \to \widetilde{G}$ be a smooth path of representations into the matrix group $\widetilde{G}$. Near $t = 0$, we may express the path ρ_t as

$$\rho_t(\gamma) = \exp(tu(\gamma) + O(t^2))\rho(\gamma),$$

where $u\colon \Gamma \to \mathfrak{g}$, defined by $\gamma \mapsto \frac{d}{dt}\big|_{t=0}\rho_t(\gamma)\rho(\gamma)^{-1}$, is a group cocycle with coefficients in $\mathfrak{g}$ twisted by the adjoint action of $\rho = \rho_0$. We denote the set of such cocycles, defined by the condition $u(\gamma_1\gamma_2) = u(\gamma_1) + \text{Ad}_{\rho(\gamma_1)}u(\gamma_2)$, by $Z^1_\rho(\Gamma, \mathfrak{g})$ and refer to its elements as *infinitesimal deformations* of ρ; it is the Zariski tangent space of $\text{Hom}(\Gamma, \widetilde{G})$ at ρ. The representation ρ is a smooth point of $\text{Hom}(\Gamma, \widetilde{G})$ if and only if each infinitesimal deformation u is *integrable*, ie u is tangent to some path ρ_t as above. The space of coboundaries, denoted by $B^1_\rho(\Gamma, \mathfrak{g})$, is the subspace of those cocycles b satisfying the *infinitesimal conjugacy* condition: that there exists $v \in \mathfrak{g}$ such that $b(\gamma) = v - \text{Ad}_{\rho(\gamma)}v$. Each such coboundary b is tangent to the conjugation path $\rho_t = c_t\rho c_t^{-1}$ at $t = 0$, where $c_t = \exp(tv)$. The first cohomology group $H^1_\rho(\Gamma, \mathfrak{g}) = Z^1_\rho(\Gamma, \mathfrak{g})/B^1_\rho(\Gamma, \mathfrak{g})$ with coefficients in $\mathfrak{g}$ twisted by the adjoint action of ρ describes the infinitesimal deformations of ρ up to infinitesimal conjugacy. In the case that ρ determines a smooth point of $\chi(\Gamma, \widetilde{G})$, this cohomology group describes its Zariski tangent space.

The higher cohomology groups, with twisted coefficients in $\mathfrak{g}$, will not be of use to us except in the following subsection. Given an infinitesimal deformation u, there is an infinite sequence of obstructions to the integrability of u, each of which is an element of the second cohomology group $H^2_\rho(\Gamma, \mathfrak{g})$.

It will be important to understand the relationship between the deformations of the representations of Γ with the deformations of representations of the peripheral subgroups $\Delta_1, \ldots, \Delta_k$. The *restriction map*

$$\text{res}\colon \text{Hom}(\Gamma, \widetilde{G}) \to \text{Hom}(\Delta_1, \widetilde{G}) \times \cdots \times \text{Hom}(\Delta_k, \widetilde{G})$$

is the product $\text{res} = \text{res}_1 \times \cdots \times \text{res}_k$, where $\text{res}_i\colon \text{Hom}(\Gamma, \widetilde{G}) \to \text{Hom}(\Delta_i, \widetilde{G})$ denotes the restriction map induced by the inclusion $\Delta_i \hookrightarrow \Gamma$ of the ith peripheral subgroup into Γ. Each such map res_i defines a restriction map on group cohomology,

$(\text{res}_i)_*\colon H^1_\rho(\Gamma, \mathfrak{g}) \to H^1_{\text{res}_i \rho}(\Delta_i, \mathfrak{g})$, and it is convenient to synthesize these into one linear map

$$\text{res}_*\colon H^1_\rho(\Gamma, \mathfrak{g}) \to \bigoplus_{i=1}^{k} H^1_{\text{res}_i \rho}(\Delta_i, \mathfrak{g}),$$

defined by

$$\text{res}_* = (\text{res}_1)_* \oplus \cdots \oplus (\text{res}_k)_*.$$

When clear, we will abuse notation, using res and res_i to denote the restriction map both on representations and on cohomology. We will also conserve space using $H^*_\rho(\Delta_i, \mathfrak{g})$ to mean $H^*_{\text{res}_i \rho}(\Delta_i, \mathfrak{g})$.

We note that, since M is aspherical, the group cohomology groups $H^*_\rho(\Gamma, \mathfrak{g})$ coincide with the de Rham cohomology groups $H^*_\rho(M, \mathfrak{g})$ with coefficients in the flat $\mathfrak{g}$–bundle over M associated to ρ. Similarly, there is a natural identification between $H^*_\rho(\Delta_i, \mathfrak{g})$ and $H^*_\rho(\partial_i, \mathfrak{g})$ for each $1 \leq i \leq k$ and between $\bigoplus_{i=1}^{k} H^*_\rho(\Delta_i, \mathfrak{g})$ and $H^*_\rho(\partial M, \mathfrak{g})$. Although it will be more convenient for us to work with group cohomology, this identification makes available the tools commonly used in the study of cohomology of manifolds, eg the long exact sequence of a relative pair and Poincaré duality (see eg Heusener and Porti [25] for details).

3.2 Infinitesimal rigidity implies smoothness

The following property, introduced and studied by Heusener and Porti [25], is the critical assumption in Theorems 1.1 and 1.3.

Definition 3.1 Let $\rho_{\text{hyp}}\colon \Gamma \to \text{SO}(3, 1) \subset \widetilde{G}$ denote the holonomy representation of the complete finite-volume hyperbolic structure on M. Then M is called *infinitesimally projectively rigid rel ∂M* if the restriction map res$\colon H^1_{\rho_{\text{hyp}}}(\Gamma, \mathfrak{g}) \to \bigoplus_{i=1}^{k} H^1_{\rho_{\text{hyp}}}(\Delta_i, \mathfrak{g})$ is an injection.

The main theorem of this section is:

Theorem 3.2 *Let M be an orientable complete finite-volume hyperbolic manifold with fundamental group Γ, and let $\rho_{\text{hyp}}\colon \Gamma \to \text{SO}(3, 1) \subset \widetilde{G}$ be the holonomy representation of the complete hyperbolic structure. If M is infinitesimally projectively rigid rel ∂M, then ρ_{hyp} is a smooth point of $\text{Hom}(\Gamma, \widetilde{G})$ and its conjugacy class is a smooth point of $\chi(\Gamma, \widetilde{G})$.*

Remark 3.3 Heusener and Porti [25] prove that the condition of infinitesimal projective rigidity rel boundary persists under infinitely many Dehn fillings. They then

show that there are infinitely many examples of one-cusped hyperbolic 3–manifolds that satisfy the condition by studying fillings of the Whitehead link, a two-cusped manifold that satisfies the condition. For example, the figure-eight knot complement as well as all but finitely many twist knots are infinitesimally projective rigid rel boundary. In future work, we hope to determine exactly which manifolds of the Hodgson–Weeks cusped census are infinitesimally projectively rigid rel boundary. A related rigidity condition in closed hyperbolic three-manifolds was studied in Cooper, Long and Thistlethwaite [13; 14] and shown experimentally to hold very often in small examples.

The proof of Theorem 3.2 requires several lemmas. Note that Lemma 3.4 does not require M to be orientable, but that Lemma 3.6 does.

Lemma 3.4 *For each $1 \leq i \leq k$, the restriction $\rho_{\mathrm{hyp}_i} = \mathrm{res}_i(\rho_{\mathrm{hyp}})$ of ρ_{hyp} to the ith peripheral subgroup is a smooth point of $\mathrm{Hom}(\Delta_i, \widetilde{G})$. Hence, $\mathrm{res}(\rho_{\mathrm{hyp}})$ is a smooth point of $\mathrm{Hom}(\Delta_1, \widetilde{G}) \times \ldots \times \mathrm{Hom}(\Delta_k, \widetilde{G})$.*

Proof We may work in $\widetilde{G}_0 = \mathrm{SL}_4\mathbb{R}$ in place of $\widetilde{G}$, since ρ_{hyp_i} has image in this smaller group. The variety $\mathrm{Hom}(\Delta_i, \widetilde{G}_0)$ is the set of real points of a complex affine variety $\mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$, which is defined over $\mathbb{R}$. It therefore suffices to prove that ρ_{hyp_i} is a smooth point of $\mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$. By Theorem C of Richardson [36], $\mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$ is an irreducible (complex) affine variety and contains a dense set of representations whose images consist of diagonalizable representations. Thus $\mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$ is an 18–dimensional complex variety. Heuristically this can be seen as follows: pick two generators γ_1 and γ_2 for Δ_i. An element $\rho \in \mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$ can map γ_1 arbitrarily and thus contributes 15 degrees of freedom. The only condition on $\rho(\gamma_2)$ is that it commute with $\rho(\gamma_1)$. Generically, the image of γ_1 will be diagonalizable with distinct eigenvalues, so the centralizer $Z(\rho(\gamma_1))$ is conjugate to the diagonal subgroup, which is three-dimensional.

To prove the lemma, we must therefore show that the Zariski tangent space to the variety $\mathrm{Hom}(\Delta_i, \mathrm{SL}_4\mathbb{C})$ is 18–dimensional at ρ_{hyp_i}. The image of Δ_i under ρ_{hyp_i} is conjugate to a lattice in the Lie group of matrices of the form

$$\begin{pmatrix} 1 & u & v & \frac{1}{2}(u^2 + v^2) \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & v \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation shows that the infinitesimal centralizer $H^0_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g})$ is 3–dimensional. By Poincaré duality (see eg Hodgson and Kerckhoff [27]),

$$\dim H^0_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g}) = \dim H^2_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g}),$$

and therefore, since the Euler characteristic of ∂M is zero, we have that $H^1_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g})$ has dimension six. Furthermore,

$$\dim H^0_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g}) + \dim B^1_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g}) = \dim \mathfrak{g} = 15,$$

so the coboundaries $B^1_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g})$ are 12–dimensional. Hence the Zariski tangent space $Z^1_{\rho_{\mathrm{hyp}_i}}(\Delta_i, \mathfrak{g})$ has dimension 18, as desired. $\qquad\square$

Remark 3.5 Despite Lemma 3.4, the representation ρ_{hyp_i} does *not* determine a smooth point of $\chi(\Delta_i, \widetilde{G})$. Indeed, the conjugacy class of ρ_{hyp_i} contains the trivial representation in its closure. Hence, the naive quotient of $\mathrm{Hom}(\Delta_i, \widetilde{G})$ by conjugation is not Hausdorff at this point. The Mumford GIT quotient avoids failure of Hausdorffness by identifying ρ_{hyp_i} with the trivial representation, which is not even a smooth point of $\mathrm{Hom}(\Delta_i, \widetilde{G})$.

Next, we derive some relevant information about second cohomology groups from the infinitesimal projective rigidity condition.

Lemma 3.6 *If M is infinitesimally rigid rel boundary then the map*

$$\mathrm{res}_*\colon H^2_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g}) \to H^2_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g})$$

is an injection.

Proof Consider the long exact sequence of the pair $(M, \partial M)$, where the first injection is by assumption,

$$0 \to H^1_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g}) \xhookrightarrow{\mathrm{res}_*} H^1_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g}) \to H^2_{\rho_{\mathrm{hyp}}}(M, \partial M, \mathfrak{g}) \to H^2_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g}) \xrightarrow{\mathrm{res}_*} H^2_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g}).$$

We showed above that $\dim H^1_{\rho_{\mathrm{hyp}_i}}(\partial_i, \mathfrak{g}) = 6$ for each i and so $\dim H^1_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g}) = 6k$. By a standard Poincaré duality argument, known as "half lives, half dies" (see eg Hodgson and Kerckhoff [27]), the image of $H^1_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g})$ under the restriction map res_* is $3k$–dimensional and so $H^1_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g})$ is itself $3k$–dimensional. By Poincaré duality, we have that the group $H^2_{\rho_{\mathrm{hyp}}}(M, \partial M, \mathfrak{g})$ is also $3k$–dimensional. We conclude that the map $H^1_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g}) \to H^2_{\rho_{\mathrm{hyp}}}(M, \partial M, \mathfrak{g})$ must be a surjection and by exactness that $\mathrm{res}_*\colon H^2_{\rho_{\mathrm{hyp}}}(M, \mathfrak{g}) \to H^2_{\rho_{\mathrm{hyp}}}(\partial M, \mathfrak{g})$ is also injective. $\qquad\square$

We now prove Theorem 3.2.

Proof of Theorem 3.2 To prove Theorem 3.2 we must show that each infinitesimal deformation $u \in Z^1_{\rho_{hyp}}(\Gamma, \mathfrak{g})$ is integrable. It is well known that integrability of u follows if we can show that the infinitely many obstructions all vanish (see eg Heusener and Porti [25, Section 8.2.4]). The obstructions to integrability are cohomology classes in $H^2_{\rho_{hyp}}(\Gamma, \mathfrak{g})$. However, if one of these obstruction classes were nonzero, it would, by Lemma 3.6, map to a nonzero class in $\bigoplus_{i=1}^{k} H^2_{\rho_{hyp}}(\Delta_i, \mathfrak{g})$, obstructing the integrability of $\text{res}_* u$ in $\bigoplus_{i=1}^{k} H^1_{\rho_{hyp}}(\Delta_i, \mathfrak{g})$. By Lemma 3.4, this is impossible. Hence, all the obstructions vanish and u is an integrable infinitesimal deformation. This proves that $\text{Hom}(\Gamma, \widetilde{G})$ is smooth at ρ_{hyp}. Since ρ_{hyp} is an irreducible representation, the orbits of ρ_{hyp} and all nearby representations are closed. Since the centralizer of all representations near ρ_{hyp} is constant (equal to $\pm I$), we conclude that $\chi(\Gamma, \widetilde{G})$ is a manifold near the conjugacy class of ρ_{hyp}. $\qquad\square$

3.3 The augmented restriction map

In this subsection, we formulate and prove some results that are needed for the main transversality argument in the next section, where the basic goal will be to find deformations of the discrete faithful SO(3, 1) representation with certain desired behavior along each of the peripheral subgroups. More specifically, we will construct a submanifold $\mathscr{S} \subset \text{Hom}(\mathbb{Z} \times \mathbb{Z}, \widetilde{G})$ consisting of representations with the desired behavior and then look for representations in $\text{Hom}(\Gamma, \widetilde{G})$ whose restriction to each peripheral subgroup are conjugate into $\mathscr{S}$. For technical reasons, in the execution of this strategy it will be more convenient to work with the following augmented restriction map.

Definition 3.7 Let M be a finite-volume hyperbolic 3–manifold with k torus cusps and with fundamental group Γ. Then we define the *augmented restriction map of M*, denoted by $\widetilde{\text{res}} \colon \text{Hom}(\Gamma, \widetilde{G}) \times \widetilde{G}^{k-1} \to \text{Hom}(\Delta_1, \widetilde{G}) \times \cdots \times \text{Hom}(\Delta_k, \widetilde{G})$, by the formula

$$(\rho, g_2, \ldots g_k) \mapsto (\text{res}_1(\rho), c(g_2) \cdot \text{res}_2(\rho), \ldots, c(g_k) \cdot \text{res}_k(\rho)),$$

where $c(g)$ denotes the conjugation action by g.

The main result about $\widetilde{\text{res}}$ that we will need for the transversality argument in Section 4 is:

Theorem 3.8 *Let M be an orientable finite-volume hyperbolic 3–manifold with k cusps and with fundamental group Γ, and let ρ_{hyp} be the holonomy representation*

of the complete hyperbolic structure. Assume that M satisfies Theorem 3.2, so that $\mathrm{Hom}(\Gamma, \widetilde{G})$ is smooth at ρ_{hyp}. Then the augmented restriction map

$$\widetilde{\mathrm{res}} \colon \mathrm{Hom}(\Gamma, \widetilde{G}) \times \widetilde{G}^{k-1} \to \mathrm{Hom}(\Delta_1, \widetilde{G}) \times \cdots \times \mathrm{Hom}(\Delta_k, \widetilde{G})$$

is a local submersion onto a submanifold of codimension $3k$ at $(\rho_{\mathrm{hyp}}, g_2, \ldots, g_k)$, where $g_2, \ldots, g_k$ are any elements of $\widetilde{G}$.

Proof Let U be a smooth neighborhood of ρ_{hyp} in $\mathrm{Hom}(\Gamma, \widetilde{G})$ whose elements are all irreducible representations, and let $W = U \times \widetilde{G}^{k-1}$. The proof proceeds by showing that the rank of the augmented restriction map $\widetilde{\mathrm{res}}$ is constant and equal to $15k$ for the points in W.

We begin by identifying the relevant tangent spaces. The tangent space $T_\rho \mathrm{Hom}(\Gamma, \widetilde{G})$ is $Z^1_\rho(\Gamma, \mathfrak{g})$ and the tangent space $T_\rho \mathrm{Hom}(\Delta_i, \widetilde{G})$ is $Z^1_\rho(\Delta_i, \mathfrak{g})$. Since any $\rho \in U$ is irreducible, we may identify $T_g \widetilde{G} = B^1_{c(g)\rho}(\Gamma, \mathfrak{g})$ for any $g \in \widetilde{G}$. As a result, for $p = (\rho, g_2, \ldots, g_k) \in W$, we may identify

$$T_p(\mathrm{Hom}(\Gamma, \widetilde{G}) \times \widetilde{G}^{k-1}) = Z^1_\rho(\Gamma, \mathfrak{g}) \oplus \bigoplus_{i=2}^{k} B^1_{c(g_i)\rho}(\Gamma, \mathfrak{g}).$$

The following diagram commutes:

$$
\begin{array}{ccc}
Z^1_\rho(\Gamma, \mathfrak{g}) \oplus \bigoplus_{i=2}^{k} B^1_{c(g_i)\rho}(\Gamma, \mathfrak{g}) & \xrightarrow{\widetilde{\mathrm{res}}_*} & Z^1_\rho(\Delta_1, \mathfrak{g}) \oplus \bigoplus_{i=2}^{k} Z^1_{c(g_i)\rho}(\Delta_i, \mathfrak{g}) \\
\Big\downarrow{\varpi_1} & & \Big\downarrow{\varpi_2} \\
H^1_\rho(\Gamma, \mathfrak{g}) & \xrightarrow{\mathrm{res}_*} & \bigoplus_{i=1}^{k} H^1_\rho(\Delta_i, \mathfrak{g})
\end{array}
$$

where ϖ_1 is projection to cohomology and ϖ_2 is projection to cohomology in each factor followed by the identification of $H^1_\rho(\Delta_i, \mathfrak{g})$ and $H^1_{c(g_i)\rho}(\Delta_i, \mathfrak{g})$ induced by $c(g_i)$. Then $\ker(\varpi_1)$ is given by $B^1_\rho(\Gamma, \mathfrak{g}) \oplus \bigoplus_{i=2}^{k} B^1_{c(g_i)\rho}(\Gamma, \mathfrak{g})$ and $\ker(\varpi_2)$ is given by $B^1_\rho(\Delta_1, \mathfrak{g}) \oplus \bigoplus_{i=2}^{k} B^1_{c(g_i)\rho}(\Delta_i, \mathfrak{g})$, so it is clear that $\widetilde{\mathrm{res}}_*(\ker(\varpi_1)) = \ker(\varpi_2)$. After possibly shrinking U we may assume for any $\rho \in U$ that, for each i, the dimension of $H^1_\rho(\Delta_i, \mathfrak{g})$ is 6, that $\mathrm{Hom}(\Delta_i, \widetilde{G})$ is smooth and 18–dimensional at $c(g_i)\mathrm{res}_i(\rho)$, and therefore that the dimension of $\ker(\varpi_2)$ is $12k$. Furthermore, the "half lives, half dies" argument from Lemma 3.6 shows that the rank of $\mathrm{res}_* \colon H^1_\rho(\Gamma, \mathfrak{g}) \to \bigoplus_{i=1}^{k} H^1_\rho(\Delta_i, \mathfrak{g})$, and hence of $\varpi_2 \circ \widetilde{\mathrm{res}}_*$, is $3k$. Combining these facts we see that the rank of $\widetilde{\mathrm{res}}_*$ must be $15k$. □

Remark 3.9 When $k = 1$, res $= \widetilde{\text{res}}$ are the same. However, for $k \geq 2$, while the local image of res is still a smooth submanifold, the codimension is larger than that of $\widetilde{\text{res}}$.

4 Diagonalizable peripheral holonomy

Recall the notation $G = \text{PGL}_4\mathbb{R}$, $\widetilde{G} = \text{SL}_4^{\pm}\mathbb{R}$ and $\widetilde{G}_0 = \text{SL}_4\mathbb{R}$ from the previous section. Also recall that M is an orientable finite-volume hyperbolic three-manifold with k torus cusps, $\Gamma = \pi_1 M$, the ith cusp is denoted by ∂_i, and Δ_i is a peripheral subgroup for ∂_i. In this section we prove:

Theorem 4.1 *Assume that M is infinitesimally projectively rigid rel ∂M. Then there exists a smooth path of representations $\rho_t \in \text{Hom}(\Gamma, \widetilde{G})$ such that $\rho_0 = \rho_{\text{hyp}}$ is the holonomy representation of the complete hyperbolic structure and $\rho_t(\Delta_i)$ is diagonalizable over the reals for all $t \neq 0$ and $i \in \{1, \ldots, k\}$.*

The proof is a transversality argument in the product of the representation spaces of the boundary tori. Let us give the rough idea in the simpler case that there is only one cusp, whose peripheral subgroup we denote by Δ: Theorem 3.8 gives that the restriction map res maps a neighborhood of ρ_{hyp} in $\text{Hom}(\Gamma, \widetilde{G})$ onto a submanifold of $\text{Hom}(\Delta, \widetilde{G})$. This submanifold has codimension three and is smoothly foliated by conjugation orbits. Now, the key ingredient for the proof is the construction of a smooth four-dimensional partial slice $\mathscr{S}$ in $\text{Hom}(\Delta, \widetilde{G})$ which is transverse to res at ρ_{hyp} and all of whose representations are either diagonalizable with real eigenvalues or lie in a unipotent (parabolic) subgroup of $\text{SO}(3, 1)$. The transverse intersection of $\mathscr{S}$ with the image of res gives a one-dimensional family in $\text{Hom}(\Delta, \widetilde{G})$ through ρ_{hyp}, in which all representations, except the restriction of ρ_{hyp}, are diagonalizable. This one-dimensional family is the image of a one-dimensional path in $\text{Hom}(\Gamma, \widetilde{G})$, as desired.

Let us begin the proof by describing the four-dimensional partial slice in the following general setting: let $\Delta \cong \mathbb{Z} \times \mathbb{Z}$ and let ρ_{hyp} be a representation taking Δ to a lattice in a unipotent subgroup of $\text{SO}(3, 1)$. The four-dimensional partial slice $\mathscr{S}$ is the image of the following map $\Phi : \mathbb{R}^4 \to \text{Hom}(\Delta, \widetilde{G}_0)$. We use coordinates (a, b, x, y) on $\mathbb{R}^4$ and generators γ_1, γ_2 for $\Delta \cong \mathbb{Z} \times \mathbb{Z}$. Define

$$\Phi(a, b, x, y)(\gamma_1) := \exp \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & a & b & 1 \\ 0 & b & -a & 0 \\ 0 & 2(a^2 + b^2) & 0 & 0 \end{pmatrix},$$

$$\Phi(a, b, x, y)(\gamma_2) := \exp \begin{pmatrix} 0 & x & y & 0 \\ 0 & ax + by & bx - ay & x \\ 0 & bx - ay & -ax - by & y \\ 0 & 2(a^2 + b^2)x & 2(a^2 + b^2)y & 0 \end{pmatrix}.$$

A simple computation checks that $\Phi(a, b, x, y)(\gamma_1)$ and $\Phi(a, b, x, y)(\gamma_2)$ commute. We may assume using conjugacy that ρ_{hyp} coincides with $\Phi(0, 0, u, v)$, where $u + iv$ is the cusp shape of the cusp $\rho_{\text{hyp}}(\Delta) \backslash \mathbb{H}^3$ with respect to the generators γ_1 and γ_2. When $a = b = 0$, and x and y are allowed to vary, Φ gives a global slice for the discrete faithful unipotent $SO_0(3, 1)$ representations of Δ; the restriction of Φ to the xy–coordinate plane is well known to be transverse to conjugation. Geometrically, $\Phi(0, 0, x, y)$ parametrizes the conjugacy classes of holonomy representations of all possible torus cusps of hyperbolic three-manifolds.

Definition 4.2 We refer to the collection of representations $\Phi(0, 0, x, y)$ for all $y \neq 0$ as the *cusp shape locus*.

The following lemma is the most important ingredient in the proof of Theorem 4.1. Its proof will be given in the following subsection.

Lemma 4.3 *Let $u, v \in \mathbb{R}$ with $v \neq 0$. Then, near the point $(a, b, x, y) = (0, 0, u, v)$, the map Φ is a local immersion which is never tangent to conjugation orbits. Further, for each $(a, b) \neq (0, 0)$, the representation $\Phi(a, b, x, y)$ is diagonalizable over the reals.*

The eigenvalues of the generators $\Phi(a, b, x, y)(\gamma_i)$ may be computed explicitly. They are most naturally described using certain branched polar coordinates around the cusp shape locus: $(a, b, x, y) = (t \cos 3\theta, t \sin 3\theta, x, y)$. In these coordinates the eigenvalues of $\Phi(t \cos 3\theta, t \sin 3\theta, x, y)(\gamma_1)$ are

$$\left(1, \exp(2t \cos \theta), \exp(-t(\cos \theta + \sqrt{3} \sin \theta)), \exp(t(-\cos \theta + \sqrt{3} \sin \theta))\right),$$

and the eigenvalues of $\Phi(t \cos 3\theta, t \sin 3\theta, x, y)(\gamma_2)$ (listed with respect to the same eigenbasis) are

$$\left(1, \exp(2t(x \cos \theta + y \sin \theta)), \exp\left(-t((x - \sqrt{3}y) \cos \theta + (\sqrt{3}x + y) \sin \theta)\right),\right.$$
$$\left.\exp\left(-t((x + \sqrt{3}y) \cos \theta + (-\sqrt{3}x + y) \sin \theta)\right)\right).$$

Note that when moving away from the cusp shape locus (ie increasing t from zero) in any direction (ie for any value of 3θ), the four eigenvalues of $\Phi(t \cos 3\theta, t \sin 3\theta, x, y)(\gamma)$

vary as smooth real-valued functions of t with distinct first derivative for at least some (generic) $\gamma \in \Delta$. However, more is required to show that Φ is not tangent to the conjugation orbit (see Remark 4.9). The complete proof of Lemma 4.3 will be given in the following subsection, together with a more geometric description of the representations $\Phi(a, b, x, y)$.

Let us now return to the context of Theorem 4.1. In order to state the next lemma, let us introduce a useful splitting of $\mathfrak{g}$ (see Johnson and Millson [28] for more details). Let $\mathfrak{g} = \mathfrak{so}(3, 1) \oplus \mathfrak{v}$ be the Killing-orthogonal splitting of the Lie algebra $\mathfrak{g}$; the splitting is invariant under the adjoint action of $\mathrm{O}(3, 1)$. Since the representation ρ_{hyp} has image in the subgroup $\mathrm{O}(3, 1) \subset \widetilde{G}$, all relevant cohomology groups split and the restriction map res: $H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{g}) \to \bigoplus_{i=1}^{k} H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{g})$ splits into the direct sum of the two maps

$$\mathrm{res}_{\mathfrak{so}(3,1)}: H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3, 1)) \to \bigoplus_{i=1}^{k} H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{so}(3, 1)),$$

$$\mathrm{res}_{\mathfrak{v}}: H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v}) \to \bigoplus_{i=1}^{k} H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{v}).$$

Note that $\dim H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{v}) = 2$ for each $i \in \{1, \ldots, k\}$ (see Heusener and Porti [25, Section 5.1]).

Lemma 4.4 *There exists a cohomology class $[z] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v})$ of infinitesimal deformations whose restriction $\mathrm{res}_i([z]) \in H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{v})$ is nontrivial for all $i \in \{1, \ldots, k\}$.*

Proof The image $L = \mathrm{res}(H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v}))$ is a Lagrangian subspace (see [25, Section 5.1]) for the cup product pairing in $\bigoplus_{i=1}^{k} H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{v})$ and thus has dimension k. Note that under the cup product pairing, the direct sum is orthogonal. The projection onto any single factor is not zero, or else L would be a Lagrangian subspace of the direct sum of the complementary $k - 1$ factors, which is impossible since $\dim L = k$. Hence, for each $j \in \{1, \ldots, k\}$, there exists $[z_j] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v})$ such that $\mathrm{res}_j([z_j]) \neq 0$. Then some linear combination $[z]$ of $\{[z_j]\}_{j=1}^{k}$ satisfies the conclusion of the lemma. □

The next lemma is a basic consequence of Calabi–Weil rigidity.

Lemma 4.5 *Let $i \in \{1, \ldots, k\}$, let μ_i be a nontrivial element of Δ_i, and let $[v] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3, 1))$ have nontrivial restriction to μ_i. Then the eigenvalues of at least one element of Δ_i (although possibly not μ_i) become complex along any path in $\mathrm{Hom}(\Gamma, \widetilde{G})$ tangent to any cocycle representative $v \in Z^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{g})$ of the class $[v]$.*

Proof Let ρ_t be any path of representations which is tangent to v. Let us first assume that the representations ρ_t remain in $\mathrm{SO}(3,1)$. Then the restriction $\mathrm{res}_i(\rho_t)$ lies in $\mathrm{SO}_0(3,1) \cong \mathrm{PSL}_2\mathbb{C}$ and we may regard the image under ρ_t of any individual element as either a 4×4 real matrix or a 2×2 complex matrix. It follows easily from Calabi–Weil rigidity [8; 40; 41] (or see Kapovich [29]) that the $\mathrm{PSL}_2\mathbb{C}$ traces of $\mathrm{res}_i \rho_t$ become complex to first order. Further, there exists an element $v_i \in \Delta_i$ such that the derivative of the $\mathrm{PSL}_2\mathbb{C}$ trace of $\rho_t(v_i)$ has imaginary part larger than its real part. It follows that the $\mathrm{SL}_4\mathbb{R}$ trace of $\rho_t(v_i)$ has strictly negative derivative. This first-order trace behavior holds for any path of representations ρ_t into $\widetilde{G} = \mathrm{SL}_4^{\pm}\mathbb{R}$ that is tangent to v. Therefore, for any sufficiently small time $t > 0$, the trace of $\rho_t(v_i)$ is strictly smaller than four and so $\rho_t(v_i)$ has at least one pair of complex eigenvalues (by the arithmetic mean vs geometric mean inequality applied to the eigenvalues). $\square$

For each $i \in \{1, \dots, k\}$, let $\mathscr{S}_i \subset \mathrm{Hom}(\Delta_i, \widetilde{G})$ be a copy of the four-dimensional partial slice $\mathscr{S}$ described above with $\Delta = \Delta_i$ and let $g_i \in \widetilde{G}$ be such that $c(g_i) \cdot \mathrm{res}_i(\rho_{\mathrm{hyp}}) \in \mathscr{S}_i$. Without loss of generality we assume $g_1 = 1$. Let

$$S = \mathscr{S}_1 \times \cdots \times \mathscr{S}_k \subset \mathrm{Hom}(\Delta_1, \widetilde{G}) \times \cdots \times \mathrm{Hom}(\Delta_k, \widetilde{G}),$$

let $V_{\mathscr{S}_i} \subset H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{g})$ denote the subspace of cohomology classes of all infinitesimal deformations tangent to $\mathscr{S}_i$, and let $V_S = V_{\mathscr{S}_1} \oplus \cdots \oplus V_{\mathscr{S}_k}$. We now prove:

Lemma 4.6 *The augmented restriction map $\widetilde{\mathrm{res}}$ is transverse to S at $(\rho_{\mathrm{hyp}}, g_2, \dots, g_k)$ with k–dimensional local intersection.*

Proof By Lemma 4.5, the intersection $V_S \cap \mathrm{res}_*\big(H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))\big)$ is trivial, since none of the representations in S have complex eigenvalues. Hence, since $\dim V_S = 4k$ by Lemma 4.3, and $H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))$ and its image in $\bigoplus_{i=1}^k H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{so}(3,1))$ have dimension $2k$, we have that

$$\bigoplus_{i=1}^k H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{g}) = V_S \oplus \mathrm{res}_*\big(H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))\big),$$

and it follows that the subspaces V_S and $\mathrm{res}_*(H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{g}))$ intersect transversely in a k–dimensional subspace. Therefore, the $4k$–dimensional tangent space to S in $\bigoplus_{i=1}^k Z^1_{c(g_i)\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{g})$ intersects the codimension-$3k$ image of the augmented restriction map transversely in a k–dimensional subspace. The result follows. $\square$

Finally we prove Theorem 4.1.

Proof of Theorem 4.1 Lemma 4.4 guarantees the existence of a cohomology class of infinitesimal deformation $[z] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v})$ whose restrictions $(\mathrm{res}_i)_*[z]$ are, for each $i \in \{1, \dots, k\}$, nontrivial in $H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{v})$. The span of $[z]$ and $H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))$ is a $(2k+1)$–dimensional subspace of $H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{g})$ whose restriction, also $(2k+1)$–dimensional by the assumption that res_* is injective, must intersect the codimension-$2k$ subspace V_S nontrivially (and indeed, transversally). Let $\mathrm{res}_*[u] = \alpha \, \mathrm{res}_*[z] + \mathrm{res}_*[w]$ be a nontrivial element of the intersection, where $[w] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))$. Since $V_S \cap \mathrm{res}_*\big(H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{so}(3,1))\big) = 0$, we must have $\alpha \neq 0$. In particular, $(\mathrm{res}_i)_*[u]$ does not lie in $H^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{so}(3,1))$ for each $i \in \{1, \dots, k\}$. Hence, $(\mathrm{res}_i)_*[u] \in V_{\mathscr{S}_i}$ is not tangent to the cusp shape locus. We may therefore find a representative cocycle $u \in Z^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{g})$ and coboundaries $b_i \in B^1_{\rho_{\mathrm{hyp}}}(\Delta_i, \mathfrak{g})$ for $i \in \{2, \dots, k\}$ such that $\mathrm{res}_{1*} u$ is tangent to $\mathscr{S}_1$ and $\mathrm{res}_{i*} u + b_i$ is tangent to $\mathscr{S}_i$ for each $i \in \{2, \dots, k\}$. By Lemma 4.6, there exists a path $\rho_t \in \mathrm{Hom}(\Gamma, \widetilde{G})$ based at $\rho_0 = \rho_{\mathrm{hyp}}$ with tangent u at $t = 0$ and paths $g_{1,t} = 1$ (constant) and $g_{2,t}, \dots, g_{k,t}$ in $\widetilde{G}$ with $g_{2,0} = g_2, \dots, g_{k,0} = g_k$ such that $c(g_{i,t}) \cdot \mathrm{res}_i \rho_t \in \mathrm{Hom}(\Delta_i, \widetilde{G})$ lies in $\mathscr{S}_i$ for each $i \in \{1, \dots, k\}$. Since $(\mathrm{res}_i)_*[u]$ is not tangent to the cusp shape locus, for sufficiently small $t > 0$, each of the representations $c(g_{i,t}) \cdot \mathrm{res}_i \rho_t$ does not lie in the cusp shape locus and is therefore diagonalizable by Lemma 4.3. The theorem is proved. $\qquad\square$

Remark 4.7 The properties characterizing the infinitesimal deformation $[z]$ from Lemma 4.4 are stable under perturbation. Therefore there is an open k–dimensional cone of $[z] \in H^1_{\rho_{\mathrm{hyp}}}(\Gamma, \mathfrak{v})$ satisfying the conclusion of Lemma 4.4. The proof of Theorem 4.1 implies that this cone parametrizes a k–dimensional family of representations satisfying the conclusion of theorem.

4.1 More on the four-dimensional partial slice $\mathscr{S}$

In this section we give the proof of Lemma 4.3, which describes the essential properties of the four-dimensional partial slice $\mathscr{S} = \mathrm{Im}(\Phi)$ used in the transversality argument for Theorem 4.1. Along the way, we will give a geometric description of the representations in $\mathscr{S}$ and indicate some of the intuition behind its construction.

Let C denote the two-dimensional abelian subgroup of $\mathrm{SO}(3,1)$ consisting of unipotent matrices fixing a point p_∞ on the ideal boundary of hyperbolic space. Let us work in the paraboloid model of $\mathbb{H}^3$, with the ideal boundary $\partial_\infty \mathbb{H}^3$ described by the paraboloid $\{[\frac{1}{2}(x_1^2 + x_2^2), x_1, x_2, 1] \in \mathbb{RP}^3 \mid x_1, x_2 \in \mathbb{R}\} \cup [1, 0, 0, 0]$, and let us take $p_\infty = [1, 0, 0, 0]$. Then each of the cusp shape representations $\Phi(0, 0, x, y)$, where $y \neq 0$, is a lattice inside C. Note that C is contained in its centralizer $Z(C)$

in $SL_4\mathbb{R}$, a three-dimensional abelian subgroup, maximal with respect to inclusion. Let $\mathfrak{a}$ denote the Cartan subalgebra of $\mathfrak{sl}_4(\mathbb{R})$ consisting of traceless diagonal matrices. Let $A = \exp \mathfrak{a}$ denote the corresponding Cartan subgroup. In order to find representations near $\mathrm{res}(\rho_{\mathrm{hyp}})$ which are diagonalizable, we must study the space of maximal (ie three-dimensional) abelian subgroups of $SL_4\mathbb{R}$ near $Z(C)$ and attempt to locate (at least some of) those that are conjugates of A.

We now construct a smooth two-dimensional family of three-dimensional abelian subgroups $A_{a,b}$, which are conjugate to A for all $(a,b) \neq (0,0)$ and such that $A_{0,0} = Z(C)$. We work in the affine chart with coordinates $[x_3, x_1, x_2, 1]$. For each $t > 0$, consider the intersection, S_t, of the paraboloid $\partial_\infty \mathbb{H}^3$ with the affine plane P_t parallel to the $x_1 x_2$–plane at height $x_3 = 1/(2t^2)$. In these coordinates, S_t is a round circle contained in P_t, invariant under rotation $R(\theta)$ by any angle θ in the $x_1 x_2$–plane about the x_3–axis. Let $p(t) = [1/(2t^2), 1/t, 0, 1]$ and let $p_1(t, \theta) = R(\theta)p(t)$, let $p_2(t, \theta) = R\left(\theta + \frac{2}{3}\pi\right)p(t)$ and let $p_3(t, \theta) = R\left(\theta + \frac{4}{3}\pi\right)p(t)$. Then $p_1(t, \theta)$, $p_2(t, \theta)$, $p_3(t, \theta)$ are the vertices of an equilateral triangle inscribed in S_t.
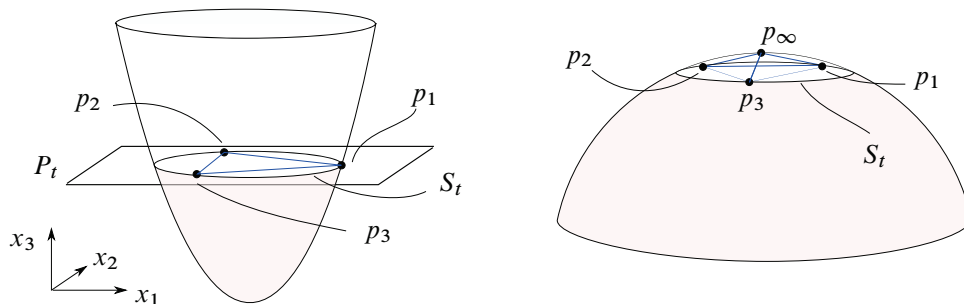

Figure 2: Left: the points $p_1(t, \theta)$, $p_2(t, \theta)$ and $p_3(t, \theta)$ form an equilateral triangle on the circle S_t at height $1/(2t^2)$ on $\partial \mathbb{H}^3$ in the paraboloid model. Right: the same picture shown in the affine chart in which $\partial \mathbb{H}^3$ is the round sphere.

Let $A_{t,\theta}$ denote the subgroup of projective transformations which fix the vertices $p_1(t, \theta)$, $p_2(t, \theta)$, $p_3(t, \theta)$, p_∞. Then $A_{t,\theta} = R(\theta)Q(t)AQ(t)^{-1}R(\theta)^{-1}$, where

$$Q(t) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2t & -t & -t & 0 \\ 0 & \sqrt{3}t & -\sqrt{3}t & 0 \\ 2t^2 & 2t^2 & 2t^2 & 0 \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $A_{t,\theta} = A_{t,\theta+2\pi/3}$. The Hausdorff limit as $t \to 0$ of $A_{t,\theta}$ is exactly $Z(C)$, independent of θ. Indeed, (t, θ) are branched polar coordinates for a smooth two-

dimensional family of three-dimensional abelian subgroups of $SL_4\mathbb{R}$. To see this, consider the three families of traceless diagonal matrices $x_{t,\theta}, y_{t,\theta}, z_{t,\theta} \in \mathfrak{a} = \mathrm{Lie}(A)$,

$$x_{t,\theta} = \begin{pmatrix} 2t\cos\theta & & & \\ & 2t\cos\left(\theta + \frac{2}{3}\pi\right) & & \\ & & 2t\cos\left(\theta + \frac{4}{3}\pi\right) & \\ & & & 0 \end{pmatrix},$$

$$y_{t,\theta} = \begin{pmatrix} 2t\sin\theta & & & \\ & 2t\sin\left(\theta + \frac{2}{3}\pi\right) & & \\ & & 2t\sin\left(\theta + \frac{4}{3}\pi\right) & \\ & & & 0 \end{pmatrix},$$

$$z_{t,\theta} = \begin{pmatrix} t^2 & & & \\ & t^2 & & \\ & & t^2 & \\ & & & -3t^2 \end{pmatrix}.$$

For all $t \neq 0$ and any θ, these three vectors form a basis of $\mathfrak{a}$. A simple computation shows that

$$x'_{t,\theta} := R(\theta)Q(t)x_{t,\theta}Q(t)^{-1}R(\theta)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & t\cos 3\theta & t\sin 3\theta & 1 \\ & t\sin 3\theta & -t\cos 3\theta & 0 \\ & 2t^2 & & 0 \end{pmatrix},$$

$$y'_{t,\theta} := R(\theta)Q(t)y_{t,\theta}Q(t)^{-1}R(\theta)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & t\sin 3\theta & -t\cos 3\theta & 0 \\ & -t\cos 3\theta & -t\sin 3\theta & 1 \\ & 0 & 2t^2 & 0 \end{pmatrix},$$

$$z'_{t,\theta} := R(\theta)Q(t)z_{t,\theta}Q(t)^{-1}R(\theta)^{-1} = \begin{pmatrix} -3t^2 & & & 2 \\ & t^2 & & \\ & & t^2 & \\ & & & t^2 \end{pmatrix}.$$

These three vectors form a basis for $\mathfrak{a}_{t,\theta} = \mathrm{Lie}(A_{t,\theta})$. We now set $a = t\cos 3\theta$ and $b = t\sin 3\theta$ and rewrite, in these coordinates,

$$x'_{a,b} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & a & b & 1 \\ 0 & b & -a & 0 \\ 0 & 2(a^2+b^2) & 0 & 0 \end{pmatrix}, \quad y'_{a,b} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & b & -a & 0 \\ 0 & -a & -b & 1 \\ 0 & 0 & 2(a^2+b^2) & 0 \end{pmatrix},$$

$$z'_{a,b} = \begin{pmatrix} -3(a^2+b^2) & & & 2 \\ & a^2+b^2 & & \\ & & a^2+b^2 & \\ & & & a^2+b^2 \end{pmatrix}.$$

In these coordinates it is transparent that the Lie algebra elements $x'_{a,b}, y'_{a,b}, z'_{a,b}$ span a smooth (in fact algebraic) family of three-dimensional abelian subalgebras $\mathfrak{a}_{a,b} = \mathfrak{a}_{t,\theta}$ with $\mathfrak{a}_{0,0} = \mathrm{Lie}(Z(C))$. More relevant for Lemma 4.3, for each $a, b \in \mathbb{R}$, the Lie algebra elements $x'_{a,b}$ and $y'_{a,b}$ span a two-dimensional Lie subalgebra $\mathfrak{c}_{a,b}$ which generates a two-dimensional abelian subgroup $C_{a,b}$ of $\mathrm{SL}_4\mathbb{R}$. Indeed both $\mathfrak{c}_{a,b}$ and $C_{a,b}$ are smooth families and, of course, $C_{0,0} = C$.

Proposition 4.8 *The maps* $(a, b) \mapsto \mathfrak{a}_{a,b}$ *and* $(a, b) \mapsto \mathfrak{c}_{a,b}$ *are both transverse to conjugation at* $(a, b) = (0, 0)$.

Proof Since $\mathfrak{c}_{a,b} \subset \mathfrak{a}_{a,b}$, it is enough to show that the map $(a, b) \mapsto \mathfrak{c}_{a,b}$ is transverse to conjugation. To do so, we must simply show that the two-parameter family of projective classes $[x'_{a,b} \wedge y'_{a,b}]$ of bivectors in $\mathbb{P}(\bigwedge^2 \mathfrak{sl}_4\mathbb{R})$ is never tangent to the conjugation orbit at $[x'_{0,0} \wedge y'_{0,0}]$. This is a straightforward calculation. First, compute

$$\partial_a(x'_{a,b} \wedge y'_{a,b})|_{(a,b)=(0,0)} = \begin{pmatrix} 0 & & & \\ & 1 & 0 & \\ & 0 & -1 & \\ & & & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \\ & & & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 \\ & -1 & 0 & 0 \\ & & & 0 \end{pmatrix},$$

$$\partial_b(x'_{a,b} \wedge y'_{a,b})|_{(a,b)=(0,0)} = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 1 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \\ & & & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & 0 & -1 & 0 \\ & & & 0 \end{pmatrix}.$$

Next, the infinitesimal action by conjugation of an arbitrary element $v = (v_{ij}) \in \mathfrak{g}$ on $x'_{0,0} \wedge y'_{0,0}$ is given by

$$\mathrm{ad}(v)x'_{0,0} \wedge y'_{0,0} + x'_{0,0} \wedge \mathrm{ad}(v)y'_{0,0} = \begin{pmatrix} -v_{21} & v_{11}-v_{22} & -v_{23} & v_{12}-v_{24} \\ -v_{41} & v_{21}-v_{42} & -v_{43} & v_{22}-v_{44} \\ 0 & v_{31} & 0 & v_{32} \\ 0 & v_{41} & 0 & v_{42} \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \\ 0 & 0 & 1 & \\ & & & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \\ & & & 0 \end{pmatrix} \wedge \begin{pmatrix} -v_{31} & -v_{32} & v_{11}-v_{33} & v_{13}-v_{34} \\ 0 & 0 & v_{21} & v_{23} \\ -v_{41} & -v_{42} & v_{31}-v_{43} & v_{33}-v_{44} \\ 0 & 0 & v_{41} & v_{43} \end{pmatrix}.$$

Now, let (e_{ij}) be the usual basis for $\mathfrak{gl}_4\mathbb{R}$, thought of as the space of 4×4 real matrices, and work in the basis for $\bigwedge^2 \mathfrak{gl}_4\mathbb{R} \supset \bigwedge^2 \mathfrak{sl}_4\mathbb{R}$ consisting of all $e_{ij} \wedge e_{mn}$ such that either $i < m$ or $i = m$ and $j < n$. Suppose that some tangent vector to the family $[x'_{a,b} \wedge y'_{a,b}]$ is tangent to the conjugation orbit at $[x'_{0,0} \wedge y'_{0,0}]$. This is equivalent to the equation

$$\alpha \partial_a (x'_{a,b} \wedge y'_{a,b}) + \beta \partial_b (x'_{a,b} \wedge y'_{a,b}) = \mathrm{ad}(v)x'_{0,0} \wedge y'_{0,0} + x'_{0,0} \wedge \mathrm{ad}(v) y'_{0,0} + \gamma x'_{0,0} \wedge y'_{0,0}.$$

The coefficient of $e_{13} \wedge e_{33}$ on the left-hand side is α while the same coefficient on the right-hand side is zero. The coefficient of $e_{12} \wedge e_{22}$ on the left-hand side is β while the same coefficient on the right-hand side is again zero. It follows that $\alpha = \beta = 0$: any vector tangent to both the family $[x'_{a,b} \wedge y'_{a,b}]$ and the conjugation orbit is trivial. The proposition is proved. $\qquad\square$

Proof of Lemma 4.3 First, let us rewrite Φ in the notation of this section:

$$\Phi(a, b, x, y)(\gamma_1) = \exp(x'_{a,b}), \quad \Phi(a, b, x, y)(\gamma_2) = \exp(xx'_{a,b} + yy'_{a,b}).$$

That the representations $\Phi(a, b, x, y)$ are diagonalizable with real eigenvalues whenever $(a, b) \neq (0, 0)$ has already been demonstrated. We must show that Φ is not tangent to the conjugation orbit in any direction at any point $(0, 0, x, y)$.

Consider the tangent vector

$$w = \dot{a}\partial_a \Phi + \dot{b}\partial_b \Phi + \dot{x}\partial_x \Phi + \dot{y}\partial_y \Phi$$

at the point $\Phi(0, 0, x, y)$ and suppose $w \in B^1_{\Phi(0,0,x,y)}(\Delta, \mathfrak{sl}_4\mathbb{R})$. By Proposition 4.8, the two-dimensional subgroup $C_{a,b}$ generated by $\Phi(a, b, x, y)$ is changing to first order in the direction of w if and only if $(\dot{a}, \dot{b}) \neq (0, 0)$. So we must have $(\dot{a}, \dot{b}) = (0, 0)$. Hence, w is tangent to the cusp shape locus. Since w is a coboundary, the cusp shape must not change to first order. It follows that $\dot{x} = 0$ and $\dot{y} = 0$. This completes the proof of Lemma 4.3. $\qquad\square$

Remark 4.9 For fixed $(a_0, b_0) \neq (0, 0)$, the eigenvalues of $\Phi(ta_0, tb_0, u, v)$ are smooth functions of t with nonzero derivative at $t = 0$. It is tempting to conclude that $\mathrm{d}\Phi_{(0,0,u,v)}(a_0, b_0, 0, 0)$ is not an infinitesimal conjugation. Although this turns out to be the case, it is fallacious to conclude so purely from the given behavior of eigenvalues; the above proof of Lemma 4.3 is needed. We give an example in the simpler setting of 2×2 matrices. Consider two paths of matrices

$$M_1(t) = \begin{pmatrix} e^t & 1 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad M_2(t) = \begin{pmatrix} 1+t & 1 \\ -t^2 & 1-t \end{pmatrix}.$$

Note that $M_1(0) = M_2(0)$ and that $M_1'(0) = M_2'(0)$. However, the eigenvalues of $M_1(t)$, which are $e^t = 1 + t + O(t^2)$ and $e^{-t} = 1 - t + O(t^2)$, vary to first order in t, while the eigenvalues of $M_2(t)$ are both constant equal to one for all t. Indeed, $M_2(t)$ is a conjugation path.

5   Geometry of manifolds with totally geodesic boundary

This section is dedicated to the geometry of convex projective three-manifolds with diagonalizable peripheral holonomy. We will complete the proof of Theorem 1.3, the algebraic side of which was given in the previous section. Recall that we denote the Lie groups $\mathrm{PGL}_4\mathbb{R}$, $\mathrm{SL}_4^{\pm}\mathbb{R}$ and $\mathrm{SL}_4\mathbb{R}$ by G, $\widetilde{G}$ and $\widetilde{G}_0$, respectively.

Let M be an open three-manifold which is the interior of a compact boundary-incompressible manifold with k torus boundary components $\partial M = \partial_1 \sqcup \cdots \sqcup \partial_k$. Let

$$\mathrm{dev}\colon \widetilde{M} \xrightarrow{\cong} \Omega \subset \mathbb{RP}^3$$

be the developing map of an indecomposable properly convex projective structure on M and denote the holonomy representation by $\rho\colon \Gamma \to \widetilde{G}$, where $\Gamma = \pi_1 M$ denotes the fundamental group. As usual, denote the peripheral subgroups by $\Delta_i = \pi_1 \partial_i$. We will assume that $\rho(\Delta_i)$ is diagonalizable over the reals with eigenvectors $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, p_4^{(i)} \in \mathbb{RP}^3$ and further that $p_4^{(i)}$ is never an attracting fixed point (not even weakly attracting) for any nontrivial element of $\rho(\Delta_i)$. Then ρ is said to satisfy the *middle eigenvalue condition*, namely that for any nontrivial $\gamma \in \Delta_i$, the eigenvalue of $\rho(\gamma)$ associated to eigenvector $p_4^{(i)}$ is never the largest nor the smallest. The middle eigenvalue condition is a slight weakening of the uniform middle eigenvalue condition defined by Choi (see [10] for more details).

The holonomy representation ρ for this structure is also the holonomy representation for other (related) convex projective structures defined by different domains. Let us begin by constructing a minimal convex domain for the action of $\rho(\Gamma)$. Any strongly attracting fixed point of an element of $\rho(\Gamma)$ must lie on $\partial\Omega$. The closure of the set of strongly attracting fixed points is called the *limit set* of $\rho(\Gamma)$ and we define $\Omega_{\min}$ to be the interior of the closed convex hull of the limit set. Then $\Omega_{\min}$ is contained in Ω, is nonempty, open, ρ–invariant and convex, and is minimal with respect to these properties. Pulling back $\Omega_{\min}$ via dev determines a convex projective structure on a submanifold $M_{\min} \cong \rho(\Gamma)\backslash\Omega_{\min}$ whose inclusion into M is a homotopy equivalence. We now investigate the geometry of $M_{\min}$, in particular of its ends.

Lemma 5.1 *For any $i \in \{1, \ldots, k\}$, each of the three fixed points $p_1^{(i)}$, $p_2^{(i)}$ and $p_3^{(i)}$ is a strongly attracting fixed point for some element of $\rho(\Delta_i)$. Therefore, $p_1^{(i)}$, $p_2^{(i)}$ and $p_3^{(i)}$ lie on $\partial\Omega$ and $\partial\Omega_{\min}$.*

Proof The peripheral holonomy group $\rho(\Delta_i)$ is obtained by exponentiating a lattice Λ inside a two-dimensional Lie subalgebra $\mathfrak{c}^{(i)}$ of the Cartan subalgebra $\mathfrak{a}^{(i)} \subset \mathfrak{g}$ corresponding to the basis $p_1^{(i)}, p_2^{(i)}, p_3^{(i)}, p_4^{(i)} \in \mathbb{RP}^3$. Let $A_{123}^{(i)} = \exp \mathfrak{a}_{123}^{(i)}$ be the (2–dimensional) subgroup of $\mathrm{SL}^{\pm}(\mathrm{span}\{p_1^{(i)}, p_2^{(i)}, p_3^{(i)}\}) \cong \mathrm{SL}_3^{\pm}\mathbb{R}$ of elements that fix each of $p_1^{(i)}$, $p_2^{(i)}$ and $p_3^{(i)}$. Consider the natural projection $\varpi_{123}: \mathfrak{a}^{(i)} \to \mathfrak{a}_{123}^{(i)}$. It follows from the middle eigenvalue condition that the restriction of ϖ_{123} to $\mathfrak{c}^{(i)}$ is injective. To see this, observe that if $A \in \ker(\varpi_{123}) \cap \mathfrak{c}^{(i)}$ then the eigenvalues of A corresponding to the eigenvectors $p_1^{(i)}$, $p_2^{(i)}$, $p_3^{(i)}$ and $p_4^{(i)}$ are λ, λ, λ and μ, respectively. As a result we see that μ is either the smallest or largest eigenvalue of A, which contradicts the middle eigenvalue condition. Thus, by dimensional considerations, we conclude that $\varpi_{123}(\mathfrak{c}^{(i)}) = \mathfrak{a}_{123}^{(i)}$. Furthermore, if $A \in \mathfrak{c}^{(i)}$, $1 \leq j \leq 3$ and the $p_j^{(i)}$–eigenvalue of $\varpi_{123}(A)$ is the largest eigenvalue, then the $p_j^{(i)}$–eigenvalue of A is also the largest eigenvalue for A.

Let $1 \leq j \leq 3$ and let D_j be the subset of $\mathfrak{a}_{123}^{(i)}$ consisting of elements where the $p_j^{(i)}$–eigenvalue is the largest. It is easy to see that D_j is a nonempty open cone, which implies that D_j has nontrivial intersection with $\varpi_{123}(\Lambda)$. As a result we can find an element of $\rho(\Delta_i)$ such that the $p_j^{(i)}$–eigenvalue is the largest. Such an element has $p_j^{(i)}$ as an attracting fixed point, which completes the proof of the lemma. $\quad\square$

We make the following definition, following Goldman [22] in the two-dimensional setting.

Definition 5.2 For each i, there is a unique (open) triangle $T^{(i)} \subset \overline{\Omega}_{\min}$ spanned by the points $p_1^{(i)}$, $p_2^{(i)}$, $p_3^{(i)}$. Any $\rho(\Gamma)$ translate of $T^{(i)}$ will be called a *principal totally geodesic triangle*. The group $\rho(\Delta_i)$ acts properly on $T^{(i)}$ and the quotient is called a *principal totally geodesic torus*.

We will show that $M_{\min}$ admits a natural compactification whose boundary consists of principal totally geodesic tori. First, we introduce a third convex domain $\Omega_{\max}$ defined as follows. Let Ω^* denote the convex domain dual to Ω. Then Γ acts properly discontinuously on Ω^* via ρ with diagonalizable peripheral holonomy, and we may perform the same construction as above: Let $(\Omega^*)_{\min}$ denote the interior

of the closed convex hull of the limit set for the $\rho(\Gamma)$ action on $\mathbb{RP}^{3*}$. We define $\Omega_{\max}$ to be the convex domain dual to $(\Omega^*)_{\min}$. It is the maximal properly convex, ρ–invariant domain because its dual is minimal. Next, observe that the fixed points of the dual action of $\rho(\Delta_i)$ are the hyperplanes $P_1^{(i)}$, $P_2^{(i)}$, $P_3^{(i)}$ and $P_4^{(i)}$ spanned by $\{p_2^{(i)}, p_3^{(i)}, p_4^{(i)}\}$, $\{p_1^{(i)}, p_3^{(i)}, p_4^{(i)}\}$, $\{p_1^{(i)}, p_2^{(i)}, p_4^{(i)}\}$ and $\{p_1^{(i)}, p_2^{(i)}, p_3^{(i)}\}$, respectively. The hyperplanes $P_1^{(i)}$, $P_2^{(i)}$ and $P_3^{(i)}$ are the attracting fixed points of $\rho(\Delta_i)$ in $\mathbb{RP}^{3*}$. Hence, $P_1^{(i)}$, $P_2^{(i)}$ and $P_3^{(i)}$ are points on the boundary of any convex domain in $\mathbb{RP}^{3*}$ preserved by ρ, including on $\partial(\Omega^*)_{\min}$. Dually, they are three support hyperplanes for any convex domain in $\mathbb{RP}^3$ preserved by ρ, in particular for Ω, $\Omega_{\min}$ and $\Omega_{\max}$. They bound a convex (but not properly convex) open triangular prism $U^{(i)}$ which is separated by $T^{(i)}$ into two components $\mathcal{T}_+^{(i)}$ and $\mathcal{T}_-^{(i)}$, each of which is an open tetrahedron.

Lemma 5.3   (1)   *Each principal totally geodesic triangle is contained in $\partial\Omega_{\min}$.*

   (2) $\Omega_{\min}$ *is the interior of the intersection of the positive half-spaces bounded by the planes $\rho(\Gamma)P_4^{(i)}$ containing principal totally geodesic triangles.*

   (3) $\Omega_{\max} \setminus \overline{\Omega}_{\min}$ *is the disjoint union over all $i \in \{1, \ldots, k\}$ and all $\gamma \in \Gamma/\Delta_i$ of open tetrahedra $\rho(\gamma)\mathcal{T}_-^{(i)}$, where $\mathcal{T}_-^{(i)}$ is the tetrahedron lying on the opposite side from $\Omega_{\min}$ of the principal triangle $T^{(i)}$.*

Proof   Let $S^{(1)}, \ldots, S^{(k)}$ be a pairwise disjoint collection of embedded tori in M which are parallel to the k boundary components $\partial_1, \ldots, \partial_k$, respectively. Let $D^{(i)}$ denote the lift to the universal cover $\widetilde{M} = \Omega$ of $S^{(i)}$ that is invariant under Δ_i. Then $D^{(i)}$ is (Γ, Δ_i) *precisely invariant*, meaning that $\rho(\gamma)D^{(i)} \cap D^{(i)} \neq \varnothing$ if and only if $\gamma \in \Delta_i$. It follows from Lemma 5.1 that $\partial D^{(i)} = \partial T^{(i)}$ and therefore $T^{(i)}$ is also (Γ, Δ_i) precisely invariant. Next, $D^{(i)}$ divides Ω into two components and one of these, the one which covers the end of M bounded by $S^{(i)}$, is also (Γ, Δ_i) precisely invariant. As a result, the limit set of $\rho(\Gamma)$ lies on one side of $\partial D^{(i)} = \partial T^{(i)}$ on $\partial\Omega$. Hence the limit set lies entirely in one of the two closed tetrahedra $\overline{\mathcal{T}_+^{(i)}}$ or $\overline{\mathcal{T}_-^{(i)}}$ and we take the labeling convention that it is $\overline{\mathcal{T}_+^{(i)}}$. Let $\Omega_+^{(i)} = \Omega \cap \mathcal{T}_+^{(i)}$. It then follows that the intersection

$$\bigcap_{\gamma \in \Gamma} \bigcap_{i=1}^{k} \rho(\gamma)\overline{\Omega_+^{(i)}}$$

is a closed convex set containing the limit set of $\rho(\Gamma)$. We denote its interior, which must be nonempty, by Ω'. In fact, we will show soon that Ω' is the minimal convex
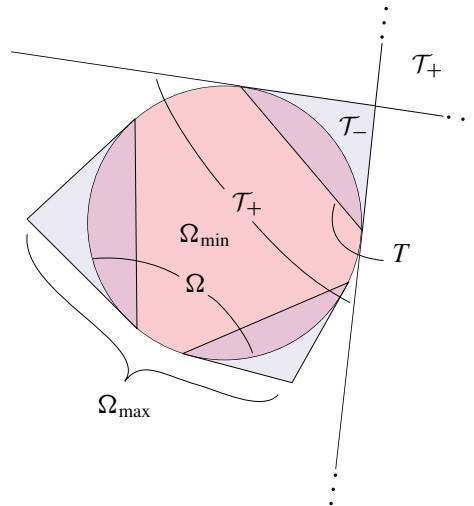
Figure 3: A schematic of the configuration $\Omega_{\min} \subset \Omega \subset \Omega_{\max}$ in Lemma 5.3. Here $\Omega_{\min}$ is red, Ω is the union of red and purple, and $\Omega_{\max}$ is the union of red, purple and blue.

domain $\Omega_{\min}$. Until then we conclude the simple fact that $T^{(i)}$ is contained in $\partial\Omega_{\min}$ because the vertices of $T^{(i)}$ must be contained in the boundary of any invariant convex domain and because $T^{(i)}$ lies in a support plane for $\partial\Omega'$.

Since the hyperplane $P_4^{(i)}$ containing $T^{(i)}$ is a support plane for $\Omega_{\min}$, it is also a point in the boundary of the dual convex domain $(\Omega_{\min})^* = (\Omega^*)_{\max}$. By applying the above to the dual convex domain Ω^* in place of Ω, we see therefore that the point $p_4^{(i)}$ belongs to the boundary of $\Omega_{\max}$. In particular, the entire tetrahedron $\mathcal{T}_-^{(i)}$ is contained in $\Omega_{\max}$; see Figure 3. Since the action of $\rho(\Gamma)$ is properly discontinuous on $\Omega_{\max}$ (or on any invariant open properly convex domain), we may now conclude that $\rho(\Gamma)$ acts properly on the subdomain Ω'_∂ consisting of the union of Ω' with all of the principal totally geodesic (open) triangles, and similarly on the union $\Omega_{\min,\partial}$ of $\Omega_{\min}$ with all of the principal totally geodesic triangles. The quotient of either set by $\rho(\Gamma)$ is a submanifold with boundary of $M_{\max} := \rho(\Gamma)\backslash\Omega_{\max}$. The boundary of either is the collection of principal totally geodesic tori $\{\rho(\Delta_i)\backslash T^{(i)}\}_{i=1}^k$. We conclude that $\Omega_{\min} = \Omega'$. It follows that the construction of Ω' is independent of the domain Ω. The proof of (2) is thus completed by applying the above argument in the case that Ω is the interior of the intersection of the positive half-spaces bounded by the hyperplanes containing principal triangles. The third statement of the lemma then follows immediately. $\qquad\square$
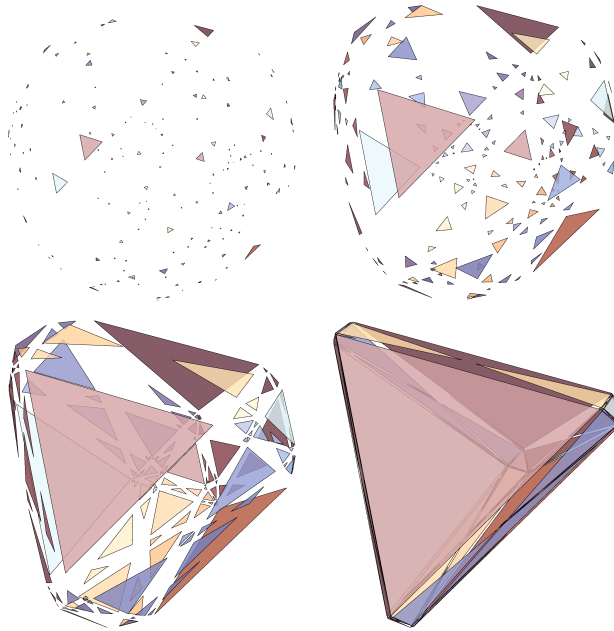
Figure 4: The principal triangles on the boundary of $\Omega_{\min}$ as the representation ρ deforms away from the holonomy representation of the complete hyperbolic structure.

Definition 5.4 The manifold $\overline{M}_{\min} = \rho(\Gamma)\backslash\Omega_{\min,\partial}$ is called a *properly convex projective manifold with totally geodesic torus boundary*. By abuse, $M_{\min} = \rho(\Gamma)\backslash\Omega_{\min}$ will also be said to have totally geodesic boundary; see Figure 4.

Definition 5.5 The manifold $M_{\max} = \rho(\Gamma)\backslash\Omega_{\max}$ is the *maximal thickening* of M. The tetrahedron $\mathcal{T}_{-}^{(i)}$ (or any of its orbits) is a *principal tetrahedron* and its quotient by $\rho(\Delta_i)$ is a *principal collar* of $M_{\max}$; see Figure 5.

Proof of Theorem 1.3 By assumption, the complete hyperbolic structure on M is infinitesimally rigid rel ∂M. By Theorem 4.1 there is a path ρ_t of representations through $\rho_0 = \rho_{\mathrm{hyp}}$ such that $\rho_t(\Delta_i)$ is diagonalizable over the reals for all $t \neq 0$ and $i \in \{1, \ldots, k\}$. The holonomy principle of Cooper, Long and Tillmann (Theorem 2.3) or Choi [11, Corollary 1.1] guarantees that there is some $\varepsilon > 0$ such that, for $t \in (0, \varepsilon)$, the representation ρ_t is the holonomy representation of a properly convex projective structure, near the complete hyperbolic structure. Further, the ends of this structure have generalized cusps, although we will not use this fact here.
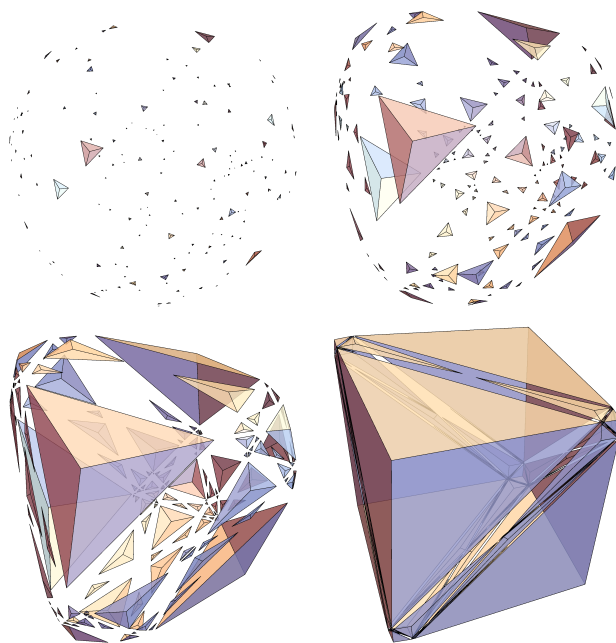
Figure 5: The principal tetrahedra of $\Omega_{\max}$ as the representation ρ deforms away from the holonomy representation of the complete hyperbolic structure.

By construction, $\operatorname{res}_i(\rho_t)$ belongs to $\mathscr{S}_i$. Hence, for $t > 0$, $\rho_t(\Delta_i)$ is a lattice in a two-dimensional diagonalizable subgroup conjugate to $C_{a,b}$, as defined in Section 4.1, where $(a, b) = (a_t^{(i)}, b_t^{(i)}) \neq (0, 0)$ depend on i and continuously on t. Since all elements of $C_{a,b}$ have a common fixed point with eigenvalue one, the middle eigenvalue condition is satisfied for $\rho_t(\Delta_i)$ for all $t > 0$ and $i \in \{1, \ldots, k\}$. For each t, the minimal convex submanifold $M_{\min,t} \subset M$ is a properly convex projective manifold with totally geodesic boundary. Further, the inclusion $M_{\min,t} \hookrightarrow M$ is a homotopy equivalence which is isotopic to a homeomorphism. Such an isotopy may be constructed explicitly in the universal cover by flowing the points lying in a principal tetrahedron of $\widetilde{M}$ (realized as the convex domain Ω_t) radially toward the principal triangle bounding that tetrahedron. $\qquad\square$

Remark 5.6 In the context of the proof of Theorem 1.3, we note that for each end, the continuous deformation of the principal totally geodesic triangles, which open up from the parabolic fixed point of the hyperbolic structure, may be observed explicitly in terms of the parameters (a, b) of the last paragraph of the proof. Indeed, the eigenvectors of $C_{a,b}$ were described explicitly in Section 4.1.

We close this subsection with one more lemma, which will be needed in the next section. The region $U^{(i)} = \mathcal{T}_+^{(i)} \cup T^{(i)} \cup \mathcal{T}_-^{(i)}$ is a triangular prism. Its boundary $\partial U^{(i)}$ is the union of three totally geodesic bigons.

Lemma 5.7 *The intersection $\partial \Omega_{\max} \cap \partial U^{(i)}$ is equal to $\partial \mathcal{T}_-^{(i)} \setminus T^{(i)}$. In particular, $\partial \Omega_{\min} \cap \partial \mathcal{T}_+^{(i)}$ contains $\overline{T^{(i)}}$ but does not contain any point of any other (open) face of $\mathcal{T}_+^{(i)}$ of positive codimension that is adjacent to $p_4^{(i)}$.*

Proof Suppose $p \in \partial \Omega_{\max}$ lies on a face $\mathscr{F}$ of $\mathcal{T}_+^{(i)}$ of positive codimension adjacent to $p_4^{(i)}$. Although $p_4^{(i)}$ is never an attractor for the action of any $\rho(\gamma)$ on $\mathcal{T}_+^{(i)}$, it is, by dimension count, an attractor for the action of some $\rho(\gamma)$ on $\mathscr{F}$. Then, by convexity, $\Omega_{\max}$ contains $\mathcal{T}_+^{(i)}$ and therefore all of $U^{(i)}$. However this is impossible, since $U^{(i)}$ is not properly convex and $\Omega_{\max}$ is properly convex. $\qquad\square$

6 Gluing convex projective structures

In this section we prove Theorem 1.4, which states that convex projective structures may be glued along principal totally geodesic torus boundary components whenever the holonomy matching condition (1-1) is satisfied. This will complete the proof of Theorem 1.1. Theorem 1.4 is proved by induction using the following two lemmas:

Lemma 6.1 *Let $M_1 = \Gamma_1 \backslash \Omega_1$ and $M_2 = \Gamma_2 \backslash \Omega_2$ be two properly convex projective three-manifolds. Let $f \colon \partial_1 \to \partial_2$ be a homeomorphism between principal totally geodesic torus boundary components ∂_1 and ∂_2 of M_1 and M_2, respectively, satisfying the holonomy matching condition: there exists $g \in \widetilde{G}$ such that*

$$\text{(6-1)} \qquad\qquad f_* \gamma = g \gamma g^{-1}$$

for all $\gamma \in \Delta_1$, where $f_* \colon \Delta_1 \to \Delta_2$ is the group homomorphism induced by f on the fundamental groups Δ_1 and Δ_2 of ∂_1 and ∂_2, respectively. Then the topological manifold $M_1 \cup_f M_2$ admits a properly convex structure which restricts to the given properly convex projective structures on M_1 and M_2.

Lemma 6.2 *Let $M = \Gamma \backslash \Omega$ be a properly convex projective three-manifold and let $f \colon \partial_1 \to \partial_2$ be a homeomorphism between principal totally geodesic boundary components ∂_1, ∂_2 of M that satisfies the holonomy matching condition: There exists $g \in \widetilde{G}$ such that*

$$\text{(6-2)} \qquad\qquad f_* \gamma = g \gamma g^{-1}$$

for all $\gamma \in \Delta_1$, where $f_*\colon \Delta_1 \to \Delta_2$ is the group homomorphism induced by f on the fundamental groups Δ_1 and Δ_2 of ∂_1 and ∂_2 respectively. Then the topological manifold M_f admits a properly convex structure which restricts to the given properly convex projective structure on M.

We now give a detailed proof of Lemma 6.1. The proof of Lemma 6.2 is nearly identical.

For convenience, throughout this subsection, we will identify the universal cover of M_i with Ω_i and the fundamental group $\pi_1 M_i$ with $\Gamma_i \subset \widetilde{G}$ for both $i \in \{1, 2\}$. For $i \in \{1, 2\}$, let $\partial_i = \Delta_i \backslash T^{(i)}$ be the given principal totally geodesic boundary torus of M_i, where $T^{(i)}$ is a totally geodesic triangle contained in $\partial \Omega_i$. For each i, let $\mathcal{T}_+^{(i)}$ and $\mathcal{T}_-^{(i)}$ be defined as in the previous section, so that $\Omega_i \subset \mathcal{T}_+^{(i)}$ and $\mathcal{T}_-^{(i)}$ is a principal tetrahedron for Ω_i. By translating Ω_1 by g and possibly a reflection in the centralizer $Z(\Delta_2)$, we may assume henceforth that $g = 1$ and that Ω_1 and Ω_2 are positioned so that $T^{(1)} = T^{(2)} =: T$, $\mathcal{T}_-^{(2)} = \mathcal{T}_+^{(1)} =: \mathcal{T}^{(1)}$ and $\mathcal{T}_-^{(1)} = \mathcal{T}_+^{(2)} =: \mathcal{T}^{(2)}$. Then, by (6-1), the identity projective transformation $g = 1$ descends to a *projective gluing map* in the isotopy class of f that glues M_1 to M_2 along their totally geodesic boundary components. Any neighborhood in $\Omega_1 \cup T \cup \Omega_2$ of a point $p \in T$ serves as a chart defining a projective structure on a neighborhood of $\partial_1 = \partial_2$ in the glued-up manifold $N = M_1 \cup_f M_2$. The charts on M_1 and M_2 are compatible with these new charts, hence N is endowed with a projective structure in which $\partial_1 = \partial_2$ is an embedded totally geodesic torus. To prove Theorem 1.4, we now show that this projective structure is properly convex.

The fundamental group of N naturally identifies with the free product $\Gamma = \Gamma_1 *_{f_*} \Gamma_2$ amalgamated over the identification $f_*\colon \Delta_1 \to \Delta_2$. Let Δ denote the inclusions of Δ_1 and Δ_2, which are identified, in Γ. We will denote the product of two elements α and β in the abstract group Γ by the notation $\alpha \star \beta$ in order to avoid confusion with matrix multiplication in $\widetilde{G}$. The universal cover $\widetilde{N}$ is described combinatorially as

$$\widetilde{N} = \Gamma \times \Omega_{1,\partial}/{\sim_1} \cup \Gamma \times \Omega_{2,\partial}/{\sim_2},$$

where $\Omega_{1,\partial}$ denotes the union of Ω_1 with the Γ_1 orbit of T and $\sim_1$ is the equivalence relation generated by $(\gamma, p) \sim_1 (\gamma\gamma_1^{-1}, \gamma_1 p)$ for all $\gamma_1 \in \Gamma_1$, and similarly for $\Omega_{2,\partial}$ and $\sim_2$. We refer to each $\{\gamma\} \times \Omega_{i,\partial}$ as a *tile*. If $p \in T = \Omega_{1,\partial} \cap \Omega_{2,\partial}$, then we consider the points $(\gamma, p) \in \Gamma \times \Omega_{1,\partial}$ and $(\gamma, p) \in \Gamma \times \Omega_{2,\partial}$ to be identified. The developing map dev for the natural projective structure on N is defined by the formula $\mathrm{dev}([\gamma, p]) = \rho(\gamma)p$ for any $\gamma \in \Gamma$ and $p \in \Omega_{1,\partial} \cup \Omega_{2,\partial}$, where $\rho\colon \Gamma \to \widetilde{G}$, the

holonomy representation, is defined by the property that its restriction to Γ_i is the inclusion map for each $i \in \{1, 2\}$. In other words, if $\gamma_1, \ldots, \gamma_m$ are elements of $\Gamma_1 \cup \Gamma_2$, then $\rho(\gamma_1 \star \cdots \star \gamma_m) = \gamma_1 \cdots \gamma_m$. We will also consider an augmented version of $\widetilde{N}$, defined by

$$\widetilde{N}_{\mathrm{aug}} = \Gamma \times \overline{\Omega}_1 / {\sim_1} \cup \Gamma \times \overline{\Omega}_2 / {\sim_2},$$

which includes the full boundaries of the convex tiles. The developing map dev extends naturally to an augmented developing map dev: $\widetilde{N}_{\mathrm{aug}} \to \mathbb{RP}^3$. It is a local embedding, even at points of the tile boundaries $[\gamma, \partial \Omega_i]$.

We show that the developing map is injective with image contained in an affine chart via a ping-pong lemma.

Lemma 6.3 Let $\gamma_1 \in \Gamma_1 \setminus \Delta$ and $\gamma_2 \in \Gamma_2 \setminus \Delta$. Then:

  (1) $\gamma_1 \overline{\mathcal{T}^{(2)}} \subset \mathcal{T}^{(1)} \setminus \Omega_1$.

  (2) $\gamma_2 \overline{\mathcal{T}^{(1)}} \subset \mathcal{T}^{(2)} \setminus \Omega_2$.

Proof We prove only the first statement as the second follows by symmetry. Note that $\mathcal{T}^{(2)} = \mathcal{T}_-^{(1)}$ is contained in $(\Omega_1)_{\mathrm{max}}$ (Definition 5.5). Hence, $\gamma_1 \mathcal{T}^{(2)} \subset (\Omega_1)_{\mathrm{max}}$ as well. $(\Omega_1)_{\mathrm{max}}$ is contained in the triangular prism $U = \mathcal{T}^{(1)} \cup T \cup \mathcal{T}^{(2)}$, which is convex (but not properly convex) and bounded by three bigons. Since $\gamma_1 \notin \Delta$, $\gamma_1 \mathcal{T}^{(2)}$ is not equal to $\mathcal{T}^{(2)}$ and therefore the two tetrahedra do not intersect. Hence, $\gamma_1 \mathcal{T}^{(2)}$ must lie in $\mathcal{T}^{(1)}$. By Lemma 5.7, we also have that $\overline{\gamma_1 \mathcal{T}^{(2)}} = \gamma_1 \overline{\mathcal{T}^{(2)}}$ lies in $\mathcal{T}^{(1)}$, that is, no point of $\gamma_1 \overline{\mathcal{T}^{(2)}}$ intersects $\partial \mathcal{T}^{(1)}$. Of course γ_1 preserves Ω_1, so $\gamma_1 \overline{\mathcal{T}^{(2)}}$ does not intersect Ω_1 because $\overline{\mathcal{T}^{(2)}}$ does not intersect Ω_1. □

Lemma 6.4 *The augmented developing map* dev: $\widetilde{N}_{\mathrm{aug}} \to \mathbb{RP}^3$ *is injective and its image is contained in* $\overline{(\Omega_1)}_{\mathrm{max}}$.

Proof We already know that dev is an embedding when restricted to the union of any two adjacent closed tiles. To show that dev is a global embedding, it suffices to show that $\mathrm{dev}([\{\gamma\} \times \overline{\Omega}_1]) = \rho(\gamma)\overline{\Omega}_1$ does not intersect $\overline{\Omega}_2$ nor $\overline{\Omega}_1$ as long as $\gamma \notin \Gamma_1 \cup \Gamma_2$. Assume that $\gamma \notin \Gamma_1 \cup \Gamma_2$. Then γ may be expressed as an alternating product $\gamma = \gamma_1 \star \cdots \star \gamma_m$ of $m \geq 2$ elements $\gamma_i \in \Gamma_1 \cup \Gamma_2 \setminus \Delta$ such that, for $i = 1, \ldots, m-1$, $\gamma_i \in \Gamma_1$ if and only if $\gamma_{i+1} \in \Gamma_2$. There are two possibilities to consider. First, assume γ_m lies in $\Gamma_2 \setminus \Delta$. Then, by Lemma 6.3,

$$\rho(\gamma)\overline{\Omega}_1 = \gamma_1 \cdots \gamma_{m-1} \gamma_m \overline{\Omega}_1 \subset \gamma_1 \cdots \gamma_{m-1} (\mathcal{T}^{(2)} \setminus \Omega_2)$$
$$\subset \gamma_1 \cdots \gamma_{m-2} (\mathcal{T}^{(1)} \setminus \Omega_1) \subset \cdots \subset (\mathcal{T}^{(j)} \setminus \Omega_j),$$

where $j = 1$ if $\gamma_1 \in \Gamma_1$ (equivalently if m is even) or $j = 2$ if $\gamma_1 \in \Gamma_2$. Hence, $\rho(\gamma)\overline{\Omega}_1$ does not intersect either $\overline{\Omega}_1$ or $\overline{\Omega}_2$. The other possibility is that $\gamma_m \in \Gamma_1 \setminus \Delta$. In this case, $\rho(\gamma)\overline{\Omega}_1 = \gamma_1 \cdots \gamma_{m-1}\overline{\Omega}_1$, with $\gamma_{m-1} \in \Gamma_2 \setminus \Delta$, and we proceed as in the previous case, replacing γ with $\gamma_1 \star \cdots \star \gamma_{m-1}$. This completes the proof that dev is injective on $\widetilde{N}_{\mathrm{aug}}$. Indeed, dev is a closed map, so dev is an embedding.

We may also see from the above ping-pong argument that the image of dev is contained in $\overline{(\Omega_1)}_{\max}$. For any $\gamma \in \Gamma_1$, we have that $\rho(\gamma)\overline{\Omega}_1 = \overline{\Omega}_1 \subset \overline{(\Omega_1)}_{\max}$ and $\rho(\gamma)\overline{\Omega}_2 \subset \rho(\gamma)\overline{\mathcal{T}^{(2)}} \subset \overline{(\Omega_1)}_{\max}$. If $\gamma \in \Gamma_2 \setminus \Delta$, then $\rho(\gamma)\overline{\Omega}_2 = \overline{\Omega}_2 \subset \overline{(\Omega_1)}_{\max}$ and $\rho(\gamma)\overline{\Omega}_1 \subset \rho(\gamma)\overline{\mathcal{T}^{(1)}} \subset \mathcal{T}^{(2)} \subset (\Omega_1)_{\max}$. Finally, let $\gamma \notin \Gamma_1 \cup \Gamma_2$. Then, as above, $\gamma = \gamma_1 \star \gamma_2 \star \cdots \star \gamma_m$ and $\rho(\gamma)\overline{\Omega}_1 \subset \mathcal{T}^{(2)} \subset (\Omega_1)_{\max}$ if $\gamma_1 \in \Gamma_2$. If $\gamma_1 \in \Gamma_1$, then $\rho(\gamma_2 \star \cdots \star \gamma_m)\overline{\Omega}_1$ is contained in $\mathcal{T}^{(2)}$, so $\rho(\gamma_1)\rho(\gamma_2 \star \cdots \star \gamma_m)\overline{\Omega}_1 = \rho(\gamma)\overline{\Omega}_1$ is contained in $(\Omega_1)_{\max}$. It follows similarly that if $\gamma \notin \Gamma_1 \cup \Gamma_2$ then $\rho(\gamma)\overline{\Omega}_2 \subset (\Omega_1)_{\max}$. $\qquad\square$

Finally, we prove:

Lemma 6.5 *The projective manifold N is properly convex.*

To prove Lemma 6.5, we will need the following basic result about convex sets in Euclidean space. It is similar to a well-known theorem of Nakajima [33] and Tietze [38]. We include a proof for convenience.

Lemma 6.6 *Suppose $A, B \subset \mathbb{R}^d$ are closed convex subsets with nonempty interior and nonempty intersection. Suppose that $A \cup B$ satisfies the following local convexity condition along $C = \partial(A \cup B) \cap (A \cap B)$: at each point $z \in C$ there is a local support plane, ie a hyperplane containing z and bounding a closed half-space that contains a neighborhood of z in $A \cup B$. Suppose further there is a point of $A \cap B$ in the interior of $A \cup B$. Then $A \cup B$ is convex.*

Proof We consider first the interior $\mathrm{Int}(A \cup B)$. Fix a point $x \in \mathrm{Int}(A)$ and let S be the set of all $y \in \mathrm{Int}(A \cup B)$ such that $[x, y] \subset \mathrm{Int}(A \cup B)$. S is clearly an open subset of $\mathrm{Int}(A \cup B)$. We show S is also closed in $\mathrm{Int}(A \cup B)$. Suppose $y_n \in S$ converges to $y \in \mathrm{Int}(A \cup B)$ but $y \notin S$. The open interval (x, y) is contained in $A \cup B$ and intersects $\partial(A \cup B)$ in at least one point z. Let H be a hyperplane supporting $A \cup B$ locally near z, guaranteed to exist by the local convexity assumption if $z \in C$ or by the convexity of A (resp. of B) if $z \in \partial A \setminus C$ (resp. if $z \in \partial B \setminus C$). Since (x, y) does not cross H transversely at z, we must have $[x, y] \subset H$. However, this is a contradiction: if $z \in A$, the local support plane H does not intersect the convex set $\mathrm{Int}(A)$ so it cannot

contain x, and if $z \in B$, then H does not intersect $\mathrm{Int}(B)$ so it does not contain y. Hence, S is closed in $\mathrm{Int}(A \cup B)$. We note that $\mathrm{Int}(A \cup B)$ is connected, since by assumption a point of $A \cap B$ is contained in $\mathrm{Int}(A \cup B)$. Hence, $S = \mathrm{Int}(A \cup B)$. Hence, $[x, y] \subset \mathrm{Int}(A \cup B)$ for all $x \in \mathrm{Int}(A)$ and $y \in \mathrm{Int}(A \cup B)$, and similarly for $x \in \mathrm{Int}(B)$ by symmetry. It now follows by taking limits that $A \cup B$ is convex. $\qquad\square$

Proof of Lemma 6.5 By Lemma 6.4, the augmented developing map is an embedding into an affine chart $\mathbb{A}$ of $\mathbb{RP}^3$. The image of each tile of $\widetilde{N}_{\mathrm{aug}}$ is convex. The union of two adjacent tiles is locally convex at the boundary of the interface between them. To check this, it suffices to examine $\overline{\Omega}_1 \cup \overline{\Omega}_2$. We may assume that $\overline{U^{(1)}} = \mathcal{T}_+^{(1)} \cup \overline{\mathcal{T}_-^{(1)}}$ intersects the affine chart $\mathbb{A}$ in an infinite triangular prism (with vertex at infinity). Since $\overline{\Omega}_1 \cup \overline{\Omega}_2$ is contained in $\overline{U^{(1)}}$ and since $\partial(\overline{\Omega}_1 \cup \overline{\Omega}_2) \cap (\overline{\Omega}_1 \cap \overline{\Omega}_2) = \partial \overline{T}$ is contained in $\partial \overline{U^{(1)}}$, it follows that $\overline{\Omega}_1 \cup \overline{\Omega}_2$ is locally convex at the boundary of the interface $\overline{T}$ between the two tiles: one of the three planes bounding $\overline{U^{(1)}}$ supports $\overline{\Omega}_1 \cup \overline{\Omega}_2$ at each point of ∂T. The union $\overline{\Omega}_1 \cup \overline{\Omega}_2$ is convex by Lemma 6.6. Next, since no three of the closed tiles of $\widetilde{N}_{\mathrm{aug}}$ meet nontrivially, it follows by induction and Lemma 6.6 that the image under the augmented developing map of any finite connected union of tiles of $\widetilde{N}_{\mathrm{aug}}$ is convex. The image of $\widetilde{N}_{\mathrm{aug}}$ is an increasing union of such convex sets and hence is convex. The image of $\widetilde{N}$, which is the interior of the image of $\widetilde{N}_{\mathrm{aug}}$, is therefore also convex, and indeed properly convex because it is contained in $\overline{(\Omega_1)}_{\mathrm{max}}$. $\qquad\square$

This completes the proof of Lemma 6.1. The proof of Lemma 6.2 is nearly the same and so we include only the following sketch, which highlights the required modifications.

(1) Possibly after modifying g by a reflection in the centralizer $Z(\Delta_2)$, we may assume that Ω and $g\Omega$ are positioned so that $g\mathcal{T}_{\mp}^{(1)} = \mathcal{T}_{\pm}^{(2)}$, where for both $i \in \{1, 2\}$, $\mathcal{T}_+^{(i)} \supset \Omega$ and $\mathcal{T}_-^{(i)}$ is the principal tetrahedron for Ω preserved by Δ_i as above.

(2) The fundamental group of M_f is the HNN extension $\Gamma *_{f_*}$. The universal cover of M_f is combinatorially a union of tiles $\widetilde{M}_f = \Gamma *_{f_*} \times \Omega_\partial /{\sim}$, where Ω_∂ is the union of Ω and the principal triangles covering the boundary component ∂_1 and ∂_2 and the equivalence relation is generated by $(\gamma, p) \sim (\gamma \gamma_1^{-1}, \gamma_1 p)$ for all $\gamma_1 \in \Gamma$.

(3) The developing map dev for the natural projective structure on M_f is defined by the formula $\mathrm{dev}([\gamma, p]) = \rho(\gamma) p$, for any $\gamma \in \Gamma *_{f_*}$ and $p \in \Omega_\partial$, where $\rho \colon \Gamma *_{f_*} \to \widetilde{G}$, the holonomy representation, is defined by the property that its

restriction to Γ is the inclusion map and ρ applied the stable letter of $\Gamma *_{f_*}$ is the gluing transformation g. The developing map dev extends to the augmented domain $\widetilde{M}_{f,\mathrm{aug}} = \Gamma *_{f_*} \times \overline{\Omega}/\sim$.

(4) The following analogue of Lemma 6.3 holds:

- $g\mathcal{T}_{\mp}^{(1)} = \mathcal{T}_{\pm}^{(2)}$ (already arranged above).
- For any $i \in \{1,2\}$ and $\gamma \in \Gamma \setminus \Delta_i$, $\gamma\overline{\mathcal{T}_{-}^{(i)}} \subset \mathcal{T}_{+}^{(i)} \setminus \Omega$ (this follows immediately because $\Omega_{\max}$ contains $\gamma\mathcal{T}_{-}^{(i)}$ and is contained in the union $U^{(i)} = \mathcal{T}_{+}^{(i)} \cup T^{(i)} \cup \mathcal{T}_{-}^{(i)}$).

(5) Plugging (4) into the argument from Lemma 6.4 shows that dev is an embedding with image contained in $\overline{\Omega}_{\max}$. Finally, as in Lemma 6.5, the image of $\widetilde{M}_{f,\mathrm{aug}}$ under dev is locally convex, showing that the projective structure on M_f is properly convex.

Proof of Theorem 1.4 The theorem follows by induction by applying Lemmas 6.1 and 6.2 in succession to glue together the given collection of convex projective manifolds along various pairs of boundary components, using the given gluing maps in any order. □

Suppose we are given a collection of cusped hyperbolic manifolds, along with gluing homeomorphisms between some of their torus boundary components. In general, it is not known whether projective structures with totally geodesic boundary satisfying the appropriate holonomy matching hypotheses (1-1) of Theorem 1.4 can be found. To find such glueable structures would seem to require a description of the global deformation space of properly convex projective structures with principal totally geodesic boundary on a given manifold. As of the writing of this article, there is no cusped hyperbolic manifold for which this global deformation space has been computed, abstractly or computationally (although there are certain Coxeter orbifolds for which the deformation space has been computed; see Section 7). Nonetheless, the matching condition is automatically satisfied in the case of doubling a convex projective manifold with principal totally geodesic boundary. Indeed:

Proof of Theorem 1.1 Since M is assumed to be infinitesimally projectively rigid rel boundary, Theorem 1.3 tells us that we can deform the complete hyperbolic structure on M to a convex projective structure where each boundary component is a principal totally geodesic torus. The gluing condition (1-1) is trivially satisfied for the identity gluing, and thus Theorem 1.4 ensures that the double $2M$ admits a properly convex projective structure. □
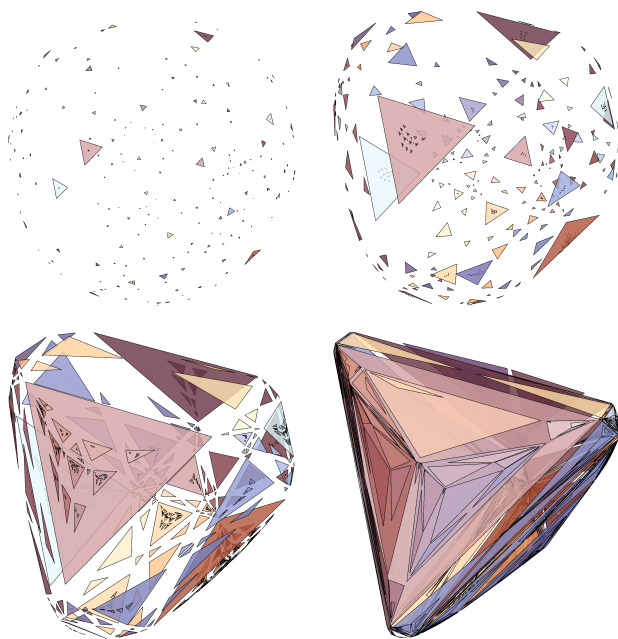
Figure 6: A collection of properly embedded triangles in the convex domains defining convex projective structures on the double of the figure eight knot complement as the projective structure on each of the two pieces deforms away from the complete hyperbolic structure.

7 Gluing covering manifolds of reflection 3–orbifolds

We conclude the paper by giving examples of properly convex projective structures on nonhyperbolic manifolds which are not obtained by doubling. To do this, we apply the convex gluing theorem, Theorem 1.4, to a collection of highly symmetric properly convex projective structures with principal totally geodesic boundary. The symmetry of these structures, which are covers of certain convex projective reflection orbifolds, greatly restricts their boundary holonomy groups. This allows us to determine which combinations of these manifolds admit structures satisfying the matching hypothesis (1-1) of Theorem 1.4.

7.1 Euclidean hex tori

We will be interested in three-dimensional hyperbolic reflection orbifolds with cusps isomorphic to the Euclidean $(3, 3, 3)$–triangle orbifold. We now recall the geometric features of this two-orbifold and describe the geometry of its torus covers.

Let $\Delta_{3,3,3}$ denote the subgroup of the isometries $\mathrm{Isom}\,\mathbb{E}^2$ of the Euclidean plane generated by reflections in the sides of an equilateral triangle T in the Euclidean plane. We denote the resulting quotient orbifold by $\mathcal{S}_{3,3,3} = \Delta_{3,3,3}\backslash\mathbb{E}^2$.

If Δ_H denotes the subgroup of all translations in $\Delta_{3,3,3}$, then Δ_H is the maximal torsion-free subgroup of $\Delta_{3,3,3}$ and all other torsion-free subgroups Δ of $\Delta_{3,3,3}$ are contained in Δ_H. In other words, the torus $\mathcal{S}_H = \Delta_H\backslash\mathbb{E}^2$ is the minimal torus cover of $\mathcal{S}_{3,3,3}$, and every orientable surface $\mathcal{S} = \Delta\backslash\mathbb{E}^2$ covering $\mathcal{S}_{3,3,3}$ is a Euclidean torus which may be decomposed into regular hexagons each made up of six copies of the equilateral triangle T. We call such a torus a *Euclidean regular hex torus*, or just *hex torus* for short. The fundamental group Δ of a Euclidean regular hex torus $\mathcal{S}$ is called a *hex torus group*.

Let k be a natural number and let Δ' be a hex torus group. Then the group $\Delta = k\Delta'$ of all k^{th} powers of elements of Δ' is a subgroup of Δ'. If $\mathcal{S}'$ and $\mathcal{S}$ are the hex tori associated to Δ' and Δ, then there is a natural homeomorphism, denoted by $k\colon \mathcal{S}' \to \mathcal{S}$, which, in the universal cover $\mathbb{E}^2$, simply scales by k.

Definition 7.1 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be hex tori corresponding to the subgroups Δ_1 and Δ_2 of Δ_H. Then a homeomorphism $f\colon \mathcal{S}_1 \to \mathcal{S}_2$ is said to be a *lattice $\mathbb{Q}$–isometry* or a *lattice k_2/k_1–isometry* if there exists a hex torus $\mathcal{S}'$ with associated group Δ' and natural numbers k_1 and k_2 such that $\Delta_1 = k_1\Delta'$, $\Delta_2 = k_2\Delta'$, and such that the map $f' = k_2^{-1}fk_1\colon \mathcal{S}' \to \mathcal{S}'$ is induced by conjugation by an element of $\Delta_{3,3,3}$. When $k_1 = k_2 = 1$, we call f a *lattice isometry*.

We now define an invariant, called the *hex shape equivalence class* which may be used to determine when a lattice $\mathbb{Q}$–isometry exists between two hex tori. Let v_1 and v_2 be a pair of vectors of minimal length such that the translations t_{v_1} and t_{v_2} generate Δ_H. Assume further that the angle between v_1 and v_2 is equal to $\frac{2}{3}\pi$. Then v_1 and v_2 are unique up to the action of the dihedral group D_6 of order 12, generated by the order six rotation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and the reflection

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Let $w_1 = p_{11}v_1 + p_{12}v_2$ and $w_2 = p_{21}v_1 + p_{22}v_2$ be an ordered pair of vectors such that the translations t_{w_1} and t_{w_2} generate a hex torus group Δ with corresponding

hex torus denoted by $\mathcal{S}$. Then we can encode the shape of Δ with the 2×2 integer matrix A_Δ appearing in the equation

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The matrix A_Δ is only well defined up to multiplication on the left by $\mathrm{SL}_2^\pm \mathbb{Z}$ and multiplication on the right by the matrices of the dihedral group D_6 above. We call the equivalence class of matrices A_Δ the *hex cusp shape* of $\mathcal{S}$ (or of Δ). The following is elementary.

Lemma 7.2 Let $\Delta_1, \Delta_2 \subset \Delta_{3,3,3}$ be the fundamental groups of two regular Euclidean hex tori $\mathcal{S}_1$ and $\mathcal{S}_2$. Let A_{Δ_1} and A_{Δ_2} be matrices representing the respective hex cusp shapes of $\mathcal{S}_1$ and $\mathcal{S}_2$. Then there exists a lattice k_2/k_1–isometry $f: \mathcal{S}_1 \to \mathcal{S}_2$ if and only if $k_2[A_{\Delta_1}] = k_1[A_{\Delta_2}]$, ie if there exists $B \in \mathrm{SL}_2^\pm \mathbb{Z}$ and $C \in D_6$ such that $k_2 A_{\Delta_1} = k_1 B A_{\Delta_2} C$.

For example, if $\Delta_1 = \langle 5v_1 - 5v_2, 5v_1 + 5v_2 \rangle$ and $\Delta_2 = \langle 8v_1, 4v_2 \rangle$, then representatives of the hex cusp shape of the corresponding hex tori $\mathcal{S}_1$ and $\mathcal{S}_2$ are given by $A_{\Delta_1} = \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix}$ and $A_{\Delta_2} = \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}$. There exists a lattice $\frac{4}{5}$–isometry $\mathcal{S}_1 \to \mathcal{S}_2$ because

$$4 A_{\Delta_1} = 5 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A_{\Delta_2} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Remark 7.3 Given a hex torus $\mathcal{S}$, the *area* a of $\mathcal{S}$, defined as the index of the covering $\mathcal{S} \to \mathcal{S}_H$, is simply the absolute value of the determinant of any matrix in the hex cusp shape. It follows from the lemma that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are lattice k_2/k_1–isometric, then their respective areas a_1 and a_2 satisfy $k_2^2 a_1 = k_1^2 a_2$. Therefore the ratio a_2/a_1 is a square, and the conformal factor k_2/k_1 of the $\mathbb{Q}$–isometry is given by $\sqrt{a_2/a_1}$. Further, there are only finitely many equivalence classes of hex tori whose area is at most a. Hence, it is quite easy in practice to determine whether a lattice $\mathbb{Q}$–isomorphism $\mathcal{S}_1 \to \mathcal{S}_2$ exists as long as the areas of $\mathcal{S}_1$ and $\mathcal{S}_2$ are small. If one exists, then at most $12 = |D_6|$ exist.

7.2 Projective hex tori

The deformation space $\mathfrak{C}(\mathcal{S}_{3,3,3})$ of marked convex real projective structures on the triangle orbifold $\mathcal{S}_{3,3,3}$ is homeomorphic to $\mathbb{R}$ and we now briefly describe this correspondence (see Goldman [20] and Vinberg [39] for more details). Let r_1, r_2 and r_3

denote the reflections in the three sides of the triangle T, generating $\Delta_{3,3,3}$. We identify the triangle T with the positive octant in the projective 2–sphere $\mathbb{S}^2 = (\mathbb{R}^3 \setminus \{0\})/\mathbb{R}^+$, which is the interior of the convex hull of three directions $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ of $\mathbb{S}^2$. For each $\tau \in \mathbb{R}$, let $s = e^{\tau/3}$, and define the representation $\zeta_\tau \colon \Delta_{3,3,3} \to \mathrm{SL}_3^\pm \mathbb{R}$ by

$$\zeta_\tau(r_1) = \begin{pmatrix} -1 & 0 & 0 \\ s & 1 & 0 \\ \frac{1}{s} & 0 & 1 \end{pmatrix}, \quad \zeta_\tau(r_2) = \begin{pmatrix} 1 & \frac{1}{s} & 0 \\ 0 & -1 & 0 \\ 0 & s & 1 \end{pmatrix}, \quad \zeta_\tau(r_3) = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & \frac{1}{s} \\ 0 & 0 & -1 \end{pmatrix}.$$

For each τ, these three elements are projective reflections in the sides of T defining a representation ζ_τ which is discrete and injective. The union of tiles, $\Omega_\tau = \bigcup_{\gamma \in \Delta_{3,3,3}} \zeta_\tau(\gamma)\overline{T}$, is a convex open domain in $\mathbb{S}^2$. When $\tau = 0$, the convex domain Ω_0 is the affine chart defined by $x_1 + x_2 + x_3 > 0$, where (x_1, x_2, x_3) are coordinates with respect to the standard basis, and the representation ζ_0 preserves a Euclidean metric on this affine chart; we identify Ω_0 with the Euclidean plane $\mathbb{E}^2$ and think of ζ_0 as the inclusion into $\mathrm{Isom}\,\mathbb{E}^2 \subset \mathrm{SL}_3^\pm \mathbb{R}$. For $\tau \neq 0$, Ω_τ is the interior of the convex hull of the directions of the three vectors

$$l_1(\tau) = \epsilon_\tau(-1, s, 0), \quad l_2(\tau) = \epsilon_\tau(0, -1, s) \quad \text{and} \quad l_3(\tau) = \epsilon_\tau(s, 0, -1),$$

where $\epsilon_\tau = \tau/|\tau|$; see Figure 7. Any ζ_τ–equivariant homeomorphism of the universal cover $\widetilde{S}_{3,3,3} = \mathbb{E}^2$ with Ω_τ is a developing map for the unique convex projective structure on $S_{3,3,3}$ associated to the representation ζ_τ. Of course, for any regular hex torus S covering $S_{3,3,3}$ and for any $\tau \in \mathbb{R}$, we obtain a convex projective structures on S by pullback. If $\Delta \subset \Delta_{3,3,3}$ denotes the fundamental group of S, then the convex projective torus $\zeta_\tau(\Delta)\backslash\Omega_\tau$ is called a *convex projective hex torus*; indeed, such a torus may be decomposed into "regular" hexagons, each of which is the union of six copies of the fundamental triangle T.

The maximal torus subgroup Δ_H of $\Delta_{3,3,3}$ is generated by three elements (which are translations in the Euclidean structure)

$$g_1 = r_3 r_2 r_3 r_1, \quad g_2 = r_1 r_3 r_1 r_2, \quad g_3 = r_2 r_1 r_2 r_3.$$

For $\tau \neq 0$, let h_τ denote the change of basis taking $(l_1(\tau), l_2(\tau), l_3(\tau))$ to the standard basis. Then $h_\tau \zeta_\tau(g_i) h_\tau^{-1} = \exp(\tau \alpha_i)$ for $i \in \{1, 2, 3\}$, where

$$\alpha_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
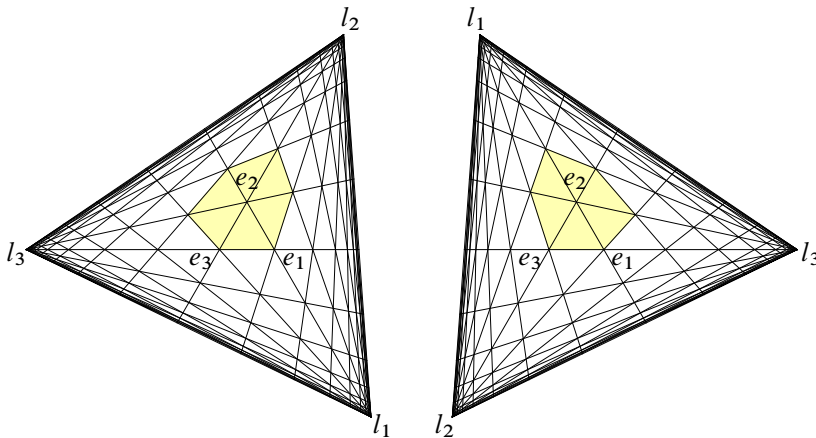
Figure 7: Fundamental domains of dual minimal hex tori for $\tau = -\frac{3}{4}$ (left) and $\tau = \frac{3}{4}$ (right)

It is then clear that, for any $\gamma \in \Delta_H$ and natural number k,

$$(7\text{-}1) \qquad\qquad h_\tau \zeta_\tau (k\gamma) h_\tau^{-1} = h_{k\tau} \zeta_{k\tau}(\gamma) h_{k\tau}^{-1}.$$

Therefore, if Δ is any hex torus group, the path of representations ζ_τ restricted to $k\Delta$ looks the same, up to conjugation, as the k–times faster path of representations $\zeta_{k\tau}$ restricted to Δ. Using this we prove:

Lemma 7.4 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Euclidean regular hex tori with fundamental groups $\Delta_1, \Delta_2 \subset \Delta_{3,3,3}$ and assume that $f: \mathcal{S}_1 \to \mathcal{S}_2$ is a lattice k_2/k_1–isometry, with $f_*: \Delta_1 \to \Delta_2$ denoting the isomorphism at the level of fundamental group. Then, for any $\tau \neq 0$, there exists $g \in \mathrm{SL}_3^{\pm}\mathbb{R}$ such that $g\zeta_{\tau/k_1}(\gamma)g^{-1} = \zeta_{\tau/k_2}(f_*(\gamma))$ for all $\gamma \in \Delta_1$.

Proof By assumption, there exists Δ' such that $\Delta_1 = k_1 \Delta'$ and $\Delta_2 = k_2 \Delta'$ and $k_2^{-1} f_* k_1: \Delta' \to \Delta'$ is conjugation by an element $\delta \in \Delta_{3,3,3}$. Then, for any $\gamma \in \Delta_1$, write $\gamma = k_1 \gamma'$, where $\gamma' \in \Delta'$ and observe

$$k_2^{-1} f_* k_1 \gamma' = \delta \gamma' \delta^{-1} \quad \Longrightarrow \quad f_*(\gamma) = k_2 \delta \gamma' \delta^{-1}$$
$$\Longrightarrow \quad \zeta_{\tau/k_2}(f_*(\gamma)) = \zeta_{\tau/k_2}\left(\frac{k_2}{k_1} k_1 \delta \gamma' \delta^{-1}\right)$$
$$= h\zeta_{\tau/k_1}(\delta k_1 \gamma' \delta^{-1}) h^{-1}$$
$$= g\zeta_{\tau/k_1}(\gamma) g^{-1},$$

where $h \in \mathrm{SL}_3^{\pm}\mathbb{R}$ is determined by (7-1) and $g = h\zeta_{\tau/k_1}(\delta)$. $\square$

7.3 Gluing 3–manifolds which cover reflection orbifolds

Let $\mathcal{O}$ be a finite-volume hyperbolic three-orbifold. Each cusp of $\mathcal{O}$ is naturally the product $\partial \times \mathbb{R}$ of a Euclidean two-orbifold ∂ with an interval. We think of $\mathcal{O}$ as the interior of an "orbifold with boundary", where the boundary includes a copy of each cusp cross-section ∂ at infinity. Note that if $\mathcal{O}$ is a reflection orbifold, then any convex projective structure on $\mathcal{O}$ induces a convex projective structure on its boundary components ∂. Indeed, the fundamental group Δ of ∂ fixes a unique point with eigenvalue one under the holonomy representation of any convex projective structure on $\mathcal{O}$; the convex projective structure induced on ∂ may be seen in the link of this point. Now, let $\partial \cong \mathcal{S}_{3,3,3}$ be a boundary component isomorphic to the Euclidean $(3, 3, 3)$–triangle orbifold. Then, for any manifold covering $M \to \mathcal{O}$, the torus boundary components $\widetilde{\partial}_1, \dots, \widetilde{\partial}_n$ of M which cover ∂ are each naturally endowed with a regular Euclidean hex structure.

Theorem 7.5 *For $i \in \{1, \dots, m\}$, let M_i be a hyperbolic three-manifold of finite volume which covers a reflection orbifold $\mathcal{O}_i$. For $\alpha \in \{1, \dots, n_i\}$, let $(\widetilde{\partial}_i)_\alpha$ be a torus component of ∂M_i which covers a component $\partial_i \cong \mathcal{S}_{3,3,3}$ of $\partial \mathcal{O}_i$. Let $\mathscr{F}$ be a collection of homeomorphisms of the form $f_{i,j}^{\alpha,\beta} \colon (\widetilde{\partial}_i)_\alpha \to (\widetilde{\partial}_j)_\beta$ which identify the boundary components $\{(\widetilde{\partial}_i)_\alpha\}_{i,\alpha}$ in disjoint pairs. Assume that:*

(1) *There are natural numbers $k_1, \dots, k_m$ such that any $f_{i,j}^{\alpha,\beta} \in \mathscr{F}$ is a lattice k_j/k_i–isometry of the induced regular hex torus structure on $(\widetilde{\partial}_i)_\alpha$, and $(\widetilde{\partial}_j)_\beta$.*

(2) *For each $i \in \{1, \dots, m\}$, there exists a nontrivial continuous path $c_i \colon (-\epsilon_i, \epsilon_i) \to \mathfrak{C}(\mathcal{O}_i)$ passing through the finite-volume complete hyperbolic structure on $\mathcal{O}_i$ at $c_i(0)$ such that the convex projective structure on ∂_i induced by $c_i(t)$ is not constant.*

Then there exists a properly convex projective structure on the manifold obtained by gluing together the building blocks $M_1, \dots, M_m$ using the gluing maps in $\mathscr{F}$.

Proof By assumption, the convex projective structure on $\partial_i \cong \mathcal{S}_{3,3,3}$ is not constant, so for some $t_i \neq 0$, the structure is isomorphic to the non-Euclidean convex projective structure associated to some $\tau_i \neq 0$ described in the previous section. It follows easily from the fact that the fundamental group of $\mathcal{O}_i$ is generated by reflections that the restriction of the holonomy representation ρ_{t_i} for $c_i(t_i)$ to the fundamental group $(\Delta_{3,3,3})_i$ of ∂_i has a fixed point p_i with eigenvalue one and that, in a suitable basis,

$\rho_{t_i}|_{(\Delta_{3,3,3})_i} = \zeta_{\tau_i} \oplus \mathrm{id}_1$, where id_1 denotes the one-dimensional trivial representation. Hence, for each $\alpha \in \{1, \dots, n_i\}$, the restriction of ρ_{t_i} to the fundamental group $(\Delta_i)_\alpha$ of $(\widetilde{\partial}_i)_\alpha$ is diagonalizable and satisfies the middle eigenvalue condition. Therefore, the pullback of $c_i(t_i)$ to M_i induces a convex projective structure which, after removing a collar neighborhood, as in Section 5, has principal totally geodesic boundary along its (relevant) components.

Further, by continuity, τ_i may be varied continuously in a small neighborhood $[0, \delta_i)$. Hence, we may choose $t_1, \dots, t_m$ so that $(\tau_1, \dots, \tau_m) = \mu(1/k_1, \dots, 1/k_m)$ for some small $\mu \in \mathbb{R}^+$. It now follows from the assumption (1) and Lemma 7.4 that the maps $f_{i,j}^{\alpha,\beta}$ of $\mathscr{F}$ satisfy the holonomy matching condition (1-1) and Theorem 1.4 implies the result. $\square$

7.4 Application: two-colorable tetrahedral hyperbolic manifolds

A finite-volume hyperbolic three-manifold is *tetrahedral* if it admits an ideal triangulation consisting of regular ideal tetrahedra. A triangulated three-manifold is *two-colorable* if the tetrahedra can be colored using two colors so that no two adjacent tetrahedra have the same color. Two-colorability may be easily checked by `Regina` [6]. For example, the figure eight knot complement and its sister manifold are tetrahedral and two-colorable. Recently, Fominykh, Garoufalidis, Goerner, Tarkaev and Vesnin [19] gave a census of all orientable tetrahedral manifolds with at most 25 tetrahedra.

It is well known (see [19, Remark 5.5] for instance) that every two-colorable tetrahedral manifold M is a cover of the Bianchi orbifold $\mathcal{O}_3 = \mathrm{PSL}(2, \mathbb{Z}[\zeta]) \backslash \mathbb{H}^3$ of discriminant $D = -3$, where $\zeta = \frac{1}{2}(1 + \sqrt{-3})$. Further, $\mathcal{O}_3$ covers the reflection orbifold $\mathcal{O}_{3,3,3}^{2,2,3}$ determined by reflections in the faces of the partially ideal tetrahedron with dihedral angles $\frac{\pi}{2}$, $\frac{\pi}{2}$ and $\frac{\pi}{3}$ at the three edges bounding one face and $\frac{\pi}{3}$ along the other edges, incident on the unique ideal vertex. The reflection orbifold $\mathcal{O}_{3,3,3}^{2,2,3}$ has one cusp and its cross-section is the Euclidean $(3, 3, 3)$–triangle orbifold. Further, it follows from Benoist's work [2] that the hyperbolic structure on the orbifold $\mathcal{O}_{3,3,3}^{2,2,3}$ may be deformed nontrivially to nearby convex projective structures. So the deformation hypothesis (2) of Theorem 7.5 is satisfied for tetrahedral two-colorable manifolds, and therefore the theorem may be applied given appropriate gluing maps between torus boundary components that respect the hex structure as in hypothesis (1). Indeed, such gluing maps seem to be easy to find. We now give some examples.

Of the 29 orientable tetrahedral manifolds with at most 8 tetrahedra, 20 are two-colorable, and each of those has at most 2 cusps. In Table 1 below, these 20 manifolds

are listed along with the hex cusp shapes of the cusps. The hex torus structure at the cusps of tetrahedral manifolds can be calculated easily using `SnapPy` [16]. In Figure 8, left, a fundamental triangle for the cusp of $\mathcal{O}^{2,2,3}_{3,3,3}$ is drawn in yellow, while the cusp pattern of the tessellation of $\mathbb{H}^3$ by regular ideal tetrahedra is drawn in black. The hex torus structure of one boundary component ∂_1 of the tetrahedral two-colorable manifold m207 is shown in Figure 8, right: Δ_H (resp. the hex lattice Δ_1 of ∂_1) corresponds to vertices of triangle with black edges (resp. red dots), and two generators of Δ_1 are shown as pink arrows; we easily read of the hex cusp shape: $A_{\Delta_1} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Name	Name [19]	∂_1	∂_2	Name	Name [19]	∂_1	∂_2
m003	otet02$_{0000}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$		t12845	otet08$_{0001}$	$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$
m004	otet02$_{0001}$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$		t12840	otet08$_{0002}$	$\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$
m202	otet04$_{0000}$	$\begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	t12842	otet08$_{0003}$	$\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$
m203	otet04$_{0001}$	$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	t12843	otet08$_{0004}$	$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
m206	otet04$_{0002}$	$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$		t12844	otet08$_{0005}$	$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
m207	otet04$_{0003}$	$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$		t12837	otet08$_{0006}$	$\begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix}$	
s959	otet06$_{0002}$	$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	t12839	otet08$_{0007}$	$\begin{bmatrix} 4 & 2 \\ 0 & 4 \end{bmatrix}$	
s961	otet06$_{0003}$	$\begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$		t12838	otet08$_{0008}$	$\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$	
s960	otet06$_{0004}$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$		t12836	otet08$_{0009}$	$\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$
s958	otet06$_{0006}$	$\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$		t12841	otet08$_{0010}$	$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Table 1: Hex torus cusps of orientable, two-colorable tetrahedral manifolds with at most 8 tetrahedra

From Table 1, we may easily read off several new examples of closed three-manifolds that admit properly convex projective structures.

Theorem 7.6 *In each line of Table 2, there exist gluing maps identifying the boundary components of the given building blocks in pairs so that the resulting closed three-manifold admits a properly convex projective structure. The gluing maps may be read off from Table 1.*

Proof Examination of Table 1 verifies that each given identification of boundary tori satisfies the hypotheses of Lemma 7.2 and therefore a lattice $\mathbb{Q}$–isometry exists
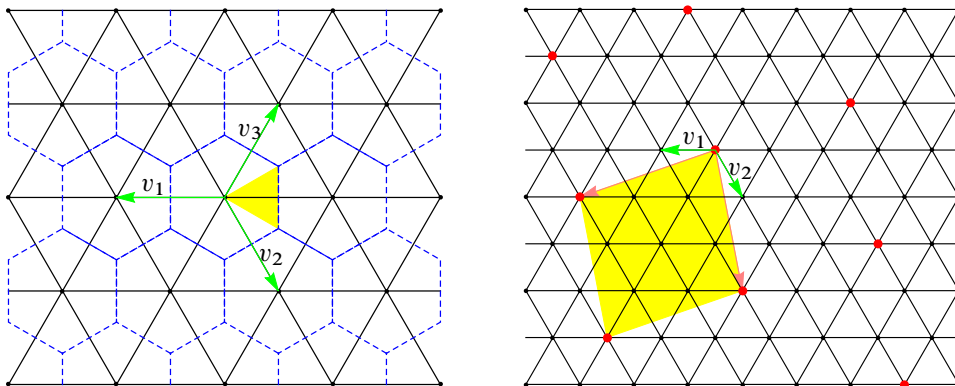
Figure 8: Hex torus cusps of orientable two-colorable tetrahedral manifolds (left) and a hex torus cusp ∂_1 of m207 (right)

between the given boundary components. For example, $\partial_1(\texttt{m003})$ is lattice $\frac{3}{2}$–isometric to $\partial_1(\texttt{s959})$ and $\partial_2(\texttt{s959})$ is lattice $\frac{2}{1}$–isometric to $\partial_1(\texttt{s960})$, and so on. The result follows from Theorem 7.5.   □

Remark 7.7 As in Remark 7.3, there may be multiple (but no more than 12) gluing maps identifying a given pair of boundary components in Theorem 7.6. Different gluing maps might or might not produce homeomorphic manifolds. We do not check carefully here the number of different homeomorphism types of closed three-manifolds for which Theorem 7.6 gives properly convex projective structures. It is at least five, but seems likely to be more.

Building blocks	Gluing maps
m003, s959, s960	$\partial_1\texttt{m003} \to \partial_1\texttt{s959}$ $\partial_2\texttt{s959} \to \partial_1\texttt{s960}$
m003, t12841, s960	$\partial_1\texttt{m003} \to \partial_2\texttt{t12841}$ $\partial_1\texttt{t12841} \to \partial_1\texttt{s960}$
m004, t12840, s961	$\partial_1\texttt{m004} \to \partial_2\texttt{t12840}$ $\partial_1\texttt{t12840} \to \partial_1\texttt{s961}$
t12843, t12844	$\partial_1\texttt{t12843} \to \partial_1\texttt{t12844}$ $\partial_2\texttt{t12843} \to \partial_2\texttt{t12844}$
t12842, t12839, s961	$\partial_1\texttt{t12842} \to \partial_1\texttt{s961}$ $\partial_2\texttt{t12842} \to \partial_1\texttt{t12839}$

Table 2: Building blocks and boundary pairings for Theorem 7.6

7.5 Application: the cusp covering conjecture

Finally, we briefly mention an application of the cusp covering conjecture, proven by Wise (see [42; 43, Corollary 16.15]).

Theorem 7.8 (cusp covering conjecture; Wise [42; 43]) *Let M be a finite-volume hyperbolic 3–manifold and let $\partial M = \partial_1 \sqcup \partial_2 \sqcup \cdots \sqcup \partial_r$. There exist finite covers $\partial_i^\circ \to \partial_i$ such that, for any further finite covers $\partial_i^c \to \partial_i^\circ$, there is a finite cover $\widehat{M} \to M$ such that, for each i, each cover of ∂_i appearing on the boundary of $\widehat{M}$ is isomorphic to $\partial_i^c \to \partial_i$.*

Let $\mathcal{O}$ be a reflection orbifold with one cusp $\partial \cong \mathcal{S}_{3,3,3}$ satisfying the hypothesis (2) of Theorem 7.5 and let M be a manifold cover. It follows easily from the cusp covering conjecture that there is a finite cover $\widehat{M}$ of M (in fact infinitely many such) such that the inclusion of the fundamental group Δ_i^c of each boundary component ∂_i^c into the fundamental group $\Delta_{3,3,3}$ of ∂ has the same image. Hence, we may glue together in pairs the boundary components of $\widehat{M}$ via a lattice isometry of the hex torus structure. The resulting manifolds admit convex projective structures by Theorem 7.5. We may also arrange that, for each i, the image of some $\Delta_i^c \hookrightarrow \Delta_{3,3,3}$ is a multiple $k\Delta'$ of the fundamental group of any given hex torus subgroup Δ'. Let Δ' be the hex lattice isomorphic to, for example, the hex torus fundamental group of the boundary torus $\partial M'$ of some one-cusped manifold M' in Table 1. Then any boundary component of the resulting cover $\widehat{M}$ of $\mathcal{O}$ may be glued to the boundary of a copy of M' by a lattice k–isometry. Let N be a manifold obtained by gluing together some pairs of boundary components of $\widehat{M}$ via hex lattices isometries and by gluing on copies of M' with hex lattice k–isometries to the remaining boundary components. Then N admits a properly convex projective structure by Theorem 7.5. This gives one way to obtain many interesting examples of closed nonhyperbolic three-manifolds N which admit convex projective structures.

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