A_∞ FUNCTORS FOR LAGRANGIAN CORRESPONDENCES

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Abstract. We construct A_∞ functors between Fukaya categories associated to monotone Lagrangian correspondences between compact symplectic manifolds. We then show that the composition of A_∞ functors for correspondences is homotopic to the functor for the composition, in the case that the composition is smooth and embedded.

Contents

1. Introduction

Recall that to any compact symplectic manifold (M, ω) is a *Fukaya category* $\mathrm{Fuk}(M)$ whose objects are Lagrangian submanifolds, morphism spaces are Lagrangian Floer cochain groups, and composition maps count pseudoholomorphic polygons with boundary in a given sequence of Lagrangians [10]. A construction of Kontsevich [15] constructs a triangulated *derived Fukaya category* which is related via the homological mirror symmetry conjecture to the derived category of bounded complexes of coherent sheaves. The latter admits natural *Mukai functors* associated to correspondences which play an important role in, for example, the McKay correspondence [25], the work of Nakajima [22], etc.

The main result of this paper constructs A_∞ functors associated to monotone Lagrangian correspondences which are meant to be mirror analogs to the Mukai functors. We learned the idea of constructing functors associated to Lagrangian correspondences from Fukaya, who suggested an approach using duality. In his construction the functor maps a Fukaya category of the domain of the correspondence to the dual of the codomain. This makes composition of functors problematic; the approach here avoids that problem by enlarging the Fukaya category. The results of this paper are chain-level versions of an earlier paper [39] in which the second two authors constructed cohomology-level functors between categories for Lagrangian correspondences. We also showed in [40] that composition of these functors agrees with the geometric composition in the case that the Lagrangian correspondences have embedded composition. Applications of the calculus of A_∞ functors developed in this paper can be found in Abouzaid and Smith [1] and Smith [30], as well as in Wehrheim-Woodward [36], [37], [32].

1.1. **Summary of results.** Given A_∞ categories $\mathcal{C}_0, \mathcal{C}_1$, let $\operatorname{Func}(\mathcal{C}_0, \mathcal{C}_1)$ denote the A_∞ category of functors from $\mathcal{C}_0$ to $\mathcal{C}_1$ (see Definition 8.1 for our conventions on A_∞ categories and functors). We construct for any pair of monotone symplectic manifolds M_0, M_1 a Fukaya category of admissible correspondences $\operatorname{Fuk}^\#(M_0, M_1)$. The objects of $\operatorname{Fuk}^\#(M_0, M_1)$ are *sequences* of compact Lagrangian correspondences with a brane structure, which we call *generalized Lagrangian correspondences*. The *brane structure* consists of an orientation, grading, and relative spin structure. The correspondences are also required to be *admissible* in the sense that the minimal Maslov numbers are at least three, or vanishing disk invariant, and the fundamental groups are torsion for any choice of base point. Denote by $\operatorname{Fuk}^\#(M) := \operatorname{Fuk}^\#(\mathrm{pt}, M)$ the natural enlargement of the Fukaya category $\operatorname{Fuk}(M)$ whose objects are admissible generalized Lagrangian correspondences with brane structures from points to a compact monotone symplectic manifold M. Our first main result is:

Theorem 1.1. (Functors for Lagrangian correspondences) *Suppose that M_0, M_1 are compact monotone symplectic manifolds with the same monotonicity constant. There exists an A_∞ functor*

$$\operatorname{Fuk}^\#(M_0, M_1) \to \operatorname{Func}(\operatorname{Fuk}^\#(M_0), \operatorname{Fuk}^\#(M_1))$$

inducing the functor of cohomology categories in [39, Definition 5.1].

In particular, for each admissible Lagrangian correspondence $L_{01} \subset M_0^- \times M_1$ equipped with a brane structure we construct an A_∞ functor

$$\Phi(L_{01}) : \operatorname{Fuk}^\#(M_0) \to \operatorname{Fuk}^\#(M_1)$$

acting in the expected way on Floer cohomology: for Lagrangian branes $L_0 \subset M_0, L_1 \subset M_1$ there is an isomorphism with $\mathbb{Z}_2$-coefficients

$$H \operatorname{Hom}(\Phi(L_{01})L_0, L_1) \cong HF(L_0^- \times L_1, L_{01})$$

where the right-hand-side is the Floer cohomology of the pair $(L_0^- \times L_1, L_{01})$. For a pair of Lagrangian correspondences $L_{01}, L_{01}' \in M_0^- \times M_1$ and a Floer cocycle $\alpha \in CF(L_{01}, L_{01}')$ we construct a natural transformation

$$\mathcal{T}_\alpha : \Phi(L_{01}) \to \Phi(L_{01}')$$

of the corresponding A_∞ functors.

The behavior of the A_∞ functors for Lagrangian correspondences under embedded geometric composition as defined in [40] is our second main result. To state it, we recall that the geometric composition of Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \quad L_{12} \subset M_1^- \times M_2$$

is

$$L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{M_1} L_{12}) \tag{1}$$

where $\pi_{02} : M_0 \times M_1^2 \times M_2 \to M_0 \times M_2$ is the projection onto the product of the first and last factors. If the fiber product is transverse and embedded by π_{02} then $L_{01} \circ L_{12}$ is a smooth Lagrangian correspondence.

Theorem 1.2. (Geometric composition theorem) *Suppose that M_0, M_1, M_2 are monotone symplectic manifolds with the same monotonicity constant. Let $L_{01} \subset M_0^- \times M_1, L_{12} \subset M_1^- \times M_2$ be admissible Lagrangian correspondences with spin structures and gradings such that $L_{01} \circ L_{12}$ is smooth, embedded by π_{02} in $M_0^- \times M_2$, and admissible. Then there exists a homotopy of A_∞ functors*

$$\Phi(L_{12}) \circ \Phi(L_{01}) \simeq \Phi(L_{01} \circ L_{12}).$$

There is a slightly more complicated statement in the case that the correspondences are only relatively spin, which involves a shift in the background class. In particular, the theorem implies that the associated derived functors

$$D(\Phi(L_{12})) \circ D(\Phi(L_{01})) \cong D(\Phi(L_{01} \circ L_{12})) : D\mathrm{Fuk}^\#(M_0) \to D\mathrm{Fuk}^\#(M_2)$$

are canonically isomorphic. The result extends to generalized Lagrangian correspondences, in particular the empty correspondence. In the last case the result shows that the Fukaya categories constructed using two different systems of perturbation data are homotopy equivalent.

A complete chain-level version of the earlier work is still missing. Namely, one would like to construct a *Weinstein-Fukaya* A_∞ 2-category whose objects are symplectic manifolds and morphism categories are the extended Fukaya categories of correspondences. Furthermore one would like an A_∞ categorification functor given by the extended Fukaya categories on objects and the functor of Theorem 1.1 on morphisms. This theory would be the chain level version of the Weinstein-Floer 2-category and categorification functor constructed in [39]. Some steps in this direction have been taken by Bottman [5], [6]. Batanin has pointed out to us a possibly-relevant construction of homotopy higher categories in [2, Definition 8.7].

The A_∞ structures, functors, and natural transformations are defined using a general theory of family quilt invariants that count pseudoholomorphic quilts with varying domain. This theory includes families of quilts associated to the associahedron, multiplihedron, and other polytopes underlying the various A_∞ structures. Unfortunately these families of quilted surfaces come with the rather inconvenient (for analysis) property that degeneration is not given by "neck stretching" but rather by "nodal degeneration". Our first step is to replace these families by ones that are more analytically convenient, see Section 2 for the precise definitions. We say that a stratified space is *labelled by quilt data* if for each stratum there is given a combinatorial type of quilted surface, and each pair of strata there is given a subset of gluing parameters for the strip like ends as in Definition 2.17 below. For technical reasons (contractibility of various choices) it is helpful to restrict to the case that each patch of each quilt is homeomorphic to a disk with at least one marking, and so has homotopically trivial automorphism group. Such quilt data are called *irrotatable*; the general case could be handled with more complicated data associated to the stratified space.

Theorem 1.3. (Existence of families of quilts with strip-like ends) *Given a stratified space $\mathcal{R}$ equipped with irrotatable quilt data, there exists a family of quilted surfaces $\mathcal{S} = (\underline{S}_r)_{r\in\mathcal{R}}$ with strip-like ends over $\mathcal{R}$ with the given data in which degeneration is given by neck-stretching.*

The next step is to define pseudoholomorphic quilt invariants associated to these families. Let $\mathcal{S} = (\underline{S}_r, r \in \mathcal{R})$ be a family of quilted surfaces with strip-like ends over a stratified space $\mathcal{R}$, $\underline{M}$ a collection of admissible monotone symplectic manifolds associated to the patches, and $\underline{L}$ a collection of admissible monotone Lagrangian correspondences associated to the seams and boundary components. Given a family $\underline{J}$ of compatible almost complex structures on the collection $\underline{M}$ and a Hamiltonian perturbation $\underline{K}$, a *holomorphic quilt* from a fiber of $\mathcal{S}$ to $\underline{M}$ is pair

$$(r \in \mathcal{R}, \underline{u} : \underline{S}_r \to \underline{M})$$

consisting of a point $r \in \mathcal{R}$ together with a $(\underline{J}, \underline{K})$-holomorphic map $\underline{u} : \underline{S}_r \to \underline{M}$ taking values in $\underline{L}$ on the seams and boundary, see Definition 3.2 for the precise equation. The necessary regularity statement is the following, proved in Theorem 3.4 in Section 2.

Theorem 1.4. (Transversality for families of holomorphic quilts) *Suppose that $\mathcal{S} \to \mathcal{R}$ is a family of quilted surfaces with strip-like ends equipped with compact monotone symplectic manifolds $\underline{M}$ for the patches and admissible Lagrangian correspondences $\underline{L}$ for the seams/boundaries. Suppose over the boundary of $\mathcal{R}$ a collection of perturbation data $(\underline{J}, \underline{K})$ is given making all pseudoholomorphic quilts of formal dimension at most one regular. Then for a generic extension of $(\underline{J}, \underline{K})$ agreeing with the extensions given by gluing near the boundary $\mathcal{S}|_{\partial\mathcal{R}}$, every pseudoholomorphic quilt $u : \underline{S}_r \to \underline{M}$ of formal dimension at most one with strip-like ends is parametrized regular.*

Using Theorem 1.4 we construct moduli spaces of pseudoholomorphic quilts and, using these, chain level *family quilt invariants* given as counts of isolated elements in the moduli space. As in the standard topological field theory philosophy, these invariants map the tensor product of cochain groups for the incoming ends $\mathcal{E}_-(\mathcal{S})$ to that for the outgoing ends $\mathcal{E}_+(\mathcal{S})$:

$$\Phi_{\mathcal{S}} : \bigotimes_{\underline{e}\in\mathcal{E}_-(\mathcal{S})} CF(\underline{L}_{\underline{e}}) \to \bigotimes_{\underline{e}\in\mathcal{E}_+(\mathcal{S})} CF(\underline{L}_{\underline{e}}).$$

These chain-level family invariants satisfy a *master equation* arising from the study of one-dimension components of the moduli spaces of pairs above:

Theorem 1.5. (Master equation for family quilt invariants) *Suppose that, in the setting of Theorem 1.4, $\mathcal{S} \to \mathcal{R}$ is a family of quilted surfaces with strip-like ends over an oriented stratified space $\mathcal{R} = \cup_\Gamma \mathcal{R}_\Gamma$ (here the strata are indexed by Γ) with boundary multiplicities $m_\Gamma \in \mathbb{Z}, \operatorname{codim}(\mathcal{R}_\Gamma) = 1$. Then the chain level invariant $\Phi_{\mathcal{S}}$*

and the coboundary operators ∂ on the tensor products of Floer cochain complexes satisfy the relation

$$\partial \circ \Phi_{\mathcal{S}} - \Phi_{\mathcal{S}} \circ \partial = \sum_{\Gamma, \mathrm{codim}(\mathcal{R}_\Gamma)=1} m_\Gamma \Phi_{\mathcal{S}_\Gamma}.$$

In other words, if $\partial\mathcal{S}$ denotes the contribution from boundary components of $\mathcal{R}$ counted with multiplicity and $\partial\Phi_{\mathcal{S}} = [\partial, \Phi_{\mathcal{S}}]$ denotes the boundary of $\Phi_{\mathcal{S}}$ considered as a morphism of chain complexes then

$$\partial\Phi_{\mathcal{S}} = \Phi_{\partial\mathcal{S}}. \tag{2}$$

The master equation (2) specializes to the A_∞ associativity, functor, natural transformation, and homotopy axioms for the various families of quilts we consider.

The paper is divided into two parts. The first part covers the general theory of parametrized pseudoholomorphic quilts and the construction of family quilt invariants. The second part covers the application of this general theory to specific families of quilts. These applications include the construction of the generalized Fukaya category, A_∞ functors between generalized Fukaya categories, as well as natural transformations and homotopies of A_∞ functors. The reader is encouraged to look at the constructions of Section 4 while reading Sections 2 and 3, in order to have concrete examples of families of quilts in mind.

The present paper is an updated and more detailed version of a paper the authors have circulated since 2007. The second and third authors have unreconciled differences over the exposition in the paper, and explain their points of view at math.berkeley.edu/∼katrin/wwpapers/ resp. christwoodwardmath.blogspot.com/. The publication in the current form is the result of a mediation.

2. Families of quilted surfaces with strip-like ends

In this and the following section we construct invariants of families of pseudoholomorphic quilts over stratified spaces, mapping tensor products of the Floer cochain groups for the incoming ends to those for the outgoing ends. We also show that Theorems 1.3, 1.4 and 1.5 from the introduction hold.

First we define a surface with strip-like ends. The definition below is essentially the same as the definition given in Seidel's book [27], except that each strip-like end comes with an extra parameter prescribing its width.

Definition 2.1. (Surfaces with strip-like ends) A *surface with strip-like ends* consists of the following data:

(a) A compact oriented surface $\overline{S}$ with boundary $\partial\overline{S}$ the disjoint union of circles $\partial\overline{S} = C_1 \sqcup \ldots \sqcup C_m$ and $d_n \geq 0$ distinct points $\underline{z}_n = (z_{n,1}, \ldots, z_{n,d_n}) \subset C_n$ in cyclic order on each boundary circle $C_n \cong S^1$ for each $n = 1, \ldots, m$. We use the indices on C_n modulo d_n, and index all marked points by

$$\mathcal{E} = \mathcal{E}(S) = \bigl\{ e = (n,l) \,\big|\, n \in \{1, \ldots, m\}, l \in \{1, \ldots, d_n\} \bigr\}. \tag{3}$$

Here we use the notation $e \pm 1 := (n, l \pm 1)$ for the cyclically adjacent indices to $e = (n,l)$. Denote by $I_e := I_{n,l} \subset C_n$ the component of ∂S between

$z_e := z_{n,l}$ and $z_{e+1} := z_{n,l+1}$. However, the boundary ∂S may also have compact components $I = C_n \cong S^1$;

(b) A complex structure j_S on $S := \overline{S} \setminus \{z_e \,|\, e \in \mathcal{E}\}$;

(c) A set of *strip-like ends* for S, that is a set of embeddings with disjoint images

$$\epsilon_e : \mathbb{R}^{\pm} \times [0, \delta_e] \to S$$

for all $e \in \mathcal{E}$ such that the following hold:

$$\begin{aligned} &\epsilon_e(\mathbb{R}^{\pm} \times \{0, \delta_e\}) \subset \partial S \\ &\lim_{s \to \pm\infty}(\epsilon_e(s,t)) = z_e, \quad \forall t \in [0, \delta_e] \\ &\epsilon_e^* j_S = j_0 \end{aligned}$$

where in the first item $\mathbb{R}^{\pm} = (0, \pm\infty)$ and in the third item j_0 is the canonical complex structure on the half-strip $\mathbb{R}^{\pm} \times [0, \delta_e]$ of width $\delta_e > 0$. Denote the set of incoming ends $\epsilon_e : \mathbb{R}^- \times [0, \delta_e] \to S$ by $\mathcal{E}_- = \mathcal{E}_-(S)$ and the set of outgoing ends $\epsilon_e : \mathbb{R}^+ \times [0, \delta_e] \to S$ by $\mathcal{E}_+ = \mathcal{E}_+(S)$;

(d) An ordering of the set of (compact) boundary components of $\overline{S}$ and orderings

$$\mathcal{E}_- = (e_1^-, \dots, e_{N_-}^-), \quad \mathcal{E}_+ = (e_1^+, \dots, e_{N_+}^+)$$

of the sets of incoming and outgoing ends; Here $e_i^{\pm} = (n_i^{\pm}, l_i^{\pm})$ denotes the incoming or outgoing end at $z_{e_i^{\pm}}$.

A *nodal surface with strip-like ends* consists of a surface with strip-like ends S, together with a set of pairs of ends (the *nodes* of the nodal surface)

$$\underline{w} = \{\{w_1^+, w_1^-\}, \dots, \{w_m^+, w_m^-\}\}, w_i^{\pm} \in \mathcal{E} = \sqcup_{k \in K} \mathcal{E}_k$$

such that for each w_j^+, w_j^-, the widths satisfy $\delta_j^+ = \delta_j^-$ (the widths of the strips are the same). A nodal surface $(S, \underline{w})$ give rise to a topological space obtained from the union $S \cup \{w_j^{\pm}, j = 1, \dots, m\} \subset \overline{S}$ by identifying w_j^+ with w_j^- for each $j = 1, \dots, m$. The resulting surface is still denoted S.

The structure maps of the Fukaya category, according to the definition in Seidel [27], are defined by counting points in a parametrized moduli space in family of surfaces with strip-like ends. These are defined as follows:

Definition 2.2. (Families of nodal surfaces with strip-like ends) A *smooth family* of nodal surfaces with strip-like ends over a smooth base $\mathcal{R}$ consists of

(a) a smooth manifold with boundary $\mathcal{S}$,

(b) a fiber bundle $\pi : \mathcal{S} \to \mathcal{R}$ and

(c) a structure of a nodal surface with strip-like ends on each fiber $\mathcal{S}_r := \pi^{-1}(r)$, whose diffeomorphism type is independent of r;

such that $\mathcal{S}_r$ varies smoothly with r (that is, the complex structures j_{S_r} fit together to smooth maps $T^{\text{vert}}\mathcal{S} \to T^{\text{vert}}\mathcal{S}$) and each $r \in \mathcal{R}$ contains a neighborhood U in which the seam maps extend to smooth maps $\varphi_\sigma : I_\sigma \times U \xrightarrow{\sim} I'_\sigma$.

Example 2.3. (Gluing strip-like ends) A typical example of a family of surfaces with strip-like ends is obtained by gluing strip-like ends by a neck of varying length. Given a nodal surface S with strip-like ends and m nodes and a pair of ends with the same width δ_e, define a family of surfaces with strip-like ends over $\mathcal{R} = \mathbb{R}^m_{\geq 0}$ by the following *gluing construction*: For any $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m_{\geq 0}$ define

$$G_\gamma(S) = \Big(S - \cup_{k=1}^m \epsilon_{k,\pm}^{-1}(\pm(1/\gamma_k, \infty) \times [0,1])\Big) / \sim \tag{4}$$

by identifying the ends $w_k^\pm, k = 1, \ldots, m$ of S by the gluing in a neck of length $1/\gamma_k$, if $\gamma_k \neq 0$. That is, if one end is outgoing and one end is incoming then one removes the ends $w_k^\pm$ with coordinate $\pm s > 1/\gamma_k$ and identifies

$$\epsilon_{w_k^+}(s,t) \sim \epsilon_{w_k^-}(s - 1/\gamma_k, t)$$

for $s \in (0, 1/\gamma_k)$ and $t \in [0, \delta_e]$. If $\gamma_k = 0$ then the gluing construction leaves the node in place. This construction gives a family of surfaces with the same number of strip-like ends and one less node than S over $\mathcal{R}$ called the *glued surface*. More generally, given a family $\mathcal{S} = (\mathcal{S}_r, r \in \mathcal{R})$ of nodal surfaces with strip-like ends with m nodes over a base $\mathcal{R}$, we obtain via the gluing construction a family

$$G(\mathcal{S}) = \bigcup_{(r,\gamma)} G_\gamma(S_r)$$

over the base $\mathcal{R} \times (\mathbb{R}_{\geq 0})^m$ whose fiber at (r, γ) is the glued surface $G_\gamma(S_r)$.

In our earlier papers [34], [33] we associated invariants to Lagrangian correspondences by counting maps from *quilted surfaces*. The notions of family and gluing construction generalize naturally to the quilted setting. Recall the definition of quilted surface from [33].

Definition 2.4. (Quilted surfaces with strip-like ends) A *quilted surface* $\underline{S}$ with strip-like ends consists of the following data:

(a) (Patches) A collection $\underline{S} = (S_k)_{k=1,\ldots,m}$ of *patches*, that is surfaces with strip-like ends as in Definition 2.1 (a)-(c). In particular, each S_k carries a complex structure j_k and has strip-like ends $(\epsilon_{k,e})_{e \in \mathcal{E}(S_k)}$ of widths $\delta_{k,e} > 0$ near marked points:

$$\lim_{s \to \pm\infty} \epsilon_{k,e}(s,t) = z_{k,e} \in \partial \overline{S}_k, \quad \forall t \in [0, \delta_e].$$

Denote by $I_{k,e} \subset \partial S_k$ the noncompact boundary component between $z_{k,e-1}$ and $z_{k,e}$.

(b) (Seams) A collection of *seams*, pairwise-disjoint pairs

$$\mathscr{S} = \big(\{(k_\sigma, I_\sigma), (k'_\sigma, I'_\sigma)\}\big)_{\sigma \in \mathscr{S}}, \quad \sigma \subset \bigcup_{k=1}^m \{k\} \times \pi_0(\partial S_k),$$

and for each $\sigma \in \mathscr{S}$, a diffeomorphism of boundary components

$$\varphi_\sigma : \partial S_{k_\sigma} \supset I_\sigma \xrightarrow{\sim} I'_\sigma \subset \partial S_{k'_\sigma}$$

that satisfy the conditions:

(i) (Real analytic) Every $z \in I_\sigma$ has an open neighborhood $\mathcal{U} \subset S_{k_\sigma}$ such that the restriction $\varphi_\sigma|_{\mathcal{U} \cap I_\sigma}$ extends to an embedding

$$\psi_z : \mathcal{U} \to S_{k'_\sigma}, \quad \psi_z^* j_{k'_\sigma} = -j_{k_\sigma}.$$