TESTS FOR INJECTIVITY OF MODULES OVER COMMUTATIVE RINGS

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Abstract. It is proved that a module M over a commutative noetherian ring R is injective if $\operatorname{Ext}^i_R((R/\mathfrak{p})_{\mathfrak{p}}, M) = 0$ holds for every $i \geqslant 1$ and every prime ideal $\mathfrak{p}$ in R. This leads to the following characterization of injective modules: If F is faithfully flat, then a module M such that $\operatorname{Hom}_R(F, M)$ is injective and $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geqslant 1$ is injective. A limited version of this characterization is also proved for certain non-noetherian rings.

1. Introduction

Let R be a commutative ring. In terms of cohomology, Baer's criterion asserts that an R-module M is injective if (and only if) $\operatorname{Ext}^1_R(R/\mathfrak{a}, M) = 0$ holds for every ideal $\mathfrak{a}$ in R. When R is also noetherian, it suffices to test against prime ideals and locally, namely, M is injective if either of the following conditions holds:

- $\operatorname{Ext}^1_R(R/\mathfrak{p}, M) = 0$ for every prime ideal $\mathfrak{p}$ in R;
- $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) = 0$ for every prime ideal $\mathfrak{p}$ in R.

Here, and henceforth, $k(\mathfrak{p})$ denotes the field $(R/\mathfrak{p})_{\mathfrak{p}}$. The main result of this paper is that injectivity can be detected by vanishing of Ext globally against these fields.

Theorem 1.1. *Let R be a commutative noetherian ring and let M be an R-complex. If for some integer d, one has*

$$\operatorname{Ext}^i_R(k(\mathfrak{p}), M) = 0 \quad \textit{for every prime ideal } \mathfrak{p} \textit{ in } R \textit{ and all } i > d\,,$$

then the injective dimension of M is at most d.

As recalled in Example 2.2, the module $\operatorname{Ext}^1_R(k(\mathfrak{p}), M)$ can be quite different from $\operatorname{Ext}^1_R(R/\mathfrak{p}, M)$ and $\operatorname{Ext}^1_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}})$. Nevertheless the appearance of $\operatorname{Ext}_R(k(\mathfrak{p}), -)$ in this context is not unexpected in the light of the recent work on cosupport of complexes in [4]; see also the discussion around Corollary 3.3.

The proof of the theorem above is given in Section 2, and applications are presented in Section 3. One such, discussed in Remark 3.2, is a characterization of injectivity of an R-module M in terms of that of $\operatorname{Hom}_R(F, M)$, where F is a faithfully flat R-module. In Section 4, we establish a partial extension of this last result to certain non-noetherian rings.

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2. Proof of Theorem 1.1

Our standard reference for basic definitions and constructions involving complexes is [1]. We first recall that as a consequence of Baer's criterion, the injective dimension of an R-complex is detected by vanishing of Ext against cyclic modules.

Baer's criterion. Let R be a commutative noetherian ring, M an R-complex, and d an integer. One has $\operatorname{inj\,dim}_R M \leqslant d$ if and only if

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, M) = 0 \quad \text{for every ideal } \mathfrak{a} \text{ in } R \text{ and all } i > d\,.$$

This result is contained in [1, Theorem 2.4.I].

Consider the collection of ideals

$$\mathcal{U} := \{\mathfrak{a} \subset R \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \neq 0 \;\; \text{for some } i > d\}\,.$$

If this collection is empty, then the desired inequality, $\operatorname{inj\,dim}_R M \leqslant d$, holds by Bear's criterion. Thus, we assume that $\mathcal{U}$ is non-empty and aim for a contradiction. It is achieved by establishing a sequence of claims, the first of which is standard but included for convenience.

Claim 1. With respect to inclusion, $\mathcal{U}$ is a poset and its maximal elements are prime ideals.

Proof. Let $\mathfrak{a}$ be a maximal element in $\mathcal{U}$. Choose a prime ideal $\mathfrak{p} \supseteq \mathfrak{a}$ such that $\mathfrak{p}/\mathfrak{a}$ is an associated prime of $R/\mathfrak{a}$, and pick an element $r \in R$ be such that $\mathfrak{p} = (\mathfrak{a} : r)$. The ideal $\mathfrak{a}+(r)$ properly contains $\mathfrak{a}$ and hence is not in $\mathcal{U}$. From the exact sequence of Ext modules associated to the standard exact sequence

$$0 \longrightarrow R/\mathfrak{p} \longrightarrow R/\mathfrak{a} \longrightarrow R/(\mathfrak{a} + (r)) \longrightarrow 0$$

it follows that $\mathfrak{p}$ is in $\mathcal{U}$. Since $\mathfrak{a}$ is maximal in $\mathcal{U}$, the equality $\mathfrak{a} = \mathfrak{p}$ holds. □

Fix a maximal element $\mathfrak{p}$ in $\mathcal{U}$; by Claim 1 it is a prime ideal. Set $S := R/\mathfrak{p}$ and let Q be the field of fractions of the domain S. We proceed to analyze the S-complex

$$X := \mathbf{R}\mathrm{Hom}_R(S, M)\,.$$

Claim 2. The natural map $\mathrm{H}^i(X) \to Q \otimes_S \mathrm{H}^i(X)$ is an isomorphism for all $i > d$.

Proof. Fix an element $s \neq 0$ in S. Let x be an element in R whose residue class mod $\mathfrak{p}$ is s. By the maximality of $\mathfrak{p}$, the ideal $\mathfrak{p} + (x)$ is not in $\mathcal{U}$. As one has $S/(s) \cong R/(\mathfrak{p} + (x))$, it follows that $\operatorname{Ext}^i_R(S/(s), M) = 0$ holds for all $i > d$. Thus, applying $\mathbf{R}\mathrm{Hom}_R(-, M)$ to the exact sequence

$$0 \longrightarrow S \xrightarrow{\;s\;} S \longrightarrow S/(s) \longrightarrow 0\,,$$

shows that multiplication $\mathrm{H}^i(X) \xrightarrow{s} \mathrm{H}^i(X)$ is an isomorphism for $i > d$. □

In the derived category over S, consider the triangle defining (soft) truncations

$$\tau^{\leqslant d} X \longrightarrow X \longrightarrow \tau^{>d} X \longrightarrow \;. \tag{2.1}$$

Claim 3. There is an isomorphism $\tau^{>d}X \cong \mathrm{H}(\tau^{>d}X)$ in the derived category over S, and the action of S on $\mathrm{H}(\tau^{>d}X)$ factors through the embedding $S \to Q$.

Proof. It follows from Claim 2 that the canonical morphism $\tau^{>d}X \to Q \otimes_S \tau^{>d}X$ yields an isomorphism in the derived category over S. The right-hand complex is one of Q-vector spaces, so it is isomorphic to its homology, and another invocation of Claim 2 yields the claim. □

Claim 4. One has $\operatorname{inj\,dim}_S(\tau^{\leqslant d}X) \leqslant d$.

Proof. By Baer's criterion it suffices to show that $\operatorname{Ext}^i_S(S/\mathfrak{b}, \tau^{\leqslant d}X)$ vanishes for every ideal $\mathfrak{b}$ in S and all $i > d$. Notice first that we may assume that $\mathfrak{b}$ is non-zero, because for $i > d$ one has

$$\operatorname{Ext}^i_S(S, \tau^{\leqslant d}X) \cong \mathrm{H}^i(\tau^{\leqslant d}X) = 0\,,$$

where the vanishing is by construction. For $\mathfrak{b} \neq 0$ one has $Q \otimes_S S/\mathfrak{b} = 0$, and Claim 3 together with Hom-tensor adjunction yields

$$\begin{aligned}\operatorname{Ext}^*_S(S/\mathfrak{b}, \tau^{>d}X) &\cong \operatorname{Ext}^*_S(S/\mathfrak{b}, \mathrm{H}(\tau^{>d}X))\\ &\cong \operatorname{Ext}^*_Q(Q \otimes_S S/\mathfrak{b}, \mathrm{H}(\tau^{>d}X))\\ &= 0\,.\end{aligned}$$

For $i > d$ the exact sequence in homology associated to (2.1) now gives the first isomorphism below

$$\begin{aligned}\operatorname{Ext}^i_S(S/\mathfrak{b}, \tau^{\leqslant d}X) &\cong \operatorname{Ext}^i_S(S/\mathfrak{b}, X)\\ &\cong \operatorname{Ext}^i_R(S/\mathfrak{b}, M)\\ &\cong \operatorname{Ext}^i_R(R/\mathfrak{a}, M)\\ &= 0\,.\end{aligned}$$

The second isomorphism follows from Hom-tensor adjunction and the definition of X. The next isomorphism holds for any choice of an ideal $\mathfrak{a}$ in R that reduces to $\mathfrak{b}$ in S, i.e. $S/\mathfrak{b} \cong R/\mathfrak{a}$ as R-modules. Since $\mathfrak{b} \subset S$ is non-zero, the ideal $\mathfrak{a}$ properly contains $\mathfrak{p}$ and hence it is not in $\mathcal{U}$. That explains the vanishing of Ext. □

Claim 5. One has $\mathrm{H}(\tau^{>d}X) = 0$.

Proof. By construction one has $\mathrm{H}^i(\tau^{>d}X) = 0$ for $i \leqslant d$. Apply $\mathbf{R}\mathrm{Hom}_S(Q, -)$ to the exact triangle (2.1). By Claim 3, using that Q-vector spaces are injective S-modules, one has

$$\begin{aligned}\operatorname{Ext}^*_S(Q, \tau^{>d}X) &\cong \operatorname{Ext}^*_S(Q, \mathrm{H}(\tau^{>d}X))\\ &\cong \mathrm{Hom}_S(Q, \mathrm{H}(\tau^{>d}X))\\ &\cong \mathrm{H}(\tau^{>d}X)\,.\end{aligned}$$

For $i > d$, Claim 4 yields $\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_S(Q, \tau^{\leqslant d}X)) = 0$, and together with the computation above, this explains the first two isomorphisms in the next chain

$$\begin{aligned}\mathrm{H}^i(\tau^{>d}X) &\cong \operatorname{Ext}^i_S(Q, \tau^{>d}X)\\ &\cong \operatorname{Ext}^i_S(Q, X)\\ &\cong \operatorname{Ext}^i_R(k(\mathfrak{p}), M)\\ &= 0\,.\end{aligned}$$

The third isomorphism follows from Hom-tensor adjunction, recalling that $Q = S_{(0)}$ as an R-module is $(R/\mathfrak{p})_{\mathfrak{p}/\mathfrak{p}} \cong k(\mathfrak{p})$. The vanishing of Ext is by hypothesis. □

Finally, from Claim 5 and (2.1) one gets the second isomorphism below

$$\operatorname{Ext}^i_R(R/\mathfrak{p}, M) \cong \mathrm{H}^i(X) \cong \mathrm{H}^i(\tau^{\leqslant d} X)\,;$$

the first one holds by the definition of X. Thus one has $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for all $i > d$, and this contradicts the assumption that $\mathfrak{p}$ is in $\mathcal{U}$.

This completes the proof of Theorem 1.1. □

To use Theorem 1.1 to verify injectivity of an R-module M one would have to check vanishing of $\operatorname{Ext}^i_R(k(\mathfrak{p}), M)$, not only for all prime ideals $\mathfrak{p}$ but also for all $i > 0$. However, building on this result, in recent work with Marley [8] we have been able to prove that it suffices to verify the vanishing for a single i, *as long as* i *is large enough.* The example below illustrates that such a restriction is needed.

Example 2.1. If R is a complete local ring with $\operatorname{depth} R \geqslant 2$, then one has

$$\operatorname{Ext}^1_R(k(\mathfrak{p}), R) = 0 \quad \text{for every prime ideal } \mathfrak{p} \text{ in } R\,.$$

Indeed, if $\mathfrak{p}$ is the maximal ideal of R, then vanishing holds by the assumption $\operatorname{depth} R \geqslant 2$, and for every non-maximal prime $\mathfrak{p}$ one has $\operatorname{Ext}^i_R(k(\mathfrak{p}), R) = 0$ for all i; see [4, Example 4.20] and (3.1).

The next example illustrates that the vanishing of $\operatorname{Ext}^i_R(k(\mathfrak{p}), R)$ does not imply that of $\operatorname{Ext}^i_R(R/\mathfrak{p}, R)$ and $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p})$, and vice versa. Thus Theorem 1.1 is not obviously a consequence of Baer's criterion, nor does it subsume it.

Example 2.2. Let R be as in Example 2.1 and $\mathfrak{p}$ a prime ideal minimal over (r) where r is not a zero divisor. In this case, both $\operatorname{Ext}^1_R(R/\mathfrak{p}, R)$ and $\operatorname{Ext}^1_{R_\mathfrak{p}}(k(\mathfrak{p}), R_\mathfrak{p})$ are nonzero, whilst $\operatorname{Ext}^1_R(k(\mathfrak{p}), R) = 0$.

On the other hand, $\operatorname{Ext}^1_\mathbb{Z}(\mathbb{Q}, \mathbb{Z})$ is nonzero, whilst $\operatorname{Ext}^1_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}) = 0 = \operatorname{Ext}^1_\mathbb{Q}(\mathbb{Q}, \mathbb{Q})$.

The analogue of Theorem 1.1 for flat dimension is well-known and easier to verify.

Remark 2.3. Let M be an R-complex. For each prime ideal $\mathfrak{p}$ and integer i there is a natural isomorphism

$$\operatorname{Tor}_i^R(k(\mathfrak{p}), M) \cong \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})\,.$$

It thus follows from [1, Proposition 5.3.F] that if there exists an integer d such that $\operatorname{Tor}_i^R(k(\mathfrak{p}), M) = 0$ for $i > d$ and each prime $\mathfrak{p}$, then the flat dimension of M is at most d. However, Theorem 1.1 does not follow, it seems, from this result by standard injective–flat duality.

3. Applications

We present some applications of Theorem 1.1. The first one improves [7, Theorem 2.2] in two directions: There is no assumption on the projective dimension of flat modules, and an extension ring is replaced by a module.

Corollary 3.1. *For every R-complex M and every faithfully flat R-module F there is an equality*

$$\operatorname{inj\,dim}_R \mathbf{R}\mathrm{Hom}_R(F, M) = \operatorname{inj\,dim}_R M\,.$$

In particular, M is acyclic if and only if $\mathbf{R}\mathrm{Hom}_R(F, M)$ is acyclic.

Proof. For every prime ideal $\mathfrak{p}$ in R and every integer i one has

$$\mathrm{Ext}^i_R(k(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_R(F, M)) \cong \mathrm{Ext}^i_R(F \otimes_R k(\mathfrak{p}), M)$$

by adjunction and flatness of F. Observe that as an R-module $F \otimes_R k(\mathfrak{p})$ is a direct sum of copies of $k(\mathfrak{p})$; it is non-zero because F is faithfully flat. It follows that $\mathrm{Ext}^i_R(k(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_R(F, M))$ is zero if and only if $\mathrm{Ext}^i_R(k(\mathfrak{p}), M)$ is zero. The equality of injective dimensions now follows from Theorem 1.1.

In view of the equality, the statement about acyclicity is trivial as M is acyclic if and only if 0 is a semi-injective resolution of M if and only if $\operatorname{inj\,dim}_R M$ is $-\infty$. □

Let F be a flat R-module. A module $F \otimes_R M$ is flat if M is flat, and the converse holds if F is faithfully flat; this is standard. It is equally standard that the module $\mathrm{Hom}_R(F, M)$ is injective if M is injective. The next remark provides something close to a converse; Example 2.1 suggests that the hypotheses are optimal.

Remark 3.2. Let F be a faithfully flat R-module. If M is an R-module with $\mathrm{Ext}^i_R(F, M) = 0$ for all $i > 0$, then $\mathbf{R}\mathrm{Hom}_R(F, M)$ is isomorphic to $\mathrm{Hom}_R(F, M)$ in the derived category over R. Thus, for such a module Corollary 3.1 asserts that $\mathrm{Hom}_R(F, M)$ is injective if and only if M is injective. This improves the Main Theorem in [7]; see also Theorem 4.3.

The only other result in this direction we are aware of is the Main Theorem in [7]. It deals with the special case where F is a faithfully flat R-algebra, and the proof relies heavily on [4, Theorem 4.5] in the form recovered by Corollary 3.3.

This points to our next application, which involves the notion of cosupport introduced in [4], in a form justified by [4, Proposition 4.4]. The *cosupport* of an R-complex M is the subset of $\operatorname{Spec} R$ given by

$$\operatorname{cosupp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k(\mathfrak{p}), M)) \neq 0\}\,. \tag{3.1}$$

The next result is [4, Theorem 4.5] applied to the derived category over R. The proof of *op. cit.* builds on the techniques developed in [3, 4] to apply to triangulated categories equipped with ring actions.

Corollary 3.3. *An R-complex M has* $\operatorname{cosupp}_R M = \emptyset$ *if and only if* $\mathrm{H}(M) = 0$.

Proof. The "if" is trivial, and the converse holds by Theorem 1.1 when one recalls that $\mathrm{H}^i(M) \neq 0$ implies $\operatorname{inj\,dim}_R M \geqslant i$. □

Remark 3.4. One can deduce the preceding corollary also from Neeman's classification [11, Theorem 2.8] of the localizing subcategories of the derived category over R. Indeed, the subcategory of the derived category consisting of R-complexes X with $\mathrm{Ext}^*_R(X, M) = 0$ is localizing. Thus, if it contains $k(\mathfrak{p})$ for each $\mathfrak{p}$ in $\operatorname{Spec} R$, then it must contain R, by *op. cit.*, that is to say, $\mathrm{H}(M) = 0$.

Conversely, Corollary 3.3 can be used to deduce Neeman's classification, by mimicking the proof of [5, Theorem 6.1]. The crucial additional observation needed to do so is that for R-complexes M and N, there is an equality

$$\operatorname{cosupp}_R \mathbf{R}\mathrm{Hom}_R(M,N) = \operatorname{supp}_R M \cap \operatorname{cosupp}_R N\,.$$

It follows from two applications of the standard adjunction:

$$\begin{aligned}\mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_R(M,N)))&\\ &\cong \mathrm{H}(\mathbf{R}\mathrm{Hom}_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M, \mathbf{R}\mathrm{Hom}_R(k(\mathfrak{p}),N)))\\ &\cong \mathrm{Hom}_{k(\mathfrak{p})}(\mathrm{H}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R M), \mathrm{H}(\mathbf{R}\mathrm{Hom}_R(k(\mathfrak{p}),N)))\,.\end{aligned}$$

4. Non-noetherian rings

In this section we establish, over certain not necessarily noetherian rings, a characterization of injective modules in the vein of [7]; see also Remark 3.2. This involves the following invariant:

$$\operatorname{splf} R = \sup\{\operatorname{proj\,dim}_R F \mid F \text{ is a flat } R\text{-module}\}\,.$$

A direct sum of flat modules is flat with $\operatorname{proj\,dim}(\bigoplus_{i\in I} F_i) = \sup_{i\in I}\{\operatorname{proj\,dim} F_i\}$, so the invariant $\operatorname{splf} R$ is finite if and only if every flat R-module has finite projective dimension. With a nod to Bass' [2, Theorem P], a ring with $\operatorname{splf} R \leqslant d$ is also called a *d-perfect* ring. If R has cardinality at most $\aleph_n$ for some natural number n, then one has $\operatorname{splf} R \leqslant n+1$ by a result of Gruson and Jensen [9, Theorem 7.10]. Osofsky [13, 3.1] has examples of rings for which the splf invariant is infinite.

Lemma 4.1. *Let R be a commutative ring with $\operatorname{splf} R < \infty$ and let S be a faithfully flat R-algebra. An R-complex M with $\mathrm{H}^i(M) = 0$ for all $i \gg 0$ is acyclic if and only if $\mathbf{R}\mathrm{Hom}_R(S,M)$ is acyclic.*

Proof. The "only if" is trivial, so assume that $\mathbf{R}\mathrm{Hom}_R(S,M)$ is acyclic. As $\mathrm{H}(M)$ is bounded above, we may assume that $\mathrm{H}^i(M) = 0$ holds for all $i > 0$, and it suffices to prove that also $\mathrm{H}^0(M) = 0$. Set $d := \operatorname{splf} R$.

Application of $\mathbf{R}\mathrm{Hom}_R(-,M)$ to the exact sequence $0 \to R \to S \to S/R \to 0$ yields $M \cong \Sigma\,\mathbf{R}\mathrm{Hom}_R(S/R,M)$ in the derived category over R. Repeated use of this isomorphism and adjunction yields $M \cong \Sigma^{d+1}\,\mathbf{R}\mathrm{Hom}_R((S/R)^{\otimes d+1},M)$. As S is faithfully flat over R, the module S/R is flat, and hence so are its tensor powers. Thus, the module $(S/R)^{\otimes d+1}$ has projective dimension at most d and, therefore, $\mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R((S/R)^{\otimes d+1},M)) = 0$ holds for all $i > d$. In particular,

$$\begin{aligned}\mathrm{H}^0(M) &\cong \mathrm{H}^0(\Sigma^{d+1}\,\mathbf{R}\mathrm{Hom}_R((S/R)^{\otimes d+1},M))\\ &= \mathrm{H}^{d+1}(\mathbf{R}\mathrm{Hom}_R((S/R)^{\otimes d+1},M))\\ &= 0\,.\end{aligned}$$

□

Proposition 4.2. *Let R be a commutative ring with $\operatorname{splf} R < \infty$ and let S be a faithfully flat R-algebra of projective dimension at most 1. An R-complex M is acyclic if and only if $\mathbf{R}\mathrm{Hom}_R(S,M)$ is acyclic.*

Proof. The "only if" is trivial, so assume that $\mathbf{R}\mathrm{Hom}_R(S,M)$ is acyclic. To prove that M is acyclic, we show that $\mathrm{H}^0(\Sigma^n M) = 0$ holds for all $n \in \mathbb{Z}$. Fix n and let

$\Sigma^n M \to I$ be a semi-injective resolution; the assumption is now $\mathrm{H}(\mathrm{Hom}_R(S,I)) = 0$ and the goal is to prove $\mathrm{H}^0(I) = 0$.

The soft truncation

$$\tau^{\leqslant 1}\mathrm{Hom}_R(S,I) = \cdots \to \mathrm{Hom}_R(S,I)^{-1} \to \mathrm{Hom}_R(S,I)^0 \to \mathrm{Z}^1(\mathrm{Hom}_R(S,I)) \to 0$$

is acyclic, and by left-exactness of Hom one has $\tau^{\leqslant 1}\mathrm{Hom}_R(S,I) = \mathrm{Hom}_R(S,\tau^{\leqslant 1}I)$. Further, still by acyclicity of $\mathrm{Hom}_R(S,I)$, there is an equality

$$\mathrm{B}^2(\mathrm{Hom}_R(S,I)) = \mathrm{Hom}_R(S,\mathrm{B}^2(I))\,.$$

Thus, the functor $\mathrm{Hom}_R(S,-)$ leaves the sequence $0 \to \mathrm{Z}^1(I) \to I^1 \to \mathrm{B}^2(I) \to 0$ exact, and that implies vanishing of $\mathrm{Ext}^1_R(S,\mathrm{Z}^1(I))$.

Let $\pi\colon P \to S$ be a projective resolution over R with $P_i = 0$ for $i > 1$. Consider its mapping cone

$$A = 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S \longrightarrow 0\,.$$

As $\mathrm{Hom}_R(A,I^n)$ is exact for every n and $\mathrm{Hom}_R(A,\mathrm{Z}^1(I))$ is exact by vanishing of $\mathrm{Ext}^1_R(S,\mathrm{Z}^1(I))$, it follows from [6, Lemma (2.5)] that $\mathrm{Hom}_R(A,\tau^{\leqslant 1}I)$ is acyclic. Thus, $\mathrm{Hom}_R(\pi,\tau^{\leqslant 1}I)$ yields an isomorphism $\mathbf{R}\mathrm{Hom}_R(S,\tau^{\leqslant 1}I) \cong \mathrm{Hom}_R(S,\tau^{\leqslant 1}I)$ in the derived category, and the latter complex is acyclic. Now Lemma 4.1 yields $\mathrm{H}(\tau^{\leqslant 1}I) = 0$, in particular $\mathrm{H}^0(I) = \mathrm{H}^0(\tau^{\leqslant 1}I) = 0$. □

Theorem 4.3. *Let R be a commutative ring with* $\mathrm{splf}\, R < \infty$*, let S be a faithfully flat R-algebra of projective dimension at most* 1*, and let M be an R-module. If* $\mathrm{Ext}^1_R(S,M) = 0$ *and the S-module* $\mathrm{Hom}_R(S,M)$ *is injective, then M is injective.*

Proof. The proof of [7, Theorem 1.7] applies with one modification: in place of [7, 1.5]—at heart a reference to [4, Theorem 4.5]—one invokes Proposition 4.2. □

Remark 4.4. The assumption in Theorem 4.3 that the flat R-algebra S has projective dimension at most 1 is satisfied if

- R is countable; see [9, Theorem 7.10].
- S is countably related; in particular, if every ideal in R is countably generated, and S is countably generated as an R-module; see Osofsky [12, Lemma 1.2] and Jensen [10, Lemma 2].

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