

# TORSION IN TENSOR POWERS OF MODULES

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ABSTRACT. Tensor products usually have nonzero torsion. This is a central theme of Auslander’s paper [1]; the theme continues in the work of Huneke and Wiegand [7]–[9]. The main focus in this note is on tensor powers of a finitely generated module over a local ring. Also, we study torsion-free modules  $N$  with the property that  $M \otimes_R N$  has nonzero torsion unless  $M$  is very special. An important example of such a module  $N$  is the Frobenius power  ${}^p R$  over a complete intersection domain  $R$  of characteristic  $p > 0$ .

## 1. INTRODUCTION

In a 1961 paper [1], Auslander studied torsion in tensor products of nonzero finitely generated modules  $M$  and  $N$  over unramified regular local rings  $R$ . Under the assumption that  $M \otimes_R N$  is torsion-free, he proved:

- (1)  $M$  and  $N$  must be torsion-free, and
- (2)  $M$  and  $N$  are *Tor-independent*, that is,  $\mathrm{Tor}_i^R(M, N) = 0$  for all  $i \geq 1$ .

The two conclusions are cleverly intertwined in his proof, which we revisit in Section 3 of the present paper. We show, over a reduced complete intersection ring  $R$  of positive characteristic  $p$ , that  $M \otimes_R {}^\varphi R$  is torsion-free if and only if  $M$  is torsion-free and of finite projective dimension, in which case  $\mathrm{Tor}_i^R(M, {}^\varphi R) = 0$  for all  $i \geq 1$ . (Here  $\varphi : R \rightarrow R$  is the Frobenius endomorphism and  ${}^\varphi R$  is the module obtained from  $R$  by restriction of scalars along  $\varphi$ .) When  $R$  is F-finite, we obtain a criterion for regularity:  $R$  is regular if and only if  $({}^\varphi M) \otimes_R {}^\varphi R$  is torsion-free for some (equivalently, every) nonzero finitely generated  $R$ -module  $M$ .

Our main results are in Section 2, where we study torsion in tensor powers. We obtain detailed information on annihilators of elements in  $\otimes_R^n M$  and draw several conclusions. Suppose, for example, that  $\underline{r} = r_1, \dots, r_d$  is a regular sequence in  $R$  and  $M$  is the cokernel of the  $d \times 1$  matrix  $[\underline{r}]^t$ . We show in Theorem 2.8 that  $\otimes_R^t M$  is torsion-free if and only if  $t \leq d$ . This result should be compared with Auslander’s observation [1, p. 638] that the same holds when  $M$  is the  $d - 1^{\mathrm{st}}$  syzygy of a module of projective dimension  $d$  over a  $d$ -dimensional regular local ring. Cf. also [9, Proposition 3.1]. If  $R$  is local, the “only if” direction holds much more generally: if we write  $M$  as the cokernel of an  $m \times n$  matrix  $\theta$  with entries in the maximal

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ideal of  $R$ , and if some entry of  $\theta$  is a non-zero-divisor, we show in Theorem 2.10 that  $\otimes_R^t M$  has nonzero torsion for every  $t \geq m$ .

Throughout this paper,  $R$  is a commutative, Noetherian ring.

## 2. TORSION IN TENSOR POWERS

In this section we establish results on annihilators of elements in tensor powers of modules.

**Notation 2.1.** Given elements  $\underline{m} := m_1, \dots, m_d$  in an  $R$ -module  $M$ , we consider the element in  $\otimes_R^d M$  defined by

$$\tau(\underline{m}) := \sum_{\sigma \in S_d} \text{sign}(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)}.$$

**Proposition 2.2.** *Let  $M$  be an  $R$ -module. If elements  $m_1, \dots, m_d$  in  $M$  and  $r_1, \dots, r_d$  in  $R$  satisfy*

$$(2.2.1) \quad r_1 m_1 + \cdots + r_d m_d = 0,$$

*then  $(r_1, \dots, r_d) \cdot \tau(\underline{m}) = 0$  in  $\otimes_R^d M$ .*

*Proof.* The twisted shuffle product gives the graded  $R$ -algebra  $\bigoplus_{n \geq 0} \otimes_R^n M$  a strictly skew-commutative structure; see [13, Chapter X, (12.4)]. Strictly skew-commutative means that for any  $a \in \otimes_R^i M$  and  $b \in \otimes_R^j M$ , there are equalities

$$a \star b = (-1)^{ij} b \star a, \quad \text{and} \quad a \star a = 0 \text{ when } i \text{ is odd.}$$

By definition of the shuffle product,  $\tau(\underline{m}) = m_1 \star \cdots \star m_d$ . Thus for each  $j$  we have

$$\begin{aligned} r_j \cdot \tau(\underline{m}) &= m_1 \star \cdots \star m_{j-1} \star r_j m_j \star m_{j+1} \star \cdots \star m_d \\ &= - \sum_{i \neq j} r_i (m_1 \star \cdots \star m_{j-1} \star m_i \star m_{j+1} \star \cdots \star m_d) = - \sum_{i \neq j} r_i 0 = 0. \quad \square \end{aligned}$$

There is a “universal” source for the element  $\tau(\underline{m})$  in the following sense:

**Remark 2.3.** Consider the polynomial ring  $\mathbb{Z}[\underline{x}]$  on indeterminates  $\underline{x} := x_1, \dots, x_d$ , and let  $U$  be the  $\mathbb{Z}[\underline{x}]$ -module with presentation

$$0 \longrightarrow \mathbb{Z}[\underline{x}] \xrightarrow{[x_1, \dots, x_d]^t} \mathbb{Z}[\underline{x}]^d \longrightarrow U \longrightarrow 0.$$

Let  $u_1, \dots, u_d$  the the generators of  $U$  corresponding to the standard basis for  $\mathbb{Z}[\underline{x}]^d$ , so that  $x_1 u_1 + \cdots + x_d u_d = 0$ , i.e.,  $\underline{x}$  and  $\underline{u}$  satisfy (2.2.1). Then  $\text{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) \supseteq (\underline{x})$  by Proposition 2.2; we shall see, in Theorem 2.8 below, that in fact  $\text{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) = (\underline{x})$ .

Given any  $R$ -module  $M$  with a syzygy relation (2.2.1), consider the ring homomorphism  $\mathbb{Z}[\underline{x}] \rightarrow R$  taking  $x_i$  to  $r_i$ , for each  $i$ , and extending the structure homomorphism  $\mathbb{Z} \rightarrow R$ . The hypothesis on  $M$  implies that there is a homomorphism of  $\mathbb{Z}[\underline{x}]$ -modules

$$f: U \longrightarrow M \quad \text{with } f(u_i) = m_i \text{ for } i = 1, \dots, d.$$

Under the induced map  $\otimes^d f: \otimes_{\mathbb{Z}[\underline{x}]}^d U \rightarrow \otimes_R^d M$ , the element  $\tau(\underline{u})$  maps to  $\tau(\underline{m})$ .

This remark prompts the discussion below, culminating in Theorem 2.8. First we review some notions regarding depth. For details, see [4, Chapter 1].

**2.4. Depth.** Let  $M$  be a finitely generated  $R$ -module and  $I$  an ideal of  $R$  satisfying  $IM \neq M$ . The  $I$ -depth of  $M$  is the number

$$\text{depth}_R(I, M) = \inf\{n \geq 0 \mid \text{Ext}_R^n(R/I, M) \neq 0\}.$$

The  $I$ -depth of  $M$  is always finite and is equal to the length of every maximal  $M$ -regular sequence in  $I$ .

If  $\underline{x} := x_1, \dots, x_d$  is a sequence of elements in  $R$ , and  $K$  is the Koszul complex on  $\underline{x}$ , then the  $(\underline{x})$ -depth of  $M$  may be computed from its Koszul homology:

$$\text{depth}_R((\underline{x}), M) = d - \sup\{i \geq 0 \mid H_i(K \otimes_R M) \neq 0\}$$

This is the *depth sensitivity* of the Koszul complex.

Suppose now that  $\underline{x}$  is  $R$ -regular. Then  $K$  is a free resolution of  $R/(\underline{x})$ , and hence  $H_*(K \otimes_R M) \cong \text{Tor}_*^R(R/(\underline{x}), M)$ . In this case, we have

$$(2.4.1) \quad \text{depth}_R((\underline{x}), M) = d - \sup\{i \geq 0 \mid \text{Tor}_i^R(R/(\underline{x}), M) \neq 0\}.$$

If  $R$  is local with maximal ideal  $\mathfrak{m}$ , we write  $\text{depth}_R M$  for the  $\mathfrak{m}$ -depth of  $M$  and call it the *depth* of  $M$ .

**2.5. A Koszul syzygy module.** Let  $R$  be a Noetherian ring and  $\underline{r} := r_1, \dots, r_d$  a regular sequence in  $R$  with  $(\underline{r}) \neq R$ . Consider the complex

$$F := 0 \longrightarrow R \xrightarrow{[r_1, \dots, r_d]^t} R^d \longrightarrow 0$$

concentrated in degrees 0 and 1. Set  $M = H_0(F)$ ; as  $r_1$  is a non-zerodivisor,  $F$  is a free resolution of  $M$ .

**Lemma 2.6.** *Let  $M$ ,  $d$ , and  $F$  be as in 2.5. For each  $n = 1, \dots, d$ , the following statements hold:*

- (1)  $M$  and  $\otimes_R^{n-1} M$  are Tor-independent.
- (2)  $\otimes_R^n F$  is a free resolution of  $\otimes_R^n M$ , and  $\text{pd}_R(\otimes_R^n M) = n$ .

*Proof.* The base case is  $n = 1$ , and then (1) and (2) are clear. Fix an integer  $n$  with  $2 \leq n \leq d$ , and assume these statements hold for all integers  $\leq n - 1$ . Set  $I = (\underline{r})$ . Since  $\otimes_R^{n-1} F$  is a free resolution of  $\otimes_R^{n-1} M$ , we have

$$\text{Tor}_*^R(R/I, \otimes_R^{n-1} M) = H_*((R/I) \otimes_R (\otimes_R^{n-1} F)) \cong (\otimes_R^{n-1}((R/I) \otimes_R F))_*,$$

where the last isomorphism holds because the complex in question has zero differential. In particular,  $\text{Tor}_{n-1}^R(R/I, \otimes_R^{n-1} M) \cong R/I \neq 0$ , so that

$$(2.6.1) \quad \sup\{i \geq 0 \mid \text{Tor}_i^R(R/(\underline{r}), \otimes_R^{n-1} M) \neq 0\} = n - 1.$$

We can now complete the induction step.

(1) The induction hypothesis implies that  $\otimes_R^{n-1} F$  is a free resolution of  $\otimes_R^{n-1} M$ , so (2.6.1) and (2.4.1) show that

$$(2.6.2) \quad \text{depth}_R(I, \otimes_R^{n-1} M) = d - (n - 1) \geq 1.$$

Moreover,  $\text{Tor}_*^R(M, \otimes_R^{n-1} M)$  is the homology of the complex

$$F \otimes_R (\otimes_R^{n-1} M) : 0 \longrightarrow \otimes_R^{n-1} M \xrightarrow{[\underline{r}]^t} (\otimes_R^{n-1} M)^d \longrightarrow 0$$

(concentrated in degrees 0 and 1). By (2.6.2), some  $r_i$  is a non-zerodivisor on  $\otimes_R^{n-1} M$ , and it follows that  $M$  and  $\otimes_R^{n-1} M$  are Tor-independent.

(2) By hypothesis,  $F$  and  $\otimes_R^{n-1} F$  are free resolutions of  $M$  and  $\otimes_R^{n-1} M$ , respectively. We have already proved, in (1), that these modules are Tor-independent, so

the complex  $F \otimes_R (\otimes_R^{n-1} F)$ , that is to say,  $\otimes_R^n F$ , is a free resolution of  $\otimes_R^n M$ . In particular,  $\text{pd}_R(\otimes_R^n M) \leq n$ ; that equality holds follows from (2.6.1).  $\square$

**2.7. Torsion submodule.** Let  $Q(R)$  be the total quotient ring of  $R$ . The *torsion submodule*  $\mathsf{T}_R M$  of  $M$  is the kernel of the natural homomorphism  $M \rightarrow Q(R) \otimes_R M$ . The inclusion  $\mathsf{T}_R M \subseteq M$  gives rise to an exact sequence

$$(2.7.1) \quad 0 \longrightarrow \mathsf{T}_R M \longrightarrow M \longrightarrow \perp_R M \longrightarrow 0.$$

The module  $M$  is *torsion* if  $\mathsf{T}_R M = M$  (that is,  $M_{\mathfrak{p}} = 0$  for each  $\mathfrak{p} \in \text{Ass}(R)$ ), and  $M$  is *torsion-free* if  $\mathsf{T}_R M = 0$ . Thus  $M$  is torsion-free if and only if  $\bigcup \text{Ass } M \subseteq \bigcup \text{Ass } R$ . The stronger condition, that  $\text{Ass } M \subseteq \text{Ass } R$ , is therefore a sufficient condition for  $M$  to be torsion-free. We will invoke this criterion twice in the proof of the next theorem.

Part (1) of the next result is reminiscent of Auslander's discussion on p. 638 of [1]. Cf. also [9, Proposition 3.1].

**Theorem 2.8.** *Let  $M$  and  $\underline{r}$  be as in 2.5. The following statements hold:*

- (1)  $\otimes_R^n M$  is torsion-free if and only if  $n \leq d-1$ .
- (2) The element  $\tau(\underline{m})$  in  $\otimes_R^d M$  satisfies  $\text{ann}_R \tau(\underline{m}) = (r)$ .
- (3) The map  $R/(r) \rightarrow \otimes_R^d M$  of  $R$ -modules with  $1 \mapsto \tau(\underline{m})$  induces a splitting

$$\otimes_R^d M \cong (R/(r)) \bigoplus W$$

where  $W$  is torsion-free; in particular, we have

$$\text{Hom}_R(R/(r), \otimes_R^d M) = R\tau(\underline{m}) \neq 0.$$

*Proof.* Set  $I = (r)$ , let  $n \leq d-1$ , and fix a prime  $\mathfrak{p} \in \text{Ass}(\otimes_R^n M)$ . If  $I \subseteq \mathfrak{p}$ , it follows from Lemma 2.6 that  $(\otimes_R^n F)_{\mathfrak{p}}$  is a minimal free resolution of  $(\otimes_R^n M)_{\mathfrak{p}}$ ; therefore

$$\text{depth}_{R_{\mathfrak{p}}}(\otimes_R^n M)_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} - n \geq d - n \geq 1,$$

contradiction. Thus  $I \not\subseteq \mathfrak{p}$  and then the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  is a nonzero free module; hence so is  $(\otimes_R^n M)_{\mathfrak{p}}$ . Therefore  $\text{depth } R_{\mathfrak{p}} = \text{depth}_{R_{\mathfrak{p}}}(\otimes_R^n M)_{\mathfrak{p}}$ . We have shown that  $\text{Ass}(\otimes_R^n M) \subseteq \text{Ass } R$ , and hence that  $\otimes_R^n M$  is torsion-free. The “only if” direction of (1) will follow from (3).

As for parts (2) and (3), by construction  $r_1 m_1 + \cdots + r_d m_d = 0$ , so Proposition 2.2 gives an inclusion  $I \subseteq \text{ann}_R \tau(\underline{m})$ . The reverse inclusion will follow, once we ascertain that the map in (3) splits. Consider the homomorphisms of  $R$ -modules

$$\otimes_R^d(F_0) \twoheadrightarrow \otimes_R^d M \twoheadrightarrow (\otimes_R^d M) \otimes_R R/I \cong H_0((\otimes_R^d F) \otimes_R R/I) = \otimes_R^d(F_0 \otimes_R R/I),$$

where the surjections are the natural ones; the isomorphism holds because  $\otimes_R^d F$  is a free resolution of  $\otimes_R^d M$ , and the equality holds because the differential on  $F$  has its image in  $IF$ . Let  $\underline{e} = e_1, \dots, e_d$  be the standard basis for  $F_0 = R^d$ , in 2.5, and let  $\underline{e}'$  be the induced basis of the free  $R/I$ -module  $F_0 \otimes_R R/I$ . Under the composite map, the element  $\tau(\underline{e})$  maps to  $\tau(\underline{e}')$ , and  $\{\tau(\underline{e}')\}$  extends to a basis of the  $R/I$ -module  $\otimes_R^d(F_0 \otimes_R R/I)$ . Since  $\tau(\underline{e})$  maps to  $\tau(\underline{m})$  in  $\otimes_R^d M$ , the map in (2) splits and gives a decomposition

$$\otimes_R^d M \cong (R/I) \bigoplus W.$$

It remains to verify that  $W$  is torsion-free; given the decomposition above, the other parts of (3) are a consequence of this fact.

For  $\mathfrak{p} \in \operatorname{Spec} R$  with  $I \not\subseteq \mathfrak{p}$ , the  $R_{\mathfrak{p}}$ -module  $M_{\mathfrak{p}}$  is free, and hence so is  $W_{\mathfrak{p}}$ . Assume now that  $I \subseteq \mathfrak{p}$ . The Koszul complex on  $\underline{r}$ , viewed as elements in  $R_{\mathfrak{p}}$ , is a minimal resolution of  $(R/I)_{\mathfrak{p}}$ , and so it is a direct summand of  $(\otimes_R^n F)_{\mathfrak{p}}$ , the minimal free resolution of  $(\otimes_R^n M)_{\mathfrak{p}}$ . The ranks of the free modules in the top degree,  $d$ , of these complexes coincide (and equal 1), whence  $\operatorname{pd}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} \leq d - 1$  and

$$\operatorname{depth}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{pd}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} \geq 1$$

These observations show that  $\operatorname{Ass} W \subseteq \operatorname{Ass} R$ , so  $W$  is torsion-free as claimed.  $\square$

**Local rings.** Next we focus on local rings, where the preceding results can be strengthened to some extent.

**Lemma 2.9.** *Let  $M$  be a finitely generated module over a local ring  $(R, \mathfrak{m})$ , and let  $m_1, \dots, m_d \in M$ . If the images of  $\{m_1, \dots, m_d\}$  in  $M/\mathfrak{m}M$  are linearly independent, then  $\tau(\underline{m})$  is not in  $\mathfrak{m}(\otimes_R^d M)$ .*

*Proof.* Let  $m'_i$  be the image of  $m_i$  in the  $k$ -vector space  $M/\mathfrak{m}M$ . Since  $\{m'_1, \dots, m'_d\}$  is linearly independent,  $\tau(\underline{m}') \neq 0$ . Hence  $\tau(\underline{m}) \notin \mathfrak{m}(\otimes_R^d M)$ .  $\square$

Given an  $R$ -module  $M$ , we write  $I(M)$  for the ideal  $(r_{ij})$  defined by the entries in a matrix in some minimal presentation

$$R^{\mu} \xrightarrow{[r_{ij}]} R^{\nu} \longrightarrow M \longrightarrow 0 \quad \text{where } \nu = \nu_R(M).$$

This ideal is independent of the presentation. Moreover,  $I(M)$  contains a non-zero-divisor if and only if, over  $\operatorname{Q}(R)$ , the total quotient ring of  $R$ , the module  $\operatorname{Q}(R) \otimes_R M$  can be generated by fewer than  $\nu$  elements. To see this we note that, since  $\operatorname{Q}(R)$  is semilocal, the module  $\operatorname{Q}(R) \otimes_R M$  needs  $\nu$  generators if and only if  $\nu_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \nu$  for some  $\mathfrak{p} \in \operatorname{Ass} R$ ; moreover,  $\nu_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \nu$  if and only if the presentation remains minimal when localized at  $\mathfrak{p}$ , that is, if and only if  $I(M) \subseteq \mathfrak{p}$ . Thus  $\operatorname{Q}(R) \otimes_R M$  needs  $\nu$  generators if and only if  $I(M) \subseteq \mathfrak{p}$  for some  $\mathfrak{p} \in \operatorname{Ass} R$ , that is, if and only if  $I(M)$  consists of zero-divisors.

Recall that  $M$  is said to have rank  $r$  if  $\operatorname{Q}(R) \otimes_R M$  is free over  $\operatorname{Q}(R)$  of rank  $r$ ; see [4, Proposition 1.4.3] for different characterizations of this property.

**Theorem 2.10.** *Let  $R$  be a local ring and  $M$  a nonzero finitely generated  $R$ -module satisfying one of the following conditions:*

- (1)  *$I(M)$  contains a non-zero-divisor; in this case, set  $b = \nu_R(M)$ ; or*
- (2)  *$M$  has rank; in this case, set  $b = \operatorname{rank}_R(M) + 1$ .*

*If  $M$  is not free, then for each nonzero finitely generated  $R$ -module  $N$  one has*

$$\top_R((\otimes_R^n M) \otimes_R N) \neq 0 \quad \text{for each } n \geq b.$$

*Proof.* It suffices to prove the statement for  $n = b$ , since

$$(\otimes_R^n M) \otimes_R N \cong (\otimes_R^b M) \otimes_R ((\otimes_R^{n-b} M) \otimes_R N),$$

and  $N \neq 0$  implies  $(\otimes_R^i M) \otimes_R N \neq 0$  for each  $i \geq 0$ , by Nakayama's lemma.

(1) Let  $m_1, \dots, m_b$  be a minimal generating set for the  $R$ -module  $M$ . The element  $\tau(\underline{m})$  in  $\otimes_R^b M$  is annihilated by  $I(M)$ , by Proposition 2.2, and is not in  $\mathfrak{m}(\otimes_R^b M)$ , by Lemma 2.9. It follows that, for each  $x$  in  $N \setminus \mathfrak{m}N$ , the element  $\tau(\underline{m}) \otimes x$  in  $(\otimes_R^b M) \otimes_R N$  is nonzero and is annihilated by  $I(M)$ , and hence is in the torsion submodule; this where the hypothesis that  $I(M)$  contains a non-zero-divisor is used.

(2) We claim that there exists a syzygy relation (2.2.1) with  $\underline{m}$  a minimal generating set for  $M$ ,  $(\underline{r}) \subseteq \mathfrak{m}$ , and some  $r_i$  a non-zero-divisor.

Indeed,  $\nu_R(M) \geq b$  since  $M$  is not free. Choose elements  $m_1, \dots, m_b$  that form part of a minimal generating set for  $M$  and such that  $m_1, \dots, m_{b-1}$  form a basis for  $\mathcal{Q}(R) \otimes_R M$  over  $\mathcal{Q}(R)$ . Then there is a syzygy relation as in (2.2.1) in which  $r_b$  is a non-zero-divisor.

The element  $\tau(\underline{m})$  in  $\otimes_R^b M$  is annihilated by  $(\underline{r})$ , by Proposition 2.2, and is not in  $\mathfrak{m}(\otimes_R^b M)$ , by Lemma 2.9. Since  $(\underline{r})$  has a non-zero-divisor, it follows as in (1) that the torsion submodule of  $(\otimes_R^b M) \otimes_R N$  is nonzero.  $\square$

We learned recently that in 2011, in response to a query on MathOverflow, David Speyer gave a proof (quite similar to ours) of part (1) when  $R$  is a domain. (See <http://mathoverflow.net/questions/73120/torsion-free-tensor-powers>.)

One cannot always expect torsion in tensor powers of non-free modules:

**Example 2.11.** Let  $R = k[[x, y]]/(xy)$ , where  $k$  is a field. The torsion-free  $R$ -module  $M := R/(x)$  is not free; however  $\otimes_R^n M$  is isomorphic to  $R/(x)$  for every  $n \geq 1$ , and hence is torsion-free.

The preceding results bring to the fore the following:

**Question 2.12.** Let  $R$  be a local domain. Is there an integer  $b$ , depending only on  $R$ , such that  $\otimes_R^n M$  has torsion for every finitely generated non-free  $R$ -module  $M$  and every integer  $n \geq b$ ?

The condition that  $R$  be a domain is to avoid the situation of Example 2.11. When  $R$  is regular, one can take  $b = \dim R$ , by results of Auslander [1, Theorem 3.2] and Lichtenbaum [11, Corollary 3].

### 3. TORSION “CARRIERS”

Some modules, even though they are torsion-free, usually generate torsion in tensor products. For example, over a local ring  $(R, \mathfrak{m}, k)$  of positive depth, the maximal ideal  $\mathfrak{m}$  is such a module: for any finitely generated non-free  $R$ -module  $M$ , the tensor product  $\mathfrak{m} \otimes_R M$  has torsion. To see this, observe that the short exact sequence

$$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$$

yields an injection from the torsion module  $\mathrm{Tor}_1^R(k, M)$  into  $\mathfrak{m} \otimes_R M$ ; moreover,  $\mathrm{Tor}_1^R(k, M) \neq 0$  because  $M$  is not free.

We will give two more examples of torsion carriers: the integral closure  $\overline{R}$  of a one-dimensional analytically unramified ring  $R$ , and the Frobenius powers  $\varphi^e R$  of a complete intersection  $R$  of characteristic  $p$ . Recall that a local ring is *analytically unramified* provided its completion is reduced. If  $R$  is one-dimensional, an equivalent condition is that  $R$  be Cohen-Macaulay with finitely generated integral closure  $\overline{R}$ , [12, Theorem 4.6].

**Theorem 3.1.** *Let  $R$  be a one-dimensional analytically unramified local ring, and let  $\overline{R}$  be the integral closure of  $R$  in its total quotient ring. If  $M$  is a finitely generated  $R$ -module for which  $\overline{R} \otimes_R M$  is torsion-free, then  $M$  is free.*

*Proof.* Let  $\mathfrak{p}_1, \dots, \mathfrak{p}_s$  be the minimal prime ideals of  $R$ , and for each  $i$  let  $r_i$  be the dimension of the  $R_{\mathfrak{p}_i}$ -vector space  $M_{\mathfrak{p}_i}$ . Put  $n = \nu_R M$ , the minimal number of generators of the  $R$ -module  $M$ , and choose an exact sequence

$$0 \rightarrow K \rightarrow R^{(n)} \rightarrow M \rightarrow 0.$$

If we can show that  $r_i = n$  for each  $i$ , we'll know that  $K$  is torsion and hence zero, and we'll be done.

Put  $D_i = \overline{R/\mathfrak{p}_i}$ , the integral closure of the domain  $R/\mathfrak{p}_i$ . Since  $R$  is reduced, we have inclusions

$$R \hookrightarrow \prod_{i=1}^s R/\mathfrak{p}_i \hookrightarrow \prod_{i=1}^s D_i \hookrightarrow \prod_{i=1}^s Q(R/\mathfrak{p}_i) = Q(R).$$

We see that  $\overline{R} = \prod_{i=1}^s D_i$ ; moreover, each  $D_i$  is a semilocal Dedekind domain and therefore a principal ideal domain. Since  $\overline{R} \otimes_R M$  is torsion-free, it is projective, in fact free of rank  $r_i$  on the component  $D_i$ . Therefore, setting  $e_i = \nu_R D_i$ , we have the equations

$$r_1 e_1 + \dots + r_s e_s = \nu_R(\overline{R} \otimes_R M) = (\nu_R \overline{R}) \cdot (\nu_R M) = (e_1 + \dots + e_s)n.$$

Since  $r_i \leq n$  for each  $i$ , it follows from these equations that  $r_i = n$  for each  $i$ .  $\square$

Let  $R$  be a Noetherian ring of positive characteristic  $p$  and  $\varphi: R \rightarrow R$  the Frobenius endomorphism  $r \mapsto r^p$ . Given an  $R$ -module  $M$  and a positive integer  $e$ , we write  $\varphi^e M$  for the  $R$ -module obtained from  $M$  by restriction of scalars along  $\varphi^e$ ; thus  $r \cdot m = r^{p^e} m$  for  $r \in R$  and  $m \in M$ . Observe that  $M$  is torsion-free if and only if  $\varphi^e M$  is torsion-free for some (equivalently, all)  $e \geq 1$ . Following [14], we write  $F^e(M)$  for the tensor product  $M \otimes_R \varphi^e R$ . One views  $F^e(M)$  as a *right*  $R$ -module: the action of  $R$  on  $F^e(M)$  comes from the right (ordinary) action of  $R$  on  $\varphi^e R$ . Thus  $F^e(R) \cong R$  as  $R$ -modules, and it follows that  $F^e(M)$  is finitely generated if  $M$  is finitely generated.

The following result follows immediately from [14, Corollary 1.10]:

**Theorem 3.2** (Peskin and Szpiro, 1973). *Let  $R$  be a local ring of characteristic  $p$ , and let  $M$  be a finitely generated  $R$ -module. If  $M$  has finite projective dimension, then  $\mathrm{Tor}_i^R(M, \varphi^e R) = 0$  for all  $e \geq 1$  and all  $i \geq 1$ .*

The converse of Theorem 3.2 is true and was proved by Herzog [5, Theorem 3.1]. For complete intersections, the following strong converse was proved by Avramov and Miller [3, Theorem]:

**Theorem 3.3** (Avramov and Miller, 2001). *Let  $(R, \mathfrak{m})$  be a complete intersection of characteristic  $p$ , and let  $M$  be a finitely generated  $R$ -module. If  $\mathrm{Tor}_i^R(M, \varphi^e R) = 0$  for some  $e \geq 1$  and some  $i \geq 1$ , then  $M$  has finite projective dimension.*

The proof that (1)  $\implies$  (2) in the next theorem follows many of the same steps Auslander used in his proof of [1, Lemma 3.1]. The main differences are that we have to allow for the possibility that  $\varphi^e R$  is not finitely generated, and that we appeal to Theorems 3.2 and 3.3 for a replacement of rigidity of Tor over regular local rings. Recall that a module  $M$  is *generically free* provided  $M_{\mathfrak{p}}$  is a free  $R_{\mathfrak{p}}$ -module for each  $\mathfrak{p} \in \mathrm{Ass} R$ .

**Theorem 3.4.** *Let  $(R, \mathfrak{m})$  be a complete intersection of characteristic  $p$  and  $M$  a finitely generated, generically free  $R$ -module. Fix a positive integer  $e$ . The following conditions are equivalent:*

- (1)  $F^e(M)$  is torsion-free
- (2)  $M$  is torsion-free and of finite projective dimension.

*Proof.* Suppose (1) holds, and apply  $-\otimes_R {}^{\varphi^e}R$  to the short exact sequence (2.7.1), getting an exact sequence

$$F^e(\mathbb{T}_R M) \xrightarrow{\alpha} F^e(M) \xrightarrow{\beta} F^e(\perp_R M) \longrightarrow 0.$$

Since  $F^e(\mathbb{T}_R M)$  is torsion and  $F^e(M)$  is torsion-free, we see that  $\alpha = 0$ , whence  $\beta$  is an isomorphism. In particular  $F^e(\perp_R M)$  is torsion-free. Next, consider the universal pushforward [6, §1]:

$$(3.4.1) \quad 0 \longrightarrow \perp_R M \longrightarrow R^{(m)} \longrightarrow N \longrightarrow 0.$$

Applying  $-\otimes_R {}^{\varphi^e}R$  to this sequence, we obtain an injection

$$\mathrm{Tor}_1^R(N, {}^{\varphi^e}R) \hookrightarrow F^e(\perp_R M).$$

Now  $\perp_R M$  is clearly generically free, and from the construction of the universal pushforward [6, §1] one checks that  $N$  is generically free as well. It follows that  $\mathrm{Tor}_1^R(N, {}^{\varphi^e}R)$  is torsion. Since  $F^e(\perp_R M)$  is torsion-free, we have  $\mathrm{Tor}_1^R(N, {}^{\varphi^e}R) = 0$ . Now we invoke Theorems 3.2 and 3.3 to see that

$$\mathrm{Tor}_i^R(N, {}^{\varphi^e}R) = 0 \text{ for all } i \geq 1,$$

and, moreover, that  $N$  has finite projective dimension. From (3.4.1) it follows that  $\mathrm{Tor}_i^R(\perp_R M, {}^{\varphi^e}R) = 0$  for all  $i \geq 1$  and that  $\perp_R M$  has finite projective dimension. Therefore we will have (2) once we show that  $\mathbb{T}_R M = 0$ . For this, we apply  $-\otimes_R {}^{\varphi^e}R$  once again to (2.7.1), to get an injection

$$F^e(\mathbb{T}_R M) \hookrightarrow F^e(M).$$

Since  $F^e(\mathbb{T}_R M)$  is torsion and  $F^e(M)$  is torsion-free, we have  $F^e(\mathbb{T}_R M) = 0$ . If  $\mathbb{T}_R M$  were non-zero, there would be a surjection  $\mathbb{T}_R M \twoheadrightarrow R/\mathfrak{m}$ . But then  $F^e(R/\mathfrak{m}) = 0$ , that is,  $\mathfrak{m}^{\varphi^e}R = {}^{\varphi^e}R$ , an obvious contradiction, since  $\mathfrak{m}^{\varphi^e}R \subseteq \mathfrak{m}$ . Thus  $\mathbb{T}_R M = 0$ , and the proof that (1)  $\implies$  (2) is complete.

Now assume (2) holds. Since  $M$  is torsion-free, we can build the universal pushforward [6, §1]:

$$0 \longrightarrow M \longrightarrow R^{(\nu)} \longrightarrow N \longrightarrow 0,$$

where  $\nu = \nu_R M^*$ . Then  $N$  has finite projective dimension. Now Theorem 3.2 implies that  $\mathrm{Tor}_i^R(N, {}^{\varphi^e}R) = 0$  for all  $i \geq 1$ . Therefore  $\mathrm{Tor}_1^R(M, {}^{\varphi^e}R) = 0$ , and we get an injection  $F^e(M) \hookrightarrow ({}^{\varphi^e}R)^{(\nu)}$ , whence  $F^e(M)$  is torsion-free.  $\square$

From Theorem 3.2 (alternatively, from the proof of Theorem 3.4), we get Tor-independence (item (2) in the introduction):

**Corollary 3.5.** *If  $R$  and  $M$  satisfy the equivalent conditions of Theorem 3.4, then  $\mathrm{Tor}_i^R(M, {}^{\varphi^{e'}}R) = 0$  for every  $i \geq 1$  and every  $e' \geq 1$ .*



Of course, if  $M$  is torsion-free, the converse of Corollary 3.5 holds, by Theorem 3.3. In fact, it suffices to check that  $\mathrm{Tor}_i^R(M, \varphi^{e'} R) = 0$  for a single  $e'$  and a single  $i$ .

Recall that  $R$  is  $F$ -finite provided  $\varphi$  is a finite map, that is,  $R$  is module-finite over  $\varphi(R)$ . In this case,  $\varphi^e$  is a finite map for each  $e \geq 1$ . Note that the action of  $R$  on the module  $(\varphi^e M)$  in items (1) and (2) below is the *Frobenius* action  $m \cdot r = mr^{p^e}$ .

**Corollary 3.6.** *Assume that  $(R, \mathfrak{m})$  is a reduced local ring, is  $F$ -finite, and is a complete intersection. The following conditions are equivalent:*

- (1)  $F^e(\varphi^{e'} M)$  is torsion-free for every torsion-free  $R$ -module  $M$  and every pair  $e, e'$  of positive integers.
- (2)  $F^e(\varphi^{e'} M)$  is torsion-free for some nonzero finitely generated  $R$ -module  $M$  and some pair  $e, e'$  of positive integers.
- (3)  $R$  is regular.

*Proof.* Obviously (1)  $\implies$  (2), and the implication (3)  $\implies$  (1) holds by Kunz's theorem [10, Theorem 2.1] that the  $R$ -module  $\varphi^e R$  is flat when  $R$  is regular.

To prove that (2)  $\implies$  (3), we note that  $\varphi^{e'} M$  is a finitely generated  $R$ -module, by  $F$ -finiteness. Also,  $\varphi^{e'} M$  is generically free because  $R$  is reduced. By Theorem 3.4,  $\varphi^{e'} M$  has finite projective dimension, and now [2, Theorem 1.1] implies that  $R$  is regular.  $\square$

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