CRITERIA FOR VANISHING OF TOR OVER COMPLETE INTERSECTIONS

OLGUR CELIKBAS, SRIKANTH B. IYENGAR,
GREG PIEPMEYER, AND ROGER WIEGAND

Abstract. In this paper we exploit properties of Dao's η-pairing [12] as well as techniques of Huneke, Jorgensen, and Wiegand [24] to study the vanishing of $\mathrm{Tor}_i(M,N)$ for finitely generated modules M, N over complete intersections. We prove vanishing of $\mathrm{Tor}_i(M,N)$ for all $i \geq 1$ under depth conditions on M, N, and $M \otimes N$. Our arguments improve a result of Dao [13] and establish a new connection between the vanishing of Tor and the depth of tensor products.

1. Introduction

In his seminal 1961 paper [1], Auslander proved that if R is a local ring and M and N are nonzero finitely generated R-modules such that $\mathrm{pd}(M) < \infty$ and $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, then

$$\mathrm{depth}(M) + \mathrm{depth}(N) = \mathrm{depth}(R) + \mathrm{depth}(M \otimes_R N)\,, \tag{1.0.1}$$

that is, the *depth formula* holds. Huneke and Wiegand [25, Theorem 2.5] established the depth formula for Tor-independent modules (not necessarily of finite projective dimension) over complete intersection rings. Christensen and Jorgensen [11] extended that result to AB rings [23], a class of Gorenstein rings strictly containing the class of complete intersections. The depth formula is important for the study of depths of tensor products of modules [1, 25], as well as of complexes [20, 28]. We seek conditions on the modules M, N and $M \otimes_R N$ forcing such a formula to hold, in particular, conditions implying $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. The following conjecture, implicit in [24], guides our search:

Conjecture 1.1 (see [24])**.** *Let M, N be finitely generated modules over a complete intersection R of codimension c. If $M \otimes_R N$ is a $(c+1)^{st}$ syzygy and M has rank, must* $\mathrm{Tor}_i^R(M,N) = 0$ *for all $i \geq 1$?*

The conjecture is true if $c = 0$ or 1, by [32, Corollary 1] and [25, Theorem 2.7] respectively. Without the assumption of rank, there are easy counterexamples, e.g., $R = k[\![x,y]\!]/(xy)$ and $M = N = R/(x)$; M is an n^{th} syzygy for all n, but the odd index Tors are non-zero.

A finitely generated module over a complete intersection is an n^{th} syzygy of some finitely generated module if and only if it satisfies *Serre's condition* (S_n); see (2.6). Our methods yield a sharpening of the following theorem due to Dao:

Date: 16th December 2014.

2010 *Mathematics Subject Classification.* 13D07, 13C40.

Key words and phrases. complete intersection, tensor product, torsion, vanishing of Tor.

SBI partly supported by NSF grant DMS-1201889 and a Simons Fellowship; RW partly supported by a Simons Collaboration Grant.

Theorem 1.2 (Dao [13])**.** *Let R be a complete intersection in an unramified regular local ring, of relative codimension c, and let M, N be finitely generated R-modules. Assume*

(i) *M and N satisfy (S_c),*
(ii) *$M \otimes_R N$ satisfies (S_{c+1}), and*
(iii) *$M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p}$ of height at most c.*

Then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$ (and hence the depth formula holds).

By analyzing Serre's conditions, we remove Dao's assumption that the ambient regular local ring be unramified; see Corollary 3.14. Even though complete intersections in unramified regular local rings suffice for many applications, our conclusion is of interest: Dao's proof uses the nonnegativity of partial Euler characteristics, but nonnegativity remains unknown for the ramified case; see [13, Theorem 6.3 and the proof of Lemma 7.7].

If the ambient regular local ring *is* unramified, we can replace c with $c-1$ in both hypotheses (i) and (ii), remove hypothesis (iii), and still conclude that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$ *provided* that $\eta_c^R(M,N) = 0$; see (3.1) for the definition of $\eta_c^R(-,-)$ and Theorem 3.10 for our result.

Moore, Piepmeyer, Spiroff, and Walker [36],[41] have proved vanishing of the η-pairing in several important cases. These, in turn, yield results on vanishing of Tor. See Proposition 4.1, Theorem 4.2, and Corollary 4.3.

Our proofs rely on a reduction technique using *quasi-liftings*; see (2.8). Quasi-liftings were initially defined and studied by Huneke, Jorgensen and Wiegand in [24]. Lemma 3.9 is the key ingredient for our argument. It shows that if $R = S/(f)$ and S is a complete intersection of codimension $c-1$, and if $\eta_c^R(M,N) = 0$, then $\eta_{c-1}^S(E,F) = 0$, where E and F are quasi-liftings of M and N to S, respectively. By induction, we obtain that $\operatorname{Tor}_i^S(E,F) = 0$ for all $i \geq 1$: this allows us to prove the vanishing of $\operatorname{Tor}_i^R(M,N)$ from the depth and syzygy relations between the pairs E, F and M, N.

In the Appendix we revisit the paper of Huneke and Wiegand [25] and use our work to obtain one of the main results there. Moreover, we point out an oversight in Miller's paper [34] and state her result in its corrected form as Corollary B.3.

2. Preliminaries

We review a few concepts and results, especially universal pushforwards and quasi-liftings [24, 25]. Throughout R will be a commutative noetherian ring.

Let $\nu_R(M)$ denote the minimal number of generators of the R-module M. If $(R,\mathfrak{m})$ is local, the *codimension* of R is $\operatorname{codim}(R) := \nu_R(\mathfrak{m}) - \dim(R)$; it is a nonnegative integer. We have $\operatorname{codim}(\widehat{R}) = \operatorname{codim}(R)$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.

2.1. Complete intersections. R is a *complete intersection in a local ring* $(Q,\mathfrak{n})$ if there a surjection $\pi\colon Q \twoheadrightarrow R$ with $\ker(\pi)$ generated by a Q-regular sequence in $\mathfrak{n}$; the length of this regular sequence is the *relative codimension of R in Q*. A *hypersurface in Q* is a complete intersection of relative codimension one in Q.

Assume $\widehat{R}$ is a complete intersection in a regular local ring $(Q,\mathfrak{n})$, of relative codimension c. Then $\widehat{R} = Q/(\underline{f})$ for a regular sequence $\underline{f} = f_1,\ldots,f_c$, where $\operatorname{codim}(R) \leq c$. Moreover, the codimension of R is c if and only if $(\underline{f}) \subseteq \mathfrak{n}^2$.

A ring is a *complete intersection* (resp., *hypersurface*) if it is local and its completion is a complete intersection (resp., hypersurface) in a regular local ring.

2.2. Ramified regular local rings. A regular local ring $(Q, \mathfrak{n}, k)$ is said to be *unramified* if either (i) Q is equicharacteristic, i.e., contains a field, or else (ii) $Q \supset \mathbb{Z}$, $\operatorname{char}(k) = p$, and $p \notin \mathfrak{n}^2$. In contrast, the regular local ring $R = V[x]/(x^2 - p)$, where V is the ring of p-adic integers, is *ramified.* Every localization, at a prime ideal, of an unramified regular local ring is again unramified; see [1, Lemma 3.4].

Let $(Q, \mathfrak{n}, k)$ be a d-dimensional complete regular local ring. If Q is ramified, then k has characteristic p. Further, there is a complete unramified discrete valuation ring (V, pV) such that $Q \cong T/(p - f)$, where $T = V[[x_1, \dots, x_d]]$ and f is contained in the square of the maximal ideal of T; see for example [5, Chaper IX, §3]. Hence every complete regular local ring is a hypersurface in an unramified one. Consequently, when R is a complete intersection, $\widehat{R}$ is a complete intersection in an unramified regular local ring Q such that $\operatorname{codim} R \le c \le \operatorname{codim} R + 1$, where c is the relative codimension of $\widehat{R}$ in Q.

2.3. The depth formula ([25, Theorem 2.5])**.** Let R be a complete intersection and let M, N be finitely generated R-modules. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \ge 1$, then the *depth formula* (1.0.1) holds, that is,

$$\operatorname{depth}(M) + \operatorname{depth}(N) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes_R N)\,.$$

Recall that $\operatorname{depth}(0) = \infty$, so the formula holds trivially if a zero module appears.

2.4. Torsion submodule. The *torsion submodule* $\top_R M$ of M is the kernel of the natural homomorphism $M \to \mathrm{Q}(R) \otimes_R M$, where $\mathrm{Q}(R) = \{\text{non-zerodivisors}\}^{-1} R$ is the total quotient ring of R. The module M is *torsion* if $\top_R M = M$, and *torsion-free* if $\top_R M = 0$. To restate, M is torsion-free if and only if every non-zerodivisor of R is a non-zerodivisor on M, that is, if and only if $\bigcup \operatorname{Ass} M \subseteq \bigcup \operatorname{Ass} R$. Similarly, M is torsion if and only if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Ass}(R)$. For notation, the inclusion $\top_R M \subseteq M$ has cokernel $\perp_R M$:

$$0 \longrightarrow \top_R M \longrightarrow M \longrightarrow \perp_R M \longrightarrow 0\,. \tag{2.4.1}$$

2.5. Torsionless and reflexive modules. Let M be a finitely generated R-module; M^* denotes its dual $\operatorname{Hom}_R(M, R)$. The module M is *torsionless* if it embeds in a free module, equivalently, the canonical map $M \to M^{**}$ is injective. Torsionless modules are torsion-free, and the converse holds if $R_{\mathfrak{p}}$ is Gorenstein for every associated prime $\mathfrak{p}$ of R; see [40, Theorem A.1]. The module M is *reflexive* provided the map $M \to M^{**}$ is an isomorphism.

2.6. Serre's conditions (see [31, Appendix A, §1] and [18, Theorem 3.8])**.** Let M be a finitely generated R-module and let n be a nonnegative integer. Then M is said to satisfy *Serre's condition* (S_n) provided that

$$\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \ge \min\{n, \operatorname{height}(\mathfrak{p})\} \text{ for all } \mathfrak{p} \in \operatorname{Supp}(M).$$

A finitely generated module M over a local ring R is *maximal Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(R)$; necessary for this equality is that $M \ne 0$.

If M satisfies (S_1), then M is torsion-free, and the converse holds if R has no embedded primes, e.g., is reduced or Cohen-Macaulay; see (2.4). If R is Gorenstein, M satisfies (S_2) if and only if M is reflexive; see (2.5) and [18, Theorem 3.6]. Moreover, if R is Gorenstein, M satisfies (S_n) if and only if M is an n^{th} syzygy module; see [31, Corollary A.12].

A localization of a torsion-free module need not be torsion-free; see, for example, [38, Example 3.9]. However, over Cohen-Macaulay rings, we have:

Remark 2.7. Assume R is Cohen-Macaulay and M is a finitely generated R-module. Let $\mathfrak{p}$ be a prime ideal of R. Note that, since $\top_R M$ is killed by a non-zerodivisor of R, $(\top_R M)_\mathfrak{p}$ is a torsion $R_\mathfrak{p}$-module. Next, $\perp_R M$ satisfies (S_1) as R is Cohen-Macaulay, and so $(\perp_R M)_\mathfrak{p}$ is a torsion-free $R_\mathfrak{p}$-module; see (2.6). Localizing the exact sequence (2.4.1) at $\mathfrak{p}$, we see that $(\top_R M)_\mathfrak{p} \cong \top_{R_\mathfrak{p}}(M_\mathfrak{p})$. In particular, if M is a torsion-free R-module, then $M_\mathfrak{p}$ is a torsion-free $R_\mathfrak{p}$-module.

We recall a technique from [24, §1] for lowering the codimension.

2.8. Pushforward and quasi-lifting (see [24, §1])**.** Let R be a Gorenstein local ring and let M be a finitely generated torsion-free R-module. Choose a surjection $\varepsilon\colon R^{(\nu)} \twoheadrightarrow M^*$ with $\nu = \nu_R(M^*)$. Applying $\operatorname{Hom}(-,R)$ to this surjection, we obtain an injection $\varepsilon^*\colon M^{**} \hookrightarrow R^{(\nu)}$. Let M_1 be the cokernel of the composition $M \hookrightarrow M^{**} \hookrightarrow R^{(\nu)}$. The exact sequence

$$0 \to M \to R^{(\nu)} \to M_1 \to 0 \tag{2.8.1}$$

is called a *pushforward* of M. The extension (2.8.1) and the module M_1 are unique up to non-canonical isomorphism; see [7, pp. 174–175]. We refer to such a module M_1 as the pushforward of M. Note $M_1 = 0$ if and only if M is free.

Assume $R = S/(f)$ where $(S,\mathfrak{n})$ is a local ring and f is a non-zerodivisor in $\mathfrak{n}$. Let $S^{(\nu)} \twoheadrightarrow M_1$ be the composition of the canonical map $S^{(\nu)} \twoheadrightarrow R^{(\nu)}$ and the map $R^{(\nu)} \twoheadrightarrow M_1$ in (2.8.1). The *quasi-lifting* of M to S is the module E in the exact sequence of S-modules:

$$0 \to E \to S^{(\nu)} \to M_1 \to 0\,. \tag{2.8.2}$$

The quasi-lifting of M is unique up to isomorphism of S-modules.

Proposition 2.9 is from [24, Propositions 1.6 & 1.7]; Proposition 2.10 is embedded in the proofs of [24, Propositions 1.8 & 2.4] and is recorded explicitly in [7, Proposition 3.2(3)(b)]. We will use Proposition 2.10 in the proofs of Theorem 3.10 and Theorem B.2 below.

Proposition 2.9 ([24])**.** *Let R be a Gorenstein local ring and let M be a finitely generated torsion-free R-module. Let M_1 denote the pushforward of M.*

(i) *Let $n \geq 0$. Then M satisfies (S_{n+1}) if and only if M_1 satisfies (S_n).*
(ii) *Let $\mathfrak{p}$ be a prime ideal. If $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module, then $(M_1)_\mathfrak{p}$ is either zero or a maximal Cohen-Macaulay $R_\mathfrak{p}$-module.*

Proposition 2.10 ([24])**.** *Let $R = S/(f)$ where S is a complete intersection and f is a non-zerodivisor in S. Let N be a finitely generated torsion-free R-module such that $M \otimes_R N$ is reflexive. Assume $\operatorname{Tor}_i^R(M,N)_\mathfrak{p} = 0$ for all $i \geq 1$, and for all primes $\mathfrak{p}$ of R with* $\operatorname{height}(\mathfrak{p}) \leq 1$.

(i) *Then $M_1 \otimes_R N$ is torsion-free.*
(ii) *Let E and F denote the quasi-liftings of M and N to S, respectively; see (2.8). Assume $\operatorname{Tor}_i^S(E,F) = 0$ for all $i \geq 1$. Then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*

Serre's conditions (S_n) need not ascend along flat local homomorphisms. This can be problematic:

Example 2.11. The ring $\mathbb{C}[[x,y,u,v]]/(x^2,xy)$ has depth two and therefore, by Heitmann's theorem [21, Theorem 8], it is the completion $\widehat{R}$ of a unique factorization domain $(R,\mathfrak{m})$. Then R, being normal, satisfies (S_2), but $\widehat{R}$ does not even satisfy (S_1), since the localization at the height-one prime ideal (x,y) has depth zero.

For flat local homomorphisms between Cohen-Macaulay rings, and more generally when the fibers are Cohen-Macaulay, however, (S_n) *does* ascend and descend:

Lemma 2.12. *Let R be a local ring, $\mathfrak{p}$ a prime ideal of R, and M a finitely generated R-module.*

(1) *If M is reflexive, then so is the $R_\mathfrak{p}$-module $M_\mathfrak{p}$.*
(2) *Suppose R is Cohen-Macaulay. Then $(\top_R M)_\mathfrak{p} = \top_{R_\mathfrak{p}} M_\mathfrak{p}$; in particular, if M is torsion-free, then so is $M_\mathfrak{p}$.*
(3) *Suppose $R \to S$ is a flat local homomorphism. If $S \otimes_R M$ satisfies (S_n) as an S-module, then M satisfies (S_n) as an R-module; the converse holds when the fibers of the map $R \to S$ are Cohen-Macaulay.*

Proof. For part (1), localize the isomorphism $M \to M^{**}$. Part (2) is Remark 2.7. Part (3) can be proved along the same lines as [33, Theorem 23.9]: For any $\mathfrak{q}$ in $\operatorname{Spec} S$ with $\mathfrak{p} = \mathfrak{q} \cap R$, it follows from [33, Theorem 15.1 and Theorem 23.3] that

$$\begin{gathered}\operatorname{height}(\mathfrak{q}) = \operatorname{height}(\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \quad \text{and} \\ \operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} = \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{depth}(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})\,.\end{gathered}$$

When $S \otimes_R M$ satisfies (S_n), for $\mathfrak{q}$ minimal in $S/\mathfrak{p}S$ these equalities give

$$\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} \geq \min\{n, \operatorname{height}(\mathfrak{q})\} = \min\{n, \operatorname{height}(\mathfrak{p})\}.$$

Thus M satisfies (S_n). Conversely, if $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is Cohen-Macaulay and the R-module M satisfies (S_n), one gets

$$\operatorname{depth}_{S_\mathfrak{q}}(S \otimes_R M)_\mathfrak{q} \geq \min\{n, \operatorname{height}(\mathfrak{p})\} + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) \geq \min\{n, \operatorname{height}(\mathfrak{q})\}.$$

This completes the proof of part (3). □

3. Main theorem

Our main result, Theorem 3.10, is here. We use the θ and η-pairings introduced by Hochster [22] and Dao [13]. After preliminaries on these, we focus on complete intersections; see (2.1), the setting of our applications.

3.1. The θ and η pairings (Hochster [22] and Dao [12, 13])**.** Let R be a local ring and let M and N be finitely generated R-modules. Assume that there exists an integer f (depending on M and N), such that $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \geq f$.

If R is a hypersurface, then $\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_{i+2}^R(M,N)$ for all $i \gg 0$; see [17]. Hochster [22] introduced the θ pairing as follows:

$$\theta^R(M,N) = \operatorname{length}(\operatorname{Tor}_{2n}^R(M,N)) - \operatorname{length}(\operatorname{Tor}_{2n-1}^R(M,N)) \text{ for } n \gg 0\,.$$

When R is any complete intersection, Dao [13, Definition 4.2.] defined:

$$\eta_e^R(M,N) = \lim_{n\to\infty} \frac{1}{n^e} \sum_{i=f}^{n} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M,N))\,.$$

The η-pairing is a natural extension to complete intersections of the θ-pairing. Moreover the following statements hold; see [13, 4.3].

(i) $\eta_e^R(M,-)$ and $\eta_e^R(-,N)$ are additive on short exact sequences, provided η_e^R is defined on the pairs of modules involved.
(ii) If R is a hypersurface, then $\eta_1^R(M,N) = \frac{1}{2}\theta^R(M,N)$. Hence $\eta_1^R(M,N) = 0$ if and only if $\theta^R(M,N) = 0$.

Assume R is a complete intersection.

(iii) $\eta_e^R(M,N) = 0$ if $e \geq \operatorname{codim} R$ and either M or N has finite length.
(iv) η_e^R is finite when $e = \operatorname{codim}(R)$, and η_e^R is zero when $e > \operatorname{codim} R$.

The next result (Dao [13, Theorem 6.3]), on *Tor-rigidity*, shows the utility of the η-pairing.

Theorem 3.2 (Dao [13])**.** *Let R be a local ring whose completion is a complete intersection, of relative codimension $c \geq 1$, in an* unramified *regular local ring. Let M, N be finitely generated R-modules. Assume $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$, and that $\eta_c^R(M,N) = 0$. Then the pair M, N is c-Tor-rigid, that is, if $s \geq 0$ and $\operatorname{Tor}_i^R(M,N) = 0$ for all $i = s, \dots, s+c-1$, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq s$.*

The following conjectures have received quite a bit of attention:

Conjectures 3.3. Assume R is a local ring which is an isolated singularity, i.e., $R_\mathfrak{p}$ is a regular local ring for all non-maximal prime ideals $\mathfrak{p}$ of R.

(i) (Dao [12, Conjecture 3.15]) If R is an equicharacteristic hypersurface of even dimension, then $\eta_1^R(M,N) = 0$ for all finitely generated R-modules M, N.
(ii) (Moore, Piepmeyer, Spiroff and Walker [36, Conjecture 2.4]) If R is a complete intersection of codimension $c \geq 2$, then $\eta_c^R(M,N) = 0$ for all finitely generated R-modules M, N.

Moore, Piepmeyer, Spiroff and Walker [35] have settled Conjecture 3.3(i) in the affirmative for certain types of affine algebras. Polishchuk and Vaintrob [39, Remark 4.1.5], as well as Buchweitz and Van Straten [6, Main Theorem], have since given other proofs, in somewhat different contexts, of this result; see Theorem 4.2 for a recent result of Walker [41] concerning Conjecture 3.3(ii), and Corollary 4.3 for an application of his result.

Our proofs of Lemma 3.6 and Theorem B.2 use the following (see [1, Lemma 3.1] or [25, Lemma 1.1]).

Remark 3.4. Let R be a local ring, and let M and N be nonzero finitely generated R-modules. Assume $M \otimes_R N$ is torsion-free. Then $M \otimes_R N \cong M \otimes \perp_R N$. Moreover, if $\operatorname{Tor}_1^R(M, \perp_R N) = 0$, then $\top_R N = 0$, and hence N is torsion-free.

We encounter the same hypotheses often enough to warrant a piece of notation.

Notation 3.5. Let c be a positive integer. A pair M, N of finitely generated modules over a ring R satisfies (SP_c) provided the following conditions hold:

(i) M and N satisfy Serre's condition (S_{c-1}).
(ii) $M \otimes_R N$ satisfies (S_c).
(iii) $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$.

Hypersurfaces. We begin with a lemma analogous to [14, Proposition 3.1]; however, we do not assume any depth properties on M or N; see (2.1) and (3.5).

Lemma 3.6. *Let R be a local ring whose completion is a hypersurface in an* unramified *regular local ring, and let M, N be finitely generated R-modules. Assume the following hold:*

(i) $\dim(R) \geq 1$.
(ii) *The pair M, N satisfies (SP_1).*
(iii) $\mathrm{Supp}_R(\top_R N) \subseteq \mathrm{Supp}_R(M)$.
(iv) $\theta^R(M,N) = 0$.

Then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, and N is torsion-free.

Proof. Consider the following conditions for a prime ideal $\mathfrak{p}$ of R:

$$(\top_R N)_{\mathfrak{p}} \text{ has finite length over } R_{\mathfrak{p}}, \text{ and } \dim(R_{\mathfrak{p}}) \geq 1. \tag{3.6.1}$$

Claim: If $\mathfrak{p}$ is as in (3.6.1), then $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = 0$ for all $i \geq 1$.

We may assume that $M_{\mathfrak{p}} \neq 0$. We know from (ii) that $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ has finite length over $R_{\mathfrak{p}}$ for all $i \gg 0$. Since $(\top_R N)_{\mathfrak{p}}$ has finite length, the exact sequence (2.4.1) for N, localized at $\mathfrak{p}$, shows that $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}})$ has finite length over $R_{\mathfrak{p}}$ for all $i \gg 0$.

Using the additivity of $\theta^{R_{\mathfrak{p}}}$ along the same exact sequence, we see that

$$\theta^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = -\theta^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\top_R N)_{\mathfrak{p}}) = 0\,, \tag{3.6.1}$$

the last by (3.1).

Since $\perp_R N$ is a torsionless R-module (see (2.5)), there exists an exact sequence

$$0 \to \perp_R N \to R^{(n)} \to Z \to 0\,. \tag{3.6.2}$$

Localizing this sequence at $\mathfrak{p}$, we see that, for $i \gg 0$, $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}})$ has finite length and hence (since $\dim(R_{\mathfrak{p}}) \geq 1$) is torsion. Now Corollary A.2 forces $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}})$ to be torsion for all $i \geq 1$.

From (3.6.2), we see that $\mathrm{Tor}_1^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}})$ embeds into $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (\perp_R N)_{\mathfrak{p}}$. But $\mathrm{Tor}_1^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}})$ is torsion, and (by Remarks 2.7 and 3.4) $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} (\perp_R N)_{\mathfrak{p}}$ is torsion-free; therefore $\mathrm{Tor}_1^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}}) = 0$.

Next we note that $\theta^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}}) = -\theta^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = 0$; see (3.6.2) and (3.6.1). This implies, by Theorem 3.2, that $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, Z_{\mathfrak{p}}) = 0$ for all $i \geq 1$; see (3.1). The claim now follows from (3.6.2).

If $\top_R N \neq 0$, then there is a prime $\mathfrak{p}$ minimal in $\mathrm{Supp}_R(\top_R N)$, and so $(\top_R N)_{\mathfrak{p}}$ is a nonzero module of finite length. Moreover $\dim(R_{\mathfrak{p}}) \geq 1$: otherwise $\mathfrak{p} \in \mathrm{Ass}(R)$ and hence $(\top_R N)_{\mathfrak{p}} = 0$; see (2.4). Thus $\mathfrak{p}$ satisfies (3.6.1) and, by our claim, $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (\perp_R N)_{\mathfrak{p}}) = 0$ for $i \geq 1$. The hypothesis (iii) on supports implies that $M_{\mathfrak{p}} \neq 0$, and now Remark 3.4 yields a contradiction. We conclude that $\top_R N = 0$.

Applying the claim to the maximal ideal $\mathfrak{p}$ of R yields the required vanishing. □

Remark 3.7.

(i) The hypothesis (iii) of Lemma 3.6 holds when, for example, the support of N is contained in that of M. Moreover, if R is a domain and M and N are nonzero, then, since $M \otimes_R N$ is torsion-free, we see that $\mathrm{Supp}(M \otimes_R N) = \mathrm{Spec}(R)$, whence $\mathrm{Supp}(M) = \mathrm{Spec}(R)$.
(ii) Most of the hypotheses in Lemma 3.6 are essential; see the discussion after [27, Remark 1.5]. Notice, without the assumption that $\dim(R) \geq 1$, the lemma would fail. Take, for example, $R = \mathbb{C}[x]/(x^2)$ and $M = R/(x) = N$. The

vanishing of θ is also essential: let $R = \mathbb{C}[[x, y]]/(xy)$, $M = R/(x)$ and $N = R/(x^2)$. Then the pair M, N satisfies the conditions (ii) and (iii) of Lemma 3.6. On the other hand $\operatorname{Tor}^R_{2i+1}(M, N) \cong k$ for all $i \geq 0$, and $\operatorname{Tor}^R_{2i}(M, N) = 0$ for all $i \geq 1$. (Thus $\theta^R(M, N) = -1$.)

The completion of any regular ring is a hypersurface in an unramified regular local ring; see (2.2). Hence the following consequence of Lemma 3.6 extends Lichtenbaum's [32, Corollary 3], which in turn builds on Auslander's [1, Theorem 3.2]; cf. C. Miller's result recorded as Corollary B.3 here.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a d-dimensional local ring whose completion is a hypersurface in an unramified regular local ring, with $d \geq 1$, and let M be a finitely generated R-module. Assume $\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for all prime ideals $\mathfrak{p} \neq \mathfrak{m}$ and that $\theta^R(M, -) = 0$. If $\otimes^n_R M$ is torsion-free for some integer $n \geq 2$, then $\operatorname{pd}(M) \leq (d-1)/n$. Consequently, if M is not free, then $\otimes^n_R M$ has torsion for each $n \geq \max\{2, d\}$.*

Proof. We may assume $M \neq 0$. Iterating Lemma 3.6 shows that $\otimes^p_R M$ is torsion-free for $p = 1, \ldots, n$, and that $\operatorname{Tor}^R_i(M, \otimes^{p-1}_R M) = 0$ for all $i \geq 1$. Taking $p = 2$, we see from [27, Theorem 1.9] that $\operatorname{pd}(M) < \infty$. Since $\operatorname{depth}(\otimes^n_R M) \geq 1$, one obtains, using [1, Corollary 1.3] and the Auslander-Buchsbaum formula [3, Theorem 3.7], $n \cdot \operatorname{pd}(M) = \operatorname{pd}(\otimes^n_R M) = d - \operatorname{depth}(\otimes^n_R M) \leq d - 1$. □

Complete intersections. Hypersurfaces in complete intersections give the inductive step for our proof of Theorem 3.10; see (2.8) on pushforwards.

Lemma 3.9. *Let $(S, \mathfrak{n})$ be a complete intersection, and let R be a hypersurface in S. Let M and N be finitely generated torsion-free R-modules, and let E and F be the quasi-liftings of M and N, respectively, to S. Assume $\operatorname{Tor}^R_i(M, N)$ has finite length for all $i \gg 0$. Let e be an integer with $e \geq \max\{2, \operatorname{codim}(S) + 1\}$. Then*

(i) *$\operatorname{Tor}^S_i(E, F)$ has finite length for all $i \gg 0$, and*
(ii) *$\eta^S_{e-1}(E, F) = 2 \cdot e \cdot \eta^R_e(M, N)$.*

Proof. By hypothesis, $R \cong S/(f)$, where f is a non-zerodivisor in S. The spectral sequence associated to the change of rings $S \to R$ yields the following exact sequence, see [32, pp. 223–224] or [37, p. 561], for all $n \geq 1$:

$$\cdots \to \operatorname{Tor}^R_{n-1}(M, N) \to \operatorname{Tor}^S_n(M, N) \to \operatorname{Tor}^R_n(M, N) \to \cdots$$

Consequently $\operatorname{Tor}^S_i(M, N)$ has finite length for $i \gg 0$. Let M_1 and N_1 be the pushforwards of M and N, respectively. Since $\operatorname{Tor}^S_i(R, -) = 0$ for all $i \geq 2$, the sequences (2.8.2) and (2.8.1) yield isomorphisms

$$\operatorname{Tor}^S_i(E, N) \cong \operatorname{Tor}^S_{i+1}(M_1, N) \cong \operatorname{Tor}^S_i(M, N) \text{ for all } i \geq 2\,.$$

Arguing in the same vein, one gets isomorphisms

$$\operatorname{Tor}^S_i(E, F) \cong \operatorname{Tor}^S_i(E, N) \text{ for all } i \geq 2.$$

Hence the length of $\operatorname{Tor}^S_i(E, F)$ is finite for all $i \gg 0$, and so (i) holds.

Similar arguments show the η-pairing, over both R and S, as appropriate, is defined for all pairs (X, Y) with $X \in \{M, M_1, E\}$ and $Y \in \{N, N_1, F\}$.

By hypothesis, $\operatorname{codim}(S) \leq e-1$, and hence $\operatorname{codim}(R) \leq e$; see (2.1). Additivity of η along the exact sequences (2.8.1) and (2.8.2) thus gives

$$\eta_e^R(M,N) = -\eta_e^R(M_1,N) = \eta_e^R(M_1,N_1) \text{ and}$$
$$\eta_{e-1}^S(E,F) = -\eta_{e-1}^S(M_1,F) = \eta_{e-1}^S(M_1,N_1)\,.$$

Our assumption that $e \geq \max\{2, \operatorname{codim} S + 1\}$, together with [13, Theorem 4.1(3)], allow us to invoke [13, Theorem 4.3(3)], which says that

$$2e \cdot \eta_e^R(M_1,N_1) = \eta_{e-1}^S(M_1,N_1)\,.$$

This gives (ii), completing the proof. □

The next theorem is our main result. As its hypotheses are technical, several of its consequences are discussed in section 4; see section 2 for background.

Theorem 3.10. *Let R be a local ring whose completion is a complete intersection in an* unramified *regular local ring, of relative codimension $c \geq 1$. Let M,N be finitely generated R-modules. Assume the following hold:*

(i) $\dim(R) \geq c$.
(ii) *The pair (M,N) satisfies (SP_c).*
(iii) $\operatorname{Supp}_R(\top_R N) \subseteq \operatorname{Supp}_R(M)$.
(iv) $\eta_c^R(M,N) = 0$

Then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.

Proof. The case $c = 1$ is Lemma 3.6. For $c \geq 2$, proceed by induction on c. We can assume R is complete, so that $R = Q/(\underline{f})$, where Q is an unramified regular local ring and $\underline{f} = f_1,\dots,f_c$ is a Q-regular sequence; see (2.2) and (2.12). Let $R = S/(f)$, where $S = Q/(f_1,\dots,f_{c-1})$ and $f = f_c$.

Hypothesis (ii) implies $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \gg 0$; see (3.5). Hence Corollary A.3 implies that, for all primes $\mathfrak{p}$ with $\operatorname{height}(\mathfrak{p}) \leq c-1$,

(3.10.1) $$\operatorname{Tor}_i^R(M,N)_{\mathfrak{p}} = 0 \text{ for all } i \geq 1.$$

Condition (ii) also implies M and N are torsion-free since $c \geq 2$; see (3.5). Hence quasi-liftings E and F of M and N to S exist; see (2.8). Using the vanishing of Tors in (3.10.1) and [24, Theorem 4.8] (cf. [7, Proposition 3.1(7)]), one gets that

(3.10.2) $$E \otimes_S F \text{ satisfies } (S_{c-1}) \text{ as an } S\text{-module.}$$

It follows from [24, Propositions 1.6 and 1.7] (see also [7, Proposition 3.1(2) and 3.1(6)]) that the assumptions in (i) of (SP_c) pass to E and F; see (3.5).

(3.10.3) $$E \text{ and } F \text{ satisfy } (S_{c-1}) \text{ as } S\text{-modules}\,.$$

Lemma 3.9 guarantees that $\operatorname{Tor}_i^S(E,F)$ has finite length for all $i \gg 0$ and that $\eta_{c-1}(E,F) = 0$. In particular the pair E,F satisfies (SP_{c-1}) over the ring S. Moreover, E and F, being syzygies, are torsion-free, so we indeed have that $\operatorname{Supp}_S(\top_S F) \subseteq \operatorname{Supp}_S(E)$. Now the inductive hypothesis implies that

(3.10.4) $$\operatorname{Tor}_i^S(E,F) = 0 \text{ for all } i \geq 1.$$

Condition (ii) also implies that $M \otimes_R N$ is reflexive since $c \geq 2$; see (2.6). Further $\operatorname{Tor}_i^R(M,N)_{\mathfrak{p}} = 0$ for all $i \geq 1$ and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{height}(\mathfrak{p}) \leq 1$; see (3.10.1). Thus Proposition 2.10 and (3.10.4) yield $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. □

Remark 3.11. In Theorem 3.10, if $c \geq 2$, hypothesis (ii) implies that N is torsion-free, i.e., $\top_R N = 0$; see (2.6) and (3.5). Thus, when $c \geq 2$, hypothesis (iii) of Theorem 3.10 is redundant.

When $\dim(R) > c$, the equivalence of (i) and (ii) in the following corollary seems interesting; see also (2.3). Actually, in that case the equivalence of (ii) and (iii) holds without the assumption that $\eta_c^R(M,N) = 0$. See [7, Corollary 2.4].

Corollary 3.12. *Let R be an isolated singularity whose completion is a complete intersection in an unramified regular local ring, of relative codimension c. Let M and N be maximal Cohen-Macaulay R-modules. Assume* $\dim(R) \geq c$. *Assume further that $\eta_c^R(M,N) = 0$. The following conditions are equivalent:*

(i) *$M \otimes_R N$ satisfies (S_c).*
(ii) *$M \otimes_R N$ is maximal Cohen-Macaulay.*
(iii) $\operatorname{Tor}_i^R(M,N) = 0$ *for all $i \geq 1$, and hence the depth formula holds.*

Over a complete intersection, vanishing of Ext is closely related to vanishing of Tor: $\operatorname{Ext}_R^i(M,N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \gg 0$; see [2, Remark 6.3]. Our next example shows the hypotheses of Theorem 3.10 do *not* force the vanishing of $\operatorname{Ext}_R^i(M,N)$ for all $i \geq 1$.

Example 3.13. Let $(R, \mathfrak{m}, k)$ be a complete intersection with $\operatorname{codim}(R) = 2$ and $\dim(R) \geq 3$. Let N be the dth syzygy of k, where $d = \dim(R)$, and let M be the second syzygy of $R/(\underline{x})$, where $\underline{x}$ is a maximal R-regular sequence.

Note that N is maximal Cohen-Macaulay, $\operatorname{depth}(M) = 2$ and $N_\mathfrak{p}$ is free over $R_\mathfrak{p}$ for all primes $\mathfrak{p} \neq \mathfrak{m}$. It follows, since $\operatorname{pd}(M) < \infty$, that $\eta_2^R(M,N) = 0$ and $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$; see (3.1) and Theorem A.1. Therefore the depth formula (2.3) shows that $\operatorname{depth}(M \otimes_R N) = 2$. Since M is a second syzygy, it satisfies (S_2) and hence $M \otimes_R N$ satisfies (S_2); see (2.6). In particular, the pair M, N satisfies (SP_2); see (3.5). However $\operatorname{Ext}_R^{d-2}(M,N) = \operatorname{Ext}^d(R/(\underline{x}), N) \neq 0$; see, for example, [33, Chapter 19, Lemma 1(iii)].

Here is the extension of Dao's theorem [13, Theorem 7.7] promised in the introduction (cf. Theorem 1.2):

Corollary 3.14. *Let R be a local ring that is a complete intersection, and let M and N be finitely generated R-modules. Assume that the following conditions hold for some integer $e \geq$* $\operatorname{codim}(R)$*:*

(i) *M and N satisfy (S_e).*
(ii) *$M \otimes_R N$ satisfies (S_{e+1}).*
(iii) *$M_\mathfrak{p}$ is a free for all prime ideals $\mathfrak{p}$ of R of height at most e.*

Then $\operatorname{Tor}_i^R(M,N) = 0$ *for all $i \geq 1$ and hence the depth formula holds.*

Proof. If $e = 0$ this is the theorem of Auslander [1] and Lichtenbaum [32, Corollary 2]. Assume now that $e \geq 1$. We use induction on $\dim R$. If $\dim R \leq e$, condition (iii) implies that M is free, and there is nothing to prove. Assuming $\dim R \geq e+1$, we note that the hypotheses localize, so $\operatorname{Tor}_i^R(M,N)_\mathfrak{p} = 0$ for each $i \geq 1$ and each prime ideal $\mathfrak{p}$ in the punctured spectrum of R; that is to say, $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i \geq 1$. Thus the pair M, N satisfies (SP_{e+1}). Moreover, since $\operatorname{codim} R < e+1$, we have $\eta_{e+1}^R = 0$ by item (iv) of (3.1). The completion of R can be realized as a complete intersection, of relative codimension $e+1$, in an unramified regular local ring (see 2.2). Hence the desired result follows from Theorem 3.10. □

4. Vanishing of η

In this section we apply our results to situations where the η-pairing is known to vanish. We know, from Theorem 3.10, that, as long as the critical hypothesis $\eta_c^R(M,N)=0$ holds, we can replace c with $c-1$ in the hypotheses of Theorem 1.2 and still conclude the vanishing of Tor. Although it is not easy to verify vanishing of η (see Conjectures 3.3), there are several classes of rings R for which it is known that $\eta^R(M,N)=0$ for all finitely generated R-modules M and N. For example, if R is an even-dimensional simple ("ADE") singularity in characteristic zero, then Dao [12, Corollary 3.16] observed that $\theta^R(M,N)=0$; see [12, Corollary 3.6] and also [12, §3] for more examples.

Now we give a localized version of a vanishing theorem for graded rings, due to Moore, Piepmeyer, Spiroff, and Walker [36].

Proposition 4.1. *Let k be a perfect field and $Q=k[x_1,\dots,x_n]$ the polynomial ring with the standard grading. Let $\underline{f}=f_1,\dots,f_c$ be a Q-regular sequence of homogeneous polynomials, with $c\geq 2$. Put $A=Q/(\underline{f})$ and $R=A_{\mathfrak{m}}$, where $\mathfrak{m}=(x_1,\dots,x_n)$. Assume that $A_{\mathfrak{p}}$ is a regular local ring for each $\mathfrak{p}$ in $\operatorname{Spec}(A)\setminus\{\mathfrak{m}\}$. Then $\eta_c^R(M,N)=0$ for all finitely generated R-modules M and N. In particular, if $n\geq 2c$ and the pair M,N satisfies (SP_c), then M and N are Tor-independent.*

Proof. Choose finitely generated A-modules U and V such that $U_{\mathfrak{m}}\cong M$ and $V_{\mathfrak{m}}\cong N$. For any maximal ideal $\mathfrak{n}\neq\mathfrak{m}$, the local ring $A_{\mathfrak{n}}$ is regular and hence $\operatorname{Tor}_i^A(U,V)_{\mathfrak{n}}=0$ for $i\gg 0$. It follows that the map $\operatorname{Tor}_i^A(U,V)\to\operatorname{Tor}_i^R(M,N)$ induced by the localization maps $U\to M$ and $V\to N$ is an isomorphism for $i\gg 0$. Also, for any A-module supported at $\mathfrak{m}$, its length as an A-module is equal to its length as an R-module. In conclusion, $\eta_c^R(M,N)=\eta_c^A(U,V)$.

As k is perfect, the hypothesis on A implies that the k-algebra $A_{\mathfrak{p}}$ is smooth for each non-maximal prime $\mathfrak{p}$ in A; see [19, Corollary 16.20]. Thus, the morphism of schemes $\operatorname{Spec}(R)\setminus\{\mathfrak{m}\}\to\operatorname{Spec}(k)$ is smooth. Now [36, Corollary 4.7] yields $\eta_c^A(U,V)=0$, and hence $\eta_c^R(M,N)=0$. It remains to note that if $n\geq 2c$, then $\dim R\geq c$, so Theorem 3.10 applies. □

Next, we quote a recent theorem due to Walker; it provides strong support for Conjectures 3.3, at least in equicharacteristic zero.

Theorem 4.2. (Walker [41, Theorem 1.2]) *Let k be a field of characteristic zero, and let Q a smooth k-algebra. Let $\underline{f}=f_1,\dots,f_c$ be a Q-regular sequence, with $c\geq 2$, and put $A=Q/(f_1,\dots,f_c)$. Assume the singular locus $\{\mathfrak{p}\in\operatorname{Spec}(A):A_{\mathfrak{p}}$ is not regular$\}$ is a finite set of maximal ideals of A. Then $\eta_c^A(U,V)=0$ for all finitely generated A-modules U, V.*

Corollary 4.3. *With A as in 4.2, put $R=A_{\mathfrak{m}}$ where $\mathfrak{m}$ is any maximal ideal of A. Then $\eta_c^R(M,N)=0$ for all finitely generated R-modules M and N. In particular, if $\dim R\geq c$ and the pair M,N satisfies (SP_c), then M and N are Tor-independent.*

Proof. By inverting a suitable element of Q, we may assume that $A_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}\neq\mathfrak{m}$. Now proceed as in the first paragraph of the proof of Proposition 4.1. □

Theorem 4.4. *Let $(R, \mathfrak{m}, k)$ be a two-dimensional, equicharacteristic, normal, excellent complete intersection of codimension c, with $c \in \{1, 2\}$, and let M and N be finitely generated R-modules. Assume k is contained in the algebraic closure of a finite field. Assume further that M, N satisfy the conditions (i) and (ii) of (SP_c). Then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Proof. The completion $\widehat{R}$ is an isolated singularity because R is excellent; see [31, Proposition 10.9], and so $\widehat{R}$ is a normal domain. Replacing R by $\widehat{R}$, we may assume that $R = S/(\underline{f})$, where $(S, \mathfrak{n}, k)$ is a regular local ring and $\underline{f}$ is a regular sequence in $\mathfrak{n}^2$ of length c. Let $\overline{k}$ be an algebraic closure of k, and choose a *gonflement* $S \hookrightarrow (\overline{S}, \overline{\mathfrak{n}}, \overline{k})$ lifting the field extension $k \hookrightarrow \overline{k}$; see [31, Chapter 10, §3]. This is a flat local homomorphism and is an inductive limit of étale extensions. Moreover, $\mathfrak{n}\overline{S} = \overline{\mathfrak{n}}$, so $\overline{S}$ is a regular local ring. By [31, Proposition 10.15], both $\overline{S}$ and $\overline{R} := \overline{S}/(\underline{f})$ are excellent, and $\overline{R}$ is an isolated singularity. Therefore $(\overline{R}, \overline{\mathfrak{m}}, \overline{k})$ is a normal domain. Finally, we pass to the completion $\widehat{S}$ of $\overline{S}$ and put $\Lambda = \widehat{S}/(\underline{f})$. This is still an isolated singularity, a normal domain, and a complete intersection of codimension c. Moreover, our hypotheses on M and N ascend along the flat local homomorphism $R \to \Lambda$; see (2.12). Since Λ is an isolated singularity, $\operatorname{Tor}_i^\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N)$ has finite length for $i \gg 0$; thus the pair $\Lambda \otimes_R M$, $\Lambda \otimes_R N$ satisfies (SP_c).

It follows from [8, Proposition 2.5 and Remark 2.6] that $G(\Lambda)/L$ is torsion, where $G(\Lambda)$ is the Grothendieck group of Λ and L is the subgroup generated by classes of modules of finite projective dimension. This implies that $\eta_c^\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$; see [12, Corollary 3.1] and the paragraph preceding it. Now Theorem 3.10 implies that $\operatorname{Tor}_i^\Lambda(\Lambda \otimes_R M, \Lambda \otimes_R N) = 0$ for all $i \geq 1$: the requirement on supports is automatically satisfied, since Λ is a domain; see Remark 3.7(i). Faithfully flat descent completes the proof. □

Appendix A. An application of pushforwards

In Theorem A.4 we use pushforwards to generalize a theorem due to Celikbas [7, Theorem 3.16]. We have two preparatory results. The first one is a special case of a theorem of Jorgensen:

Theorem A.1. ([29, Theorem 2.1]) *Let R be a complete intersection and let M and N be finitely generated R-modules. Assume M is maximal Cohen-Macaulay. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Corollary A.2. *Let R be a complete intersection and let M, N be finitely generated R-modules. If $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \gg 0$, then $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \geq 1$.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of R. By (2.4), it suffices to prove that $\operatorname{Tor}_i^{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$ for all $i \geq 1$. For that we may assume $M_\mathfrak{p} \neq 0$. Then, since $R_\mathfrak{p}$ is artinian, it follows that $M_\mathfrak{p}$ is a maximal Cohen-Macaulay $R_\mathfrak{p}$-module. Therefore Theorem A.1 gives the desired vanishing. □

Corollary A.3. *Let R be a complete intersection, and let M, N be finitely generated R-modules. Assume M satisfies (S_w), where w is a positive integer, and that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \gg 0$. Let $\mathfrak{p}$ be a non-maximal prime ideal of R such that* $\operatorname{height}(\mathfrak{p}) \leq w$. *Then $\operatorname{Tor}_i^R(M, N)_\mathfrak{p} = 0$ for all $i \geq 1$.*

Proof. Serre's condition (S_w) localizes, so $M_{\mathfrak{p}}$ is either zero or a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module; see (2.6). As $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for $i \gg 0$, Theorem A.1 implies that $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$ for all $i \geq 1$. □

The next theorem generalizes a result due to Celikbas [7, 3.16]; we emphasize that the ambient regular local ring in Theorem A.4 is allowed to be ramified.

Theorem A.4. *Let R be a complete intersection with $\dim R \geq \operatorname{codim} R$, and let M and N be finitely generated R-modules. Assume the pair M, N satisfies (SP_c) for some $c \geq \operatorname{codim} R$. If $c = 1$, assume further that M or N is torsion-free. If $\operatorname{Tor}_1^R(M, N) = 0$, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

Proof. Without loss of generality, one may assume that $c = \operatorname{codim} R$. When $c = 0$, the desired result is the rigidity theorem of Auslander [1] and Lichtenbaum [32], so in the remainder of the proof we assume that $c \geq 1$.

Assume first that $c = 1$. By hypotheses $\operatorname{Tor}_i^R(M, N)$ has finite length for $i \gg 0$ and $M \otimes_R N$ is torsion-free; see (3.5). Moreover, we may assume N (say) is torsion-free. Tensoring M with the pushforward (2.8) for N gives the following:

$$\operatorname{Tor}_1^R(M, N_1) \hookrightarrow M \otimes_R N \tag{A.4.1}$$

$$\operatorname{Tor}_i^R(M, N_1) \cong \operatorname{Tor}_{i-1}^R(M, N) \text{ for all } i \geq 2. \tag{A.4.2}$$

Equation (A.4.2) implies that $\operatorname{Tor}_i^R(M, N_1)$ has finite length for all $i \gg 0$. Therefore, since $\dim(R) \geq 1$, $\operatorname{Tor}_i^R(M, N_1)$ is torsion for all $i \gg 0$; see (2.4). Now Corollary A.2 implies that $\operatorname{Tor}_i^R(M, N_1)$ is torsion for all $i \geq 1$. As $M \otimes_R N$ is torsion-free, we deduce from (A.4.1) that $\operatorname{Tor}_1^R(M, N_1) = 0$. By (A.4.2) we have $\operatorname{Tor}_2^R(M, N_1) \cong \operatorname{Tor}_1^R(M, N) = 0$. Therefore $\operatorname{Tor}_2^R(M, N_1) = 0 = \operatorname{Tor}_1^R(M, N_1)$, and hence Murthy's rigidity theorem [37, Theorem 1.6] implies that $\operatorname{Tor}_i^R(M, N_1) = 0$ for all $i \geq 1$. Now (A.4.2) completes the proof for the case $c = 1$.

Assume now that $c \geq 2$. We define a sequence $M_0, M_1, \ldots, M_{c-1}$ of finitely generated modules by setting $M_0 = M$, and M_n to be the pushforward of M_{n-1}, for all $n = 1, \ldots, c-1$. These pushforwards exist: M_0 satisfies (S_{c-1}) by hypothesis (3.5)(i), and so, by Proposition 2.9(i),

(1) each M_n satisfies (S_{c-n-1}).

For the desired result, it suffices to prove that $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for all $i \geq c$. We will, in fact, prove this for all $i \geq 1$. To this end, we establish by induction that the following hold for $n = 0, \ldots, c-1$:

(2) $M_n \otimes_R N$ satisfies (S_{c-n});
(3) $\operatorname{Tor}_i^R(M_n, N)$ has finite length for all $i \gg 0$;
(4) $\operatorname{Tor}_i^R(M_n, N) = 0$ for $i = 1, \ldots, n+1$.

For $n = 0$, conditions (2) and (3) are part of (3.5), while (4) is from our hypothesis that $\operatorname{Tor}_1^R(M, N) = 0$; recall that $M_0 = M$. Assume that (2), (3) and (4) hold for some integer n with $0 \leq n \leq c-2$.

Tensor the pushforward of M_n with N, see (2.8), to obtain

$$\operatorname{Tor}_i^R(M_{n+1}, N) \cong \operatorname{Tor}_{i-1}^R(M_n, N) \text{ for all } i \geq 2, \tag{A.4.3}$$

and the following exact sequence in which F is finitely generated and free:

$$0 \to \operatorname{Tor}_1^R(M_{n+1}, N) \to M_n \otimes_R N \to F \otimes_R N \to M_{n+1} \otimes_R N \to 0\,. \tag{A.4.4}$$

Induction and (A.4.3) imply that $\operatorname{Tor}_i^R(M_{n+1}, N)$ has finite length for all $i \gg 0$, so (3) holds; furthermore, by Corollary A.2, $\operatorname{Tor}_i^R(M_{n+1}, N)$ is torsion for all $i \geq 1$. (Recall that $\dim(R) \geq \operatorname{codim}(R) = c \geq 1$ so that finite length modules are torsion.) Since $n \leq c-1$, condition (2) implies that $M_n \otimes_R N$ satisfies (S_1) and hence $M_n \otimes_R N$ is torsion-free; therefore the exact sequence (A.4.4) forces $\operatorname{Tor}_1^R(M_{n+1}, N)$ to vanish. Now (A.4.3) gives (4). It remains to verify (2), namely, that $M_{n+1} \otimes_R N$ satisfies (S_{c-n-1}). To that end, let $\mathfrak{p} \in \operatorname{Supp}(M_{n+1} \otimes_R N)$. We will verify that $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} \geq \min\{c-n-1, \operatorname{height}(\mathfrak{p})\}$; see (2.6).

Suppose $\operatorname{height}(\mathfrak{p}) \geq c-n$. Recall, by hypothesis (3.5)(i), N satisfies (S_{c-1}). Hence $F \otimes_R N$, a direct sum of copies of N, satisfies (S_{c-n-1}). In particular it follows that $\operatorname{depth}_{R_\mathfrak{p}}(F \otimes_R N)_\mathfrak{p} \geq c-n-1$. Furthermore, by (2) of the induction hypothesis, we have that $\operatorname{depth}_{R_\mathfrak{p}}(M_n \otimes_R N)_\mathfrak{p} \geq c-n$. Recall that $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$. Therefore, localizing the short exact sequence in (A.4.4) at $\mathfrak{p}$, we conclude by the depth lemma that $\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} \geq c-n-1$.

Next assume $\operatorname{height}(\mathfrak{p}) \leq c-n-1$. We want to show that $(M_{n+1} \otimes_R N)_\mathfrak{p}$ is maximal Cohen-Macaulay. By the induction hypotheses, $\operatorname{Tor}_i^R(M_n, N)$ has finite length for all $i \gg 0$. As $n \geq 0$, we see that $\dim(R) \geq \operatorname{codim}(R) = c \geq c-n$, whence $\mathfrak{p}$ is not the maximal ideal. Thus $\operatorname{Tor}_i^R(M_n, N)_\mathfrak{p} = 0$ for all $i \gg 0$. Now, setting $w = c-n-1$ and using Corollary A.3 for the pair M_n, N, we conclude that $\operatorname{Tor}_i^R(M_n, N)_\mathfrak{p} = 0$ for all $i \geq 1$. Then (A.4.3) and the already established fact that $\operatorname{Tor}_1^R(M_{n+1}, N) = 0$ give $\operatorname{Tor}_i^R(M_{n+1}, N)_\mathfrak{p} = 0$ for all $i \geq 1$. Thus the depth formula holds; see (2.3):

$$\operatorname{depth}_{R_\mathfrak{p}}(M_{n+1})_\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(M_{n+1} \otimes_R N)_\mathfrak{p} .$$

Since Serre's conditions localize, $N_\mathfrak{p}$ is maximal Cohen-Macaulay over $R_\mathfrak{p}$; see hypothesis (3.5)(i). Also, $(M_{n+1})_\mathfrak{p}$ is maximal Cohen-Macaulay whether or not $(M_n)_\mathfrak{p}$ is zero; see the pushforward sequence or Proposition 2.9(ii). By the depth formula, $(M_{n+1} \otimes_R N)_\mathfrak{p}$ is maximal Cohen-Macaulay. Thus $M_{n+1} \otimes_R N$ satisfies (2), and the induction is complete.

Now we parallel the argument for the case $c = 1$. At the end, $\operatorname{Tor}_i^R(M_{c-1}, N)$ has finite length for all $i \gg 0$, and is equal to 0 for $i = 1, \ldots, c$. Tensoring M_{c-1} with the pushforward of N, we get

$$\operatorname{Tor}_i^R(M_{c-1}, N_1) \cong \operatorname{Tor}_{i-1}^R(M_{c-1}, N) \text{ for all } i \geq 2, \tag{A.4.5}$$

$$\text{and} \quad \operatorname{Tor}_1^R(M_{c-1}, N_1) \hookrightarrow M_{c-1} \otimes_R N. \tag{A.4.6}$$

In view of (A.4.5), it suffices to show that $\operatorname{Tor}_1^R(M_{c-1}, N_1) = 0$: this will imply $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for all $i = 1, \ldots, c+1$, and hence Murthy's rigidity theorem [37, Theorem 1.6] will yield that $\operatorname{Tor}_i^R(M_{c-1}, N_1) = 0$ for all $i \geq 1$, and consequently $\operatorname{Tor}_i^R(M_{c-1}, N) = 0$ for all $i \geq 1$ by (A.4.5). We know that $M_{c-1} \otimes_R N$ is torsion-free. Therefore we use (A.4.6) and Corollary A.2, and obtain $\operatorname{Tor}_1^R(M_{c-1}, N_1) = 0$, as we did in the case $c = 1$. □

Appendix B. Amending the literature

We use Theorem A.4 to give a different proof of an important result of Huneke and Wiegand; see Theorem B.2 and the ensuing paragraph. We also point out a missing hypothesis in a result of C. Miller [34, Theorem 3.1], and state the corrected form of her theorem in Corollary B.3. At the end of the paper we indicate an alterate route to the proof of the following result [25, Theorem 3.1], the main theorem of the 1994 paper of Huneke and Wiegand:

Theorem B.1 (Huneke and Wiegand [25])**.** *Let R be a hypersurface, and let M and N be finitely generated R-modules. If M or N has rank, and $M \otimes_R N$ is maximal Cohen-Macaulay, then both M and N are maximal Cohen-Macaulay, and either M or N is free.*

Theorem B.1 and its variations have been analyzed, used, and studied in the literature; see [10] and [16] for some history and many consequences of the theorem. The following result [25, Theorem 2.7] played an important role in its proof.

Theorem B.2 (Huneke and Wiegand [25])**.** *Let R be a hypersurface and let M, N be nonzero finitely generated R-modules. Assume $M \otimes_R N$ is reflexive and that N has rank. Then the following conditions hold:*

(i) $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$.
(ii) *M is reflexive, and N is torsion-free.*

Theorem B.2 was established by Huneke and Wiegand in [25, Theorem 2.7]: however their conclusion was that *both* M and N are reflexive, and the proof of this stronger claim is flawed. Dao realized the oversight of [25, Theorem 2.7], and Huneke and Wiegand addressed it in the erratum [26]. A similar flaw can be found in Miller's paper; see [34, Theorems 1.3 and 1.4] and compare it with our correction in Corollary B.3. The version stated above reflects our current understanding and is from the paper [9]. We do not yet know whether N is forced to be reflexive, that is, the question below remains open; cf. [25, Theorem 2.7] and [34, Theorem 1.3].

Question. Let R be a hypersurface and M, N nonzero finitely generated R-modules. If N has rank and $M \otimes_R N$ is reflexive, must *both* M and N be reflexive?

This question has been recently studied in [9], which gives partial answers using the New Intersection Theorem.

We now show how Theorem B.2 follows from Theorem A.4. In fact, one needs only the case $c = 1$ of Theorem A.4.

Proof of Theorem B.2 using Theorem A.4. Set $d = \dim R$. If $d = 0$, then N is free (since it has rank), so all is well. From now on assume $d \geq 1$. We remark at the outset that neither M nor N can be torsion, i.e., $\perp_R M \neq 0$ and $\perp_R N \neq 0$. Also, by the assumption of rank, $\operatorname{Supp}(N) = \operatorname{Spec}(R)$. Suppose first that both M and N are torsion-free; we will prove (i) by induction on $d = \dim R$. Let M_1 denote the pushforward of M; see (2.8). Then $\operatorname{Tor}_1^R(M_1, N)$ is torsion as N has rank. Since $M \otimes_R N$ is torsion-free, applying $- \otimes_R N$ to (2.8.1) shows that

$$\operatorname{Tor}_1^R(M_1, N) = 0\,. \tag{B.2.1}$$

Suppose for the moment that $d = 1$. Since N has rank, there is an exact sequence

$$0 \to N \to F \to C \to 0\,,$$

in which F is free and C is torsion. (See [25, Lemma 1.3].) Note that C is of finite length since $d = 1$. Note also that $\operatorname{Tor}_2^R(M_1, C) \cong \operatorname{Tor}_1^R(M_1, N) = 0$; see (B.2.1). Therefore [25, Corollary 2.3] implies that $\operatorname{Tor}_i^R(M_1, C) = 0$ for all $i \geq 2$, and hence $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. Now (2.8.1) establishes (i).

Still assuming that both M and N are torsion-free, let $d \geq 2$. The inductive hypothesis implies that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \geq 1$. In particular

$\operatorname{Tor}_i^R(M,N)_{\mathfrak{q}} = 0$ for all prime ideals $\mathfrak{q}$ of R of height at most one. Therefore Proposition 2.10 shows that $M_1 \otimes_R N$ is torsion-free, that is, $M_1 \otimes_R N$ satisfies (S_1); see (2.5) and (2.6). Furthermore, from the pushforward exact sequence (2.8.1), we see that $\operatorname{Tor}_i^R(M_1,N)$ has finite length for all $i \geq 2$. Consequently the pair M_1, N satisfies (SP_1). Now Theorem A.4, applied to M_1, N, shows that $\operatorname{Tor}_i^R(M_1,N) = 0$ for all $i \geq 1$. By (2.8.1), we see that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. This proves (i) under the additional assumption that M and N are torsion-free.

Since $M \otimes_R N$ is torsion-free, it follows from (3.4) that there are isomorphisms

$$M \otimes_R N \cong M \otimes_R \perp_R N \cong \perp_R M \otimes_R N \cong \perp_R M \otimes_R \perp_R N\,.$$

In particular, $\perp_R M \otimes_R \perp_R N$ is also reflexive. As noted before, neither M nor N is torsion so $\perp_R M$ and $\perp_R N$ are nonzero. As N has rank so does $\perp_R N$, so the already established part of the result (applied to $\perp_R M$ and $\perp_R N$) yields

$$\operatorname{Tor}_i^R(\perp_R M, \perp_R N) = 0 \quad \text{for } i \geq 1.$$

Given this, since $\perp_R M \otimes_R N$ is torsion-free by the isomorphisms above, applying (3.4) to the R-modules $\perp_R M$ and N gives $N = \perp_R N$; then applying (3.4) to M and N yields $M = \perp_R M$. In conclusion, M and N are torsion-free, and hence $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. From the last, the depth formula holds.

The remaining step is to prove that M is reflexive. Since $\operatorname{Supp}(N) = \operatorname{Spec}(R)$, we have $\operatorname{depth}(N_{\mathfrak{p}}) \leq \operatorname{height}(\mathfrak{p})$ for all primes $\mathfrak{p}$ of R. Localizing the depth formula (2.3) shows Serre's condition (S_2) on M; see (2.6). $\square$

The next result is due to C. Miller. In her paper [34], the essential requirement — that M have rank — is missing: for example, the module $M = R/(x)$ over the node $k[\![x,y]\!]/(xy)$ is not free, yet $M \otimes_R M$, which is just M, is maximal Cohen-Macaulay and hence reflexive. We state her result here in its corrected form and include a proof for completeness.

Corollary B.3. (C. Miller [34, Theorem 3.1]) *Let R be a d-dimensional hypersurface and let M a finitely generated R-module with rank. If $\otimes_R^n M$ is reflexive for some $n \geq \max\{2, d-1\}$, then M is free.*

Proof. If $d \leq 2$, then $\otimes_R^n M$ is maximal Cohen-Macaulay, and Theorem B.1 gives the result. Assume now that $d \geq 3$. Applying Theorem B.2 and [27, Theorem 1.9] repeatedly, we conclude the following:

(i) $\otimes_R^r M$ is reflexive for all $r = 1, \ldots, n$.
(ii) $\operatorname{Tor}_i^R(M, \otimes_R^{r-1} M) = 0$ for all $i \geq 1$ and all $r = 2, \ldots, n$.
(iii) $\operatorname{pd}(M) < \infty$.

It follows from (i) that $\operatorname{depth}(\otimes_R^r M) \geq 2$ for all $r = 1, \ldots, n$; see (2.6). Also, (ii) implies the depth formula:

$$\operatorname{depth}(M) + \operatorname{depth}(\otimes_R^{r-1} M) = d + \operatorname{depth}(\otimes_R^r M)\,,$$

for all $r = 2, \ldots, n$. One checks by induction on r that

$$r \cdot \operatorname{depth}(M) = (r-1) \cdot d + \operatorname{depth}(\otimes_R^r M)\,,$$

for $r = 2, \ldots, n$. Setting $r = n$, and using the inequalities $n \geq d-1$ and $\operatorname{depth}(\otimes_R^n M) \geq 2$, we obtain:

$$n \cdot \operatorname{depth}(M) \geq (n-1) \cdot d + 2 = n \cdot (d-1) + n - d + 2 \geq n \cdot (d-1) + 1.$$

Therefore depth$(M) \geq d$, that is, M is maximal Cohen-Macaulay. Now (iii) and the Auslander-Buchsbaum formula [3, Theorem 3.7] imply that M is free. $\square$

A consequence of Theorems B.1 and B.2 is the following result [27, Theorem 1.9], observed by Huneke and Wiegand in their 1997 paper:

Proposition B.4 ([27])**.** *Let M and N be finitely generated modules over a hypersurface R, and assume that* $\operatorname{Tor}_i^R(M,N) = 0$ *for $i \gg 0$. Then at least one of the modules has finite projective dimension.*

At about the same time Miller [34] obtained the same result independently, by an elegant, direct argument. As Miller observed in [34], one can turn things around and easily deduce Theorem B.1 from Proposition B.4 and the vanishing result Theorem B.2.

Acknowledgments

We would like to thank the referee for a careful reading of the paper.

References

[1] M. Auslander, Modules over unramified regular local rings, *Illinois J. Math.* **5** (1961), 631–647.

[2] L. L. Avramov and R.-O. Buchweitz, Support varieties and cohomology over complete intersections, *Invent. Math.*, **142**, (2000), 285–318.

[3] M. Auslander and C. Buchsbaum, Homological dimension in local rings. *Trans. Amer. Math. Soc.*, **85** (1957), 390–405.

[4] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.

[5] N. Bourbaki, *Commutative algebra*, Chapters 8 and 9, Elements of Mathematics, Springer-Verlag, Berlin, 2006.

[6] R.-O. Buchweitz and D. Van Straten, An index theorem for modules on a hypersurface singularity, *Mosc. Math. J.* **12** (2012), 237–259.

[7] O. Celikbas, Vanishing of Tor over complete intersections, *J. Comm. Alg.* **3** (2011), 169–206.

[8] O. Celikbas and H. Dao, Asymptotic behavior of Ext functors for modules of finite complete intersection dimension, *Math. Z.* **269** (2011), 1005–1020.

[9] O. Celikbas and G. Piepmeyer. Syzygies and tensor product of modules. *Math. Z.*, **276** (2014), 457–468.

[10] O. Celikbas and R. Wiegand, Vanishing of Tor, and why we care about it, to appear in *J. Pure Appl. Algebra*, **219** (2015), 429–448.

[11] L. W. Christensen and D. A. Jorgensen. Vanishing of Tate homology and depth formulas over local rings. *J. Pure Appl. Algebra*, **219** (2015), 464–481.

[12] H. Dao, Decent intersection and Tor-rigidity for modules over local hypersurfaces, to appear in *Trans. Amer. Math. Soc.*, **365** (2013), 2803–2821.

[13] H. Dao, Asymptotic behaviour of Tor over complete intersections and applications, preprint, arxiv:07105818.

[14] H. Dao, Some observations on local and projective hypersurfaces, *Math. Res. Lett.* **15** (2008), 207–219.

[15] H. Dao and K. Kurano, Hochster's theta pairing and numerical equivalence, *J. K-theory*, to appear; arxiv:1208.6083.

[16] H. Dao, Homological properties of modules over a complete intersection, in *Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of His 65th Birthday* (I. Peeva, Ed.), Springer Science & Business Media, New York (2013).

[17] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, *Trans. Amer. Math. Soc.* 260 (1980), 35-64.

[18] E. G. Evans and P. Griffith. *Syzygies*, volume 106 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1985.

[19] D. Eisenbud, *Commutative Algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.
[20] H. B. Foxby. Homological dimensions of complexes of modules. In *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979)*, volume 795 of *Lecture Notes in Math.*, pages 360–368. Springer, Berlin, 1980.
[21] R. C. Heitmann, Characterization of completions of unique factorization domains, *Trans. Amer. Math. Soc.* **337** (1993), 379–387.
[22] M. Hochster, The dimension of an intersection in an ambient hypersurface, In *Algebraic geometry (Chicago, Ill., 1980)*, Lecture Notes in Math. **862**, 93–106. Springer, Berlin, 1981.
[23] C. Huneke and D. A. Jorgensen. Symmetry in the vanishing of Ext over Gorenstein rings. *Math. Scand.*, 93(2):161–184, 2003.
[24] C. Huneke, D. A. Jorgensen, and R. Wiegand, Vanishing theorems for complete intersections, *J. Algebra*, **238** (2001), 684–702.
[25] C. Huneke and R. Wiegand, Tensor products of modules and the rigidity of Tor, *Math. Ann.* **299** (1994), 449–476; `Correction:` *Math. Ann.* **338** (2007), 291–293.
[26] C. Huneke and R. Wiegand, Correction to: "Tensor products of modules and the rigidity of Tor" [Math. Ann. **299** (1994), 449–476], *Math. Ann.*, (2007), 291–293.
[27] C. Huneke and R. Wiegand, Tensor products of modules, rigidity and local cohomology, *Math. Scand.* **81** (1997), 161–183.
[28] S. Iyengar, Depth for complexes, and intersection theorems, *Math. Z.* **230** (1999), 545–567.
[29] D. A. Jorgensen, Complexity and Tor on a complete intersection, *J. Algebra* **211** (1999), 578–598.
[30] D. A. Jorgensen, Tor and torsion on a complete intersection, *J. Algebra* **195** (1997), 526–537.
[31] G. Leuschke and R. Wiegand, *Cohen-Macaulay Representations*, Mathematical Surveys and Monographs **181**, Amer. Math. Soc., Providence, RI 2012.
[32] S. Lichtenbaum, On the vanishing of Tor in regular local rings, *Illinois J. Math.* **10** (1966), 220–226.
[33] H. Matsumura, *Commutative ring theory*, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989, xiv+320 pp.
[34] C. Miller, Complexity of tensor products of modules and a theorem of Huneke-Wiegand, Proc. Amer. Math. Soc. **126** (1998), 53–60.
[35] W. F. Moore, G. Piepmeyer, S. Spiroff, and M.E. Walker, Hochster's theta invariant and the Hodge-Riemann bilinear relations, *Adv. Math.* **226** (2011), 1692–1714.
[36] W. F. Moore, G. Piepmeyer, S. Spiroff, and M.E. Walker, The vanishing of a higher codimension analog of Hochster's theta invariant, *Math. Z.*, **273** (2013), 907–920.
[37] M. P. Murthy, Modules over regular local rings, *Illinois J. Math.*, **7** (1963), 558–565.
[38] N. Epstein and Y. Yongwei Criteria for flatness and injectivity, *Math. Z.* **271** (2012), 1193–1210.
[39] A. Polishchuk and A. Vaintrob, Chern characters and Hirzebruch-Riemann-Roch formula for matrix factorizations, *Duke Math. J.* **161** (2012), 1863–1926.
[40] W. V. Vasconcelos, Reflexive modules over Gorenstein rings, *Proc. Amer. Math. Soc.*, **19** (1968), 1349–1355.
[41] M. E. Walker, Chern Characters for Twisted Matrix Factorizations and the Vanishing of the Higher Herbrand Difference, preprint, arXiv:1404.0352.
[42] Serre, Jean-Pierre, Local algebra, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin. 2000

Olgur Celikbas, University of Connecticut, Storrs, CT 06269
E-mail address: `olgur.celikbas@uconn.edu`

Srikanth B. Iyengar, University of Utah, Salt Lake City, UT 84112, U.S.A
E-mail address: `iyengar@math.utah.edu`

Greg Piepmeyer, Columbia Basin College, Pasco, WA 99301, U.S.A
E-mail address: `gpiepmeyer@columbiabasin.edu`

Roger Wiegand, University of Nebraska–Lincoln, Lincoln, NE 68588, U.S.A
E-mail address: `rwiegand1@math.unl.edu`