STRATIFYING MODULAR REPRESENTATIONS OF FINITE GROUPS

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ABSTRACT. We classify localising subcategories of the stable module category of a finite group that are closed under tensor product with simple (or, equivalently all) modules. One application is a proof of the telescope conjecture in this context. Others include new proofs of the tensor product theorem and of the classification of thick subcategories of the finitely generated modules which avoid the use of cyclic shifted subgroups. Along the way we establish similar classifications for differential graded modules over graded polynomial rings, and over graded exterior algebras.

CONTENTS

1. INTRODUCTION

Let G be a finite group and k a field of characteristic p, where p divides the order of G. Let $\mathsf{Mod}(kG)$ be the category of possibly infinite dimensional modules over the group algebra kG. In this article, a full subcategory C of $\mathsf{Mod}(kG)$ is said to be *thick* if it satisfies the following conditions.

- Any direct summand of a module in C is also in C.

- If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of kG-modules, and two of M_1, M_2, M_3 are in C then so is the third.

A thick subcategory C is *localising* when, in addition, the following property holds:

- If $\{M_\alpha\}$ is a set of modules in C then $\bigoplus_\alpha M_\alpha$ is in C.

By a version of the Eilenberg swindle the first condition follows from the others.

There is a notion of *support* in this context, introduced by Benson, Carlson and Rickard [4]. It associates to each kG-module M a subset $\mathcal{V}_G(M)$ of the set $\operatorname{Proj} H^*(G, k)$ of homogeneous prime ideals in the cohomology ring $H^*(G, k)$ other than the maximal ideal of positive degree elements.

Our main result is that $\operatorname{Proj} H^*(G, k)$ stratifies $\mathsf{Mod}(kG)$, in the following sense.

Theorem 1.1. *There is a natural one to one correspondence between non-zero localising subcategories of $\mathsf{Mod}(kG)$ that are closed under tensoring with simple kG-modules and subsets of $\operatorname{Proj} H^*(G, k)$.*

The localising subcategory corresponding to a subset $\mathcal{V} \subseteq \operatorname{Proj} H^*(G, k)$ is the full subcategory of modules M satisfying $\mathcal{V}_G(M) \subseteq \mathcal{V}$.

A more precise version of this result is given in Theorem 10.4. It is modelled on Neeman's classification [25] of the localising subcategories of the unbounded derived category $\mathsf{D}(\mathsf{Mod}\, R)$ of complexes of modules over a noetherian commutative ring R. Neeman's work in turn was inspired by Hopkins' classification [20] of the thick subcategories of the perfect complexes over R in terms of specialisation closed subsets of $\operatorname{Spec} R$. The corresponding classification problem for the stable category $\mathsf{stmod}(kG)$ of finitely generated kG-modules was solved by Benson, Carlson and Rickard [5], at least in the case where k is algebraically closed. There is also a discussion in that paper as to why one demands that the subcategories are closed under tensor products with simple modules. For a p-group this condition is automatically satisfied, but for an arbitrary finite group there is a subvariety of $\operatorname{Proj} H^*(G, k)$ called the *nucleus*, that encapsulates the obstruction to a classification of all localising subcategories, at least for the principal block.

Applications of Theorem 1.1 include a classification of smashing localisations of the stable category $\mathsf{StMod}(kG)$ of all kG-modules, a proof of the telescope conjecture in this setting, a classification of localising subcategories that are closed under products and duality, and a description of the left perpendicular category of a localising subcategory.

We also provide new proofs of the subgroup theorem, the tensor product theorem and the classification of thick subcategories of $\mathsf{stmod}(kG)$. The proofs of these results given in [4, 5] rely heavily on the use of cyclic shifted subgroups and an infinite dimensional version of Dade's lemma, which play no role in our work.

As intermediate steps in the proof of the classification theorems for modules over kG, we establish analogous results on differential graded modules over polynomial rings, and over exterior algebras. These are of independent interest.

This paper is the third in a sequence devoted to supports and localising subcategories of triangulated categories. We have tried to make this paper as easy

as possible to read without having to go through the first two papers [6, 7] in the series. In particular, some of the arguments in them have been repeated in this more restricted context for convenience.

2. Strategy

The proof of Theorem 1.1 consists of a long chain of transitions from one category to another. In this section, we give an outline of the strategy.

The first step is to reduce to the stable module category $\mathsf{StMod}(kG)$. This category has the same objects as the module category $\mathsf{Mod}(kG)$, but the morphisms in $\mathsf{StMod}(kG)$ are given by quotienting out those morphisms in $\mathsf{Mod}(kG)$ that factor through a projective module. The category $\mathsf{StMod}(kG)$ is a triangulated category, in which the triangles come from the short exact sequences of kG-modules. See for example Theorem I.2.6 of Happel [19] or §5 of Carlson [11] for further details.

A *thick subcategory* of a triangulated category is a full triangulated subcategory that is closed under direct summands. A *localising subcategory* of a triangulated category is a full triangulated subcategory that is closed under direct sums. By an Eilenberg swindle, localising subcategories are also closed under direct summands, and hence thick.

Recall that, given kG-modules M and N, one considers $M \otimes_k N$ as a kG-module with the diagonal G-action. The kG-modules $M \otimes_k N$ and $N \otimes_k M$ are isomorphic, and $P \otimes_k N$ is projective for any projective kG-module P. Thus the tensor product on $\mathsf{Mod}(kG)$ passes down to a tensor product on $\mathsf{StMod}(kG)$. In either context, we say that a thick, or localising, subcategory C is *tensor ideal* if it satisfies the following condition.

- If M is in C and S is a simple kG-module then $M \otimes_k S$ is in C.

This condition is vacuous if G is a finite p-group since the only simple module is k with trivial G-action. Furthermore, every kG-module has a finite filtration whose subquotients are direct sums of simple modules (induced by the radical filtration of kG), so the condition above is equivalent to:

- If M is in C and N is any kG-module then $M \otimes_k N$ is in C.

Proposition 2.1. *Every non-zero tensor ideal localising subcategory of* $\mathsf{Mod}(kG)$ *contains the localising subcategory of projective modules.*

The canonical functor from $\mathsf{Mod}(kG)$ *to* $\mathsf{StMod}(kG)$ *induces a one to one correspondence between non-zero tensor ideal localising subcategories of* $\mathsf{Mod}(kG)$ *and tensor ideal localising subcategories of* $\mathsf{StMod}(kG)$.

Proof. The tensor product of any module with kG is a direct sum of copies of kG. So if C is a non-zero tensor ideal localising subcategory of $\mathsf{Mod}(kG)$, it contains kG, and hence every projective kG-module. This proves the first statement of the proposition. The rest is now clear. $\square$

There are some technical disadvantages to working in $\mathsf{StMod}(kG)$ that are solved by moving to a slightly larger triangulated category: $\mathsf{K}(\mathsf{Inj}\, kG)$, the category whose objects are the complexes of injective kG-modules and whose morphisms are the homotopy classes of degree preserving maps of complexes. The tensor product of modules extends to complexes, and defines a tensor product on $\mathsf{K}(\mathsf{Inj}\, kG)$. This category was investigated in detail by Benson and Krause [8]. Taking the Tate resolution $tM = M \otimes_k tk$ of a kG-module M gives an equivalence of triangulated categories from the stable module category $\mathsf{StMod}(kG)$ to the full subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\, kG)$ of $\mathsf{K}(\mathsf{Inj}\, kG)$ consisting of acyclic complexes. This equivalence preserves the tensor product. It suffices to check that $tk \otimes_k tk \cong tk$, and an explicit isomorphism can be found in Section XII.4 of Cartan and Eilenberg [12].

The Verdier quotient of $\mathsf{K}(\mathsf{Inj}\, kG)$ by $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\, kG)$ is the unbounded derived category $\mathsf{D}(\mathsf{Mod}\, kG)$. There are left and right adjoints, forming a recollement

$$\mathsf{StMod}(kG) \xrightarrow{\sim} \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\, kG) \xrightarrow[\underset{-\otimes_k tk}{\xleftarrow{\hspace{1.5cm}}}]{\overset{\mathrm{Hom}_k(tk,-)}{\xleftarrow{\hspace{1.5cm}}}} \mathsf{K}(\mathsf{Inj}\, kG) \xrightarrow[\underset{-\otimes_k pk}{\xleftarrow{\hspace{1.5cm}}}]{\overset{\mathrm{Hom}_k(pk,-)}{\xleftarrow{\hspace{1.5cm}}}} \mathsf{D}(\mathsf{Mod}\, kG)$$

where pk denotes a projective resolution of k.

It is shown in [8] that the theory of supports for $\mathsf{StMod}(kG)$ developed in [4] extends in a natural way to $\mathsf{K}(\mathsf{Inj}\, kG)$. Exactly one more prime ideal appears in the theory, namely the maximal ideal $\mathfrak{m}$ of positive degree elements in $H^*(G,k)$. We write $\mathrm{Spec}\, H^*(G,k)$ for the set $\mathrm{Proj}\, H^*(G,k) \cup \{\mathfrak{m}\}$ of all homogeneous prime ideals in $H^*(G,k)$. If X is an object in $\mathsf{K}(\mathsf{Inj}\, kG)$ then there is associated to it a *support* $\mathcal{V}_G(X)$, which is a subset of $\mathrm{Spec}\, H^*(G,k)$; see §3.

Proposition 2.2. *For every tensor ideal localising subcategory C of $\mathsf{StMod}(kG)$ there are two tensor ideal localising subcategories of $\mathsf{K}(\mathsf{Inj}\, kG)$. One is the image of C and the other is generated by this together with pk. This sets up a two to one correspondence between tensor ideal localising subcategories of $\mathsf{K}(\mathsf{Inj}\, kG)$ and those of $\mathsf{StMod}(kG)$.*

Proof. First we claim that for any non-zero object X in $\mathsf{D}(\mathsf{Mod}\, kG)$, the tensor ideal localising subcategory generated by X is the whole of $\mathsf{D}(\mathsf{Mod}\, kG)$.

Indeed, for any non-zero kG-module M the kG-module $M \otimes_k kG$ is free and non-zero. Therefore, $H^*(X \otimes_k kG)$ is a non-zero direct sum of copies of kG, since it is isomorphic to $H^*(X) \otimes_k kG$. This implies that in $\mathsf{D}(\mathsf{Mod}\, kG)$ the complex $X \otimes_k kG$ is isomorphic to a non-zero direct sum of copies of shifts of kG. Hence kG is in the tensor ideal localising subcategory generated by X. It remains to note that every object in $\mathsf{D}(\mathsf{Mod}\, kG)$ is in the localising subcategory generated by kG.

Next observe that the full subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\, kG)$ of $\mathsf{K}(\mathsf{Inj}\, kG)$ consisting of acyclic complexes is a tensor ideal localising subcategory. Thus the image in

$\mathsf{K}(\mathsf{Inj}\,kG)$ of every tensor ideal localising subcategory of $\mathsf{StMod}(kG)$ is a tensor ideal localising subcategory.

Now let D be a tensor ideal localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ that is not in the image of $\mathsf{StMod}(kG)$. Then D contains some object X which is not acyclic. The image of X in $\mathsf{D}(\mathsf{Mod}\,kG)$ is non-zero, and hence generates $\mathsf{D}(\mathsf{Mod}\,kG)$. Thus the tensor ideal containing X also contains pk. It follows that D is generated by pk and its intersection with the image of $\mathsf{StMod}(kG)$. $\qquad\square$

A version of the main theorem for $\mathsf{K}(\mathsf{Inj}\,kG)$ is as follows; see also Theorem 10.1.

Theorem 2.3. *There is a natural one to one correspondence between tensor ideal localising subcategories* C *of* $\mathsf{K}(\mathsf{Inj}\,kG)$ *and subsets of* $\operatorname{Spec} H^*(G,k)$.

The localising subcategory corresponding to a subset $\mathcal{V}$ *of* $\operatorname{Spec} H^*(G,k)$ *is the full subcategory of complexes* X *satisfying* $\mathcal{V}_G(X) \subseteq \mathcal{V}$.

The proof of Theorems 1.1 and 2.3 occupies §§5–10. An outline of the proof is as follows. Using an appropriate version of the Quillen stratification theorem, it suffices to work with an elementary abelian p-group

$$E = \langle g_1, \ldots, g_r \rangle.$$

Since k is of characteristic p the group algebra kE is isomorphic to the algebra

$$k[z_1, \ldots, z_r]/(z_1^p, \ldots, z_r^p),$$

where the (image of the element) z_i corresponds to $g_i - 1$.

We write A for the Koszul complex on kE with respect to $z_1, \ldots, z_r$, and view it as a differential graded (dg) algebra. Thus, as a graded algebra A is the exterior algebra over kE on indeterminates $y_1, \ldots, y_r$, with each y_i of degree -1, and the differential on A is determined by setting

$$d(z_i) = 0 \qquad \text{and} \qquad d(y_i) = z_i.$$

Observe that the elements $z_1^{p-1}y_1, \ldots, z_r^{p-1}y_r$ are cycles in A of degree -1. Let Λ be an exterior algebra over k on r generators ξ_i in degree -1, and view it as a dg algebra with zero differential. The map $\Lambda \to A$ given by

$$\xi_i \mapsto z_i^{p-1}y_i$$

is a morphism of dg algebras and a quasi-isomorphism; see §7.

Let S be a graded polynomial ring $k[x_1, \ldots, x_r]$ where each variable x_i is of degree 2, and view it as a dg algebra with zero differential. One has an isomorphism of graded k-algebras $S \cong \operatorname{Ext}_\Lambda^*(k, k)$; see §6.

Let $\mathsf{K}(\mathsf{Inj}\,A)$ and $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ denote the homotopy categories of graded-injective dg modules over A and Λ respectively; see §4. Our strategy is to establish first a classification of the localising subcategories of $\mathsf{D}(S)$, the derived category of dg S-modules, and then successively for $\mathsf{K}(\mathsf{Inj}\,\Lambda)$, $\mathsf{K}(\mathsf{Inj}\,A)$, $\mathsf{K}(\mathsf{Inj}\,kE)$, $\mathsf{K}(\mathsf{Inj}\,kG)$, and finally for $\mathsf{StMod}(kG)$ and $\mathsf{Mod}(kG)$. The following diagram provides an overview

of the proof of Theorem 1.1.

$$\mathsf{D}(\overset{\S5}{S}) \overset{\S6}{\rightsquigarrow} \mathsf{K}(\mathsf{Inj}\,\Lambda) \overset{\S7}{\rightsquigarrow} \mathsf{K}(\mathsf{Inj}\,A) \overset{\S8}{\rightsquigarrow} \mathsf{K}(\mathsf{Inj}\,kE) \overset{\S9}{\rightsquigarrow} \mathsf{K}(\mathsf{Inj}\,kG) \overset{\S10}{\rightsquigarrow} \mathsf{Mod}(kG).$$

<div align="center">Leitfaden</div>

The passage from S to kE is modelled on the work of Avramov, Buchweitz, Iyengar, and Miller [1], where it is used to establish results on numerical invariants of complexes over commutative local rings by tracking them along a chain of categories as above. The focus here is on tracking structural information. A crucial idea in executing this passage from S to kG is that of a *stratification* of a tensor triangulated category, which allows one to focus on minimal localising subcategories. In §3 we describe the general theory of stratifications, and show that when a tensor triangulated category is stratified, one can classify its tensor ideal localising subcategories. This development is partly inspired by the work of Hovey, Palmieri, and Strickland [21].

In §11 we describe various applications, including a classification of smashing localisations of $\mathsf{StMod}(kG)$, and new proofs of the subgroup theorem, the tensor product theorem, and the classification of thick subcategories of $\mathsf{stmod}(kG)$.

3. Stratifications

In this section, we recall from [6, 7] those parts of the general theory of support varieties and stratifications that we wish to use in this paper.

Let T be a triangulated category admitting arbitrary coproducts. Usually, we denote Σ the shift on T. Given a subcategory C of T we write $\mathsf{Loc}(\mathsf{C})$ for the localising subcategory generated by C; this is the smallest localising subcategory of T containing C. The thick subcategory generated by C is denoted $\mathsf{Thick}(\mathsf{C})$. An object C of T is a *generator* if $\mathsf{Loc}(C) = \mathsf{T}$. The standing assumption in this article is that T is generated by a single compact object. Recall that an object C is *compact* if the functor $\mathrm{Hom}_{\mathsf{T}}(C, -)$ commutes with coproducts.

For objects X, Y in T we write $\mathrm{Hom}^*_{\mathsf{T}}(X, Y)$ for the graded abelian group with degree n component $\mathrm{Hom}_{\mathsf{T}}(X, \Sigma^n Y)$, and $\mathrm{End}^*_{\mathsf{T}}(X)$ for the graded ring $\mathrm{Hom}^*_{\mathsf{T}}(X, X)$.

Let R be a graded commutative noetherian ring; thus R is a $\mathbb{Z}$-graded noetherian ring such that $rs = (-1)^{|r||s|} sr$ for all homogeneous elements r, s in R. We say that the triangulated category T is R-*linear*, or that R *acts on* T, if we are given a homomorphism of graded rings $\phi \colon R \to Z^*\mathsf{T}$, to the graded centre of T. This means that for each object X there is a homomorphism of graded rings

$$\phi_X \colon R \to \mathrm{End}^*_{\mathsf{T}}(X)$$

such that for each pair of objects X, Y the induced left and right actions of R on $\mathrm{Hom}^*_{\mathsf{T}}(X, Y)$ agree up to the usual sign; see [6, §4].

Let T be an R-linear triangulated category.

We write $\mathrm{Spec}\, R$ for the set of homogeneous prime ideals in R. For any ideal $\mathfrak{a}$ in R the subset $\{\mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p} \supseteq \mathfrak{a}\}$ is denoted $\mathcal{V}(\mathfrak{a})$. A subset $\mathcal{V} \subseteq \mathrm{Spec}\, R$ is

specialisation closed if $\mathfrak{p} \in \mathcal{V}$ and $\mathfrak{q} \supseteq \mathfrak{p}$ imply $\mathfrak{q} \in \mathcal{V}$. Given such a subset $\mathcal{V}$, pick a compact generator C of T, and let $\mathsf{T}_\mathcal{V}$ be the full subcategory

$$\mathsf{T}_\mathcal{V} = \{X \in \mathsf{T} \mid \operatorname{Hom}^*_\mathsf{T}(C, X)_\mathfrak{p} = 0 \ \forall \, \mathfrak{p} \in \operatorname{Spec} R \setminus \mathcal{V}\}.$$

This is a localising subcategory of T. The following result is proved in [6, §4].

Proposition 3.1. *The localising subcategory* $\mathsf{T}_\mathcal{V}$ *depends only on* $\mathcal{V}$ *and not on the choice of compact generator* C. *Furthermore, there is a localisation functor* $L_\mathcal{V} \colon \mathsf{T} \to \mathsf{T}$ *such that* $L_\mathcal{V} X = 0$ *if and only if* $X \in \mathsf{T}_\mathcal{V}$. $\qquad \square$

This result and the theory of Bousfield localisation imply that there is an exact functor $\Gamma_\mathcal{V} \colon \mathsf{T} \to \mathsf{T}$ and for each object X in T an exact *localisation triangle*

$$(3.2) \qquad\qquad \Gamma_\mathcal{V} X \to X \to L_\mathcal{V} X \to .$$

Proposition 3.1 has the following useful consequence.

Corollary 3.3. *Let* $\phi, \phi' \colon R \to Z^*\mathsf{T}$ *be actions of* R *on* T. *If there exists a compact generator* C *for* T *for which the maps* $\phi_C, \phi'_C \colon R \to \operatorname{End}^*_\mathsf{T}(C)$ *agree, then for any specialisation closed set* $\mathcal{V} \subseteq \operatorname{Spec} R$ *the functors* $\Gamma_\mathcal{V}$ *and* $L_\mathcal{V}$ *defined through* ϕ *agree with those defined through* ϕ'.

Proof. The definitions of $\Gamma_\mathcal{V}$ and $L_\mathcal{V}$ only depend on the action of R on $\operatorname{Hom}^*_\mathsf{T}(C, X)$, since C is a compact generator. The action factors through the map $R \to \operatorname{End}^*_\mathsf{T}(C)$, which justifies the claim. $\qquad \square$

In this work the principal source of an R-action on T is a tensor structure on it.

Tensor triangulated categories.

Let $(\mathsf{T}, \otimes, \mathbb{1})$ be a *tensor triangulated category*. By this we mean the following structure; see [6, §8] for details.

- T is a compactly generated triangulated category with coproducts.
- T is symmetric monoidal, with *tensor product* $\otimes \colon \mathsf{T} \times \mathsf{T} \to \mathsf{T}$ and *unit* $\mathbb{1}$.
- The tensor product is exact in each variable and preserves coproducts.
- The unit $\mathbb{1}$ is compact.
- Compact objects are strongly dualisable.

In this article we make also the following assumption, for simplicity of exposition.

- T has a single compact generator.

We write $\operatorname{Loc}^\otimes(\mathsf{C})$ for the tensor ideal localising subcategory generated by a subcategory C of T. Observe that $\operatorname{Loc}(\mathsf{C}) \subseteq \operatorname{Loc}^\otimes(\mathsf{C})$ and that the two categories coincide when the unit $\mathbb{1}$ is a generator for T; this is the case in many of our contexts.

The stable category $\operatorname{StMod}(kG)$ of a finite group G is tensor triangulated, where the tensor product is $M \otimes_k N$ with diagonal G-action and the unit is k. The homotopy category $\mathsf{K}(\operatorname{Inj} kG)$ is also tensor triangulated with the same tensor product, but here the unit is ik, the injective resolution of k. In either case, the unit generates the category when G is a p-group; see [8] for details.

The ring $\mathrm{End}^*_{\mathsf{T}}(\mathbb{1})$ is graded commutative and acts on T via homomorphisms

$$\mathrm{End}^*_{\mathsf{T}}(\mathbb{1}) \xrightarrow{X \otimes -} \mathrm{End}^*_{\mathsf{T}}(X),$$

for each X in T. Thus any homomorphism of graded rings $R \to \mathrm{End}^*_{\mathsf{T}}(\mathbb{1})$ induces an R-action on T, and we call such an R-action *canonical*.

In the remainder of this section T denotes a tensor triangulated category and R a graded commutative noetherian ring that acts canonically on T.

For each specialisation closed subset $\mathcal{V}$ of $\mathrm{Spec}\,R$ there are natural isomorphisms

$$\Gamma_{\mathcal{V}} X \cong \Gamma_{\mathcal{V}} \mathbb{1} \otimes X \quad \text{and} \quad L_{\mathcal{V}} X \cong L_{\mathcal{V}} \mathbb{1} \otimes X.$$

Thus, the localisation triangle (3.2) is obtained by applying $- \otimes X$ to the triangle

$$\Gamma_{\mathcal{V}} \mathbb{1} \to \mathbb{1} \to L_{\mathcal{V}} \mathbb{1} \to .$$

These are analogues of Rickard's triangles [32] in $\mathsf{StMod}(kG)$; see [6, §§4,10].

Next we recall the construction of Koszul objects from [6, Definition 5.10]. Given a homogeneous element r of R and an object X in T, we write $X/\!/r$ for any object that appears in an exact triangle

$$X \xrightarrow{r} \Sigma^{|r|} X \to X/\!/r \to .$$

Iterating this construction on a sequence of elements $\boldsymbol{r} = r_1, \ldots, r_n$ one gets an object in T that is denoted $X/\!/\boldsymbol{r}$. Given an ideal $\mathfrak{a}$ in R, we write $X/\!/\mathfrak{a}$ for any Koszul object on a finite sequence of elements generating $\mathfrak{a}$.

While a Koszul object $X/\!/\mathfrak{a}$ depends on a choice of a generating sequence for $\mathfrak{a}$, the thick subcategory it generates does not. This follows from the second part of the following statement, where $\sqrt{\mathfrak{a}}$ denotes the radical of $\mathfrak{a}$ in R.

Lemma 3.4. *Let $\mathfrak{a}$ be an ideal in R and X an object in T.*

(1) $X/\!/\mathfrak{a}$ *is in* $\mathsf{Thick}(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$.
(2) $X/\!/\mathfrak{a}$ *is in* $\mathsf{Thick}(X/\!/\mathfrak{b})$ *for any ideal $\mathfrak{b}$ with $\sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$.*

Proof. By construction it is clear that $X/\!/\mathfrak{a}$ is in $\mathsf{Thick}(X)$. Since the functor $\Gamma_{\mathcal{V}(\mathfrak{a})}$ is exact, this implies that $\Gamma_{\mathcal{V}(\mathfrak{a})}(X/\!/\mathfrak{a})$ is in $\mathsf{Thick}(\Gamma_{\mathcal{V}(\mathfrak{a})} X)$. It remains to note that $\Gamma_{\mathcal{V}(\mathfrak{a})}(X/\!/\mathfrak{a}) \cong X/\!/\mathfrak{a}$, for $X/\!/\mathfrak{a}$ is in $\mathsf{T}_{\mathcal{V}(\mathfrak{a})}$ by [6, Corollary 5.11]. This proves (1).

The claim in (2) can be proved along the same lines as [21, Lemma 6.0.9]. $\square$

The second part of the next result improves upon [6, Theorem 6.4].

Proposition 3.5. *Let $\mathcal{V}$ be a specialisation closed subset of $\mathrm{Spec}\,R$.*

(1) *If $\mathcal{W} \supseteq \mathcal{V}$ is specialisation closed, then $\Gamma_{\mathcal{V}} X$ is in $\mathsf{Loc}^{\otimes}(\Gamma_{\mathcal{W}} X)$.*
(2) *Let C be a compact generator of T. For any decomposition $\mathcal{V} = \bigcup_{i \in I} \mathcal{V}(\mathfrak{a}_i)$, where each $\mathfrak{a}_i$ is an ideal in R, there are equalities*

$$\mathsf{T}_{\mathcal{V}} = \mathsf{Loc}(\{C/\!/\mathfrak{a}_i \mid i \in I\}) = \mathsf{Loc}(\{\Gamma_{\mathcal{V}(\mathfrak{a}_i)} C \mid i \in I\}).$$

Proof. (1) From [6, Proposition 6.1] one gets the first isomorphism below:

$$\Gamma_{\mathcal{V}} X \cong \Gamma_{\mathcal{V}}(\Gamma_{\mathcal{W}} X) \cong \Gamma_{\mathcal{V}} \mathbb{1} \otimes \Gamma_{\mathcal{W}} X.$$

Thus $\Gamma_\mathcal{V}X$ is in $\mathsf{Loc}^\otimes(\Gamma_\mathcal{W}X)$.

(2) For each $\mathfrak{p} \in \mathcal{V}$ there exists an i in I such that $\mathfrak{p} \in \mathcal{V}(\mathfrak{a}_i)$ holds, so that $C/\!\!/\mathfrak{p}$ is in $\mathsf{Thick}(C/\!\!/\mathfrak{a}_i)$, by Lemma 3.4(2). This fact and [6, Theorem 6.4] imply the last of the following inclusions:

$$\mathsf{Loc}(\{C/\!\!/\mathfrak{a}_i \mid i \in I\}) \subseteq \mathsf{Loc}(\{\Gamma_{\mathcal{V}(\mathfrak{a}_i)}C \mid i \in I\}) \subseteq \mathsf{T}_\mathcal{V} \subseteq \mathsf{Loc}(\{C/\!\!/\mathfrak{a}_i \mid i \in I\}).$$

The first inclusion holds by Lemma 3.4(1). The second one holds as $\Gamma_{\mathcal{V}(\mathfrak{a}_i)}C$ is in $\mathsf{T}_\mathcal{V}$ for each i, since $\mathsf{T}_{\mathcal{V}(\mathfrak{a}_i)} \subseteq \mathsf{T}_\mathcal{V}$. The inclusions above yield the desired result. $\square$

Support.

Fix a point $\mathfrak{p} \in \operatorname{Spec} R$ and let $\mathcal{V}$ and $\mathcal{W}$ be specialisation closed sets of primes such that $\mathcal{V} \setminus \mathcal{W} = \{\mathfrak{p}\}$. It is shown in [6, Theorem 6.2] that the functor $\Gamma_\mathfrak{p} = \Gamma_\mathcal{V}L_\mathcal{W}$ is independent of choice of $\mathcal{V}$ and $\mathcal{W}$ with these properties, and coincides with $L_\mathcal{W}\Gamma_\mathcal{V}$. There are natural isomorphisms

$$\Gamma_\mathfrak{p}X \cong \Gamma_\mathfrak{p}\mathbb{1} \otimes X.$$

Following [6, §5], we define the *support* of an object X in T to be

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_\mathfrak{p}X \neq 0\}.$$

If the R-module $\operatorname{Hom}_\mathsf{T}^*(C, X)$ is finitely generated for some compact generator C for T and $\mathfrak{a}$ is its annihilator ideal, then $\operatorname{supp}_R X$ equals the Zariski-closed subset $\mathcal{V}(\mathfrak{a})$; see [6, Theorem 5.5(1)].

The notion of support defined above is an analogue of the one for modular representations of a finite group G given in Benson, Carlson and Rickard [4]. Indeed, if $\mathsf{T} = \mathsf{StMod}(kG)$ and $\mathfrak{p}$ is a non-maximal prime in $H^*(G, k)$, then $\Gamma_\mathfrak{p}\mathbb{1}$ is the kappa module for $\mathfrak{p}$ defined in [4]; see [6, §10].

The following theorem is the "local-global principle", studied in [7] for general triangulated categories.

Theorem 3.6. *Let X be an object in T. Then*

$$X \in \mathsf{Loc}^\otimes(\{\Gamma_\mathfrak{p}X \mid \mathfrak{p} \in \operatorname{supp}_R X\}).$$

Proof. It suffices to prove that $\mathbb{1}$ is in the subcategory

$$\mathsf{C} = \mathsf{Loc}^\otimes\{\Gamma_\mathfrak{p}\mathbb{1} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Indeed, one then gets the desired result by tensoring with X, keeping in mind that if $\mathfrak{p} \notin \operatorname{supp}_R X$ then $\Gamma_\mathfrak{p}\mathbb{1} \otimes X = \Gamma_\mathfrak{p}X = 0$.

Proposition 3.5(1) implies that the set

$$\mathcal{V} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathcal{V}(\mathfrak{p})}\mathbb{1} \in \mathsf{C}\}.$$

is specialisation closed. We claim that $\mathcal{V} = \operatorname{Spec} R$. If not, then since the ring R is noetherian there exists a prime $\mathfrak{q}$ maximal in $\operatorname{Spec} R \setminus \mathcal{V}$. Setting $\mathcal{Z} = \mathcal{V}(\mathfrak{q}) \setminus \{\mathfrak{q}\}$ one gets an exact triangle

$$\Gamma_\mathcal{Z}\mathbb{1} \to \Gamma_{\mathcal{V}(\mathfrak{q})}\mathbb{1} \to \Gamma_\mathfrak{q}\mathbb{1} \to .$$

Since $\mathcal{Z} \subseteq \mathcal{V}$, it follows from Proposition 3.5(2) that

$$\Gamma_{\mathcal{Z}}\mathbb{1} \in \mathsf{T}_{\mathcal{Z}} \subseteq \mathsf{T}_{\mathcal{V}} = \mathsf{Loc}\{\Gamma_{\mathcal{V}(\mathfrak{p})}C \mid \mathfrak{p} \in \mathcal{V}\} \subseteq \mathsf{Loc}^{\otimes}\{\Gamma_{\mathcal{V}(\mathfrak{p})}\mathbb{1} \mid \mathfrak{p} \in \mathcal{V}\} \subseteq \mathsf{C},$$

where C denotes any compact generator of T. By definition, $\Gamma_{\mathfrak{q}}\mathbb{1}$ is in C so it follows from the triangle above that $\Gamma_{\mathcal{V}(\mathfrak{q})}\mathbb{1}$ is also in it. This contradicts the choice of $\mathfrak{q}$. Therefore $\mathcal{V} = \operatorname{Spec} R$ holds. It remains to note that

$$\mathbb{1} = \Gamma_{\operatorname{Spec} R}\mathbb{1} \in \mathsf{C}$$

where the equality holds because $\Gamma_{\operatorname{Spec} R}$ is the identity on T, since $\mathsf{T}_{\operatorname{Spec} R} = \mathsf{T}$. $\square$

Stratification.

Let T be an R-linear tensor triangulated category. For each $\mathfrak{p}$ in $\operatorname{Spec} R$, the full subcategory $\Gamma_{\mathfrak{p}}\mathsf{T}$ is tensor ideal and localising. It consists of objects X in T with the property that the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion, where C is some compact generator for T; see [6, Corollary 4.10].

We say that T is *stratified by* R if $\Gamma_{\mathfrak{p}}\mathsf{T}$ is either zero or minimal among tensor ideal localising subcategories for each $\mathfrak{p}$ in $\operatorname{Spec} R$.

Given a localising subcategory C of T and a subset $\mathcal{V}$ of $\operatorname{Spec} R$, we write

$$\sigma(\mathsf{C}) = \operatorname{supp}_R \mathsf{C} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}}\mathsf{C} \neq 0\}$$
$$\tau(\mathcal{V}) = \{X \in \mathsf{T} \mid \operatorname{supp}_R X \subseteq \mathcal{V}\}.$$

It follows from [6, §8] that the subcategory $\tau(\mathcal{V})$ is tensor ideal and localising, so these are maps

(3.7) $\{\text{tensor ideal localising subcategories of } \mathsf{T}\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \{\text{subsets of } \operatorname{supp}_R \mathsf{T}\}.$

The theorem below is the reason stratifications are relevant to this work.

Theorem 3.8. *If T is stratified by R, then σ and τ are mutually inverse bijections between the tensor ideal localising subcategories of T and subsets of $\operatorname{supp}_R \mathsf{T}$.*

Proof. It is clear that $\sigma\tau(\mathcal{V}) = \mathcal{V}$ for any subset $\mathcal{V}$ of $\operatorname{supp}_R \mathsf{T}$ and that $\mathsf{C} \subseteq \tau\sigma(\mathsf{C})$ for any tensor ideal localising subcategory C. It remains to check that $\tau\sigma(\mathsf{C}) \subseteq \mathsf{C}$.

For each $\mathfrak{p} \in \operatorname{supp}_R \mathsf{C}$, minimality of $\Gamma_{\mathfrak{p}}\mathsf{T}$ implies the equality below

$$\Gamma_{\mathfrak{p}}\mathsf{T} = \Gamma_{\mathfrak{p}}\mathsf{C} \subseteq \mathsf{C}$$

while the inclusion holds because $\Gamma_{\mathfrak{p}}X \cong \Gamma_{\mathfrak{p}}\mathbb{1} \otimes X$. For any X in $\tau\sigma(\mathsf{C})$ one has $\operatorname{supp}_R X \subseteq \sigma(\mathsf{C})$, so Theorem 3.6 and the inclusions above imply $X \in \mathsf{C}$. $\square$

The following characterisation of minimality is often useful.

Lemma 3.9. *Let T be a tensor triangulated category and C a non-zero tensor ideal localising subcategory of T. The following statements are equivalent:*
   (a) *The tensor ideal localising subcategory C is minimal.*
   (b) *For all non-zero objects X and Y in C there exists an object Z in T such that $\operatorname{Hom}^*_{\mathsf{T}}(X \otimes Z, Y) \neq 0$ holds.*

(c) *If C is a compact generator for T, then all non-zero objects X and Y in C satisfy $\mathrm{Hom}^*_{\mathsf{T}}(X \otimes C, Y) \neq 0$.*

Proof. (a) $\Rightarrow$ (b): Let X be a non-zero object in C. Minimality of C implies that $\mathsf{Loc}^{\otimes}(X) = \mathsf{C}$. Therefore, if there exists Y in C such that for all Z in T we have $\mathrm{Hom}^*_{\mathsf{T}}(X \otimes Z, Y) = 0$, then in particular $\mathrm{Hom}^*_{\mathsf{T}}(Y, Y) = 0$ and hence $Y = 0$.

(b) $\Leftrightarrow$ (c): This is clear, since $\mathsf{Loc}(C) = \mathsf{T}$ for any generator C.

(c) $\Rightarrow$ (a): Suppose that C is not minimal, so that it contains a non-zero proper tensor ideal localising subcategory, say C'. Let X be a non-zero object in C'. It follows from [27, Corollary 4.4.3] that there is a localisation functor with kernel $\mathsf{Loc}(X \otimes C)$, so that for each object W in T there is a triangle $W' \to W \to W'' \to$ with $W' \in \mathsf{C}'$ and $\mathrm{Hom}^*_{\mathsf{T}}(X \otimes C, W'') = 0$. Pick an object W in C but not in C' and set $Y = W''$. Since W' and W are in C, so is Y and W is not in C' one gets $Y \neq 0$. Finally, we have $\mathrm{Hom}^*_{\mathsf{T}}(X \otimes C, Y) = 0$, a contradiction. $\square$

We wish to transfer stratifications from one tensor triangulated category to another. In particular, we need to change the tensor product on a fixed triangulated category. It turns out to be inconvenient to have to keep track of the map from R into the graded centre of T. So we formulate the following principle.

Lemma 3.10. *Let T be a triangulated category admitting two tensor triangulated structures with the same unit object, $\mathbb{1}$, and let $\phi, \phi' \colon R \to Z^*\mathsf{T}$ be two actions. If $\mathbb{1}$ generates T and the maps $\phi_{\mathbb{1}}, \phi'_{\mathbb{1}} \colon R \to \mathrm{End}^*_{\mathsf{T}}(\mathbb{1})$ agree, then T is stratified by R through ϕ if it is stratified by R through ϕ'.*

Proof. It follows from Corollary 3.3 that $\Gamma_{\mathfrak{p}}X$ defined through ϕ and ϕ' agree. This justifies the claim, since localising subcategories are tensor ideal. $\square$

We require also a change of rings result for stratifications.

Lemma 3.11. *Let T be an R-linear tensor triangulated category and $\phi \colon Q \to R$ a homomorphism of rings. If T is stratified by the induced action of Q, then it is stratified by R.*

Proof. Fix a prime $\mathfrak{p}$ in $\mathrm{Spec}\, R$ and set $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. It is straightforward to verify that if an object X in T is $\mathfrak{p}$-torsion and $\mathfrak{p}$-local for the R-action, then it is $\mathfrak{q}$-torsion and $\mathfrak{q}$-local, for the Q-action. The claim is now obvious. $\square$

4. Graded-injective dg modules

This section concerns a certain homotopy category of dg modules over a dg algebra. The development is based on the work of Avramov, Foxby, and Halperin [2], which is also our general reference for this material. The main results we prove here, Theorems 4.4 and 4.11, play a critical role in §§6–8 and are tailored for ready use in them; they are not the best one can do in that direction.

Hypothesis 4.1. Let A be a dg algebra over a field k with the following properties.

(1) $A^i = 0$ for $i > 0$ and A is finite dimensional over k.

(2) A^0 is a local ring with residue field k.

(3) A has a structure of a cocommutative dg Hopf k-algebra.

For the definition of a dg Hopf algebra see [16, §21]. The main consequence of the Hopf structure used in our work is that there is an isomorphism of dg A-modules

$$\operatorname{Hom}_k(A, k) \cong \Sigma^d A \quad \text{for some integer } d.$$

This isomorphism can be verified as in [3, §3.1]. Note that A^0 is also a cocommutative Hopf algebra and the inclusion $A^0 \subseteq A$ is compatible with the Hopf structure. In particular, A is free as a graded module over A^0; see [3, §3.3].

We write $A^\natural$ for the graded algebra underlying A. When M is a dg A-module, $M^\natural$ is the underlying graded $A^\natural$-module. Following [2], we say that a dg A-module I is *graded-injective* if $I^\natural$ is an injective object in the category of graded $A^\natural$-modules. Under Hypothesis 4.1 this condition is equivalent to saying that $I^\natural$ is a graded free $A^\natural$-module. We write $\mathsf{K}(\operatorname{Inj} A)$ for the homotopy category of graded-injective dg A-modules. The objects of this category are graded-injective dg A-modules and morphisms between such dg modules are identified if they are homotopic. The category $\mathsf{K}(\operatorname{Inj} A)$ is given the usual structure of a triangulated category, where the exact triangles correspond to exact sequences of graded-injective dg modules. This is analogous to the case of the homotopy category of complexes over a ring, as described for instance in Verdier's article in SGA4$\frac{1}{2}$ [14]; see also [34].

$\mathsf{K}(\operatorname{Inj} A)$ is compactly generated.

A dg A-module I is said to be *semi-injective* if it is graded-injective and the functor $\operatorname{Hom}_\mathsf{K}(-, I)$, where K is the homotopy category of dg A-modules, takes quasi-isomorphisms to isomorphisms. Every dg module X admits a *semi-injective resolution*: a quasi-isomorphism $X \to I$ of dg A-modules with I semi-injective.

Lemma 4.2. *Let A be a dg algebra satisfying Hypothesis* 4.1. *Each graded-injective dg A-module I has a family $\{I(n)\}_{n \in \mathbb{Z}}$ of dg submodules with the properties below.*

(1) *For each integer n, one has $I(n-1) \subseteq I(n)$.*

(2) *For each integer i, there exists an n_i with $I(n_i)^{\geq i} = I^{\geq i}$.*

(3) *For each integer n, the dg module $I(n)$ is semi-injective.*

When $I^i = 0$ for $i \ll 0$ the dg A-module I is semi-injective.

Proof. Hypothesis 4.1 implies that the $A^\natural$-module $I^\natural$ is free, so $I^\natural = \bigoplus_{j \in \mathbb{Z}} A^\natural U^j$, where U^j is the set of basis elements in degree j. Set

$$I(n) = \bigoplus_{j \geq -n} A U^j.$$

Since $A^i = 0$ for $i > 0$, by Hypothesis 4.1, it follows that $d(U^j) \subseteq I(j-1)$ holds for each j, and hence each $I(n)$ is a dg A-submodule of I. Conditions (1) and (2) are immediate by construction, and it is evident that each $I(n)$ is graded-injective with $I(n)^i = 0$ for $i \ll 0$. It thus remains to verify the last claim of the lemma, for that would also imply that the $I(n)$ are semi-injective.

Assume $I^i = 0$ for $i \ll 0$ and let $I(n)$ be as above. There are canonical surjections $I \to I/I(n)$ for each n and $I \cong \lim_n I/I(n)$ as dg modules over A. It thus suffices to prove that each $I/I(n)$ is semi-injective. So we may assume that $I^{\natural} = \bigoplus_{j \in \mathbb{Z}} A^{\natural} U^j$ with $U^j = \varnothing$ for $|j| \gg 0$. By induction on the number of non-empty U^j, it suffices to consider the case when I is a free dg A-module. Thus I has the form $A \otimes_k V$ with V a graded k-vector space with zero differential. The self-duality of A then implies that I is isomorphic to a shift of $\mathrm{Hom}_k(A, V)$ and hence semi-injective. $\square$

The statement below is a special case of a result from [2]. We thank the authors for allowing us to reproduce the proof here.

Lemma 4.3. *Let A be a dg algebra satisfying Hypothesis 4.1, and let $\mathfrak{m}$ be the dg ideal $\mathrm{Ker}(A \to k)$. Each graded-injective dg A-module I is isomorphic in $\mathsf{K}(\mathsf{Inj}\,A)$ to a graded-injective dg A-module J whose differential satisfies $d(J) \subseteq \mathfrak{m}J$.*

Proof. We repeatedly use the fact that graded-injective dg A-modules are graded-free. For any dg module F we write $\mathrm{cone}(F)$ for the mapping cone of the identity map $F \xrightarrow{=} F$. When F is graded-free, $\mathrm{cone}(F)$ has the following lifting property: If $\alpha \colon M \to N$ is a surjective morphism of dg A-modules, then for any morphism $\beta \colon \mathrm{cone}(F) \to N$ there is a morphism $\gamma \colon \mathrm{cone}(F) \to M$ such that $\alpha\gamma = \beta$. This is readily verified by a diagram chase. This observation is used below.

Since $A/\mathfrak{m}A$ is the field k, the complex of k-vector spaces $I/\mathfrak{m}I$ is isomorphic to $H^*(I/\mathfrak{m}I) \oplus \mathrm{cone}(V)$, where V is a graded k-vector space with zero differential. Pick a surjective morphism $F \to V$ of dg A-modules with F a free dg A-module and $F/\mathfrak{m}F \cong V$; here V is viewed as a dg A-module via the morphism $A \to k$. One thus gets a surjective morphism $\mathrm{cone}(F) \to \mathrm{cone}(V)$ of dg A-modules. The composed morphism $\mathrm{cone}(F) \to \mathrm{cone}(V) \to I/\mathfrak{m}I$ then lifts to a morphism $\gamma \colon \mathrm{cone}(F) \to I$ of dg A-modules. It follows by construction that the map $\gamma \otimes_A k$ is injective, which implies that γ itself is split-injective. Hence its cokernel, say, J, is graded-free. It is also not hard to verify that $d(J) \subseteq \mathfrak{m}J$ holds. Finally, $\mathrm{cone}(F) \cong 0$ in $\mathsf{K}(\mathsf{Inj}\,A)$ and hence I is homotopy equivalent to J. $\square$

The result below is an analogue of [24, Proposition 2.3] for dg algebras. In it $\mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\,A)$ denotes the full subcategory of compact objects in $\mathsf{K}(\mathsf{Inj}\,A)$. As usual, $\mathsf{D}(A)$ stands for the derived category of dg modules over A. We write $\mathsf{D}^{\mathsf{f}}(A)$ for the full subcategory whose objects are dg modules M such that the $H^*(A)$-module $H^*(M)$ is finitely generated; equivalently, $H^*(M)$ is finite dimensional over k.

Theorem 4.4. *Let A be a dg algebra satisfying Hypothesis 4.1 and ik a semi-injective resolution of the dg A-module k. The triangulated category $\mathsf{K}(\mathsf{Inj}\,A)$ is compactly generated by ik and the canonical functor $\mathsf{K}(\mathsf{Inj}\,A) \to \mathsf{D}(A)$ restricts to an equivalence*

$$\mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\,A) \xrightarrow{\sim} \mathsf{D}^{\mathsf{f}}(A).$$

In particular, $\mathsf{Thick}(ik) = \mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\,A)$ and $\mathsf{Loc}(ik) = \mathsf{K}(\mathsf{Inj}\,A)$.

Proof. The dg A-module k has a semi-injective resolution I with $I^j = 0$ for $j < 0$. One way to construct it is to start with a semi-free resolution $F \to k$ with $F^i = 0$ for

$i > 0$ and apply $\mathrm{Hom}_k(-, k)$; note that $\mathrm{Hom}_k(F, k)$ is semi-injective, by adjunction. Semi-injective resolutions of ik are isomorphic in $\mathsf{K}(\mathsf{Inj}\, A)$ so we may assume that ik is concentrated in non-negative degrees.

We have to prove that ik is compact in $\mathsf{K}(\mathsf{Inj}\, A)$ and that it generates $\mathsf{K}(\mathsf{Inj}\, A)$. Let K denote the homotopy category of dg A-modules with the usual triangulated structure. Identifying $\mathsf{K}(\mathsf{Inj}\, A)$ with a full subcategory of K one gets an identification of $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, A)}(-, -)$ with $\mathrm{Hom}_{\mathsf{K}}(-, -)$.

We claim that for any graded-injective module I, the natural map $k \to ik$ induces an isomorphism $\mathrm{Hom}_{\mathsf{K}}(ik, I) \cong \mathrm{Hom}_{\mathsf{K}}(k, I)$. Indeed, the mapping cone of the canonical inclusion $k \to ik$ gives rise to an exact triangle $k \to ik \to C \to$ in K with $C^j = 0$ for $j < 0$. The desired result is that

$$\mathrm{Hom}_{\mathsf{K}}(C, I) = 0 = \mathrm{Hom}_{\mathsf{K}}(\Sigma^{-1}C, I).$$

By Lemma 4.2, there exists a semi-injective dg module $J \subseteq I$ with $J^{\geqslant -2} = I^{\geqslant -2}$. One then has isomorphisms

$$0 \cong \mathrm{Hom}_{\mathsf{K}}(\Sigma^n C, J) \cong \mathrm{Hom}_{\mathsf{K}}(\Sigma^n C, I) \quad \text{for each } n \leq 0,$$

where the second one holds for degree reasons, and the first one because $H^*(C) = 0$ and J is semi-injective. This proves the claim.

If $\{I_\alpha\}$ is a set of graded-injective dg modules over A then the claim yields the first and third isomorphisms below:

$$\mathrm{Hom}_{\mathsf{K}}\left(ik, \bigoplus_\alpha I_\alpha\right) \cong \mathrm{Hom}_{\mathsf{K}}\left(k, \bigoplus_\alpha I_\alpha\right)$$

$$\cong \bigoplus_\alpha \mathrm{Hom}_{\mathsf{K}}(k, I_\alpha)$$

$$\cong \bigoplus_\alpha \mathrm{Hom}_{\mathsf{K}}(ik, I_\alpha).$$

The second isomorphism holds because the $A^\natural$-module k is finitely generated. Therefore the dg module ik is compact in $\mathsf{K}(\mathsf{Inj}\, A)$.

Suppose I is a graded-injective dg module with $\mathrm{Hom}_{\mathsf{K}}^*(ik, I) = 0$. We wish to verify that I is homotopy equivalent to 0. We may assume that $d(I) \subseteq \mathfrak{m}I$, by Lemma 4.3, and hence that the differential on the dg module $\mathrm{Hom}_A(k, I)$ is zero. This explains the first isomorphism below:

$$\mathrm{Hom}_A(k, I) \cong H^*(\mathrm{Hom}_A(k, I)) \cong \mathrm{Hom}_{\mathsf{K}}^*(k, I) \cong \mathrm{Hom}_{\mathsf{K}}^*(ik, I) = 0.$$

The second isomorphism is standard and the third one holds by the claim established above. The equality is by hypothesis, and it follows that $I = 0$. $\qquad\square$

The following test for equivalence of triangulated categories is implicit in [22, §4.2]. The proof uses a standard dévissage argument. Recall that T^c denotes the subcategory of compact objects in T.

Lemma 4.5. *Let $F\colon \mathsf{S} \to \mathsf{T}$ be an exact functor between compactly generated triangulated categories with coproducts. If F preserves coproducts and restricts to an equivalence $\mathsf{S}^{\mathsf{c}} \xrightarrow{\sim} \mathsf{T}^{\mathsf{c}}$, then F is an equivalence of categories.*

In particular, if there exists a compact generator C of S such that $F(C)$ is a compact generator of T and the induced map $\mathrm{End}^*_{\mathsf{S}}(C) \to \mathrm{End}^*_{\mathsf{T}}(FC)$ is an isomorphism, then F is an equivalence.

Proof. Fix a compact object D of S and let S_D be the full subcategory with objects X in S for which the induced map $F_{D,X}\colon \mathrm{Hom}_{\mathsf{S}}(D, X) \to \mathrm{Hom}_{\mathsf{T}}(FD, FX)$ is a bijection. This is a localising subcategory and contains S^{c} by the assumption on F. Therefore $\mathsf{S}_D = \mathsf{S}$. Given this, a similar argument shows that for any object Y in S the subcategory $\{X \in \mathsf{S} \mid F_{X,Y} \text{ is bijective}\}$ equals S. Thus F is fully faithful. The essential image of F is a localising subcategory of T and contains a set of compact generators. We conclude that F is an equivalence. $\qquad\square$

This is used in the proof of the following result.

Proposition 4.6. *Let $\varphi\colon A \to B$ be a morphism of dg k-algebras where A and B satisfy Hypothesis 4.1. One has an exact functor $\mathrm{Hom}_A(B, -)\colon \mathsf{K}(\mathsf{Inj}\, A) \to \mathsf{K}(\mathsf{Inj}\, B)$ of triangulated categories. If φ is a quasi-isomorphism, then this functor is an equivalence and sends a semi-injective resolution of k over A to a semi-injective resolution of k over B.*

Proof. When I is a graded-injective dg module over A, the adjunction isomorphism

$$\mathrm{Hom}_B(-, \mathrm{Hom}_A(B, I)) \cong \mathrm{Hom}_A(-, I)$$

implies that $\mathrm{Hom}_A(B, I)$ is a graded-injective over B. Since $\mathrm{Hom}_A(B, -)$ is additive it defines an exact functor at the level of homotopy categories. The isomorphism above also implies that when I preserves quasi-isomorphisms so does $\mathrm{Hom}_A(-, I)$. Hence, when I is semi-injective so is $\mathrm{Hom}_A(B, I)$.

Suppose φ is a quasi-isomorphism and let ik be a semi-injective resolution of k over A. The dg B-module $\mathrm{Hom}_A(B, ik)$ is then semi-injective and has cohomology k. It is hence a semi-injective resolution of k over B. In view of Theorem 4.4 one thus gets the following commutative diagram.

$$
\begin{array}{ccc}
\mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\, A) & \xrightarrow{\mathrm{Hom}_A(B,-)} & \mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\, B) \\
\Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\
\mathsf{D}^{\mathsf{f}}(A) & \xrightarrow{\mathbf{R}\mathrm{Hom}_A(B,-)} & \mathsf{D}^{\mathsf{f}}(B)
\end{array}
$$

The functor $\mathbf{R}\mathrm{Hom}_A(B, -)$ is an equivalence, because φ is a quasi-isomorphism. Finally, since $B^{\natural}$ is finite dimensional over k, it is finite when viewed as a module over $A^{\natural}$ via φ, so the functor $\mathrm{Hom}_A(B, -)$ preserves coproducts. It remains to apply Lemma 4.5 to deduce that $\mathrm{Hom}_A(B, -)$ is an equivalence. $\qquad\square$

In the remainder of this section we discuss stratification for homotopy categories of graded-injective dg modules.

$\mathsf{K}(\mathsf{Inj}\, A)$ **is tensor triangulated.**

Given a dg Hopf algebra A and dg A-modules M and N, there is a dg A-module structure on $M \otimes_k N$, obtained by restricting the natural action of $A \otimes_k A$ along the comultiplication $A \to A \otimes_k A$. This is the *diagonal action* of A on $M \otimes_k N$.

Proposition 4.7. *Let A be a dg k-algebra satisfying Hypothesis* 4.1 *and ik a semi-injective resolution of k over A. The tensor product $\otimes_k$ with diagonal A-action endows $\mathsf{K}(\mathsf{Inj}\, A)$ with a structure of a tensor triangulated category with unit ik.*

Proof. Standard arguments show that for any dg A-module M the graded $A^\natural$-modules underlying $M \otimes_k A$ and $A \otimes_k M$ are free; see [3, §3.1]. Graded-injective dg A-modules are graded-free as $A^\natural$-modules and direct sums of graded-injectives are graded-injectives, since $A^\natural$ is noetherian. Therefore if I and J are graded-injective dg A-modules, then so is the dg A-module $I \otimes_k J$. Hence one does get a tensor product on $\mathsf{K}(\mathsf{Inj}\, A)$.

We claim that the morphism $k \to ik$ induces an isomorphism

$$I \cong k \otimes_k I \xrightarrow{\sim} ik \otimes_k I$$

in $\mathsf{K}(\mathsf{Inj}\, A)$, so that ik is the unit of the tensor product on $\mathsf{K}(\mathsf{Inj}\, A)$. Indeed, it is an isomorphism when $I = ik$ because the morphism $ik \to ik \otimes_k ik$ is a quasi-isomorphism and both ik and $ik \otimes_k ik$ are semi-injective dg modules, the first by construction and the second by Lemma 4.2. Therefore the map above is an isomorphism for any I in $\mathsf{Loc}(ik)$, which is all of $\mathsf{K}(\mathsf{Inj}\, A)$, by Theorem 4.4. For an alternative argument, see [8, Proposition 5.3].

The other requirements of a tensor triangulated structure are readily verified. $\square$

Remark 4.8. Let A be a dg k-algebra satisfying Hypothesis 4.1.

The k-algebra $\mathrm{Ext}_A^*(k, k)$ is graded commutative, as A is a Hopf k-algebra. Identifying $\mathrm{Ext}_A^*(k, k)$ with $\mathrm{End}_{\mathsf{K}}^*(ik)$ there is a *canonical action* on $\mathsf{K}(\mathsf{Inj}\, A)$ given by

$$\mathrm{Ext}_A^*(k, k) \xrightarrow{X \otimes_k -} \mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\, A)}^*(X, X)$$

for X in $\mathsf{K}(\mathsf{Inj}\, A)$. If the k-algebra $\mathrm{Ext}_A^*(k, k)$ is finitely generated, and hence noetherian, the theory of localisation and support described in §3 applies to $\mathsf{K}(\mathsf{Inj}\, A)$. Finite generation holds, for instance, when the differential on A is zero, by a result of Friedlander and Suslin [17].

Given a dg k-algebra A satisfying Hypothesis 4.1, the structure of $\mathsf{K}(\mathsf{Inj}\, A)$ which is most relevant for us does not depend on the choice of a comultiplication on A. This is made precise in the following proposition which is an immediate consequence of Corollary 3.3 and Lemma 3.10.

Proposition 4.9. *Let A be a dg k-algebra satisfying Hypothesis* 4.1. *The following structures of $\mathsf{K}(\mathsf{Inj}\, A)$ do not depend on the choice of a comultiplication on A:*

(1) *the functors $\Gamma_\mathcal{V}$, $L_\mathcal{V}$, and $\Gamma_\mathfrak{p}$;*
(2) *the maps σ and τ defined in* (3.7);
(3) *stratification of $\mathsf{K}(\mathsf{Inj}\, A)$ via the canonical action of $\mathrm{Ext}_A^*(k, k)$.* $\square$

Transfer of stratification.

The next results deal with transfer of stratification between homotopy categories of graded injective dg modules of dg algebras.

Proposition 4.10. *Let $\varphi \colon A \to B$ be a quasi-isomorphism of dg k-algebras where A, B satisfy Hypothesis 4.1. Then $\mathsf{K}(\mathsf{Inj}\, A)$ is stratified by the canonical action of $\mathrm{Ext}_A^*(k,k)$ if and only if $\mathsf{K}(\mathsf{Inj}\, B)$ is stratified by the canonical action of $\mathrm{Ext}_B^*(k,k)$.*

Observe: φ is not required to commute with the comultiplications on A and B.

Proof. Proposition 4.6 yields that $\mathrm{Hom}_A(B, -) \colon \mathsf{K}(\mathsf{Inj}\, A) \to \mathsf{K}(\mathsf{Inj}\, B)$ is an equivalence sending a semi-injective resolution of k over A to a semi-injective resolution of k over B. Thus $\mathrm{Hom}_A(B, -)$ induces an isomorphism $\mu \colon \mathrm{Ext}_A^*(k,k) \xrightarrow{\sim} \mathrm{Ext}_B^*(k,k)$. Observe that $\mathsf{K}(\mathsf{Inj}\, B)$ admits two actions of $\mathrm{Ext}_A^*(k,k)$. The first is the canonical action of $\mathrm{Ext}_B^*(k,k)$ composed with μ and the other is the canonical action on $\mathsf{K}(\mathsf{Inj}\, A)$ composed with the equivalence $\mathrm{Hom}_A(B, -)$.

Suppose $\mathsf{K}(\mathsf{Inj}\, A)$ is stratified by the canonical action of $\mathrm{Ext}_A^*(k,k)$. Then $\mathsf{K}(\mathsf{Inj}\, B)$ is stratified by $\mathrm{Ext}_A^*(k,k)$ via the second action because $\mathrm{Hom}_A(B, -)$ is an equivalence. It follows from Lemma 3.10 that $\mathsf{K}(\mathsf{Inj}\, B)$ is stratified via the first action, and hence the canonical action of $\mathrm{Ext}_B^*(k,k)$ stratifies $\mathsf{K}(\mathsf{Inj}\, B)$ since μ is an isomorphism. This argument can be reversed by using a quasi-inverse of $\mathrm{Hom}_A(B, -)$. $\quad\square$

Theorem 4.11. *Let A be a dg k-algebra satisfying Hypothesis 4.1. If $\mathsf{K}(\mathsf{Inj}\, A)$ is stratified by the canonical action of $\mathrm{Ext}_A^*(k,k)$, then $\mathsf{K}(\mathsf{Inj}\, A^0)$ is stratified by the canonical action of $\mathrm{Ext}_{A^0}^*(k,k)$.*

Proof. We write $\mathsf{K}(A^0)$ and $\mathsf{K}(A)$ for $\mathsf{K}(\mathsf{Inj}\, A^0)$ and $\mathsf{K}(\mathsf{Inj}\, A)$, respectively. Hypothesis 4.1 implies that $A^\natural$, the graded module underlying A, is free of finite rank over A^0. Therefore when I is a graded-injective (respectively, semi-injective) dg A-module the adjunction isomorphism $\mathrm{Hom}_{A^0}(-, I) \cong \mathrm{Hom}_A(A \otimes_{A^0} -, I)$ yields that I is also graded-injective (respectively, semi-injective) as a dg A^0-module. In particular, the inclusion $A^0 \to A$ gives rise to a restriction functor

$$(-)\!\downarrow \colon \mathsf{K}(A) \to \mathsf{K}(A^0).$$

Let ik be a semi-injective resolution of k over A. The dg A^0-module $ik\!\downarrow$ is then a semi-injective resolution of k over A^0, so restriction induces a homomorphism of graded k-algebras $\mathrm{Ext}_A^*(k,k) \to \mathrm{Ext}_{A^0}^*(k,k)$. In view of Lemma 3.11 it thus suffices to prove that $\mathsf{K}(A^0)$ is stratified by the action of $\mathrm{Ext}_A^*(k,k)$.

The restriction functor has a right adjoint

$$\mathrm{Hom}_{A^0}(A, -) \colon \mathsf{K}(A^0) \to \mathsf{K}(A).$$

Fix a prime $\mathfrak{p}$ in $\mathrm{Spec}\, \mathrm{Ext}_A^*(k,k)$, and let X and Y be objects in $\Gamma_\mathfrak{p}\mathsf{K}(A^0)$ with $\mathrm{Hom}_{\mathsf{K}(A^0)}^*(X, Y) = 0$. It suffices to prove that X or Y is zero; see Lemma 3.9.

As an A^0-module $A^\natural$ is free of finite rank therefore the dg A^0-module $A\!\downarrow$ is in $\mathsf{Thick}(A^0)$, the thick subcategory of $\mathsf{K}(A^0)$ generated by A^0. Thus the dg A^0-module

$\mathrm{Hom}_{A^0}(A, X)\!\downarrow$ is in $\mathsf{Thick}(X)$. This justifies the second isomorphism below:

$$\mathrm{Hom}^*_{\mathsf{K}(A)}(\mathrm{Hom}_{A^0}(A, X), \mathrm{Hom}_{A^0}(A, Y)) \cong \mathrm{Hom}^*_{\mathsf{K}(A^0)}(\mathrm{Hom}_{A^0}(A, X)\!\downarrow, Y) = 0.$$

The first one is adjunction. Observe that the adjunction isomorphism

$$\mathrm{Hom}^*_{\mathsf{K}(A^0)}(ik\!\downarrow, X) \cong \mathrm{Hom}^*_{\mathsf{K}(A)}(ik, \mathrm{Hom}_{A^0}(A, X))$$

is compatible with the action of $\mathrm{Ext}^*_A(k, k)$. It thus follows that both $\mathrm{Hom}_{A^0}(A, X)$ and $\mathrm{Hom}_{A^0}(A, Y)$ are in $\varGamma_{\mathfrak{p}}\mathsf{K}(A)$, and hence that one of them is zero, since $\mathsf{K}(A)$ is stratified by $\mathrm{Ext}^*_A(k, k)$. Assume, without loss of generality, that $\mathrm{Hom}_{A^0}(A, X) = 0$. The isomorphism above then yields $\mathrm{Hom}^*_{\mathsf{K}(A^0)}(ik\!\downarrow, X) = 0$. By Theorem 4.4 the dg A^0-module $ik\!\downarrow$ generates $\mathsf{K}(A^0)$, hence one gets that $X = 0$. $\qquad\square$

5. Graded polynomial algebras

Let k be a field and S a graded polynomial k-algebra on finitely many indeterminates. We assume that each variable is of even degree if the characteristic of k is not 2, so that S is strictly commutative. The algebra S is viewed as a dg algebra with zero differential and we write $\mathsf{D}(S)$ for the derived category of dg S-modules. The objects of this category are dg S-modules and the morphisms are obtained by inverting the quasi-isomorphisms; see for example [22].

The main result of this section is a classification of the localising subcategories of $\mathsf{D}(S)$. This is a graded analogue of the theorem of Neeman [25]. In [7], we establish this result for general graded commutative noetherian rings. The argument presented here for S is simpler because we make use of the fact that the Koszul complex of a regular local ring is quasi-isomorphic to its residue field.

The category $\mathsf{D}(S)$ is tensor triangulated where the tensor product is $\otimes^{\mathbf{L}}_S$, the derived tensor product, and the unit is S. Observe that S is compact and that it generates $\mathsf{D}(S)$. In particular, localising subcategories of $\mathsf{D}(S)$ are also tensor ideal. There is a canonical action of the ring S on $\mathsf{D}(S)$, where the homomorphism $S \to \mathrm{End}^*_{\mathsf{D}(S)}(X)$ is given by multiplication. The theory of localisation and support described in §3 thus applies.

In this context, one has also the following useful identification.

Lemma 5.1. *Let $\mathfrak{p}$ be a point in $\mathrm{Spec}\, S$ and set $\mathcal{Z} = \{\mathfrak{q} \in \mathrm{Spec}\, S \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$. For each M in $\mathsf{D}(S)$ there is a natural isomorphism $L_{\mathcal{Z}} M \cong M_{\mathfrak{p}}$.*

Proof. The functor on $\mathsf{D}(S)$ defined by $M \mapsto M_{\mathfrak{p}}$ is a localisation functor and has the same acyclic objects as $L_{\mathcal{Z}}$, since $H^*(L_{\mathcal{Z}} M) \cong H^*(M)_{\mathfrak{p}}$, by [6, Theorem 4.7]. This implies the desired result. $\qquad\square$

Theorem 5.2. *The category $\mathsf{D}(S)$ is stratified by the canonical S-action. In particular, the maps*

$$\{\textit{localising subcategories of } \mathsf{D}(S)\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \{\textit{subsets of } \mathrm{Spec}\, S\}$$

described in (3.7) are mutually inverse bijections.

Proof. By Theorem 3.8, the second part of the statement follows from the first since localising subcategories of $\mathsf{D}(S)$ are tensor ideal.

Fix a point $\mathfrak{p} \in \operatorname{Spec} S$. Since $\operatorname{Hom}^*_{\mathsf{D}(S)}(S, X)$ equals $H^*(X)$, the subcategory $\Gamma_\mathfrak{p}\mathsf{D}(S)$ consists of dg modules whose cohomology is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. Let $k(\mathfrak{p})$ be the homogeneous localisation of $S/\mathfrak{p}$ at $\mathfrak{p}$; it is a graded field. Evidently $k(\mathfrak{p})$ is in $\Gamma_\mathfrak{p}\mathsf{D}(S)$, so for the desired result it suffices to prove that there is an equality

$$\mathsf{Loc}(M) = \mathsf{Loc}(k(\mathfrak{p})).$$

for any non-zero dg module M in $\Gamma_\mathfrak{p}\mathsf{D}(S)$. We verify this first for $M = \Gamma_\mathfrak{p}S$.

Let $\boldsymbol{s} = s_1, \ldots, s_d$ be a sequence of elements in S whose images in $S_\mathfrak{p}$ are a minimal set of generators for the ideal $\mathfrak{p}S_\mathfrak{p}$ in the ring $S_\mathfrak{p}$. Let $\mathcal{V}$ denote the set of primes in $\operatorname{Spec} S$ containing $\boldsymbol{s}$ and let $\mathcal{Z} = \{\mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$; these are specialisation closed subsets. Observe that $\mathcal{V} \setminus \mathcal{Z} = \{\mathfrak{p}\}$, so one gets the first equality below:

$$\mathsf{Loc}(\Gamma_\mathfrak{p}S) = \mathsf{Loc}(L_\mathcal{Z}\Gamma_\mathcal{V}S) = \mathsf{Loc}(L_\mathcal{Z}(S/\!\!/\boldsymbol{s})) = \mathsf{Loc}((S/\!\!/\boldsymbol{s})_\mathfrak{p}) = \mathsf{Loc}(k(\mathfrak{p})).$$

The second equality is a consequence of Proposition 3.5(2), since the functor $L_\mathcal{Z}$ preserves arbitrary coproducts, while the third one follows from Lemma 5.1. The last equality holds because the dg module $(S/\!\!/\boldsymbol{s})_\mathfrak{p}$ is quasi-isomorphic to $k(\mathfrak{p})$ by the choice of $\boldsymbol{s}$, since it is a regular sequence in $S_\mathfrak{p}$; see [10, Corollary 1.6.14].

We now know that $\mathsf{Loc}(\Gamma_\mathfrak{p}S) = \mathsf{Loc}(k(\mathfrak{p}))$ holds. Applying the functor $- \otimes^{\mathbf{L}}_S M$ to it yields the second equality below:

$$\mathsf{Loc}(M) = \mathsf{Loc}(\Gamma_\mathfrak{p}M) = \mathsf{Loc}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_S M).$$

The first one holds because M is in $\Gamma_\mathfrak{p}\mathsf{D}(S)$. The action of S on $k(\mathfrak{p}) \otimes^{\mathbf{L}}_S M$ factors through the graded field $k(\mathfrak{p})$, as S is commutative. The equality above implies that $H^*(k(\mathfrak{p}) \otimes^{\mathbf{L}}_S M)$ is non-zero, so one deduces that $k(\mathfrak{p}) \otimes^{\mathbf{L}}_S M$ is quasi-isomorphic to a direct sum of shifts of $k(\mathfrak{p})$. Hence $\mathsf{Loc}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_S M) = \mathsf{Loc}(k(\mathfrak{p}))$. Combining this equality with the one above yields the desired result. $\square$

6. Exterior algebras

Let k be a field and let Λ be the graded exterior algebra over k on indeterminates $\xi_1, \ldots, \xi_c$ of negative odd degree. We view Λ as a dg algebra with zero differential. The main result of this section is a classification of the localising subcategories of the homotopy category of graded-injective dg Λ-modules. It will be deduced from Theorem 5.2, via a dg Bernstein-Gelfand-Gelfand correspondence from [1, §7].

Definition 6.1. Let S be a graded polynomial algebra over k on indeterminates $x_1, \ldots, x_c$ with $|x_i| = -|\xi_i| + 1$ for each i. The k-algebra $\Lambda \otimes_k S$ is graded commutative; view it as a dg algebra with zero differential. In it consider the element

$$\delta = \sum_{i=1}^c \xi_i \otimes_k x_i$$

of degree 1. It is easy to verify that $\delta^2 = 0$ holds. In what follows J denotes the dg module over $\Lambda \otimes_k S$ with underlying graded module and differential given by

$$J^\natural = \operatorname{Hom}_k(\Lambda, k) \otimes_k S \quad \text{and} \quad d(e) = \delta e.$$

Observe that since J is a dg module over $\Lambda \otimes_k S$, for each dg module M over Λ there is an induced structure of a dg S-module on $\operatorname{Hom}_\Lambda(J, M)$.

The result below builds on [1, Theorem 7.4]; see Remark 6.3 below. Recall that $\mathsf{K}(\operatorname{Inj} \Lambda)$ is the homotopy category of graded-injective dg modules over Λ and $\mathsf{D}(S)$ is the derived category of dg modules over S.

Theorem 6.2. *The dg $(\Lambda \otimes_k S)$-module J in Definition* 6.1 *has these properties:*

(1) *There is a quasi-isomorphism $k \to J$ of dg Λ-modules and J is semi-injective.*
(2) *The map $S \to \operatorname{Hom}_\Lambda(J, J)$ induced by right multiplication is a morphism of dg k-algebras and a quasi-isomorphism.*

For any dg $(\Lambda \otimes_k S)$-module J satisfying these conditions the functor

$$\operatorname{Hom}_\Lambda(J, -) \colon \mathsf{K}(\operatorname{Inj} \Lambda) \to \mathsf{D}(S)$$

is an equivalence of triangulated categories.

Proof. The surjection $\Lambda \to k$ is a morphism of dg Λ-modules and hence so is its dual $k \to \operatorname{Hom}_k(\Lambda, k)$. Combined with the map of k-vector spaces $k \to S$ one gets a morphism $k \to \operatorname{Hom}_k(\Lambda, k) \otimes_k S = J$ of dg Λ-modules. The module J is precisely the dg module X from [1, §7.3]. It thus follows from [1, §§7.6.2, 7.6.5] that $k \to J$ is a quasi-isomorphism and that J is semi-injective as a dg Λ-module. Moreover, the map $S \to \operatorname{Hom}_\Lambda(J, J)$ is a quasi-isomorphism by [1, Theorem 7.4]. The module J thus has the stated properties.

Let J be any dg $(\Lambda \otimes_k S)$-module satisfying conditions (1) and (2) in the statement of the theorem. It is easy to verify that the functor $\operatorname{Hom}_\Lambda(J, -)$ from $\mathsf{K}(\operatorname{Inj} \Lambda)$ to $\mathsf{D}(S)$ is exact. We claim that J is compact and generates the triangulated category $\mathsf{K}(\operatorname{Inj} \Lambda)$. One way to prove this is to note that Λ satisfies Hypothesis 4.1, with comultiplication defined by $\xi_i \mapsto \xi_i \otimes 1 + 1 \otimes \xi_i$, so that Theorem 4.4 applies. Compactness yields that the functor $\operatorname{Hom}_\Lambda(J, -)$ preserves coproducts, since a quasi-isomorphism between dg S-modules is an isomorphism in $\mathsf{D}(S)$. Furthermore, as S is a compact generator for $\mathsf{D}(S)$, condition (2) provides the hypotheses required to apply Lemma 4.5, which yields that $\operatorname{Hom}_\Lambda(J, -)$ is an equivalence. $\square$

Remark 6.3. Let $F \colon \mathsf{D}(S) \to \mathsf{K}(S)$ be a left adjoint to the canonical localisation functor $\mathsf{K}(S) \to \mathsf{D}(S)$, and $- \otimes_S^{\mathbf{L}} J$ the composite functor

$$\mathsf{D}(S) \xrightarrow{F} \mathsf{K}(S) \xrightarrow{- \otimes_S J} \mathsf{K}(\operatorname{Inj} \Lambda).$$

The proof of Theorem 6.2 shows that $- \otimes_S^{\mathbf{L}} J$ is left adjoint to $\operatorname{Hom}_\Lambda(J, -)$, so restricting the equivalence in Theorem 6.2 to compact objects yields the equivalence $\mathsf{D}^{\mathsf{f}}(\Lambda) \xrightarrow{\sim} \mathsf{D}^{\mathsf{f}}(S)$ contained in [1, Theorem 7.4].

As in the proof of Theorem 6.2 consider Λ as dg Hopf k-algebra with

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i.$$

Observe that Λ satisfies Hypothesis 4.1, so the triangulated category $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ has a canonical tensor triangulated structure, by Proposition 4.7, and hence a canonical action of $\mathrm{Ext}^*_\Lambda(k,k)$; see Remark 4.8. Moreover, the k-algebra $\mathrm{Ext}^*_\Lambda(k,k)$ is noetherian, by Theorem 6.2(2).

Theorem 6.4. *The tensor triangulated category $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is stratified by the canonical action of $\mathrm{Ext}^*_\Lambda(k,k)$. Therefore the maps*

$$\left\{ \text{localising subcategories of } \mathsf{K}(\mathsf{Inj}\,\Lambda) \right\} \; \overset{\sigma}{\underset{\tau}{\rightleftarrows}} \; \left\{ \text{subsets of } \mathrm{Spec}\,\mathrm{Ext}^*_\Lambda(k,k) \right\}$$

described in (3.7) are mutually inverse bijections.

Proof. Let S and J be as in Definition 6.1. In view of Theorem 6.2(1) we identify $\mathrm{Ext}^*_\Lambda(k,k)$ and $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\Lambda)}(J,J)$. Theorem 5.2 applies to S and yields that $\mathsf{D}(S)$ is stratified by the canonical S-action on it, and hence also by an action of $\mathrm{Ext}^*_\Lambda(k,k)$ obtained from the isomorphism $S \cong \mathrm{Ext}^*_\Lambda(k,k)$ in Theorem 6.2(2). Theorem 6.2 provides an equivalence of triangulated categories $\mathsf{K}(\mathsf{Inj}\,\Lambda) \to \mathsf{D}(S)$ that sends J, the compact generator of $\mathsf{K}(\mathsf{Inj}\,\Lambda)$, to S, the compact generator of $\mathsf{D}(S)$. Hence $\mathsf{K}(\mathsf{Inj}\,\Lambda)$ is stratified by the canonical action of $\mathrm{Ext}^*_\Lambda(k,k)$, by Lemma 3.10.

By Theorem 3.8 the stated bijection is a consequence of the stratification. $\square$

7. A Koszul dg algebra

In this section we classify the localising subcategories of the homotopy category of graded-injective dg modules over the Koszul dg algebra of the group algebra of an elementary abelian group. To this end we establish an equivalence with the corresponding homotopy category over an exterior algebra, covered by Theorem 6.4.

Let E be an elementary abelian p-group of rank r and let k be a field of characteristic p. The group algebra kE is thus of the form

$$kE = k[z_1, \ldots, z_r]/(z_1^p, \ldots, z_r^p).$$

Let A be the dg algebra with $A^\natural$ an exterior algebra over kE on generators $y_1, \ldots, y_r$ each of degree -1, and differential defined by $d(y_i) = z_i$, so that $d(z_i) = 0$. This is the Koszul complex of A, viewed as a dg algebra; see [10, §1.6].

The group algebra kE is an example of a complete intersection, and the Koszul dg algebra of such a ring is formal, with cohomology an exterior algebra; see [10, §2.3]. This computation is straightforward for the case of kE and is given below.

Lemma 7.1. *Let Λ be an exterior algebra over k on indeterminates $\xi_1, \ldots, \xi_r$ of degree -1, viewed as a dg algebra with zero differential. The morphism $\varphi \colon \Lambda \to A$ of dg k-algebras defined by $\varphi(\xi_i) = z_i^{p-1}y_i$ is a quasi-isomorphism. In particular, the k-algebra $\mathrm{Ext}^*_A(k,k)$ is a polynomial ring in r indeterminates of degree 2.*

Proof. A routine calculation shows that φ is a morphism of dg k-algebras. What needs to be verified is that it is a quasi-isomorphism, and this is determined only by the structure of Λ and A as complexes of k-vector spaces.

Let $\Lambda(i)$ be the exterior algebra on the variable ξ_i and $A(i)$ the Koszul dg algebra over $k[z_i]/(z_i^p)$, with exterior generator y_i. Observe that $\varphi = \varphi(1) \otimes_k \cdots \otimes_k \varphi(r)$ where $\varphi(i)\colon \Lambda(i) \to A(i)$ is the morphism of complexes mapping ξ_i to $z_i^{p-1}y_i$. Each $\varphi(i)$ is a quasi-isomorphism, by inspection, and hence so is φ.

Since φ is a quasi-isomorphism the k-algebras $\operatorname{Ext}_\Lambda^*(k, k)$ and $\operatorname{Ext}_A^*(k, k)$ are isomorphic. Theorem 6.2(2) implies $\operatorname{Ext}_A^*(k, k)$ has the stated structure. $\qquad\square$

We endow the dg algebra A with a comultiplication
$$\Delta(z_i) = z_i \otimes 1 + 1 \otimes z_i \quad \text{and} \quad \Delta(y_i) = y_i \otimes 1 + 1 \otimes y_i.$$
With this structure A satisfies Hypothesis 4.1. The homotopy category of graded-injective dg A-modules $\mathsf{K}(\operatorname{Inj} A)$ has thus a tensor triangulated structure and a canonical action of $\operatorname{Ext}_A^*(k, k)$; see Proposition 4.7 and Remark 4.8.

Theorem 7.2. *The tensor triangulated category $\mathsf{K}(\operatorname{Inj} A)$ is stratified by the canonical action of $\operatorname{Ext}_A^*(k, k)$. Therefore the maps*

$$\left\{ \text{localising subcategories of } \mathsf{K}(\operatorname{Inj} A) \right\} \; \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \; \left\{ \text{subsets of } \operatorname{Spec} \operatorname{Ext}_A^*(k, k) \right\}$$

described in (3.7) *are mutually inverse bijections.*

Proof. Let Λ be the exterior algebra from Lemma 7.1. With comultiplication defined by $\xi_i \mapsto \xi_i \otimes 1 + 1 \otimes \xi_i$ this dg algebra satisfies Hypothesis 4.1. The desired result thus follows from Lemma 7.1, Theorem 6.4 and Proposition 4.10. $\qquad\square$

8. Elementary abelian groups

We classify the localising subcategories of the homotopy category of injective modules over the group algebra of an elementary abelian group, by deducing it from the corresponding statement for its Koszul dg algebra in Theorem 7.2.

Let E be an elementary abelian p-group of rank r and k a field of characteristic p. In this section we view its group algebra kE, which is isomorphic to
$$k[z_1, \ldots, z_r]/(z_1^p, \ldots, z_r^p)$$
as a dg algebra over k with zero differential. The diagonal map of E endows kE with a structure of a dg Hopf k-algebra with comultiplication
$$\Delta(z_i) = z_i \otimes 1 + z_i \otimes z_i + 1 \otimes z_i.$$
We call this the *group Hopf structure* on kE; it is evidently cocommutative. This Hopf structure is needed in §9 for the passage from E to a general finite group.

We also have to consider a different cocommutative dg Hopf algebra structure on kE, where the comultiplication is defined by
$$\Delta(z_i) = z_i \otimes 1 + 1 \otimes z_i.$$

We call this the *Lie Hopf structure* on kE. It comes from regarding kE as the restricted universal enveloping algebra of an abelian p-restricted Lie algebra with zero p-power map. The Lie Hopf structure has the advantage that the inclusion of kE into its Koszul dg algebra is a map of dg Hopf algebras; this is exploited in the proof of Theorem 8.1 below.

The group Hopf structure and the Lie Hopf structure on kE both satisfy Hypothesis 4.1, and thus give rise to two tensor triangulated structures on $\mathsf{K}(\mathsf{Inj}\, kE)$. Thus there are two actions of $\mathrm{Ext}^*_{kE}(k, k)$ on $\mathsf{K}(\mathsf{Inj}\, kE)$, which we call the group action and the Lie action, respectively; see Proposition 4.7 and Remark 4.8. The k-algebra $\mathrm{Ext}^*_{kE}(k, k)$ is finitely generated and hence noetherian. Thus the theory of localisation and support described in §3 applies.

Theorem 8.1. *The triangulated category* $\mathsf{K}(\mathsf{Inj}\, kE)$ *is stratified by both the group action and the Lie action of* $\mathrm{Ext}^*_{kE}(k, k)$. *The maps* σ *and* τ *described in* (3.7) *do not depend on the action used, and give mutually inverse bijections:*

$$\{\text{localising subcategories of } \mathsf{K}(\mathsf{Inj}\, kE)\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \{\text{subsets of } \mathrm{Spec}\,\mathrm{Ext}^*_{kE}(k, k)\}.$$

Proof. Proposition 4.9 justifies the statement about independence of actions. Hence it suffices to consider the Lie Hopf structure on kE.

Let A be the Koszul dg algebra of kE with structure of dg Hopf k-algebra introduced in §7. Observe that $kE = A^0$ and that the inclusion $kE \to A$ is compatible with the Lie Hopf structure; this is the reason for working with this Hopf structure on kE. It now remains to apply Theorems 7.2 and 4.11. $\qquad\square$

9. Finite groups

In this section we prove that the tensor ideal localising subcategories of the homotopy category of complexes of injective modules over the group algebra of a finite group are stratified by the cohomology of the group. This is achieved by descending to the case of an elementary abelian group, covered by Theorem 8.1. The crucial new input required here is Quillen's stratification theorem [28, 29].

Let G be a finite group and k a field of characteristic dividing the order of G. The group algebra kG is a Hopf algebra where the comultiplication is defined by $\Delta(g) = g \otimes g$ for each $g \in G$. We consider the homotopy category of complexes of injective kG-modules $\mathsf{K}(\mathsf{Inj}\, kG)$. The diagonal action of G induces a tensor triangulated structure with unit the injective resolution ik of k; see [8, §5] and also Proposition 4.7. As is customary, $H^*(G, k)$ denotes the cohomology of G, which is the k-algebra $\mathrm{Ext}^*_{kG}(k, k)$. This algebra is finitely generated, and hence noetherian, by a result of Evens and Venkov [15, 33]; see also Golod [18]. It acts on $\mathsf{K}(\mathsf{Inj}\, kG)$ via the canonical action, and the theory described in §3 applies. We write $\mathcal{V}_G$ for $\mathrm{Spec}\, H^*(G, k)$ and $\mathcal{V}_G(X)$ for the support of any complex X in $\mathsf{K}(\mathsf{Inj}\, kG)$.

For each subgroup H of G restriction yields a homomorphism of graded rings $\mathrm{res}_{G,H} \colon H^*(G, k) \to H^*(H, k)$, and hence a map on Spec:

$$\mathrm{res}^*_{G,H} \colon \mathcal{V}_H \to \mathcal{V}_G.$$

Part of Quillen's theorem[1] is that for each $\mathfrak{p}$ in $\mathcal{V}_G$ there exists an elementary abelian subgroup E of G such that $\mathfrak{p}$ is in the image of $\mathrm{res}^*_{G,E}$. We say that $\mathfrak{p}$ *originates* in such an E if there does not exist a proper subgroup E' of E such that $\mathfrak{p}$ is in the image of $\mathrm{res}^*_{G,E'}$. In this language, [29, Theorem 10.2] reads:

Theorem 9.1. *For each $\mathfrak{p} \in \mathcal{V}_G$, the pairs $(E, \mathfrak{q})$ where $\mathfrak{p} = \mathrm{res}^*_{G,E}(\mathfrak{q})$ and such that $\mathfrak{p}$ originates in E are all G-conjugate. This sets up a one to one correspondence between primes $\mathfrak{p}$ in $\mathcal{V}_G$ and G-conjugacy classes of such pairs $(E, \mathfrak{q})$.* $\square$

To make use of this we need a subgroup theorem for elementary abelian groups. As usual given a subgroup $H \leq G$ there are exact functors

$$(-)\!\downarrow_H \colon \mathsf{K}(\mathsf{Inj}\,kG) \to \mathsf{K}(\mathsf{Inj}\,kH) \quad \text{and} \quad (-)\!\uparrow^G \colon \mathsf{K}(\mathsf{Inj}\,kH) \to \mathsf{K}(\mathsf{Inj}\,kG)$$

defined by restriction and induction, $- \otimes_{kH} kG$, respectively. The functor $(-)\!\uparrow^G$ is faithful and left adjoint to $(-)\!\downarrow_H$. For each object X in $\mathsf{K}(\mathsf{Inj}\,kG)$ and object Y in $\mathsf{K}(\mathsf{Inj}\,kH)$ there is a natural isomorphism

$$(9.2) \qquad\qquad (X\!\downarrow_H \otimes_k Y)\!\uparrow^G \cong X \otimes_k Y\!\uparrow^G$$

in $\mathsf{K}(\mathsf{Inj}\,kG)$, where an element $(x \otimes y) \otimes g$ is mapped to $xg \otimes (y \otimes g)$.

Lemma 9.3. *Let H be a subgroup of G. Fix $\mathfrak{p} \in \mathcal{V}_G$ and set $\mathcal{U} = (\mathrm{res}^*_{G,H})^{-1}\{\mathfrak{p}\}$.*

(1) *For any $X \in \mathsf{K}(\mathsf{Inj}\,kG)$ there is an isomorphism $(\Gamma_\mathfrak{p} X)\!\downarrow_H \cong \bigoplus_{\mathfrak{q} \in \mathcal{U}} \Gamma_\mathfrak{q}(X\!\downarrow_H)$.*

(2) *For any $Y \in \mathsf{K}(\mathsf{Inj}\,kH)$ there is an isomorphism $\Gamma_\mathfrak{p}(Y\!\uparrow^G) \cong \bigoplus_{\mathfrak{q} \in \mathcal{U}}(\Gamma_\mathfrak{q} Y)\!\uparrow^G$.*

Proof. (1) For $X = ik$, this follows as in [4, Lemma 8.2 and Proposition 8.4 (iv)]. The general case is deduced as follows:

$$(\Gamma_\mathfrak{p} X)\!\downarrow_H \cong (\Gamma_\mathfrak{p} ik \otimes_k X)\!\downarrow_H \cong (\Gamma_\mathfrak{p} ik)\!\downarrow_H \otimes_k X\!\downarrow_H$$
$$\cong \bigoplus_{\mathfrak{q} \in \mathcal{U}} \Gamma_\mathfrak{q} ik \otimes_k X\!\downarrow_H \cong \bigoplus_{\mathfrak{q} \in \mathcal{U}} \Gamma_\mathfrak{q}(X\!\downarrow_H).$$

(2) Using part (1) and (9.2), one gets isomorphisms

$$\Gamma_\mathfrak{p}(Y\!\uparrow^G) \cong \Gamma_\mathfrak{p} ik \otimes_k Y\!\uparrow^G \cong ((\Gamma_\mathfrak{p} ik)\!\downarrow_H \otimes_k Y)\!\uparrow^G$$
$$\cong \bigoplus_{\mathfrak{q} \in \mathcal{U}}(\Gamma_\mathfrak{q} ik \otimes_k Y)\!\uparrow^G \cong \bigoplus_{\mathfrak{q} \in \mathcal{U}}(\Gamma_\mathfrak{q} Y)\!\uparrow^G. \qquad \square$$

Proposition 9.4. *Let H be a subgroup of G. For any object X in $\mathsf{K}(\mathsf{Inj}\,kG)$ and object Y in $\mathsf{K}(\mathsf{Inj}\,kH)$ one has*

$$\mathcal{V}_G(X\!\downarrow_H\!\uparrow^G) \subseteq \mathcal{V}_G(X) \quad \text{and} \quad \mathcal{V}_G(Y\!\uparrow^G) = \mathrm{res}^*_{G,H} \mathcal{V}_H(Y).$$

Proof. Let W be an injective resolution of the permutation module, $k(G/H)$, on the cosets of H in G. We claim that the natural map $k(G/H) \to W$ of complexes of kG-modules induces an isomorphism $X\!\downarrow_H\!\uparrow^G \cong X \otimes_k W$ in $\mathsf{K}(\mathsf{Inj}\,kG)$.

Indeed, this is easy to check when $X = iM$, the injective resolution of a finite dimensional kG-module M, for then both $iM\!\downarrow_H\!\uparrow^G$ and $iM \otimes_k W$ are injective

---

[1]See the discussion following Proposition 11.2 of [29].

resolutions of $M \otimes_k k(G/H)$. It then remains to note that complexes of the form iM are a set of compactly generators for $\mathsf{K}(\mathsf{Inj}\, kG)$; see [24, Lemmas 2.1 and 2.2].

When $\mathfrak{p} \in \mathcal{V}_G(X \otimes_k W)$ holds, since $\Gamma_{\mathfrak{p}}(X \otimes_k W) \cong \Gamma_{\mathfrak{p}}X \otimes_k W$ one gets $\Gamma_{\mathfrak{p}}X \neq 0$, that is to say, $\mathfrak{p} \in \mathcal{V}_G(X)$, as desired.

By Lemma 9.3(2) the condition $\mathfrak{p} \in \mathcal{V}_G(Y{\uparrow}^G)$ is equivalent to: there exists $\mathfrak{q} \in \mathcal{V}_H$ such that $\mathrm{res}^*_{G,H}(\mathfrak{q}) = \mathfrak{p}$ and $\Gamma_{\mathfrak{q}}Y \neq 0$. Hence $\mathcal{V}_G(Y{\uparrow}^G) = \mathrm{res}^*_{G,H} \mathcal{V}_H(Y)$. $\square$

The result below is an analogue of the subgroup theorem for an elementary abelian group E. Its proof is based on the classification of localising subcategories of $\mathsf{K}(\mathsf{Inj}\, kE)$. The full version of the subgroup theorem, Theorem 11.2, will be a consequence of the classification theorem for $\mathsf{K}(\mathsf{Inj}\, kG)$.

Theorem 9.5. *Let $E' \leq E$ be elementary abelian p-groups. For any object X in $\mathsf{K}(\mathsf{Inj}\, kE)$ there is an equality*

$$\mathcal{V}_{E'}(X{\downarrow}_{E'}) = (\mathrm{res}^*_{E,E'})^{-1}\mathcal{V}_E(X).$$

Proof. Fix a prime $\mathfrak{q}$ in $\mathcal{V}_{E'}$ and set $\mathfrak{p} = \mathrm{res}^*_{E,E'}(\mathfrak{q})$. Proposition 9.4 yields an equality $\mathcal{V}_E(\Gamma_{\mathfrak{q}}ik{\uparrow}^E) = \{\mathfrak{p}\} = \mathcal{V}_E(\Gamma_{\mathfrak{p}}ik)$. It thus follows from the classification of localising subcategories for kE in Theorem 8.1 that there is an equality

$$\mathsf{Loc}(\Gamma_{\mathfrak{q}}ik{\uparrow}^E) = \mathsf{Loc}(\Gamma_{\mathfrak{p}}ik).$$

This implies that $\Gamma_{\mathfrak{q}}ik{\uparrow}^E \otimes_k X \neq 0$ holds if and only if $\Gamma_{\mathfrak{p}}ik \otimes_k X \neq 0$. The desired result follows from the chain of implications:

$$\Gamma_{\mathfrak{q}}(X{\downarrow}_{E'}) \neq 0 \iff \Gamma_{\mathfrak{q}}ik \otimes_k X{\downarrow}_{E'} \neq 0 \iff$$
$$\Gamma_{\mathfrak{q}}ik{\uparrow}^E \otimes_k X \neq 0 \iff \Gamma_{\mathfrak{p}}ik \otimes_k X \neq 0 \iff \Gamma_{\mathfrak{p}}X \neq 0.$$

The second implication follows from (9.2) and the fact that $(-){\uparrow}^E$ is faithful. $\square$

Next we formulate a version of Chouinard's theorem for $\mathsf{K}(\mathsf{Inj}\, kG)$. Recall that for any ring A, a complex P of projective A-modules is *semi-projective* if the functor $\mathrm{Hom}_{\mathsf{K}}(P, -)$, where K is the homotopy category of complexes of A-modules, takes quasi-isomorphisms to isomorphisms.

Proposition 9.6. *Let G be a finite group and $\mathfrak{m}$ the ideal of positive degree elements of $H^*(G, k)$. The following statements hold for each X in $\mathsf{K}(\mathsf{Inj}\, kG)$.*

(1) $\mathcal{V}_G(X) \subseteq \mathrm{Spec}\, H^*(G, k) \setminus \{\mathfrak{m}\}$ *if and only if X is acyclic.*
(2) $\mathcal{V}_G(X) \subseteq \{\mathfrak{m}\}$ *if and only if X is semi-projective.*
(3) $X = 0$ *if and only if $X{\downarrow}_E = 0$ for every elementary abelian subgroup $E \leq G$.*

Proof. We write pk for a projective resolution and tk for a Tate resolution of the kG-module k. These fit into an exact triangle $pk \to ik \to tk \to$ which is the localisation triangle (3.2) in $\mathsf{K}(\mathsf{Inj}\, kG)$ for ik with respect to the closed subset $\{\mathfrak{m}\}$. In particular $\Gamma_{\mathfrak{m}}ik = pk$, so that $\Gamma_{\mathfrak{m}}X \cong pk \otimes_k X$.

Claim. The induced map $pk \otimes_k X \to X$ is a semi-projective resolution of X.

Indeed, since pk is a complex of projectives, so is $pk \otimes_k X$; semi-projectivity follows from the isomorphism $\mathrm{Hom}_{\mathsf{K}}(pk \otimes_k X, -) \cong \mathrm{Hom}_{\mathsf{K}}(pk, \mathrm{Hom}_k(X, -))$.

Statements (1) and (2) are immediate from the preceding claim.

(3) Suppose X is non-zero in $\mathsf{K}(\mathsf{Inj}\, kG)$. In view of the localisation triangle $pk \otimes_k X \to X \to tk \otimes_k X \to$ we may assume that $pk \otimes_k X \neq 0$ or $tk \otimes_k X \neq 0$.

Assume that $pk \otimes_k X$ is non-zero; equivalently, that it is not acyclic, by the preceding claim. Observe that for each subgroup $H \leq G$ the restriction functor $(-)\!\downarrow_H$ sends semi-projectives to semi-projectives. Therefore $(pk \otimes_k X)\!\downarrow_H$, and hence $X\!\downarrow_H$, is non-zero.

Now assume that $tk \otimes_k X$ is non-zero. Chouinard's theorem [13] applies, because one can identify the subcategory of acyclic complexes in $\mathsf{K}(\mathsf{Inj}\, kG)$ with $\mathsf{StMod}(kG)$, and this identification is compatible with restrictions. This yields an elementary abelian subgroup $E \leq G$ with $(tk \otimes_k X)\!\downarrow_E$, and hence also $X\!\downarrow_E$, non-zero. $\qquad\square$

The following result is a culmination of the development in §§4–9. Its applications are deferred to ensuing sections.

Theorem 9.7. *Let G be a finite group. The triangulated category $\mathsf{K}(\mathsf{Inj}\, kG)$ is stratified by the canonical action of the cohomology algebra $H^*(G, k)$.*

Proof. For any subgroup $H \leq G$ we abbreviate $\mathsf{K}(\mathsf{Inj}\, kH)$ to $\mathsf{K}(kH)$. We have to prove that for each $\mathfrak{p} \in \mathcal{V}_G$, the subcategory $\Gamma_{\mathfrak{p}}\mathsf{K}(kG)$ is minimal among tensor ideal localising subcategories of $\mathsf{K}(kG)$.

Let X be a non-zero object in $\Gamma_{\mathfrak{p}}\mathsf{K}(kG)$. Proposition 9.6 provides an elementary abelian subgroup E_0 of G such that $X\!\downarrow_{E_0}$ is non-zero. Choose a prime $\mathfrak{q}_0$ in $\mathcal{V}_{E_0}(X\!\downarrow_{E_0})$. Using Proposition 9.4 one thus obtains

$$\mathrm{res}^*_{G, E_0}(\mathfrak{q}_0) \in \mathcal{V}_G(X\!\downarrow_{E_0}\!\uparrow^G) \subseteq \mathcal{V}_G(X) = \{\mathfrak{p}\}.$$

Hence $\mathrm{res}^*_{G, E_0}(\mathfrak{q}_0) = \mathfrak{p}$, so that $E_0 \geq E$ and $\mathfrak{q}_0 = \mathrm{res}^*_{E_0, E}(\mathfrak{q})$ for some pair $(E, \mathfrak{q})$ corresponding to $\mathfrak{p}$ as in Theorem 9.1. Thus $\mathfrak{q} \in \mathcal{V}_E(X\!\downarrow_E)$, by Theorem 9.5.

By Theorem 9.1 all pairs $(E, \mathfrak{q})$ where $\mathfrak{p}$ originates in E are conjugate, hence if we choose one, then each non-zero $X \in \Gamma_{\mathfrak{p}}\mathsf{K}(kG)$ has $\Gamma_{\mathfrak{q}}X\!\downarrow_E \neq 0$.

Now let Y be another non-zero object in $\Gamma_{\mathfrak{p}}\mathsf{K}(kG)$ and set Z to be the injective resolution of $k(G/E)$, the permutation module. As noted in the proof of Proposition 9.4, the complexes $X\!\downarrow_E\!\uparrow^G$ and $X \otimes_k Z$ are isomorphic in $\mathsf{K}(\mathsf{Inj}\, kG)$. This gives the first isomorphism below:

$$\mathrm{Hom}^*_{\mathsf{K}(kG)}(X \otimes_k Z, Y) \cong \mathrm{Hom}^*_{\mathsf{K}(kG)}(X\!\downarrow_E\!\uparrow^G, Y) \cong \mathrm{Hom}^*_{\mathsf{K}(kE)}(X\!\downarrow_E, Y\!\downarrow_E).$$

The second isomorphism is by Frobenius reciprocity. Lemma 9.3 implies that $\Gamma_{\mathfrak{q}}X\!\downarrow_E$ and $\Gamma_{\mathfrak{q}}Y\!\downarrow_E$ are non-zero direct summands of $X\!\downarrow_E$ and $Y\!\downarrow_E$, respectively. It follows from Theorem 8.1 and the isomorphism above that $\mathrm{Hom}^*_{\mathsf{K}(kG)}(X \otimes_k Z, Y)$ is non-zero, so $\Gamma_{\mathfrak{p}}\mathsf{K}(kG)$ is minimal, by Lemma 3.9. $\qquad\square$

10. The main theorems

Let k be a field of characteristic p and G be a finite group, where p divides the order of G. The following result implies Theorem 2.3.

Theorem 10.1. *The tensor triangulated category* $\mathsf{K}(\mathsf{Inj}\,kG)$ *is stratified by the canonical action of* $H^*(G, k)$. *The maps*

$$\left\{ \begin{array}{c} \textit{tensor ideal localising} \\ \textit{subcategories of } \mathsf{K}(\mathsf{Inj}\,kG) \end{array} \right\} \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \{\textit{subsets of } \operatorname{Spec} H^*(G, k)\}$$

described in (3.7) *are mutually inverse bijections.*

Proof. This follows from Theorems 3.8 and 9.7. Note that the support of $\mathsf{K}(\mathsf{Inj}\,kG)$ equals $\operatorname{Spec} H^*(G, k)$ since $\mathcal{V}_G(k) = \operatorname{Spec} H^*(G, k)$. $\square$

Next we prove our main result about the stable module category $\mathsf{StMod}(kG)$. We identify it with $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG)$, the full subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ consisting of acyclic complexes. Observe that $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG)$ is the tensor ideal localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$ corresponding to the subset $\operatorname{Proj} H^*(G, k)$ of non-maximal primes; see Proposition 9.6. This provides one way of classifying all tensor ideal localising subcategories of $\mathsf{StMod}(kG)$ in terms of subsets of $\operatorname{Proj} H^*(G, k)$; see Proposition 2.2.

Now we define an action of the cohomology algebra $H^*(G, k)$ on $\mathsf{StMod}(kG)$. The equivalence $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG) \xrightarrow{\sim} \mathsf{StMod}(kG)$ sends a complex X to its cycles $Z^0 X$ in degree zero. Restriction to the category of acyclic complexes induces therefore the following $H^*(G, k)$-action

$$H^*(G, k) \to Z^* \mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\mathrm{res}} Z^* \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG) \xrightarrow{\sim} Z^* \mathsf{StMod}(kG),$$

where the first map is given by the canonical action of $H^*(G, k)$ on $\mathsf{K}(\mathsf{Inj}\,kG)$. For each kG-module M, this action induces the map

$$(10.2) \qquad H^*(G, k) \hookrightarrow \widehat{\operatorname{Ext}}^*_{kG}(k, k) = \operatorname{End}^*_{\mathsf{StMod}(kG)}(k) \xrightarrow{M \otimes_k -} \operatorname{End}^*_{\mathsf{StMod}(kG)}(M).$$

Thus we are in the setting of §3. The support of any module M in $\mathsf{StMod}(kG)$ is denoted $\mathcal{V}_G(M)$; it is a subset of $\operatorname{Spec} H^*(G, k)$. By construction, this coincides with the support of a Tate resolution of M in $\mathsf{K}(\mathsf{Inj}\,kG)$. Moreover, $\mathcal{V}_G(M)$ is the support defined in [4]. Indeed if $\mathfrak{p}$ is a non-maximal prime $\mathfrak{p}$ in $H^*(G, k)$, then $\Gamma_{\mathfrak{p}} k$ equals the kappa module for $\mathfrak{p}$ defined in [4]; see [6, §10].

Theorem 10.3. *The tensor triangulated category* $\mathsf{StMod}(kG)$ *is stratified by the action of* $H^*(G, k)$ *given by* (10.2). *The maps*

$$\left\{ \begin{array}{c} \textit{tensor ideal localising} \\ \textit{subcategories of } \mathsf{StMod}(kG) \end{array} \right\} \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \{\textit{subsets of } \operatorname{Proj} H^*(G, k)\}$$

described in (3.7) *are mutually inverse bijections.*

Proof. Using the identification of $\mathsf{StMod}(kG)$ with $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG)$, the assertion follows from Theorem 10.1 and Proposition 2.2. $\square$

The result below implies Theorem 1.1 from the introduction.

Theorem 10.4. *The maps*

$$\left\{ \begin{array}{c} \textit{non-zero tensor ideal localising} \\ \textit{subcategories of } \mathsf{Mod}\,kG \end{array} \right\} \xrightleftharpoons[\tau]{\sigma} \{\textit{subsets of } \mathrm{Proj}\,H^*(G,k)\}$$

given by the obvious analogue of (3.7) *are mutually inverse bijections.*

Proof. This follows from Theorem 10.3 and Proposition 2.1. □

In [8], it was proved that if G is a finite p-group then there is an equivalence of triangulated categories

$$\mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\sim} \mathsf{D}(C^*(BG;k))$$

where $C^*(BG;k)$ denotes the dg algebra of cochains on the classifying space BG of G. This gives $\mathsf{D}(C^*(BG;k))$ the structure of a tensor triangulated category. Composing with the canonical map one gets an action

$$(10.5) \qquad\qquad H^*(G,k) \to Z^*\mathsf{K}(\mathsf{Inj}\,kG) \xrightarrow{\sim} Z^*\mathsf{D}(C^*(BG;k)).$$

Theorem 10.6. *If G is a finite p-group then the tensor triangulated category* $\mathsf{D}(C^*(BG;k))$ *is stratified by the action of $H^*(G,k)$ given by* (10.5). *The maps*

$$\{\textit{localising subcategories of } \mathsf{D}(C^*(BG;k))\} \xrightleftharpoons[\tau]{\sigma} \{\textit{subsets of } \mathrm{Spec}\,H^*(G,k)\}$$

described in (3.7) *are mutually inverse bijections.*

Proof. This follows from Theorem 10.1 and the observation that every localising subcategory is tensor ideal. □

11. Applications

In this section we deduce the principal theorems of Benson, Carlson, and Rickard [4, 5] from Theorem 10.1 without using shifted subgroups, any form of Dade's lemma, or algebraic closure of the field. Then we make various other deductions from the main theorem. We classify localising subcategories closed under products, and show that these are the same as those closed under Brown-Comenetz duality. We classify the smashing subcategories, and show that the telescope conjecture holds for $\mathsf{StMod}(kG)$ and $\mathsf{K}(\mathsf{Inj}\,kG)$. Finally, we find the left perpendicular categories to localising subcategories. Note that similar applications can be formulated for dg modules over graded polynomial algebras and graded exterior algebras; this is left to the interested reader; see [7].

Throughout this section, we abbreviate $\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,kG)}(-,-)$ to $\mathrm{Hom}_{\mathsf{K}(kG)}(-,-)$.

The tensor product theorem.

Part (2) of the result below was proved by Benson, Carlson, and Rickard [4] under the additional hypothesis that k is algebraically closed.

Theorem 11.1. *Let G be a finite group and k be a field of characteristic p.*

(1) *If X, Y are objects in $\mathsf{K}(\mathsf{Inj}\,kG)$, then $\mathcal{V}_G(X \otimes_k Y) = \mathcal{V}_G(X) \cap \mathcal{V}_G(Y)$.*

(2) *If M, N are objects in $\mathsf{StMod}(kG)$, then $\mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N)$.*

Proof. Part (2) follows from (1) via the identification of $\mathsf{StMod}\, kG$ with $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\, kG)$.

(1) One has an isomorphism $\Gamma_{\mathfrak{p}}(X \otimes_k Y) \cong \Gamma_{\mathfrak{p}} X \otimes_k \Gamma_{\mathfrak{p}} Y$, which yields an inclusion $\mathcal{V}_G(X \otimes_k Y) \subseteq \mathcal{V}_G(X) \cap \mathcal{V}_G(Y)$. Conversely, if $\mathfrak{p} \in \mathcal{V}_G(X) \cap \mathcal{V}_G(Y)$ then $\Gamma_{\mathfrak{p}} X \neq 0$ and $\Gamma_{\mathfrak{p}} Y \neq 0$. Theorem 10.1 implies that $\Gamma_{\mathfrak{p}} ik$ is in $\mathsf{Loc}^{\otimes}(\Gamma_{\mathfrak{p}} X)$, hence that $\Gamma_{\mathfrak{p}} Y$ is in $\mathsf{Loc}^{\otimes}(\Gamma_{\mathfrak{p}}(X \otimes_k Y))$. Since $\Gamma_{\mathfrak{p}} Y \neq 0$ it follows that $\Gamma_{\mathfrak{p}}(X \otimes_k Y) \neq 0$. $\qquad\square$

The subgroup theorem.

The theorem below strengthens Theorem 9.5 from elementary abelian p-groups to all finite groups. For $\mathsf{StMod}(kG)$ and k algebraically closed it was proved in [4].

Theorem 11.2. *Let $H \leq G$ be a subgroup. For any complex X in $\mathsf{K}(\mathsf{Inj}\, kG)$ there is an equality*
$$\mathcal{V}_H(X \!\downarrow_H) = (\mathrm{res}_{G,H}^*)^{-1} \mathcal{V}_G(X).$$
The analogous statement where $\mathsf{K}(\mathsf{Inj}\, kG)$ is replaced by $\mathsf{StMod}(kG)$ also holds.

Proof. The argument for $\mathsf{K}(\mathsf{Inj}\, kG)$ is the same as the proof of Theorem 9.5 except that one uses Theorem 10.1 instead of Theorem 8.1. The statement about $\mathsf{StMod}(kG)$ is deduced by identifying it with the acyclic complexes in $\mathsf{K}(\mathsf{Inj}\, kG)$. $\qquad\square$

The thick subcategory theorem.

Recall from [24, Proposition 2.3] that the compact objects in $\mathsf{K}(\mathsf{Inj}\, kG)$ are the semi-injective resolutions of bounded complexes of finitely generated kG-modules, and that the canonical functor $\mathsf{K}(\mathsf{Inj}\, kG) \to \mathsf{D}(\mathsf{Mod}\, kG)$ induces an equivalence of triangulated categories $\mathsf{K}^{\mathsf{c}}(\mathsf{Inj}\, kG) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$; we view this as an identification.

Lemma 11.3. *Let C be a direct sum of the simple kG-modules. For any complex X in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$ one has $\mathcal{V}_G(X) = \mathcal{V}(\mathfrak{a})$ where $\mathfrak{a}$ is the annihilator of the $H^*(G, k)$-module $H^*(G, C \otimes_k X)$.*

Proof. This is a restatement of [6, Theorem 5.5 (1)] in this context. $\qquad\square$

In view of the equivalence of tensor triangulated categories
$$\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)/\mathsf{D}^{\mathsf{b}}(\mathsf{proj}\, kG) \xrightarrow{\sim} \mathsf{stmod}(kG)$$
given by [31, Theorem 2.1], the following theorem generalises, with a new proof, a classification of the tensor ideal thick subcategories of $\mathsf{stmod}(kG)$ from [5].

Theorem 11.4. *There is a one to one correspondence between tensor ideal thick subcategories of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$ and specialisation closed subsets $\mathcal{V}$ of $\mathcal{V}_G$.*

The thick subcategory corresponding to $\mathcal{V}$ is the full subcategory of objects X in $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$ such that $\mathcal{V}_G(X) \subseteq \mathcal{V}$.

Proof. Let C be a tensor ideal thick subcategory of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG)$, and let C' be the tensor ideal localising subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by C. It follows using the arguments described in [32, §5] that $\mathsf{C}' \cap \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kG) = \mathsf{C}$. The supports of C and C' coincide, so the map sending C to its support is injective by Theorem 10.1.

It follows from Lemma 11.3 that the support of a tensor ideal thick subcategory is specialisation closed. On the other hand, if $\mathcal{V}$ is a specialisation closed subset of $\mathcal{V}_G$, then the support of the tensor ideal thick subcategory generated by $\{ik/\!/\mathfrak{p} \mid \mathfrak{p} \in \mathcal{V}\}$ equals $\mathcal{V}$, by Proposition 3.5(2). So every specialisation closed subset of $\mathcal{V}_G$ occurs as the support of some tensor ideal thick subcategory C of $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,kG)$. $\square$

Localising subcategories closed under products and duality.

Let k be a field and $(\mathsf{T}, \otimes, \mathbb{1})$ a k-linear tensor triangulated category. There are two notions of duality in T. The *Spanier-Whitehead dual* $X^{\vee}$ of an object X is defined as the function object $\mathcal{H}om(X, \mathbb{1})$ as in [6, §8] by the adjunction

$$\mathrm{Hom}_{\mathsf{T}}(-\otimes X, \mathbb{1}) \cong \mathrm{Hom}_{\mathsf{T}}(-, \mathcal{H}om(X, \mathbb{1})).$$

The *Brown-Comenetz dual* X^* of an object X is defined by the isomorphism

$$\mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(\mathbb{1}, -\otimes X), k) \cong \mathrm{Hom}_{\mathsf{T}}(-, X^*).$$

The language and notation is borrowed from stable homotopy theory. The commutativity of the tensor product implies that both dualities are self adjoint. Thus we have a natural isomorphism $\mathrm{Hom}_{\mathsf{T}}(X, Y^*) \cong \mathrm{Hom}_{\mathsf{T}}(Y, X^*)$ which gives rise to a natural *biduality morphism* $X \to X^{**}$.

Lemma 11.5. *The following statements hold for each X in T.*

(1) *There is a natural isomorphism $X^* \cong \mathcal{H}om(X, \mathbb{1}^*)$.*
(2) *For each compact object C, applying $\mathrm{Hom}_{\mathsf{T}}(C, -)$ to the morphism $X \to X^{**}$ yields the biduality map*

$$\mathrm{Hom}_{\mathsf{T}}(C, X) \to \mathrm{Hom}_k(\mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(C, X), k), k).$$

(3) *If $X^* = 0$, then $X = 0$.*

Proof. The first claim is a consequence of the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{T}}(-, X^*) &\cong \mathrm{Hom}_{\mathsf{T}}(-\otimes \mathbb{1}, X^*) \\
&\cong \mathrm{Hom}_{\mathsf{T}}(-, \mathcal{H}om(\mathbb{1}, X^*)) \\
&\cong \mathrm{Hom}_{\mathsf{T}}(-, \mathcal{H}om(X, \mathbb{1}^*))
\end{aligned}$$

where the last one is immediate from the definition of the Brown-Comenetz dual.

(2) Set $(-)^{\dagger} = \mathrm{Hom}_k(-, k)$. There is a natural isomorphism $\eta\colon \mathrm{Hom}_{\mathsf{T}}(C, X)^{\dagger} \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{T}}(C^{\vee}, X^*)$ which is compatible with the adjunction isomorphisms for $(-)^*$ and $(-)^{\dagger}$. This follows from the defining isomorphism of the Brown-Comenetz dual and the fact that C is a strongly dualising object. Thus the following diagram

commutes.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{T}}(X^*, X^*) & \xrightarrow{\mathrm{Hom}_{\mathsf{T}}(C^\vee, -)} & \mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(C^\vee, X^*), \mathrm{Hom}_{\mathsf{T}}(C^\vee, X^*)) \\
\Big\downarrow \sim & & \sim \Big\downarrow {\scriptstyle \mathrm{Hom}_k(\eta, \eta^{-1})} \\
& & \mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(C, X)^\dagger, \mathrm{Hom}_{\mathsf{T}}(C, X)^\dagger) \\
& & \Big\downarrow \sim \\
\mathrm{Hom}_{\mathsf{T}}(X, X^{**}) & \xrightarrow{\mathrm{Hom}_{\mathsf{T}}(C, -)} & \mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(C, X), \mathrm{Hom}_{\mathsf{T}}(C, X)^{\dagger\dagger})
\end{array}$$

This justifies the claim.

(3) When $X^* = 0$ it follows from (2) that $\mathrm{Hom}_{\mathsf{T}}(C, X) = 0$ for any compact object C, since k is a field. Thus $X = 0$ as claimed. $\qquad\square$

In $\mathsf{StMod}\, kG$ the function object is $\mathrm{Hom}_k(M, N)$, with diagonal action. The Spanier-Whitehead dual and the Brown-Comenetz dual of kG-modules are closely related. As usual for a kG-module N we write ΩN and $\Omega^{-1} N$ for the kernel of a projective cover of N and the cokernel of an injective envelope of N, respectively.

Proposition 11.6. *In* $\mathsf{StMod}(kG)$ *for each kG-module M there are isomorphisms*

$$M^\vee \cong \mathrm{Hom}_k(M, k) \quad and \quad M^* \cong \Omega\,\mathrm{Hom}_k(M, k).$$

Hence $M^\vee = 0$ if and only if $M^* = 0$ if and only if M is projective.

Proof. The expression for $M^\vee$ is by definition. Set $\mathsf{T} = \mathsf{StMod}(kG)$. For each kG-module N, Tate duality [12, Chapter XII, Theorem 6.4] gives the third isomorphism below:

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{T}}(N, \Omega\,\mathrm{Hom}_k(M, k)) &\cong \mathrm{Hom}_{\mathsf{T}}(\Omega^{-1} N, \mathrm{Hom}_k(M, k)) \\
&\cong \mathrm{Hom}_{\mathsf{T}}(\Omega^{-1} N \otimes_k M, k) \\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(k, \Omega((\Omega^{-1} N) \otimes_k M)), k) \\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathsf{T}}(k, N \otimes_k M), k).
\end{aligned}$$

The other isomorphisms are standard. Thus $M^* \cong \Omega\,\mathrm{Hom}_k(M, k)$.

In $\mathsf{StMod}(kG)$ one has $\Omega N = 0$ if and only if $N = 0$. Therefore the last claim follows from Lemma 11.5(3). $\qquad\square$

The situation in $\mathsf{K}(\mathsf{Inj}\, kG)$ is more complicated. Observe that in this category the function object of complexes X and Y is the complex $\mathrm{Hom}_k(X, Y)$ of injective kG-modules with diagonal action: $(g\phi)(x) = g(\phi(g^{-1}x))$.

Proposition 11.7. *For each X in* $\mathsf{K}(\mathsf{Inj}\, kG)$ *there are isomorphisms*

$$X^\vee = \mathrm{Hom}_k(X, ik) \quad and \quad X^* \cong \mathrm{Hom}_k(X, ik^*) \cong \mathrm{Hom}_k(X, pk).$$

Hence $X^\vee$ is semi-injective, and $X^\vee = 0$ if and only if X is acyclic. Furthermore $X^* = 0$ if and only if $X = 0$.

Proof. The expression for $X^\vee$ is by definition. For X^* use Lemma 11.5(1) and an isomorphism $ik^* \cong pk$, which is a variant of Tate duality:

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{K}(kG)}(-, ik^*) &\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{K}(kG)}(ik, -), k) \\
&\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{K}(kG)}(k, -), k) \\
&\cong \operatorname{Hom}_{\mathsf{K}(kG)}(-, k) \\
&\cong \operatorname{Hom}_{\mathsf{K}(kG)}(-, pk).
\end{aligned}$$

The adjunction isomorphism $\operatorname{Hom}_\mathsf{T}(-, X^\vee) \cong \operatorname{Hom}_\mathsf{T}(X \otimes_k -, ik)$ implies that $X^\vee$ is semi-injective, since ik is semi-injective. Given this it is clear that $X^\vee = 0$ precisely when X is acyclic. The statement about X^* is part of Lemma 11.5. $\square$

Given a subcategory C of a triangulated category T we define full subcategories

$$^\perp\mathsf{C} = \{X \in \mathsf{T} \mid \operatorname{Hom}^*_\mathsf{T}(X, Y) = 0 \text{ for all } Y \text{ in } \mathsf{C}\}$$

$$\mathsf{C}^\perp = \{X \in \mathsf{T} \mid \operatorname{Hom}^*_\mathsf{T}(Y, X) = 0 \text{ for all } Y \text{ in } \mathsf{C}\}.$$

Evidently, $^\perp\mathsf{C}$ is a localising subcategory, and $\mathsf{C}^\perp$ is a colocalising subcategory, i.e., a thick subcategory closed under products.

Theorem 11.8. *For any tensor ideal localising subcategory C of $\mathsf{K}(\operatorname{Inj} kG)$ the following are equivalent:*
 (a) *The subcategory C is closed under products.*
 (b) *The complement of the support of C in $\operatorname{Spec} H^*(G, k)$ is specialisation closed.*
 (c) *The subcategory C is equal to $\mathsf{D}^\perp$ with D a subcategory of compact objects.*
 (d) *The Brown-Comenetz dual of any object in C is also in C.*

The analogous statement where $\mathsf{K}(\operatorname{Inj} kG)$ is replaced by $\mathsf{StMod}(kG)$ and the set $\operatorname{Spec} H^*(G, k)$ is replaced by $\operatorname{Proj} H^*(G, k)$ also holds.

The proof of this result relies on a construction of objects $T(I)$ in $\mathsf{K}(\operatorname{Inj} kG)$ which we recall from [8, §11]. Given an injective $H^*(G, k)$-module I, the object $T(I)$ is defined in terms of the following natural isomorphism

(11.9) $$\operatorname{Hom}_{H^*(G,k)}(H^*(G, -), I) \cong \operatorname{Hom}_{\mathsf{K}(kG)}(-, T(I)).$$

For each prime ideal $\mathfrak{p}$ of $H^*(G, k)$ let $I(\mathfrak{p})$ be the injective envelope of $H^*(G, k)/\mathfrak{p}$.

Lemma 11.10. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in $H^*(G, k)$ with $\mathfrak{q} \subseteq \mathfrak{p}$, and let I be an injective $H^*(G, k)$-module.*
 (1) $\mathcal{V}_G(T(I(\mathfrak{p}))) = \{\mathfrak{p}\}$.
 (2) $T(I(\mathfrak{q}))$ *is a direct summand of a direct product of shifts of $T(I(\mathfrak{p}))$.*
 (3) *The natural morphism $T(I) \to T(I)^{**}$ is a split monomorphism.*

Proof. (1) Let C be a compact object of $\mathsf{K}(\operatorname{Inj} kG)$. Then $H^*(G, C)$ is finitely generated as a module over $H^*(G, k)$. It follows that $\operatorname{Hom}^*_{\mathsf{K}(kG)}(C, T(I(\mathfrak{p})))$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion as a $H^*(G, k)$-module, for it is isomorphic to

$$\operatorname{Hom}^*_{H^*(G,k)}(H^*(G, C), I(\mathfrak{p})).$$

Now apply [6, Corollary 5.9].

(2) The module $I(\mathfrak{q})$ is $\mathfrak{p}$-local and the shifted copies of $I(\mathfrak{p})$ form a set of injective cogenerators for the category of $\mathfrak{p}$-local modules. Thus $I(\mathfrak{q})$ is a direct summand of a product of shifted copies of $I(\mathfrak{p})$. Now apply the functor T and observe that T preserves products.

(3) The induced map $H^*(G, T(I)) \to H^*(G, T(I)^{**})$ is a monomorphism, by Lemma 11.5(2). Applying $\mathrm{Hom}_{H^*(G,k)}(-, I)$ to this map and using (11.9), one gets an epimorphism

$$\mathrm{Hom}_{\mathsf{K}(kG)}(T(I)^{**}, T(I)) \to \mathrm{Hom}_{\mathsf{K}(kG)}(T(I), T(I))$$

which provides an inverse for $T(I) \to T(I)^{**}$. $\qquad\square$

Proof of Theorem 11.8. First we prove the theorem for $\mathsf{K}(\mathsf{Inj}\, kG)$.

(a) $\Rightarrow$ (b): Let $\mathfrak{p}$ be a prime ideal in the support of C. Theorem 10.1 implies that $T(I(\mathfrak{p}))$ is in C, since its support is $\{\mathfrak{p}\}$, by Lemma 11.10(1). Therefore $T(I(\mathfrak{q}))$ is in C for every $\mathfrak{q} \subseteq \mathfrak{p}$ by Lemma 11.10, since C is closed under products. Thus the complement of the support of C is specialisation closed.

(b) $\Rightarrow$ (c): Let $\mathcal{V}$ denote the complement of the support of C. Since it is specialisation closed the localising subcategory $\mathsf{K}_{\mathcal{V}}$ of $\mathsf{K}(\mathsf{Inj}\, kG)$ corresponding to $\mathcal{V}$ is generated by compact objects, by Proposition 3.5. Therefore $\mathsf{K}_{\mathcal{V}}^{\perp} = (\mathsf{K}_{\mathcal{V}}^{\mathsf{c}})^{\perp}$ where $\mathsf{K}_{\mathcal{V}}^{\mathsf{c}}$ denotes the full subcategory consisting of the compact objects in $\mathsf{K}_{\mathcal{V}}$. On the other hand, $\mathsf{K}_{\mathcal{V}}^{\perp}$ is the localising subcategory consisting of all objects with support contained in the complement of $\mathcal{V}$, by [6, Corollary 5.7]. Thus Theorem 10.1 implies $\mathsf{C} = (\mathsf{K}_{\mathcal{V}}^{\mathsf{c}})^{\perp}$.

(c) $\Rightarrow$ (a): This implication is clear.

(c) $\Rightarrow$ (d): Suppose that $\mathsf{C} = \mathsf{D}^{\perp}$ with D a subcategory of compact objects. Fix objects D in D and X in C. We need to show that $\mathrm{Hom}_{\mathsf{K}(kG)}(D, X^*) = 0$. For any compact object C there are isomorphisms

$$\mathrm{Hom}_{\mathsf{K}(kG)}(D \otimes_k C, X) \cong \mathrm{Hom}_{\mathsf{K}(kG)}(D, \mathrm{Hom}_k(C, X))$$
$$\cong \mathrm{Hom}_{\mathsf{K}(kG)}(D, C^{\vee} \otimes_k X) = 0$$

where the last one holds because C is tensor ideal. Hence $D \otimes_k C$ is in $^{\perp}\mathsf{C}$ for any compact object C. Thus $D^{\vee} \otimes_k C$ is in $^{\perp}\mathsf{C}$ as well, since $D^{\vee}$ is a direct summand of $D^{\vee} \otimes_k D \otimes_k D^{\vee}$. A similar argument now yields

$$\mathrm{Hom}_{\mathsf{K}(kG)}(C, D \otimes_k X) \cong \mathrm{Hom}_{\mathsf{K}(kG)}(C \otimes_k D^{\vee}, X) = 0$$

for any compact object C. Therefore $D \otimes_k X = 0$, so that

$$\mathrm{Hom}_{\mathsf{K}(kG)}(D, X^*) \cong \mathrm{Hom}_{\mathsf{K}(kG)}(D, \mathrm{Hom}_k(X, ik^*))$$
$$\cong \mathrm{Hom}_{\mathsf{K}(kG)}(D \otimes_k X, ik^*) = 0.$$

(d) $\Rightarrow$ (b): Suppose that C is closed under Brown-Comenetz duality, and let $\mathfrak{p}$ be a prime in the support of C. We apply Lemma 11.10 several times. First notice

that $T(I(\mathfrak{p}))$ is in C, by Theorem 10.1. Given any set $\{X_\alpha\}$ where each X_α is a shift of $T(I(\mathfrak{p}))$, it follows that the complex

$$\prod_\alpha X_\alpha^{**} = \left(\bigoplus_\alpha X_\alpha^*\right)^*$$

is in C. The natural map $\prod_\alpha X_\alpha \to \prod_\alpha X_\alpha^{**}$ is a split monomorphism, and hence $\prod_\alpha X_\alpha$ is in C. If $\mathfrak{q} \subseteq \mathfrak{p}$ then $T(I(\mathfrak{q}))$ is a direct summand of a direct product of shifts of $T(I(\mathfrak{p}))$. It follows that $\mathfrak{q}$ is also in the support of C. Thus the complement of this set is specialisation closed.

This completes the proof of the theorem for $\mathsf{K}(\mathsf{Inj}\,kG)$. It remains to consider the category $\mathsf{StMod}(kG)$. We use the same arguments as before for the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (c) $\Rightarrow$ (a), because they are formal, given Theorem 10.3. For the other implications, we identify $\mathsf{StMod}(kG)$ with the category $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG)$ of acyclic complexes and view each tensor ideal localising subcategory of $\mathsf{StMod}(kG)$ as a tensor ideal localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$. Then we use the fact that the inclusion into $\mathsf{K}(\mathsf{Inj}\,kG)$ preserves products and Brown-Comenetz duals. Moreover, the support of $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\,kG)$ equals $\mathrm{Proj}\,H^*(G,k)$, by Proposition 9.6. $\square$

The telescope conjecture for $\mathsf{StMod}(kG)$ and $\mathsf{K}(\mathsf{Inj}\,kG)$.

A localising subcategory C of a triangulated category T is *strictly localising* if the inclusion has a right adjoint. This is equivalent to the statement that there is a localisation functor $L\colon \mathsf{T} \to \mathsf{T}$ such that an object X of T is in C if and only if $LX = 0$; see for example [23, Lemma 3.5]. The obstruction to constructing the right adjoint is that the collections of morphisms in the Verdier quotient may be too large to be sets.

Lemma 11.11. *Tensor ideal localising subcategories of $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{StMod}(kG)$ are strictly localising.*

Proof. Let C be a tensor ideal localising subcategory of $\mathsf{K}(\mathsf{Inj}\,kG)$, and let $\mathcal{V}$ be its support, which is a subset of $\mathrm{Spec}\,H^*(G,k)$. Fix a compact generator C of $\mathsf{K}(\mathsf{Inj}\,kG)$ and consider the functor from $\mathsf{K}(\mathsf{Inj}\,kG)$ to the category of graded abelian groups sending an object X to $\bigoplus_{\mathfrak{p}\notin\mathcal{V}} \mathrm{Hom}^*_{\mathsf{K}(kG)}(C, \Gamma_\mathfrak{p} X)$. Theorem 10.1 implies that its kernel is C. The functor is cohomological and preserves coproducts. Thus there exists a localisation functor $\mathsf{K}(\mathsf{Inj}\,kG) \to \mathsf{K}(\mathsf{Inj}\,kG)$ with kernel C; see for example [6, Proposition 3.6].

An analogous argument works for $\mathsf{StMod}(kG)$. $\square$

A strictly localising subcategory C of a triangulated category T is *smashing* if the localisation functor $\mathsf{T} \to \mathsf{T}$ with kernel C preserves coproducts. If T is tensor triangulated and generated by its tensor unit $\mathbb{1}$, then a localisation functor $L\colon \mathsf{T} \to \mathsf{T}$ preserves coproducts if and only if the natural morphism $L\mathbb{1} \otimes X \to LX$ is an isomorphism for all X in T. This fact explains the term "smashing", because in algebraic topology the smash product plays the role of the tensor product.

Next we discuss the telescope conjecture which is due to Bousfield and Ravenel [9, 30]. In its general form, the conjecture asserts for a compactly generated triangulated category T that every smashing localising subcategory is generated by objects that are compact in T; see [26]. The following result confirms this conjecture for $\mathsf{K}(\mathsf{Inj}\,kG)$ and $\mathsf{StMod}(kG)$, at least for all smashing subcategories that are tensor ideal.

Theorem 11.12. *Let* C *be a tensor ideal localising subcategory of* $\mathsf{K}(\mathsf{Inj}\,kG)$. *The following conditions are equivalent:*
  (a) *The localising subcategory* C *is smashing.*
  (b) *The localising subcategory* C *is generated by objects compact in* $\mathsf{K}(\mathsf{Inj}\,kG)$.
  (c) *The support of* C *is a specialisation closed subset of* $\operatorname{Spec} H^*(G, k)$.
A similar result holds for $\mathsf{StMod}(kG)$ *with* $\operatorname{Spec} H^*(G, k)$ *replaced by* $\operatorname{Proj} H^*(G, k)$.

Proof. We prove the theorem for $\mathsf{K}(\mathsf{Inj}\,kG)$; an analogous argument works for the stable module category. Let $L\colon \mathsf{K}(\mathsf{Inj}\,kG) \to \mathsf{K}(\mathsf{Inj}\,kG)$ be a localisation functor with kernel C. For each object X in $\mathsf{K}(\mathsf{Inj}\,kG)$, there exists a localisation triangle $\Gamma X \to X \to LX \to$ with ΓX in C and LX in $\mathsf{C}^\perp$. Using this exact triangle one shows that C is smashing if and only if $\mathsf{C}^\perp$ is closed under coproducts.

(a) $\Rightarrow$ (b): If C is smashing then $\mathsf{C}^\perp$ is a localising subcategory closed under products. Thus $\mathsf{C}^\perp = \mathsf{D}^\perp$ for some category D consisting of compact objects by Theorem 11.8. It follows that C is generated by D.

(b) $\Rightarrow$ (c): Let D be a subcategory of compact objects such that $\mathsf{C} = \mathsf{Loc}(\mathsf{D})$. Then C and D have same support. Now one uses that the support of each compact object is specialisation closed by Lemma 11.3.

(c) $\Rightarrow$ (a): Let $\mathcal{V}$ be the support of C. Then C consists of all objects with support contained in $\mathcal{V}$, by Theorem 10.1. This implies that $\mathsf{C}^\perp$ is the localising subcategory consisting of all objects with support disjoint from $\mathcal{V}$, since $\mathcal{V}$ is specialisation closed; see [6, Corollary 5.7]. In particular, $\mathsf{C}^\perp$ is closed under coproducts, and therefore C is smashing. $\qquad\square$

Left perpendicular categories.

If $\mathcal{V}$ is a subset of $\operatorname{Spec} H^*(G, k)$, we write $\operatorname{cl}(\mathcal{V})$ for the specialisation closure of $\mathcal{V}$, namely the smallest specialisation closed subset containing it.

Theorem 11.13. *For X and Y in* $\mathsf{K}(\mathsf{Inj}\,kG)$ *the following are equivalent:*
  (a) $\operatorname{Hom}^*_{\mathsf{K}(kG)}(X, Y') = 0$ *for all* $Y' \in \mathsf{Loc}^\otimes(Y)$.
  (b) $\operatorname{cl}(\mathcal{V}_G(X)) \cap \mathcal{V}_G(Y) = \varnothing$

Proof. The implication (b) $\Rightarrow$ (a) is part of [6, Corollary 5.8]. Assume (a) holds. Choose primes $\mathfrak{q} \in \mathcal{V}_G(Y)$ and $\mathfrak{p} \in \mathcal{V}_G(X)$, and a compact object C in $\mathsf{K}(\mathsf{Inj}\,kG)$ with $\operatorname{Hom}_k(C, \Gamma_\mathfrak{p} X) \neq 0$. Since $\mathcal{V}_G(T(I(\mathfrak{q}))) = \{\mathfrak{q}\}$, by Lemma 11.10, Theorem 10.1 yields $C \otimes_k T(I(\mathfrak{q})) \in \mathsf{Loc}^\otimes(Y)$. Since $\Gamma_\mathfrak{p} X = \Gamma_\mathfrak{p} ik \otimes_k X \in \mathsf{Loc}^\otimes(X)$ holds, one has

$$\operatorname{Hom}^*_{H^*(G,k)}(H^*(G, \operatorname{Hom}_k(C, \Gamma_\mathfrak{p} X)), I(\mathfrak{q})) \cong \operatorname{Hom}^*_{\mathsf{K}(kG)}(\operatorname{Hom}_k(C, \Gamma_\mathfrak{p} X), T(I(\mathfrak{q})))$$
$$\cong \operatorname{Hom}^*_{\mathsf{K}(kG)}(\Gamma_\mathfrak{p} X, C \otimes_k T(I(\mathfrak{q}))) = 0.$$

The $H^*(G, k)$-module $H^*(G, \mathrm{Hom}_k(C, \Gamma_{\mathfrak{p}}X))$ is non-zero and $\mathfrak{p}$-local, so it follows that $\mathfrak{p} \not\subseteq \mathfrak{q}$ as required. $\qquad\square$

If $\mathcal{V}$ is a subset of $\mathcal{V}_G$, we write $^\perp\mathcal{V}$ for the set of primes $\mathfrak{q} \in \mathcal{V}_G$ such that for all $\mathfrak{p} \in \mathcal{V}$ we have $\mathfrak{q} \not\supseteq \mathfrak{p}$. In other words, $^\perp\mathcal{V}$ is the largest specialisation closed subset of $\mathcal{V}_G$ that has trivial intersection with $\mathcal{V}$. The statement of the result below was suggested by a question of Jeremy Rickard.

Corollary 11.14. *Let* C *be a tensor ideal localising subcategory of* $\mathsf{K}(\mathsf{Inj}\, kG)$. *If* $\mathcal{V}$ *is the support of* C, *then* $^\perp\mathcal{V}$ *is the support of* $^\perp\mathsf{C}$.

Proof. This follows from Theorem 11.13. $\qquad\square$

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