

COLOCALIZING SUBCATEGORIES AND COSUPPORT

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ABSTRACT. The Hom closed colocalizing subcategories of the stable module category of a finite group are classified. Along the way, the colocalizing subcategories of the homotopy category of injectives over an exterior algebra, and the derived category of a formal commutative differential graded algebra, are classified. To this end, and with an eye towards future applications, a notion of local homology and cosupport for triangulated categories is developed, building on earlier work of the authors on local cohomology and support.

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1. INTRODUCTION

Let G be a finite group and k a field of characteristic p , dividing the order of G , and $\mathbf{StMod}(kG)$ the stable module category of possibly infinite dimensional kG -modules. We write \mathcal{V}_G for the set of all homogeneous prime ideals except the maximal ideal in the cohomology algebra $H^*(G, k)$ of G , and $\mathcal{V}_G(M)$ for the support of any $M \in \mathbf{StMod}(kG)$, defined by Benson, Carlson and Rickard [3] when k is algebraically closed, and extended in [5] to all fields. One of the main results in this work is a classification of the colocalizing subcategories of $\mathbf{StMod}(kG)$:

Theorem 1.1. *The map that assigns to each subset $\mathcal{U} \subseteq \mathcal{V}_G$ the subcategory*

$$\{N \in \mathbf{StMod}(kG) \mid \underline{\mathrm{Hom}}_{kG}(M, N) = 0 \text{ for all } M \text{ with } \mathcal{V}_G(M) \subseteq \mathcal{U}\}$$

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gives a bijection between subsets of \mathcal{V}_G and colocalizing subcategories of $\mathbf{StMod}(kG)$ that are closed under tensor product with simple kG -modules.

A *colocalizing subcategory* \mathbf{C} is by definition a full triangulated subcategory that is closed under set-indexed products. Such a subcategory is closed under tensor product with simples if and only if it is Hom closed: If N is in \mathbf{C} , so is $\mathrm{Hom}_k(M, N)$ for any $M \in \mathbf{StMod}(kG)$. Theorem 1.1 complements the classification of the localizing subcategories of $\mathbf{StMod}(kG)$ from [5, Theorem 10.3]. Combining them gives a remarkable bijection:

Corollary 1.2. *The map sending a localizing subcategory \mathbf{S} of $\mathbf{StMod}(kG)$ to \mathbf{S}^\perp induces a bijection*

$$\left\{ \begin{array}{l} \text{tensor closed localizing} \\ \text{subcategories of } \mathbf{StMod}(kG) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Hom closed colocalizing} \\ \text{subcategories of } \mathbf{StMod}(kG) \end{array} \right\}.$$

The inverse map sends a colocalizing subcategory \mathbf{S} to ${}^\perp\mathbf{S}$.

Theorem 1.1 and Corollary 1.2, proved in Section 11, are analogues of recent results of Neeman [21] on the derived category of a noetherian commutative ring.

The definition of the inverse of the map in Theorem 1.1 involves a notion of *cosupport* for a module M in $\mathbf{StMod}(kG)$, introduced in this work to be the subset

$$\mathrm{cosupp}_G M = \{\mathfrak{p} \in \mathcal{V}_G \mid \mathrm{Hom}_k(\kappa_{\mathfrak{p}}, M) \text{ is not projective}\},$$

with $\kappa_{\mathfrak{p}}$ the Rickard idempotent module associated to \mathfrak{p} , constructed in [3]. Recall that the support of M is $\{\mathfrak{p} \in \mathcal{V}_G \mid \kappa_{\mathfrak{p}} \otimes M \text{ is not projective}\}$. The inverse map in Theorem 1.1 assigns to a subcategory \mathbf{C} of $\mathbf{StMod}(kG)$ the complement in \mathcal{V}_G of the set $\bigcup_{M \in \mathbf{C}} \mathrm{cosupp}_G M$.

The proof of Theorem 1.1 is modelled on that of [5, Theorem 10.3], where localizing subcategories of $\mathbf{StMod}(kG)$ are classified. It involves a sequence of changes of category, for which reason it has been necessary to develop a theory of cosupport for objects in triangulated categories, along the lines for the one for support in our earlier work [4, 5, 6].

The context is a compactly generated triangulated category \mathbf{T} with set-indexed coproducts endowed with an action of a graded commutative noetherian ring R ; meaning, a homomorphism $R \rightarrow Z^*(\mathbf{T})$ of graded rings from R to the graded center of \mathbf{T} . For each \mathfrak{p} in $\mathrm{Spec} R$, the set of homogeneous prime ideals in R , we introduce a *local homology functor* $\Lambda^{\mathfrak{p}}$, constructed as a right adjoint to the local cohomology functor, $\Gamma_{\mathfrak{p}}$, from [4], and define the *cosupport* of an object X of \mathbf{T} by

$$\mathrm{cosupp}_R X = \{\mathfrak{p} \in \mathrm{Spec} R \mid \Lambda^{\mathfrak{p}} X \neq 0\}.$$

In the first part of this paper, Sections 2 to 5, we establish salient properties of local homology and cosupport; for instance, that the maximal elements with respect to inclusion in the cosupport and the support of any object X in \mathbf{T} coincide:

$$\max(\mathrm{cosupp}_R X) = \max(\mathrm{supp}_R X).$$

This is proved as part of Theorem 4.13. It follows that $\mathrm{cosupp}_R X = \emptyset$ if and only if $\mathrm{supp}_R X = \emptyset$, which is equivalent to $X = 0$ by [4, Theorem 5.2]. These results suggest a close connection between the support and cosupport. However, while the support of an object is well-understood, the cosupport remains a mysterious entity. For instance, the only complete results we could obtain for finitely generated modules over commutative noetherian rings are given in Propositions 4.18 and 4.19.

From Section 8 onwards we turn to colocalizing subcategories of \mathbf{T} , focusing on the case when \mathbf{T} is tensor triangulated with a *canonical R -action*, meaning an

action induced by a homomorphism $R \rightarrow \text{End}_T^*(\mathbb{1})$, where $\mathbb{1}$ is the unit for the tensor product on T . This is the context of the main results of this work, and the rest of this introduction. The category T admits an internal function object, denoted $\mathcal{H}om(X, Y)$, and it is natural to examine the Hom closed colocalizing subcategories of T . A useful result concerning these is that for each $X \in T$ there is an equality

$$\text{Coloc}_T^{\mathcal{H}om}(X) = \text{Coloc}_T^{\mathcal{H}om}(\{\Lambda^{\mathfrak{p}}X \mid \mathfrak{p} \in \text{Spec } R\})$$

which is a form of local-global principle for colocalizing subcategories. This statement is Theorem 8.6 and an analogue of such a local-global principle for localizing subcategories in [5, Theorem 3.6]. The theorem is a first step in our approach to the problem of classifying the Hom closed colocalizing subcategories of T , for it permits one to reduce it to the classification problem for $\Lambda^{\mathfrak{p}}T$, the essential image of the functor $\Lambda^{\mathfrak{p}}$, for each $\mathfrak{p} \in \text{Spec } R$; see Proposition 9.1. We note that $\Lambda^{\mathfrak{p}}T$ is itself colocalizing and Hom closed; see Propositions 4.16 and 8.3.

The following definition thus naturally emerges: *the tensor triangulated category T is costratified by R* if for each $\mathfrak{p} \in \text{Spec } R$ there are no non-trivial Hom closed colocalizing subcategories in $\Lambda^{\mathfrak{p}}T$. Given the discussion above, it is clear that when this property holds the map assigning to a subcategory C the subset $\bigcup_{X \in C} \text{cosupp}_R X$ of $\text{Spec } R$ sets up a bijection

$$\left\{ \begin{array}{c} \text{Hom closed colocalizing} \\ \text{subcategories of } T \end{array} \right\} \xrightarrow{\sim} \{ \text{subsets of } \text{supp}_R T \}.$$

This bijection is Corollary 9.2 and was the main reason for our interest in the costratification condition. However, there are other remarkable consequences that follow from it. For instance, we prove in Theorem 9.7: *if T is costratified by R , it is also stratified by R* , meaning that there are no proper tensor closed localizing subcategories of $\Gamma_{\mathfrak{p}}T$; see [5, 6]. One consequence is that if T is costratified by R then there is a bijection, analogous to the one in Corollary 1.2, between the tensor closed localizing subcategories and the Hom closed colocalizing subcategories of T , via left and right perp; see Corollary 9.9.

In Theorem 9.5 we prove that if T is stratified by R there is an equality

$$\text{cosupp}_R \mathcal{H}om(X, Y) = \text{supp}_R X \cap \text{cosupp}_R Y \quad \text{for all } X, Y \in T.$$

This is an analogue of the tensor product theorem for support [6, Theorem 7.3]. It follows that one gets

$$\text{Hom}_T^*(X, Y) = 0 \quad \Longleftrightarrow \quad \text{supp}_R X \cap \text{cosupp}_R Y = \emptyset$$

provided that the tensor identity generates T ; see Corollary 9.6. This is a surprisingly complete result, for it is often difficult to obtain precise conditions under which there are non-zero maps between objects in a triangulated category.

In Section 11 we prove Theorem 1.1, by establishing that the tensor triangulated category $\text{StMod}(kG)$ is costratified by the canonical action of $H^*(G, k)$. Along the way we prove that the following tensor triangulated categories are costratified:

- The derived category of a formal dg algebra whose cohomology is graded commutative and noetherian; see Theorem 10.3.
- The homotopy category of graded injectives over an exterior algebra, viewed as a dg algebra with zero differential; see Theorem 10.4.
- The homotopy category of complexes of injective kG -modules, where G is a finite group; see Theorem 11.10.

The proofs of these results use much of the material on cosupport and local homology in the preceding sections, as well as results on their behavior under changes

of categories, studied in Section 7. Specialized to the case of a commutative noetherian ring, viewed as a dg algebra concentrated in degree 0, the first item in the preceding list is Neeman's theorem, mentioned at the beginning, which was the inspiration for the results described in this article.

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2. (Co)LOCALIZATION FUNCTORS ON TRIANGULATED CATEGORIES

In this section we collect basic facts about localization and colocalization functors on triangulated categories required in this work; see [4, §3] for details.

Let \mathcal{T} be a triangulated category which admits set-indexed products and coproducts. We write Σ for the suspension functor on \mathcal{T} . The *kernel* of an exact functor $F: \mathcal{T} \rightarrow \mathcal{T}$ is the full subcategory

$$\mathrm{Ker} L = \{X \in \mathcal{T} \mid LX = 0\},$$

while the *essential image* of F is the full subcategory

$$\mathrm{Im} F = \{X \in \mathcal{T} \mid X \cong FY \text{ for some } Y \text{ in } \mathcal{T}\}.$$

A *localizing subcategory* of \mathcal{T} is a full triangulated subcategory that is closed under taking all coproducts. We write $\mathrm{Loc}_{\mathcal{T}}(\mathcal{C})$ for the smallest localizing subcategory containing a given class of objects \mathcal{C} in \mathcal{T} , and call it the localizing subcategory *generated* by \mathcal{C} . Analogously, a *colocalizing subcategory* of \mathcal{T} is a full triangulated subcategory that is closed under taking all products, and $\mathrm{Coloc}_{\mathcal{T}}(\mathcal{C})$ denotes the colocalizing subcategory of \mathcal{T} that is *cogenerated* by \mathcal{C} .

A *localization functor* $L: \mathcal{T} \rightarrow \mathcal{T}$ is an exact functor that admits for each X in \mathcal{T} a natural morphism $\eta X: X \rightarrow LX$, called *adjunction*, such that $L(\eta X)$ is an isomorphism and $L(\eta X) = \eta(LX)$. A functor $\Gamma: \mathcal{T} \rightarrow \mathcal{T}$ is a *colocalization functor* if its opposite functor $\Gamma^{\mathrm{op}}: \mathcal{T}^{\mathrm{op}} \rightarrow \mathcal{T}^{\mathrm{op}}$ is a localization functor; the corresponding natural morphism $\theta X: \Gamma X \rightarrow X$ is called *coadjunction*.

A localization functor $L: \mathcal{T} \rightarrow \mathcal{T}$ is essentially determined by its kernel, which is a localizing subcategory of \mathcal{T} , for it coincides with the kernel of a functor that admits a right adjoint; see [4, Lemma 3.1]. The natural transformation $\eta: \mathrm{Id}_{\mathcal{T}} \rightarrow L$ induces for each object X in \mathcal{T} a natural exact *localization triangle*

$$\Gamma X \longrightarrow X \longrightarrow LX \longrightarrow$$

This exact triangle gives rise to an exact functor $\Gamma: \mathcal{T} \rightarrow \mathcal{T}$ with

$$\mathrm{Ker} L = \mathrm{Im} \Gamma \quad \text{and} \quad \mathrm{Ker} \Gamma = \mathrm{Im} L.$$

The functor Γ is a colocalization, and each colocalization functor on \mathcal{T} arises in this way. This yields a natural bijection between localization and colocalization functors on \mathcal{T} . Note that $\mathrm{Ker} \Gamma$ is a colocalizing subcategory of \mathcal{T} .

Given a subcategory \mathcal{C} of a triangulated category \mathcal{T} we define full subcategories

$${}^{\perp}\mathcal{C} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0 \text{ for all } Y \in \mathcal{C} \text{ and } n \in \mathbb{Z}\}$$

$$\mathcal{C}^{\perp} = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}^*(\Sigma^n Y, X) = 0 \text{ for all } Y \in \mathcal{C} \text{ and } n \in \mathbb{Z}\}.$$

Evidently, ${}^{\perp}\mathcal{C}$ is a localizing subcategory, and \mathcal{C}^{\perp} is a colocalizing subcategory.

The next lemma summarizes the basic facts about localization and colocalization.

Lemma 2.1. *Let \mathcal{T} be a triangulated category and \mathcal{S} a triangulated subcategory. Then the following are equivalent:*

- (1) *There exists a localization functor $L: \mathcal{T} \rightarrow \mathcal{T}$ such that $\text{Ker } L = \mathcal{S}$.*
- (2) *There exists a colocalization functor $\Gamma: \mathcal{T} \rightarrow \mathcal{T}$ such that $\text{Im } \Gamma = \mathcal{S}$.*

In that case both functors are related by a functorial exact triangle

$$\Gamma X \longrightarrow X \longrightarrow LX \longrightarrow .$$

Moreover, there are equalities $\mathcal{S}^\perp = \text{Im } L = \text{Ker } \Gamma$ and ${}^\perp(\mathcal{S}^\perp) = \mathcal{S}$.

Proof. See [4, Lemma 3.3]. □

Remark 2.2. There is a dual version of Lemma 2.1 whose formulation is left to the reader. Note that $\text{Im } F^{\text{op}} = \text{Im } F$ and $\text{Ker } F^{\text{op}} = \text{Ker } F$ for any functor F .

Adjoints. We discuss the formal properties of right adjoints of (co)localization functors. This material is the foundation for local homology and cosupport.

Proposition 2.3. *Let $L, \Gamma: \mathcal{T} \rightarrow \mathcal{T}$ be exact functors such that L is a localization functor, Γ is a colocalization functor, and both induce a functorial exact triangle $\Gamma X \rightarrow X \rightarrow LX \rightarrow$. Then L admits a right adjoint if and only if Γ admits a right adjoint. In that case let Λ and V denote right adjoints of Γ and L , respectively.¹ Then the following holds.*

- (1) *The functor Λ is a localization functor and V is a colocalization functor. They induce a functorial exact triangle*

$$VX \longrightarrow X \longrightarrow \Lambda X \longrightarrow .$$

- (2) *There are identities*

$$(\text{Im } \Gamma)^\perp = \text{Ker } \Gamma = \text{Im } L = \text{Im } V = \text{Ker } \Lambda = {}^\perp(\text{Im } \Lambda) .$$

- (3) *There are isomorphisms*

$$\Lambda \Gamma \xrightarrow{\sim} \Lambda, \quad \Gamma \xrightarrow{\sim} \Gamma \Lambda, \quad V L \xrightarrow{\sim} L, \quad \text{and} \quad V \xrightarrow{\sim} L V .$$

- (4) *The functors Γ and Λ induce mutually quasi-inverse equivalences*

$$\text{Im } \Lambda \xrightarrow{\sim} \text{Im } \Gamma \quad \text{and} \quad \text{Im } \Gamma \xrightarrow{\sim} \text{Im } \Lambda .$$

Remark 2.4. The functors L, Γ, Λ, V occurring in the preceding proposition induce the following recollement

$$\mathcal{S} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{\text{inc}} \\ \xleftarrow{V} \end{array} \mathcal{T} \begin{array}{c} \xleftarrow{\bar{\Gamma}} \\ \xrightarrow{Q} \\ \xleftarrow{\bar{\Lambda}} \end{array} \mathcal{T}/\mathcal{S}$$

where $\mathcal{S} = \text{Im } L = \text{Im } V$ and $Q: \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$ denotes the quotient functor so that $\bar{\Gamma} = \Gamma Q$ and $\bar{\Lambda} = \Lambda Q$.

Proof of Proposition 2.3. It follows from Proposition A.4 that L admits a right adjoint if and only if there exists a colocalization functor $V: \mathcal{T} \rightarrow \mathcal{T}$ with $\text{Im } V = \text{Im } L$. Using Lemma 2.1 and the fact that $\text{Im } L = \text{Ker } \Gamma$, it follows that the existence of V is equivalent to the existence of a localization functor $\Lambda: \mathcal{T} \rightarrow \mathcal{T}$ with $\text{Ker } \Lambda = \text{Ker } \Gamma$. Proposition A.5 and Remark A.6 imply that the existence of Λ is equivalent to the existence of a right adjoint of Γ .

(1) The properties of Λ and V are explained above. The existence of the functorial exact triangle then follows from Lemma 2.1 as $\text{Ker } \Lambda = \text{Im } V$.

¹It is customary to write L for a localization functor. A colocalization functor is a localization functor for the opposite category; we denote it $\bar{\Gamma}$, thought of as L turned upside down. The interpretation of local cohomology in the sense of Grothendieck as colocalization provides another reason for the use of $\bar{\Gamma}$. Local homology in the sense of Greenlees and May is denoted $\bar{\Lambda}$; it is a right adjoint of $\bar{\Gamma}$ and hence a localization. The corresponding colocalization is thus denoted V .

- (2) The identities follow from the first part of the proof and Lemma 2.1.
- (3) Combine the localization triangles for L and Λ with the identities in (2).
- (4) The isomorphisms in (3) induce isomorphisms

$$\Lambda\Gamma\Lambda \cong \Lambda^2 \cong \Lambda \quad \text{and} \quad \Gamma\Lambda\Gamma \cong \Gamma^2 \cong \Gamma.$$

Thus $\Lambda\Gamma$ is isomorphic to the identity on $\text{Im } \Lambda$, while $\Gamma\Lambda$ is isomorphic to the identity on $\text{Im } \Gamma$. \square

3. LOCAL COHOMOLOGY AND SUPPORT

In this section we recall the construction, and basic properties, of local cohomology functors and support for triangulated categories, from [4, 6].

Compact generation. An object C in a triangulated category T admitting set-indexed coproducts is *compact* if the functor $\text{Hom}_{\mathsf{T}}(C, -)$ commutes with all coproducts. We write T^c for the full subcategory of compact objects in T . The category T is *compactly generated* if it is generated by a set of compact objects.

Recall that we write Σ for the suspension on T . For objects X and Y in T , let

$$\text{Hom}_{\mathsf{T}}^*(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathsf{T}}(X, \Sigma^i Y)$$

be the graded abelian group of morphisms. Set $\text{End}_{\mathsf{T}}^*(X) = \text{Hom}_{\mathsf{T}}^*(X, X)$; this is a graded ring, and $\text{Hom}_{\mathsf{T}}^*(X, Y)$ is a right $\text{End}_{\mathsf{T}}^*(X)$ and left $\text{End}_{\mathsf{T}}^*(Y)$ -bimodule.

Central ring actions. Let R be a graded-commutative ring; thus R is \mathbb{Z} -graded and satisfies $rs = (-1)^{|r||s|}sr$ for each pair of homogeneous elements r, s in R . We say that a triangulated category T is *R -linear*, or that R *acts* on T , if there is a homomorphism $\phi: R \rightarrow Z^*(\mathsf{T})$ of graded rings, where $Z^*(\mathsf{T})$ is the graded center of T . This yields for each object X a homomorphism $\phi_X: R \rightarrow \text{End}_{\mathsf{T}}^*(X)$ of graded rings such that for all objects $X, Y \in \mathsf{T}$ the R -module structures on $\text{Hom}_{\mathsf{T}}^*(X, Y)$ induced by ϕ_X and ϕ_Y agree, up to the usual sign rule.

Henceforth T will be a compactly generated triangulated category with set-indexed coproducts, and R a graded-commutative noetherian ring acting on T .

Since T is compactly generated with set-indexed coproducts, it follows from the Brown representability theorem that T also admits set-indexed products; see [20, Proposition 8.4.6]. This fact is used without further comment.

Local cohomology and support. We write $\text{Spec } R$ for the set of homogeneous prime ideals of R . Fix $\mathfrak{p} \in \text{Spec } R$ and let M be a graded R -module. The homogeneous localization of M at \mathfrak{p} is denoted by $M_{\mathfrak{p}}$ and M is called *\mathfrak{p} -local* when the natural map $M \rightarrow M_{\mathfrak{p}}$ is bijective.

Given a homogeneous ideal \mathfrak{a} in R , we set

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{p} \supseteq \mathfrak{a}\}.$$

A graded R -module M is *\mathfrak{a} -torsion* if each element of M is annihilated by a power of \mathfrak{a} ; equivalently, if $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \text{Spec } R \setminus \mathcal{V}(\mathfrak{a})$.

The *specialization closure* of a subset \mathcal{U} of $\text{Spec } R$ is the set

$$\text{cl } \mathcal{U} = \{\mathfrak{p} \in \text{Spec } R \mid \text{there exists } \mathfrak{q} \in \mathcal{U} \text{ with } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

The subset \mathcal{U} is *specialization closed* if $\text{cl}\mathcal{U} = \mathcal{U}$; equivalently, if \mathcal{U} is a union of Zariski closed subsets of $\text{Spec } R$. For each specialization closed subset \mathcal{V} of $\text{Spec } R$, we define the full subcategory of \mathbf{T} of \mathcal{V} -torsion objects as follows:

$$\mathbf{T}_{\mathcal{V}} = \{X \in \mathbf{T} \mid \text{Hom}_{\mathbf{T}}^*(C, X)_{\mathfrak{p}} = 0 \text{ for all } C \in \mathbf{T}^c, \mathfrak{p} \in \text{Spec } R \setminus \mathcal{V}\}.$$

This is a localizing subcategory and there exists a localization functor $L_{\mathcal{V}}: \mathbf{T} \rightarrow \mathbf{T}$ such that $\text{Ker } L_{\mathcal{V}} = \mathbf{T}_{\mathcal{V}}$; see [4, Lemma 4.3, Proposition 4.5].

The localization functor $L_{\mathcal{V}}$ induces a colocalization functor on \mathbf{T} , which we denote $\Gamma_{\mathcal{V}}$, and call the *local cohomology functor* with respect to \mathcal{V} ; see Section 2. For each object X in \mathbf{T} there is then an exact localization triangle

$$\Gamma_{\mathcal{V}}X \longrightarrow X \longrightarrow L_{\mathcal{V}}X \longrightarrow .$$

In [4] we established a number of properties of these functors; for instance, that they commute with all coproducts in \mathbf{T} , see [4, Corollary 6.5].

For each \mathfrak{p} in $\text{Spec } R$ and each object X in \mathbf{T} set

$$X_{\mathfrak{p}} = L_{\mathcal{Z}(\mathfrak{p})}X, \quad \text{where } \mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \text{Spec } R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

The notation is justified by the fact that, by [4, Theorem 4.7], the adjunction morphism $X \rightarrow X_{\mathfrak{p}}$ induces for any compact object C an isomorphism of R -modules

$$\text{Hom}_{\mathbf{T}}^*(C, X)_{\mathfrak{p}} \xrightarrow{\sim} \text{Hom}_{\mathbf{T}}^*(C, X_{\mathfrak{p}}).$$

We say X is \mathfrak{p} -local if the adjunction morphism $X \rightarrow X_{\mathfrak{p}}$ is an isomorphism; this is equivalent to the condition that there exists *some* isomorphism $X \cong X_{\mathfrak{p}}$ in \mathbf{T} .

Consider the exact functor $\Gamma_{\mathfrak{p}}: \mathbf{T} \rightarrow \mathbf{T}$ obtained by setting

$$\Gamma_{\mathfrak{p}}X = \Gamma_{\mathcal{V}(\mathfrak{p})}(X_{\mathfrak{p}}) \quad \text{for each object } X \text{ in } \mathbf{T},$$

and let $\Gamma_{\mathfrak{p}}\mathbf{T}$ denote its essential image. One has a natural isomorphism $\Gamma_{\mathfrak{p}}^2 \cong \Gamma_{\mathfrak{p}}$, and an object X from \mathbf{T} is in $\Gamma_{\mathfrak{p}}\mathbf{T}$ if and only if the R -module $\text{Hom}_{\mathbf{T}}^*(C, X)$ is \mathfrak{p} -local and \mathfrak{p} -torsion for every compact object C ; see [4, Corollary 4.10].

The *support* of an object X in \mathbf{T} is by definition the set

$$\text{supp}_R X = \{\mathfrak{p} \in \text{Spec } R \mid \Gamma_{\mathfrak{p}}X \neq 0\}.$$

One has $\text{supp}_R X = \emptyset$ if and only if $X = 0$ holds; see [4, Theorem 5.2].

Koszul objects. For each object X in \mathbf{T} and each homogeneous ideal \mathfrak{a} in R , we denote $X//\mathfrak{a}$ a Koszul object on a finite sequence of elements generating the ideal \mathfrak{a} ; see [4, §5]. Its construction depends on a choice of a generating sequence, but the localizing subcategory generated by it is independent of choice, and depends only on the radical ideal of \mathfrak{a} ; this follows from [6, Proposition 2.11(2)]. Set

$$X(\mathfrak{p}) = (X//\mathfrak{p})_{\mathfrak{p}} \quad \text{for each } \mathfrak{p} \in \text{Spec } R.$$

The following computations will be used often:

$$(3.1) \quad \text{supp}_R(X//\mathfrak{p}) = \mathcal{V}(\mathfrak{p}) \cap \text{supp}_R X \quad \text{and} \quad \text{supp}_R X(\mathfrak{p}) = \{\mathfrak{p}\} \cap \text{supp}_R X$$

For the first one, see [6, Lemma 2.6]; the second follows, given [4, Theorem 5.6].

The first part of the result below is [4, Theorem 6.4], see also [6, Proposition 2.7]; the second one is part of [6, Proposition 3.9].

Theorem 3.2. *Suppose \mathbf{G} is a set of compact generators for \mathbf{T} . For each specialization closed subset \mathcal{V} and $\mathfrak{p} \in \text{Spec } R$, there are equalities*

$$\mathbf{T}_{\mathcal{V}} = \text{Loc}_{\mathbf{T}}(C//\mathfrak{p} \mid C \in \mathbf{G} \text{ and } \mathfrak{p} \in \mathcal{V}) \quad \text{and} \quad \Gamma_{\mathfrak{p}}\mathbf{T} = \text{Loc}_{\mathbf{T}}(C(\mathfrak{p}) \mid C \in \mathbf{G}),$$

where both generating sets consist of compact objects. \square

4. LOCAL HOMOLOGY AND COSUPPORT

Let T denote a compactly generated R -linear triangulated category, as in Section 3. We introduce local homology functors and a notion of cosupport for T .

Local homology. Fix a specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec} R$. The functors $L_{\mathcal{V}}$ and $\Gamma_{\mathcal{V}}$ on T preserve coproducts by [4, Corollary 6.5] and hence have right adjoints, by Brown representability. Following the notation in Proposition 2.3, this yields adjoint pairs $(L_{\mathcal{V}}, V^{\mathcal{V}})$ and $(\Gamma_{\mathcal{V}}, \Lambda^{\mathcal{V}})$, and, for each $X \in \mathsf{T}$, an exact triangle

$$(4.1) \quad V^{\mathcal{V}}X \longrightarrow X \longrightarrow \Lambda^{\mathcal{V}}X \longrightarrow$$

We call $\Lambda^{\mathcal{V}}$ the *local homology functor* with respect to \mathcal{V} ; see Remark 4.17.

The commutation rules for the functors $L_{\mathcal{V}}$ and $\Gamma_{\mathcal{V}}$ given in [4, Proposition 6.1] carry over to their right adjoints: For any specialization closed subset \mathcal{W} of $\operatorname{Spec} R$ there are isomorphisms:

$$(4.2) \quad \begin{aligned} \Lambda^{\mathcal{V}}\Lambda^{\mathcal{W}} &\cong \Lambda^{\mathcal{V} \cap \mathcal{W}} \cong \Lambda^{\mathcal{W}}\Lambda^{\mathcal{V}} \\ V^{\mathcal{V}}V^{\mathcal{W}} &\cong V^{\mathcal{V} \cup \mathcal{W}} \cong V^{\mathcal{W}}V^{\mathcal{V}} \\ V^{\mathcal{V}}\Lambda^{\mathcal{W}} &\cong \Lambda^{\mathcal{W}}V^{\mathcal{V}} \end{aligned}$$

If $\mathcal{V} \supseteq \mathcal{W}$ holds, then these isomorphisms and Proposition 2.3(2) yield:

$$(4.3) \quad V^{\mathcal{V}}\Lambda^{\mathcal{W}} \cong 0 \cong \Lambda^{\mathcal{W}}V^{\mathcal{V}}.$$

These facts will be used without comment. For each $\mathfrak{p} \in \operatorname{Spec} R$ set

$$\Lambda^{\mathfrak{p}} = \Lambda^{\mathcal{V}(\mathfrak{p})}V^{\mathcal{Z}(\mathfrak{p})}.$$

Note that $\Lambda^{\mathfrak{p}} \cong V^{\mathcal{Z}(\mathfrak{p})}\Lambda^{\mathcal{V}(\mathfrak{p})}$; that $\Lambda^{\mathfrak{p}} \cong (\Lambda^{\mathfrak{p}})^2$; and that $(\Gamma_{\mathfrak{p}}, \Lambda^{\mathfrak{p}})$ is an adjoint pair.

Cosupport. The *cosupport* of an object X in T is the set

$$\operatorname{cosupp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Lambda^{\mathfrak{p}}X \neq 0\}.$$

Cosupport can be computed using Koszul objects, recalled in Section 3.

Proposition 4.4. *For each object X in T and $\mathfrak{p} \in \operatorname{Spec} R$, one has*

$$\mathfrak{p} \in \operatorname{cosupp}_R X \iff \operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), X) \neq 0 \text{ for some } C \in \mathsf{T}.$$

Moreover, the object C can be chosen from any set of compact generators for T .

Proof. If $\Lambda^{\mathfrak{p}}X \neq 0$, then $\operatorname{Hom}_{\mathsf{T}}(\Lambda^{\mathfrak{p}}X, \Lambda^{\mathfrak{p}}X) \neq 0$, and hence $\operatorname{Hom}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}\Lambda^{\mathfrak{p}}X, X) \neq 0$, since $(\Gamma_{\mathfrak{p}}, \Lambda^{\mathfrak{p}})$ form an adjoint pair. Since $\Gamma_{\mathfrak{p}}\Lambda^{\mathfrak{p}}X$ is evidently in $\Gamma_{\mathfrak{p}}\mathsf{T}$, the last condition implies $\operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), X) \neq 0$ for some compact object C which is part of a generating set for T , by Theorem 3.2.

Conversely, if $\operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), X) \neq 0$ for some $C \in \mathsf{T}$, then since $\Gamma_{\mathfrak{p}}C(\mathfrak{p}) \cong C(\mathfrak{p})$, by, for instance, Theorem 3.2, one obtains that

$$\operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), \Lambda^{\mathfrak{p}}X) \cong \operatorname{Hom}_{\mathsf{T}}(\Gamma_{\mathfrak{p}}C(\mathfrak{p}), X) \cong \operatorname{Hom}_{\mathsf{T}}(C(\mathfrak{p}), X) \neq 0.$$

Thus, $\Lambda^{\mathfrak{p}}X \neq 0$, that is to say, \mathfrak{p} is in $\operatorname{cosupp}_R X$. \square

A important property of cosupport is that it is non-empty for non-zero objects. We deduce this result from the corresponding statement for supports and the result above. For another perspective, see Theorem 4.13.

Theorem 4.5. *For any $X \in \mathsf{T}$, one has $\operatorname{cosupp}_R X = \emptyset$ if and only if $X = 0$.*

Proof. Clearly, $X = 0$ implies $\text{cosupp}_R X = \emptyset$.

If $X \neq 0$, then $\text{supp}_R X \neq \emptyset$, by [4, Theorem 5.2]. Pick a prime \mathfrak{p} in $\text{supp}_R X$, maximal with respect to inclusion. Then $\text{supp}_R(X//\mathfrak{p}) = \{\mathfrak{p}\}$, by (3.1), which implies, in particular, that $X//\mathfrak{p} \neq 0$; equivalently, $\text{Hom}_T(X//\mathfrak{p}, X//\mathfrak{p}) \neq 0$. Moreover, $X//\mathfrak{p}$ is \mathfrak{p} -local, hence isomorphic to $X(\mathfrak{p})$, which explains the isomorphism below:

$$\text{Hom}_T(X(\mathfrak{p}), X//\mathfrak{p}) \cong \text{Hom}_T(X//\mathfrak{p}, X//\mathfrak{p}) \neq 0.$$

Since $X//\mathfrak{p}$ is in $\text{Thick}_T(X)$, by construction, it follows that $\text{Hom}_T(X(\mathfrak{p}), X) \neq 0$. Hence $\mathfrak{p} \in \text{cosupp}_R X$, by Proposition 4.4. \square

Next we describe further basic properties of local homology functors and cosupports. The one below is immediate from the exactness of the functor $\Lambda^{\mathfrak{p}}$.

Proposition 4.6. *For each exact triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ in T , one has*

$$\text{cosupp}_R Y \subseteq \text{cosupp}_R X \cup \text{cosupp}_R Z \quad \text{and} \quad \text{cosupp}_R \Sigma X = \text{cosupp}_R X. \quad \square$$

There is a more precise result for exact triangles (4.1).

Proposition 4.7. *Let \mathcal{V} be a specialization closed subset of $\text{Spec } R$. For each X in T the following equalities hold:*

$$\begin{aligned} \text{cosupp}_R \Lambda^{\mathcal{V}} X &= \mathcal{V} \cap \text{cosupp}_R X \\ \text{cosupp}_R V^{\mathcal{V}} X &= (\text{Spec } R \setminus \mathcal{V}) \cap \text{cosupp}_R X. \end{aligned}$$

Proof. Fix \mathfrak{p} in $\text{Spec } R$. If $\mathfrak{p} \in \mathcal{V}$, then $\Lambda^{\mathcal{V}(\mathfrak{p})} \Lambda^{\mathcal{V}} = \Lambda^{\mathcal{V}(\mathfrak{p})}$, by (4.2), and $V^{\mathcal{Z}(\mathfrak{p})} \Lambda^{\mathcal{V}} = 0$ if $\mathfrak{p} \notin \mathcal{V}$, by (4.3). Hence one gets that:

$$\Lambda^{\mathfrak{p}} \Lambda^{\mathcal{V}} = \begin{cases} \Lambda^{\mathfrak{p}} & \text{if } \mathfrak{p} \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

The identity for $\text{cosupp}_R \Lambda^{\mathcal{V}} X$ follows. The proof of the second one is similar. \square

The preceding result and Theorem 4.5 yield:

Corollary 4.8. *Let $\mathcal{V} \subseteq \text{Spec } R$ be specialization closed and X an object in T . The following conditions are equivalent:*

- (1) $\text{cosupp}_R X \subseteq \mathcal{V}$.
- (2) $X \in \text{Im } \Lambda^{\mathcal{V}}$.
- (3) *the natural map $X \rightarrow \Lambda^{\mathcal{V}} X$ is an isomorphism.* \square

This result above is complemented by:

Corollary 4.9. *Let $\mathcal{V} \subseteq \text{Spec } R$ be specialization closed and X an object in T . The following conditions are equivalent:*

- (1) $\text{cosupp}_R X \subseteq \text{Spec } R \setminus \mathcal{V}$.
- (1') $\text{supp}_R X \subseteq \text{Spec } R \setminus \mathcal{V}$.
- (2) $X \in \text{Im } V^{\mathcal{V}}$.
- (2') $X \in \text{Im } L_{\mathcal{V}}$.
- (3) *The natural map $V^{\mathcal{V}} X \rightarrow X$ is an isomorphism.*
- (3') *The natural map $X \rightarrow L_{\mathcal{V}} X$ is an isomorphism.*

In particular, $\text{cosupp}_R X \subseteq \text{Spec } R \setminus \mathcal{Z}(\mathfrak{p})$ if and only if X is \mathfrak{p} -local.

Proof. The equivalence of (1), (2), and (3) follows from Proposition 4.7 and Theorem 4.5, while the equivalence of (1'), (2'), and (3') is part of [4, Corollary 5.7]. It remains to note that (2) \Leftrightarrow (2'), by Proposition 2.3(2).

The last assertion is (1) \Leftrightarrow (3'), applied to $\mathcal{V} = \mathcal{Z}(\mathfrak{p})$. \square

The next result is an analogue of [4, Corollary 5.8].

Corollary 4.10. *Let X, Y be objects in T . Then $\operatorname{cosupp}_R X \cap \operatorname{cl}(\operatorname{cosupp}_R Y) = \emptyset$ implies $\operatorname{Hom}_{\mathsf{T}}^*(X, Y) = 0$.*

Proof. Set $\mathcal{V} = \operatorname{cl}(\operatorname{cosupp}_R Y)$. Then X is in $\operatorname{Im} V^{\mathcal{V}}$, by Corollary 4.9, and Y is in $\operatorname{Im} \Lambda^{\mathcal{V}}$, by Corollary 4.8, so $\operatorname{Hom}_{\mathsf{T}}^*(X, Y) = 0$, by Proposition 2.3(2). \square

The next goal is Theorem 4.13; the following two results prepare for its proof. The one below is extracted from [4, Lemma 5.11].

Lemma 4.11. *Let \mathfrak{a} be a homogeneous ideal in R . For any objects X and Y in T , the following statements hold.*

- (1) *The R -modules $\operatorname{Hom}_{\mathsf{T}}^*(X//\mathfrak{a}, Y)$ and $\operatorname{Hom}_{\mathsf{T}}^*(X, Y//\mathfrak{a})$ are \mathfrak{a} -torsion.*
- (2) *$\operatorname{Hom}_{\mathsf{T}}^*(X, Y) = 0$ implies $\operatorname{Hom}_{\mathsf{T}}^*(X, Y//\mathfrak{a}) = 0$ and the converse holds when the R -module $\operatorname{Hom}_{\mathsf{T}}^*(X, Y) = 0$ is \mathfrak{a} -torsion.*
- (3) *$\operatorname{Hom}_{\mathsf{T}}^*(X//\mathfrak{a}, Y) = 0$ if and only if $\operatorname{Hom}_{\mathsf{T}}^*(X, Y//\mathfrak{a}) = 0$.* \square

The equality below is a version of (3.1) for cosupport.

Lemma 4.12. *Let \mathfrak{a} be a homogeneous ideal in R and X an object in T . Then*

$$\operatorname{cosupp}_R(X//\mathfrak{a}) = \mathcal{V}(\mathfrak{a}) \cap \operatorname{cosupp}_R X.$$

Proof. From (3.1) one gets an equality

$$\operatorname{supp}_R(C(\mathfrak{p})//\mathfrak{a}) = \mathcal{V}(\mathfrak{a}) \cap \{\mathfrak{p}\} \cap \operatorname{supp}_R C \quad \text{for any } C \in \mathsf{T}.$$

If $\operatorname{Hom}^*(C(\mathfrak{p}), X//\mathfrak{a}) \neq 0$ for some $C \in \mathsf{T}$, then $\operatorname{Hom}^*(C(\mathfrak{p})//\mathfrak{a}, X) \neq 0$; this follows from Lemma 4.11(3). In particular, $C(\mathfrak{p})//\mathfrak{a} \neq 0$, so $\mathfrak{p} \in \mathcal{V}(\mathfrak{a})$, by the equality above. One thus obtains from Proposition 4.4 that $\operatorname{cosupp}_R(X//\mathfrak{a}) \subseteq \mathcal{V}(\mathfrak{a})$.

When $\mathfrak{p} \in \mathcal{V}(\mathfrak{a})$ holds, it follows from Lemma 4.11(1), and the observation that $C(\mathfrak{p})$ is isomorphic to $C_{\mathfrak{p}}//\mathfrak{p}$, that the R -module $\operatorname{Hom}_{\mathsf{T}}^*(C(\mathfrak{p}), X)$ is \mathfrak{p} -torsion, and hence also \mathfrak{a} -torsion. Therefore, Lemma 4.11(2) yields:

$$\operatorname{Hom}_{\mathsf{T}}^*(C(\mathfrak{p}), X) \neq 0 \iff \operatorname{Hom}_{\mathsf{T}}^*(C(\mathfrak{p}), X//\mathfrak{a}) \neq 0.$$

The desired equality involving cosupport now follows from Proposition 4.4. \square

Given $\mathcal{U} \subseteq \operatorname{Spec} R$, we write $\max \mathcal{U}$ for the set of elements $\mathfrak{p} \in \mathcal{U}$ such that $\mathfrak{q} \in \mathcal{U}$ and $\mathfrak{q} \supseteq \mathfrak{p}$ imply $\mathfrak{q} = \mathfrak{p}$. Recall that $\Lambda^{\mathfrak{p}} \mathsf{T}$ denotes the essential image of $\Lambda^{\mathfrak{p}}$.

Theorem 4.13. *For each object X in T there is an equality:*

$$\max(\operatorname{supp}_R X) = \max(\operatorname{cosupp}_R X).$$

Moreover, $\operatorname{cosupp}_R X(\mathfrak{p}) \subseteq \{\mathfrak{p}\}$ and $X(\mathfrak{p}) \in \Lambda^{\mathfrak{p}} \mathsf{T} \cap \Gamma_{\mathfrak{p}} \mathsf{T}$, for each $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. We prove $\max(\operatorname{supp}_R X) \subseteq \operatorname{cosupp}_R X$ and $\max(\operatorname{cosupp}_R X) \subseteq \operatorname{supp}_R X$; the first equality would then follow.

Fix \mathfrak{p} in $\max(\operatorname{supp}_R X)$. Then $\operatorname{supp}_R(X//\mathfrak{p}) = \{\mathfrak{p}\}$, by (3.1), so $X//\mathfrak{p}$ is \mathfrak{p} -local; see Corollary 4.9. It is always \mathfrak{p} -torsion, so $X//\mathfrak{p} = X(\mathfrak{p})$, and then

$$\operatorname{Hom}_{\mathsf{T}}^*(X(\mathfrak{p}), X//\mathfrak{p}) \cong \operatorname{Hom}_{\mathsf{T}}^*(X//\mathfrak{p}, X//\mathfrak{p}) \neq 0.$$

This implies that \mathfrak{p} is in $\operatorname{cosupp}_R(X//\mathfrak{p})$ by Proposition 4.4, hence also that it is in $\operatorname{cosupp}_R X$, by Lemma 4.12.

If $\mathfrak{p} \in \max(\operatorname{cosupp}_R X)$, then $\operatorname{cosupp}_R(X//\mathfrak{p}) = \{\mathfrak{p}\}$, by Lemma 4.12. Therefore the object $X//\mathfrak{p}$ is \mathfrak{p} -local, by Corollary 4.9, and \mathfrak{p} -torsion, so $\operatorname{supp}_R(X//\mathfrak{p}) = \{\mathfrak{p}\}$. It remains to recall (3.1) to conclude that $\mathfrak{p} \in \operatorname{supp}_R X$.

Finally, $\text{cosupp}_R X(\mathfrak{p}) \subseteq \mathcal{V}(\mathfrak{p})$ by Lemma 4.12, since $X(\mathfrak{p}) \cong X_{\mathfrak{p}}//\mathfrak{p}$. Thus the inclusion $\text{supp}_R X(\mathfrak{p}) \subseteq \{\mathfrak{p}\}$ from (3.1) implies the corresponding inclusion for co-support. It then follows from Corollaries 4.8 and 4.9 that $X(\mathfrak{p})$ is in $\Lambda^{\mathfrak{p}}\mathbb{T}$. On the other hand, $X(\mathfrak{p})$ is also in $\Gamma_{\mathfrak{p}}\mathbb{T}$, by [4, Corollary 5.7]. \square

To round off this material, we prove an analogue of Proposition 4.4 for supports.

Proposition 4.14. *For each object X in \mathbb{T} and $\mathfrak{p} \in \text{Spec } R$, one has*

$$\mathfrak{p} \in \text{supp}_R X \iff \text{Hom}_{\mathbb{T}}(X, Y(\mathfrak{p})) \neq 0 \text{ for some } Y \in \mathbb{T}.$$

Proof. When \mathfrak{p} is in $\text{supp}_R X$ it follows from (3.1) that $X(\mathfrak{p}) \neq 0$, hence

$$\text{Hom}_{\mathbb{T}}^*(X//\mathfrak{p}, X(\mathfrak{p})) \cong \text{Hom}_{\mathbb{T}}^*(X(\mathfrak{p}), X(\mathfrak{p})) \neq 0;$$

where the first isomorphism holds as $X(\mathfrak{p})$ is \mathfrak{p} -local. Since $X//\mathfrak{p}$ is in $\text{Thick}_{\mathbb{T}}(X)$, one obtains that $\text{Hom}_{\mathbb{T}}(X, X(\mathfrak{p})) \neq 0$. This settles one implication.

The other implication follows from the chain of isomorphisms

$$\text{Hom}_{\mathbb{T}}(X, Y(\mathfrak{p})) \cong \text{Hom}_{\mathbb{T}}(X, \Lambda^{\mathfrak{p}}Y(\mathfrak{p})) \cong \text{Hom}_{\mathbb{T}}(\Gamma_{\mathfrak{p}}X, Y(\mathfrak{p})),$$

where the first one holds because $\Lambda^{\mathfrak{p}}Y(\mathfrak{p}) \cong Y(\mathfrak{p})$, by Theorem 4.13. \square

Discrete sets. Recall that a subset $\mathcal{U} \subseteq \text{Spec } R$ is *discrete* if $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mathfrak{p} = \mathfrak{q}$ for each pair of primes $\mathfrak{p}, \mathfrak{q} \in \mathcal{U}$.

Proposition 4.15. *Let X and Y be objects of \mathbb{T} and \mathcal{U} a discrete subset of $\text{Spec } R$. When $\text{supp}_R X \subseteq \mathcal{U}$ and $\text{cosupp}_R Y \subseteq \mathcal{U}$ there are natural isomorphisms*

$$X \xleftarrow{\sim} \prod_{\mathfrak{p} \in \mathcal{U}} \Gamma_{\mathcal{V}(\mathfrak{p})} X \xrightarrow{\sim} \prod_{\mathfrak{p} \in \mathcal{U}} \Gamma_{\mathfrak{p}} X \quad \text{and} \quad Y \xrightarrow{\sim} \prod_{\mathfrak{p} \in \mathcal{U}} \Lambda^{\mathcal{V}(\mathfrak{p})} Y \xleftarrow{\sim} \prod_{\mathfrak{p} \in \mathcal{U}} \Lambda^{\mathfrak{p}} Y.$$

Proof. The statement for X is proved in [6, Proposition 3.3]; see also [4, Theorem 7.1]. Modifying the arguments by taking adjunctions, and taking into account Proposition 4.16 below, yields the proof of the statement for Y . \square

Analogues of the next statement hold for $V^{\mathcal{V}}$ and $\Lambda^{\mathcal{V}}$ also.

Proposition 4.16. *Given objects $\{X_i\}_{i \in I}$ in \mathbb{T} there is a natural isomorphism*

$$\Lambda^{\mathfrak{p}} \left(\prod_{i \in I} X_i \right) \xrightarrow{\sim} \prod_{i \in I} \Lambda^{\mathfrak{p}} X_i.$$

In particular, for any subset $\mathcal{U} \subseteq \text{Spec } R$ the full subcategory with objects

$$\{X \in \mathbb{T} \mid \text{cosupp}_R X \subseteq \mathcal{U}\}$$

is a colocalizing subcategory of \mathbb{T} .

Proof. Right adjoints distribute over products. This applies to $\Lambda^{\mathfrak{p}}$, which is right adjoint to $\Gamma_{\mathfrak{p}}$, to yield the desired isomorphism. Given this, the second part of the statement follows, for the subcategory in question equals $\bigcap_{\mathfrak{p} \notin \mathcal{U}} \text{Ker } \Lambda^{\mathfrak{p}}$. \square

Commutative noetherian rings. Let A be a commutative noetherian ring and $D(A)$ the derived category of the category of all A -modules. The category $D(A)$ is triangulated and compactly generated; indeed, A is a compact generator. It is also A -linear, where for each $M \in D(A)$, the homomorphism $A \rightarrow \text{Hom}_D(M, M)$ is given by scalar multiplication.

Remark 4.17. Fix an ideal \mathfrak{a} in A . In [4, Theorem 9.1] it is proved that $\Gamma_{\mathcal{V}(\mathfrak{a})}$ is the derived functor of the \mathfrak{a} -torsion functor, which assigns an A -module M to the module $\varinjlim \mathrm{Hom}_A(A/\mathfrak{a}^n, M)$. Greenlees and May [14, §2], see also Lipman [19, §4], proved that the right adjoint of the latter is local homology and that it coincides with left derived functor of the \mathfrak{a} -adic completion functor, which assigns M to $\varprojlim M/\mathfrak{a}^n M$. In commutative algebra literature (as in [19]), the \mathfrak{a} -adic completion functor itself would usually be denoted $\Lambda_{\mathfrak{a}}$. Our choice of notation, $\Lambda^{\mathcal{V}(\mathfrak{a})}$ for the right adjoint of $\Gamma_{\mathcal{V}(\mathfrak{a})}$ is based on this connection.

The functor $V^{\mathcal{V}}$ too has, in this context, a familiar avatar, at least in the special case $\mathcal{V} = \mathcal{Z}(\mathfrak{p})$ for some \mathfrak{p} in $\mathrm{Spec} A$: The morphism $M \rightarrow M_{\mathfrak{p}} = L_{\mathcal{Z}(\mathfrak{p})} M$ is the usual localization map, see the proof of [4, Theorem 9.1], so it follows from the classical Hom-tensor adjunction isomorphism that its right adjoint is

$$V^{\mathcal{Z}(\mathfrak{p})} M \cong \mathbf{R}\mathrm{Hom}_A(A_{\mathfrak{p}}, M),$$

and the morphism $\mathbf{R}\mathrm{Hom}_A(A_{\mathfrak{p}}, M) \rightarrow M$ is the one induced by $A \rightarrow A_{\mathfrak{p}}$. Hence

$$\mathrm{cosupp}_A M = \{\mathfrak{p} \in \mathrm{Spec} A \mid \mathbf{R}\mathrm{Hom}_A(A_{\mathfrak{p}}, \Lambda^{\mathcal{V}(\mathfrak{p})} M) \neq 0\}.$$

In practice however, the cosupport seems hard to compute, even for $M = A$. What little we know about this is contained in the following results.

Proposition 4.18. *For each M in $\mathrm{D}(\mathbb{Z})$ with H^*M finitely generated, one has*

$$\mathrm{supp}_{\mathbb{Z}} M = \{\mathfrak{p} \in \mathrm{Spec} \mathbb{Z} \mid (H^*M)_{\mathfrak{p}} \neq 0\} = \mathrm{cosupp}_{\mathbb{Z}} M.$$

Proof. The equality on the left is easily verified. For the one on the right, given Theorem 4.13, it suffices to prove that the zero ideal is in $\mathrm{cosupp}_{\mathbb{Z}} M$ if and only if it is also in $\mathrm{supp}_{\mathbb{Z}} M$. By Proposition 4.4, this amounts to verifying that

$$\mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, M) \neq 0 \iff \mathbb{Q} \otimes_{\mathbb{Z}}^{\mathbf{L}} M \neq 0.$$

Since $M \cong H^*M$ in $\mathrm{D}(\mathbb{Z})$ it suffices to verify the equivalence above when M is an indecomposable finitely generated \mathbb{Z} -module, hence of the form $\mathbb{Z}/n\mathbb{Z}$, for some $n \geq 0$. When $n \geq 1$, one has $\mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/n\mathbb{Z}) = 0 = \mathbb{Q} \otimes_{\mathbb{Z}}^{\mathbf{L}} \mathbb{Z}/n\mathbb{Z}$. It remains to verify that $\mathbf{R}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \neq 0$; see [11, §51, Exercise 7]. \square

For complete local rings on the other hand, the difference between cosupport and support can be as large as Theorem 4.13 permits.

Proposition 4.19. *Let A be a commutative noetherian ring and \mathfrak{a} an ideal in A . The following conditions are equivalent:*

- (1) *A is \mathfrak{a} -adically complete.*
- (2) *$\mathrm{cosupp}_A A \subseteq \mathcal{V}(\mathfrak{a})$ holds.*
- (3) *$\mathrm{cosupp}_A M \subseteq \mathcal{V}(\mathfrak{a})$ holds for any $M \in \mathrm{D}(A)$ with H^*M finitely generated.*

Proof. The equivalence of (1) and (2) is contained in Corollary 4.8. Thus it remains to prove that (1) implies (3).

When A is \mathfrak{a} -adically complete so is any finitely generated module. Since the \mathfrak{a} -adic completion functor is exact on finitely generated modules, one obtains an isomorphism $M \xrightarrow{\sim} \Lambda^{\mathcal{V}(\mathfrak{a})} M$ for any such module M , and hence also for any complex M with H^*M finitely generated. Now apply Corollary 4.8. \square

For any object X in \mathbf{T} and $\mathfrak{p} \in \mathrm{Spec} R$, the support of $X_{\mathfrak{p}}$ is contained in that of X ; see [4, Theorem 5.6]. The corresponding statement for cosupports does not hold, which speaks to one of the difficulties in computing this invariant.

Example 4.20. Let (A, \mathfrak{m}) be a local ring that is \mathfrak{m} -adically complete. Then for any $\mathfrak{p} \in \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ one has

$$\{\mathfrak{p}\} \subseteq \operatorname{cosupp}_A(A_{\mathfrak{p}}) \subsetneq \operatorname{cosupp}_A A = \{\mathfrak{m}\}.$$

Indeed, the equality is by Proposition 4.19. That \mathfrak{p} is in $\operatorname{cosupp}_A(A_{\mathfrak{p}})$ can be checked directly, or via Theorem 4.13 and the equality $\operatorname{supp}_A(A_{\mathfrak{p}}) = \operatorname{Spec} A_{\mathfrak{p}}$.

Remark 4.21. The notion of cosupport for an A -module M is not the same as the one introduced by Richardson [23], which is denoted $\operatorname{coSupp} M$. For instance, it follows from [23, Theorem 2.7(vi)] that $\operatorname{coSupp} \mathbb{Z} = \operatorname{Spec} \mathbb{Z} \setminus \{(0)\}$, while Proposition 4.18 yields $\operatorname{cosupp}_{\mathbb{Z}} \mathbb{Z} = \operatorname{Spec} \mathbb{Z}$. When (A, \mathfrak{m}) is local, each non-zero finitely generated module M satisfies $\operatorname{coSupp} M = \{\mathfrak{m}\}$, by [23, Theorem 2.7(i)], while Proposition 4.19 implies $\operatorname{cosupp}_A M = \{\mathfrak{m}\}$ if and only if M is \mathfrak{m} -adically complete.

5. \mathfrak{p} -LOCAL AND \mathfrak{p} -COMPLETE OBJECTS

As before, let \mathbf{T} denote a compactly generated R -linear triangulated category. In this section we investigate the subcategories $\Lambda^{\mathfrak{p}}\mathbf{T}$, for each $\mathfrak{p} \in \operatorname{Spec} R$. They contain local information about \mathbf{T} and are important in classifying its colocalizing subcategories; see the discussion in the last part of this section.

We begin by noting that an object X is in $\Lambda^{\mathfrak{p}}\mathbf{T}$ if and only if $\operatorname{cosupp}_R X \subseteq \{\mathfrak{p}\}$, if and only if X is \mathfrak{p} -local (that is, $X \rightarrow X_{\mathfrak{p}}$ is an isomorphism) and also \mathfrak{p} -complete, meaning that the natural map $X \rightarrow \Lambda^{\vee(\mathfrak{p})} X$ is an isomorphism; see Corollaries 4.9 and 4.8. Also $\Lambda^{\mathfrak{p}}\mathbf{T}$ is a colocalizing subcategory of \mathbf{T} , by Proposition 4.16.

Recall that X is in $\Gamma_{\mathfrak{p}}\mathbf{T}$ if and only if $\operatorname{supp}_R X \subseteq \{\mathfrak{p}\}$, if and only if X is \mathfrak{p} -local and \mathfrak{p} -torsion; see [4, Corollaries 4.9, 5.10], and that $\Gamma_{\mathfrak{p}}\mathbf{T}$ is a localizing subcategory.

Dwyer Greenlees correspondence. Fix a prime \mathfrak{p} in $\operatorname{Spec} R$. The result below, which establishes an equivalence between the category of \mathfrak{p} -local and \mathfrak{p} -complete objects and the category of \mathfrak{p} -local and \mathfrak{p} -torsion objects, may be viewed as extension of such an equivalence discovered by Dwyer and Greenlees [9, Theorem 2.1], see also Hovey, Palmieri, and Strickland [15, Theorem 3.3.5], to our setting.

Proposition 5.1. *The functors $\Gamma_{\mathfrak{p}}: \Lambda^{\mathfrak{p}}\mathbf{T} \rightarrow \Gamma_{\mathfrak{p}}\mathbf{T}$ and $\Lambda^{\mathfrak{p}}: \Gamma_{\mathfrak{p}}\mathbf{T} \rightarrow \Lambda^{\mathfrak{p}}\mathbf{T}$ form an adjoint pair, that are mutually quasi-inverse to each other.*

Proof. Apply the identities in Proposition 2.3. □

As a triangulated category, $\Gamma_{\mathfrak{p}}\mathbf{T}$ is compactly generated by $\{C(\mathfrak{p}) \mid C \in \mathbf{T}^c\}$; see Theorem 3.2. Given Theorem 4.13 and the equivalence $\Gamma_{\mathfrak{p}}\mathbf{T} \xrightarrow{\sim} \Lambda^{\mathfrak{p}}\mathbf{T}$ in Proposition 5.1, it follows that the same set generates $\Lambda^{\mathfrak{p}}\mathbf{T}$, where the coproduct in $\Lambda^{\mathfrak{p}}\mathbf{T}$ is the one induced from $\Gamma_{\mathfrak{p}}\mathbf{T}$.

Next we describe a set of cogenerators for $\Lambda^{\mathfrak{p}}\mathbf{T}$.

Perfect cogeneration. Let \mathbf{U} be a triangulated category with set-indexed products. A set of objects \mathbf{S} *perfectly cogenerates* \mathbf{U} if the following conditions hold:

- (1) If X is an object in \mathbf{U} and $\operatorname{Hom}_{\mathbf{U}}(X, S) = 0$ for all $S \in \mathbf{S}$ then $X = 0$.
- (2) If a countable family of maps $X_i \rightarrow Y_i$ in \mathbf{U} is such that

$$\operatorname{Hom}_{\mathbf{U}}(Y_i, S) \rightarrow \operatorname{Hom}_{\mathbf{U}}(X_i, S)$$

is surjective for all i and all $S \in \mathbf{S}$, then so are the induced maps:

$$\operatorname{Hom}_{\mathbf{U}}\left(\prod_i Y_i, S\right) \rightarrow \operatorname{Hom}_{\mathbf{U}}\left(\prod_i X_i, S\right).$$

Proposition 5.2. *If a set of objects \mathbf{S} perfectly cogenerates \mathbf{U} then $\operatorname{Coloc}_{\mathbf{U}}(\mathbf{S}) = \mathbf{U}$.*

Proof. The proof is akin to that of the corollary in [17, §1]. \square

Injective objects. Let now \mathcal{T} be an R -linear triangulated category as before. For each compact object C in \mathcal{T} and each injective R -module I , Brown representability yields an object $T_C(I)$ in \mathcal{T} and a natural isomorphism:

$$(5.3) \quad \mathrm{Hom}_{\mathcal{T}}(-, T_C(I)) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{T}}^*(C, -), I).$$

For $\mathfrak{p} \in \mathrm{Spec} R$, let $I(\mathfrak{p})$ denote the injective envelope of R/\mathfrak{p} .

Proposition 5.4. *Fix $\mathfrak{p} \in \mathrm{Spec} R$. For each compact object C in \mathcal{T} , one has*

$$\mathrm{cosupp}_R T_C(I(\mathfrak{p})) = \mathrm{supp}_R C \cap \{\mathfrak{q} \in \mathrm{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Moreover, the set $\{T_{C//\mathfrak{p}}(I(\mathfrak{p})) \mid C \in \mathcal{T}^c\}$ perfectly cogenerates $\Lambda^{\mathfrak{p}}\mathcal{T}$.

Proof. The shifts of $I(\mathfrak{p})$ form a set of injective cogenerators for the category of \mathfrak{p} -local R -modules. For any $\mathfrak{q} \in \mathrm{Spec} R$ and object D in \mathcal{T} , one has equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}^*(D(\mathfrak{q}), T_C(I(\mathfrak{p}))) \neq 0 &\iff \mathrm{Hom}_R^*(\mathrm{Hom}_{\mathcal{T}}^*(C, D(\mathfrak{q})), I(\mathfrak{p})) \neq 0 \\ &\iff \mathrm{Hom}_{\mathcal{T}}^*(C, D(\mathfrak{q})) \neq 0 \text{ and } \mathfrak{q} \subseteq \mathfrak{p}. \end{aligned}$$

The first one is by (5.3); the second holds as the R -module $\mathrm{Hom}_{\mathcal{T}}^*(C, D(\mathfrak{q}))$ is \mathfrak{q} -local and \mathfrak{q} -torsion, by (3.1). Propositions 4.4 and 4.14 now yield the stated equality.

Let X be a non-zero object in $\Lambda^{\mathfrak{p}}\mathcal{T}$, so that $\mathrm{cosupp}_R X = \{\mathfrak{p}\}$, and pick a compact object C with $\mathrm{Hom}_{\mathcal{T}}^*(C(\mathfrak{p}), X) \neq 0$; see Proposition 4.4. It then follows that:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}^*(C//\mathfrak{p}, X) &\cong \mathrm{Hom}_{\mathcal{T}}^*(C//\mathfrak{p}, \Lambda^{\mathfrak{p}}X) \\ &\cong \mathrm{Hom}_{\mathcal{T}}^*(\Gamma_{\mathfrak{p}}(C//\mathfrak{p}), X) \\ &\cong \mathrm{Hom}_{\mathcal{T}}^*(C(\mathfrak{p}), X) \\ &\neq 0 \end{aligned}$$

Replacing C by an appropriate suspension $\Sigma^n C$, if necessary, from the computation above and (5.3) one gets

$$\mathrm{Hom}_{\mathcal{T}}(X, T_{C//\mathfrak{p}}(I(\mathfrak{p}))) \cong \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{T}}^*(C//\mathfrak{p}, X), I(\mathfrak{p})) \neq 0.$$

The other condition for perfect cogeneration holds because, for any map $X \rightarrow Y$ in $\Lambda^{\mathfrak{p}}\mathcal{T}$, the induced map

$$\mathrm{Hom}_{\mathcal{T}}^*(Y, T_{C//\mathfrak{p}}(I(\mathfrak{p}))) \rightarrow \mathrm{Hom}_{\mathcal{T}}^*(X, T_{C//\mathfrak{p}}(I(\mathfrak{p})))$$

is surjective if and only if $\mathrm{Hom}_{\mathcal{T}}^*(C//\mathfrak{p}, X) \rightarrow \mathrm{Hom}_{\mathcal{T}}^*(C//\mathfrak{p}, Y)$ is injective. \square

Classifying colocalizing subcategories. We say that the *local-global principle* holds for colocalizing subcategories of \mathcal{T} if for each object X in \mathcal{T} there is an equality

$$\mathrm{Coloc}_{\mathcal{T}}(X) = \mathrm{Coloc}_{\mathcal{T}}(\{\Lambda^{\mathfrak{p}}X \mid \mathfrak{p} \in \mathrm{Spec} R\}).$$

The corresponding notion for localizing subcategories is investigated in [6, §3]. The reformulation below of the local-global principle is easy to prove.

Lemma 5.5. *The local-global principle for colocalizing subcategories is equivalent to the statement: For any $X \in \mathcal{T}$ and any colocalizing subcategory \mathcal{S} of \mathcal{T} , one has*

$$X \in \mathcal{S} \iff \Lambda^{\mathfrak{p}}X \in \mathcal{S} \text{ for each } \mathfrak{p} \in \mathrm{Spec} R. \quad \square$$

Colocalizing subcategories of \mathcal{T} are related to subsets of $\mathrm{Spec} R$ via maps

$$\left\{ \begin{array}{c} \text{colocalizing} \\ \text{subcategories of } \mathcal{T} \end{array} \right\} \xrightleftharpoons[\tau]{\sigma} \left\{ \begin{array}{c} \text{families } (\mathcal{S}(\mathfrak{p}))_{\mathfrak{p} \in \mathrm{Spec} R} \text{ with } \mathcal{S}(\mathfrak{p}) \\ \text{a colocalizing subcategory of } \Lambda^{\mathfrak{p}}\mathcal{T} \end{array} \right\}$$

which are defined by $\sigma(\mathcal{S}) = (\mathcal{S} \cap \Lambda^{\mathfrak{p}}\mathcal{T})$ and $\tau(\mathcal{S}(\mathfrak{p})) = \mathrm{Coloc}_{\mathcal{T}}(\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec} R)$.

Proposition 5.6. *If the local-global principle for colocalizing subcategories of \mathbf{T} holds, then the map σ is bijective, with inverse τ .*

Proof. We use the fact that $\Lambda^{\mathbf{p}}$ is an idempotent exact functor preserving products.

First observe that for each colocalizing subcategory \mathbf{S} of \mathbf{T} there is an inclusion

$$(*) \quad \mathbf{S} \cap \Lambda^{\mathbf{p}}\mathbf{T} \subseteq \Lambda^{\mathbf{p}}\mathbf{S} \quad \text{for each } \mathbf{p} \in \text{Spec } R.$$

We prove that $\sigma\tau$ is the identity, that is to say, that for any family $(\mathbf{S}(\mathbf{p}))_{\mathbf{p} \in \text{Spec } R}$ of colocalizing subcategories with $\mathbf{S}(\mathbf{p}) \subseteq \Lambda^{\mathbf{p}}\mathbf{T}$ the colocalizing subcategory cogenerated by all the $\mathbf{S}(\mathbf{p})$, call it \mathbf{S} , satisfies

$$\mathbf{S} \cap \Lambda^{\mathbf{p}}\mathbf{T} = \mathbf{S}(\mathbf{p}) \quad \text{for each } \mathbf{p} \in \text{Spec } R.$$

Note that $\Lambda^{\mathbf{p}}\mathbf{S} = \mathbf{S}(\mathbf{p})$ holds, since $\Lambda^{\mathbf{p}}\Lambda^{\mathbf{q}} = 0$ when $\mathbf{p} \neq \mathbf{q}$. Therefore $(*)$ yields an inclusion $\mathbf{S} \cap \Lambda^{\mathbf{p}}\mathbf{T} \subseteq \mathbf{S}(\mathbf{p})$. The reverse inclusion is obvious.

For any localizing subcategory \mathbf{S} of \mathbf{T} , the reformulation of the local-global principle in Lemma 5.5 gives $\mathbf{S} = \text{Coloc}_{\mathbf{T}}(\mathbf{S} \cap \Lambda^{\mathbf{p}}\mathbf{T} \mid \mathbf{p} \in \text{Spec } R)$. Thus $\tau\sigma = \text{id}$. \square

Remark 5.7. In analogy with the notion of stratification for localizing subcategories of \mathbf{T} introduced in [6, §4], we say that \mathbf{T} is *costratified* by R if

- (C1) The local-global principle holds for colocalizing subcategories of \mathbf{T} ;
- (C2) For each $\mathbf{p} \in \text{Spec } R$, the colocalizing subcategory $\Lambda^{\mathbf{p}}\mathbf{T}$ contains no proper non-zero colocalizing subcategories.

It is immediate from Proposition 5.6 that when these conditions hold, the maps σ and τ induce a bijection

$$\left\{ \begin{array}{c} \text{Colocalizing} \\ \text{subcategories of } \mathbf{T} \end{array} \right\} \xrightleftharpoons[\text{cosupp}_R^{-1}]{\text{cosupp}_R} \{ \text{subsets of } \text{supp}_R \mathbf{T} \}.$$

For the main results of this work, it suffices to consider a version of costratification for tensor triangulated categories, see Section 9; so we do not study the general notion in any great detail.

6. AXIOMS FOR SUPPORT AND COSUPPORT

In this section we give an axiomatic description of cosupport, analogous to the one for support in [4, Theorem 5.15]; see also Theorem 6.4 below. This material is not used elsewhere in this paper.

As before, we fix a compactly generated R -linear triangulated category \mathbf{T} . The starting point is the following cohomological addendum to Corollary 4.9.

Lemma 6.1. *Let $\mathcal{V} \subseteq \text{Spec } R$ be a specialization closed subset. For each object X in \mathbf{T} , the following conditions are equivalent:*

- (1) $\mathcal{V} \cap \text{cosupp}_R X = \emptyset$.
- (1') $\mathcal{V} \cap \text{supp}_R X = \emptyset$.
- (2) $\text{Hom}_{\mathbf{T}}^*(C//\mathbf{p}, X) = 0$ for all $C \in \mathbf{T}^c$ and $\mathbf{p} \in \mathcal{V}$.
- (3) $\text{Hom}_{\mathbf{T}}^*(C, X)$ is either zero or not \mathbf{p} -torsion for each $C \in \mathbf{T}^c$ and $\mathbf{p} \in \mathcal{V}$.

Proof. (1) \Leftrightarrow (1') is part of Corollary 4.9, while (1') \Leftrightarrow (2) is a consequence of Theorem 3.2. To prove that (2) \Leftrightarrow (3), use Lemma 4.11. \square

Remark 6.2. As noted above, for each specialization closed subset \mathcal{V} of $\text{Spec } R$ there is an equality of subcategories

$$\{X \in \mathbf{T} \mid \text{cosupp}_R X \cap \mathcal{V} = \emptyset\} = \{X \in \mathbf{T} \mid \text{supp}_R X \cap \mathcal{V} = \emptyset\}.$$

The subcategory on the left is colocalizing, by Proposition 4.16, while the one on the right is localizing, since $\Gamma_{\mathfrak{p}}$ preserves set-indexed coproducts; see [4, Corollary 6.6].

Given a specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec} R$, Lemma 6.1 yields cohomological criteria for the (co)support of any object X to be contained in $\operatorname{Spec} R \setminus \mathcal{V}$. This leads to axiomatic descriptions of cosupport and support.

Theorem 6.3. *There exists a unique assignment sending each object X in T to a subset $\operatorname{cosupp}_R X$ of $\operatorname{Spec} R$ such that the following properties hold:*

- (1) Cohomology: *For each object X in T and each \mathfrak{p} in $\operatorname{Spec} R$, one has*

$$\mathcal{V}(\mathfrak{p}) \cap \operatorname{cosupp}_R X \neq \emptyset$$

if and only if $\operatorname{Hom}_{\mathsf{T}}^(C, X)$ is non-zero and \mathfrak{p} -torsion for some C in T^c .*

- (2) Orthogonality: *For objects X and Y in T , one has that*

$$\operatorname{cosupp}_R X \cap \operatorname{cl}(\operatorname{cosupp}_R Y) = \emptyset \quad \text{implies} \quad \operatorname{Hom}_{\mathsf{T}}(X, Y) = 0.$$

- (3) Exactness: *For every exact triangle $X \rightarrow Y \rightarrow Z \rightarrow$ in T , one has*

$$\operatorname{cosupp}_R Y \subseteq \operatorname{cosupp}_R X \cup \operatorname{cosupp}_R Z.$$

- (4) Separation: *For any specialization closed subset \mathcal{V} of $\operatorname{Spec} R$ and object X in T , there exists an exact triangle $X' \rightarrow X \rightarrow X'' \rightarrow$ in T such that*

$$\operatorname{cosupp}_R X' \subseteq \operatorname{Spec} R \setminus \mathcal{V} \quad \text{and} \quad \operatorname{cosupp}_R X'' \subseteq \mathcal{V}.$$

Proof. Lemma 6.1 implies (1). Corollary 4.10 is (2), and Proposition 4.6 is (3). Proposition 4.7 implies (4). Here one uses for any specialization closed subset \mathcal{V} of $\operatorname{Spec} R$ the localization triangle (4.1).

Now let $\sigma: \mathsf{T} \rightarrow \operatorname{Spec} R$ be a map satisfying properties (1)–(4).

Fix a specialization closed subset $\mathcal{V} \subseteq \operatorname{Spec} R$ and an object $X \in \mathsf{T}$. It suffices to verify that the following equalities hold:

$$(*) \quad \sigma(\Lambda^{\mathcal{V}} X) = \sigma(X) \cap \mathcal{V} \quad \text{and} \quad \sigma(V^{\mathcal{V}} X) = \sigma(X) \cap (\operatorname{Spec} R \setminus \mathcal{V}).$$

Indeed, for any point \mathfrak{p} in $\operatorname{Spec} R$ one then obtains that

$$\begin{aligned} \sigma(\Lambda^{\mathfrak{p}} X) &= \sigma(V^{\mathcal{Z}(\mathfrak{p})} \Lambda^{\mathcal{V}(\mathfrak{p})} X) \\ &= \sigma(\Lambda^{\mathcal{V}(\mathfrak{p})} X) \cap (\operatorname{Spec} R \setminus \mathcal{Z}(\mathfrak{p})) \\ &= \sigma(X) \cap \mathcal{V}(\mathfrak{p}) \cap (\operatorname{Spec} R \setminus \mathcal{Z}(\mathfrak{p})) \\ &= \sigma(X) \cap \{\mathfrak{p}\}. \end{aligned}$$

Therefore, $\mathfrak{p} \in \sigma(X)$ if and only if $\sigma(\Lambda^{\mathfrak{p}} X) \neq \emptyset$; this last condition is equivalent to $\Lambda^{\mathfrak{p}} X \neq 0$, by the cohomology property. The upshot is that $\mathfrak{p} \in \sigma(X)$ if and only if $\mathfrak{p} \in \operatorname{supp}_R X$, which is the desired conclusion.

It thus remains to prove (*).

Let $X' \rightarrow X \rightarrow X'' \rightarrow$ be the triangle associated to \mathcal{V} , provided by property (4). It suffices to verify the following statements:

- (i) $\sigma(X'') = \sigma(X) \cap \mathcal{V}$ and $\sigma(X') = \sigma(X) \cap (\operatorname{Spec} R \setminus \mathcal{V})$;
- (ii) $\Lambda^{\mathcal{V}} X \cong X'$ and $L_{\mathcal{V}} X \cong X''$.

The equalities in (i) are immediate from properties (3) and (4). In verifying (ii), the crucial observation is that, by the cohomology property, for any Y in T one has

$$\mathcal{V} \cap \sigma(Y) = \emptyset \iff Y \in \operatorname{Im} V^{\mathcal{V}}.$$

Thus X' is in $\text{Im } V^\vee$. On the other hand, property (2) and Lemma 2.1 imply that X'' is in $\text{Im } \Lambda^\vee$. One thus obtains the following morphism of triangles

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & \\ \downarrow \alpha & & \parallel & & \downarrow \beta & & \\ V^\vee X & \longrightarrow & X & \longrightarrow & \Lambda^\vee X & \longrightarrow & \end{array}$$

where the object $\text{Cone}(\alpha) \cong \text{Cone}(\Sigma^{-1}\beta)$ belongs to $\text{Im } V^\vee \cap \text{Im } \Lambda^\vee$, hence is trivial. Therefore, α and β are isomorphisms, which yields (ii). \square

The following axiomatic description of support is a slight modification of [4, Theorem 5.15]. Note that conditions (1) and (3) coincide for support and cosupport. The crucial difference appears in conditions (2) and (4).

Theorem 6.4. *There exists a unique assignment sending each object X in T to a subset $\text{supp}_R X$ of $\text{Spec } R$ such that the following properties hold:*

- (1) Cohomology: *For each object X in T and each \mathfrak{p} in $\text{Spec } R$, one has*

$$\mathcal{V}(\mathfrak{p}) \cap \text{supp}_R X \neq \emptyset$$

if and only if $\text{Hom}_{\mathsf{T}}^(C, X)$ is non-zero and \mathfrak{p} -torsion for some C in T^c .*

- (2) Orthogonality: *For objects X and Y in T , one has that*

$$\text{cl}(\text{supp}_R X) \cap \text{supp}_R Y = \emptyset \quad \text{implies} \quad \text{Hom}_{\mathsf{T}}(X, Y) = 0.$$

- (3) Exactness: *For every exact triangle $X \rightarrow Y \rightarrow Z \rightarrow$ in T , one has*

$$\text{supp}_R Y \subseteq \text{supp}_R X \cup \text{supp}_R Z.$$

- (4) Separation: *For any specialization closed subset \mathcal{V} of $\text{Spec } R$ and object X in T , there exists an exact triangle $X' \rightarrow X \rightarrow X'' \rightarrow$ in T such that*

$$\text{supp}_R X' \subseteq \mathcal{V} \quad \text{and} \quad \text{supp}_R X'' \subseteq \text{Spec } R \setminus \mathcal{V}.$$

Proof. Adapt the proof of [4, Theorem 5.15], using Lemma 6.1. \square

7. CHANGE OF RINGS AND CATEGORIES

In this section we discuss how support and cosupport is affected by the change of rings and categories. Throughout R is a graded-commutative noetherian ring.

Linear functors. Let T and U be R -linear triangulated categories. We say that a functor $F: \mathsf{T} \rightarrow \mathsf{U}$ is R -linear if it is an exact functor such that for each X in T the following diagram is commutative:

$$\begin{array}{ccc} & R & \\ \phi_X \swarrow & & \searrow \phi_{FX} \\ \text{End}_{\mathsf{T}}^*(X) & \xrightarrow{F} & \text{End}_{\mathsf{U}}^*(FX) \end{array}$$

Let $F: \mathsf{T} \rightarrow \mathsf{U}$ be an R -linear functor, and let X and Y be objects in T and U , respectively. The structure homomorphisms ϕ_X and ϕ_{FX} provide two R -module structures on $\text{Hom}_{\mathsf{U}}^*(FX, Y)$, and the R -linearity of F is equivalent to the claim that these coincide.

We are grateful to the referee for suggesting the following lemma.

Lemma 7.1. *Let $F: \mathsf{T} \rightarrow \mathsf{U}$ be an R -linear functor and G a right adjoint. Then the following statements hold.*

- (1) *The adjunction isomorphism $\text{Hom}_{\mathsf{T}}^*(X, GY) \xrightarrow{\sim} \text{Hom}_{\mathsf{U}}^*(FX, GY)$ is R -linear.*

(2) *The functor G is R -linear.*

Proof. For (1), observe that the adjunction isomorphism can be factored as

$$\mathrm{Hom}_{\mathbb{T}}^*(X, GY) \xrightarrow{F} \mathrm{Hom}_{\mathbb{U}}^*(FX, FGY) \xrightarrow{(FX, \theta Y)} \mathrm{Hom}_{\mathbb{U}}^*(FX, Y)$$

where the second map is induced by the counit $\theta: FG \rightarrow \mathrm{Id}_{\mathbb{U}}$. For (2), note that the map $\mathrm{Hom}_{\mathbb{U}}(Y, Y) \xrightarrow{G} \mathrm{Hom}_{\mathbb{T}}(GY, GY)$ can be factored as

$$\mathrm{Hom}_{\mathbb{U}}^*(Y, Y) \xrightarrow{(\theta Y, Y)} \mathrm{Hom}_{\mathbb{U}}^*(FGY, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}}^*(GY, GY)$$

where the second map is the adjunction isomorphism. \square

Proposition 7.2. *Let $F: \mathbb{T} \rightarrow \mathbb{U}$ be a functor between compactly generated R -linear triangulated categories which preserves set-indexed coproducts and products. Let E be a left adjoint of F , and suppose that F or E is R -linear. For any specialization closed subset \mathcal{V} of $\mathrm{Spec} R$ there are then natural isomorphisms*

$$F\Gamma_{\mathcal{V}} \cong \Gamma_{\mathcal{V}}F, \quad FL_{\mathcal{V}} \cong L_{\mathcal{V}}F, \quad \Gamma_{\mathcal{V}}E \cong E\Gamma_{\mathcal{V}}, \quad \text{and} \quad L_{\mathcal{V}}E \cong EL_{\mathcal{V}}.$$

Note that the functor F admits a left adjoint by Brown representability, because F preserves set-indexed products.

Proof. For each object X in \mathbb{T} , one has an exact triangle

$$F\Gamma_{\mathcal{V}}X \longrightarrow FX \longrightarrow FL_{\mathcal{V}}X \longrightarrow$$

induced by the localization triangle for \mathcal{V} . It thus suffices to verify that $F\Gamma_{\mathcal{V}}X$ is in $\mathrm{Im} \Gamma_{\mathcal{V}}$ and that $FL_{\mathcal{V}}X$ is in $\mathrm{Im} L_{\mathcal{V}}$; see [4, §4]. We use the adjunction isomorphisms

$$(7.3) \quad \begin{aligned} \mathrm{Hom}_{\mathbb{U}}^*(C, F\Gamma_{\mathcal{V}}X) &\cong \mathrm{Hom}_{\mathbb{T}}^*(EC, \Gamma_{\mathcal{V}}X) \\ \mathrm{Hom}_{\mathbb{U}}^*(C, FL_{\mathcal{V}}X) &\cong \mathrm{Hom}_{\mathbb{T}}^*(EC, L_{\mathcal{V}}X) \end{aligned}$$

which are R -linear because F or E is R -linear; see Lemma ???. Note that EC is compact if C is compact, since F preserves set-indexed coproducts.

An object Y in \mathbb{U} belongs to $\mathrm{Im} \Gamma_{\mathcal{V}}$ if and only if $\mathrm{Hom}_{\mathbb{U}}^*(C, Y)_{\mathfrak{p}} = 0$ for all compact $C \in \mathbb{U}$ and all $\mathfrak{p} \in \mathrm{Spec} R \setminus \mathcal{V}$. Applying this characterization to $F\Gamma_{\mathcal{V}}X$ and $\Gamma_{\mathcal{V}}X$, the adjunction (7.3) implies that $F\Gamma_{\mathcal{V}}X$ is in $\mathrm{Im} \Gamma_{\mathcal{V}}$.

An object Y in \mathbb{U} belongs to $\mathrm{Im} L_{\mathcal{V}}$ if and only if $\mathrm{Hom}_{\mathbb{U}}^*(C//\mathfrak{p}, Y) = 0$ for all compact $C \in \mathbb{U}$ and all $\mathfrak{p} \in \mathcal{V}$, by Corollary 4.9 and Lemma 6.1. Applying this characterization to $FL_{\mathcal{V}}X$ and $L_{\mathcal{V}}X$, the adjunction (7.3) implies that $FL_{\mathcal{V}}X$ is in $\mathrm{Im} L_{\mathcal{V}}$. Here, one uses that $E(C//\mathfrak{p}) \cong (EC)//\mathfrak{p}$, and this completes the proof of the first pair of isomorphisms.

The isomorphism $FL_{\mathcal{Z}(\mathfrak{p})} \cong L_{\mathcal{Z}(\mathfrak{p})}F$ implies $F(X(\mathfrak{p})) \cong (FX)(\mathfrak{p})$ for each X in \mathbb{T} and each \mathfrak{p} in $\mathrm{Spec} R$. Given this, the proof of the isomorphisms involving E is similar: For each object Y in \mathbb{U} , it follows from Proposition 4.14 and adjunction that there are inclusions

$$\mathrm{supp}_R E\Gamma_{\mathcal{V}}Y \subseteq \mathcal{V} \quad \text{and} \quad \mathrm{supp}_R EL_{\mathcal{V}}Y \subseteq \mathrm{Spec} R \setminus \mathcal{V}.$$

Thus $E\Gamma_{\mathcal{V}}Y$ is in $\mathrm{Im} \Gamma_{\mathcal{V}}$ and $EL_{\mathcal{V}}Y$ is in $\mathrm{Im} L_{\mathcal{V}}$. \square

Remark 7.4. In the preceding proof, the assumption on F or E to be R -linear is only used for the R -linearity of the adjunction isomorphisms (7.3).

Change of rings. Let S be a graded-commutative noetherian ring, and \mathcal{U} an S -linear triangulated category. Given a homomorphism of rings $\alpha: R \rightarrow S$, there is a natural R -linear structure on \mathcal{U} induced by homomorphisms

$$R \xrightarrow{\alpha} S \xrightarrow{\phi_X} \text{End}_{\mathcal{U}}^*(X) \quad \text{for } X \in \mathcal{U}.$$

As usual, α induces a map $\alpha^*: \text{Spec } S \rightarrow \text{Spec } R$, with $\alpha^*(\mathfrak{q}) = \alpha^{-1}(\mathfrak{q})$ for each \mathfrak{q} in $\text{Spec } S$. Observe that if $\mathcal{V} \subseteq \text{Spec } R$ is specialization closed, then so is the subset $(\alpha^*)^{-1}\mathcal{V}$ of $\text{Spec } S$.

Proposition 7.5. *Let $\alpha: R \rightarrow S$ be a homomorphism of rings, and \mathcal{U} an S -linear triangulated category, with induced R -linear structure via α . Let $\mathcal{V} \subseteq \text{Spec } R$ be a specialization closed set and $\mathcal{W} = (\alpha^*)^{-1}\mathcal{V}$. Then there are isomorphisms*

$$\Gamma_{\mathcal{V}} \cong \Gamma_{\mathcal{W}} \quad \text{and} \quad L_{\mathcal{V}} \cong L_{\mathcal{W}}.$$

Proof. It suffices to prove that any object $X \in \mathcal{U}$ is in $\mathcal{U}_{\mathcal{V}}$ if and only if it is in $\mathcal{U}_{\mathcal{W}}$. Thus one needs to show for any compact object $C \in \mathcal{U}$ that $\text{Hom}_{\mathcal{U}}^*(C, X)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \text{Spec } R \setminus \mathcal{V}$ if and only if $\text{Hom}_{\mathcal{U}}^*(C, X)_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \text{Spec } S \setminus \mathcal{W}$. This one finds in Lemma 7.6 below. \square

Lemma 7.6. *Let $\alpha: R \rightarrow S$ be a homomorphism of graded-commutative noetherian rings and $\mathcal{V} \subseteq \text{Spec } R$ a specialization closed subset. Given an S -module M , one has $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \text{Spec } R \setminus \mathcal{V}$ if and only if $M_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \text{Spec } S \setminus (\alpha^*)^{-1}\mathcal{V}$.*

Proof. Suppose first that $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin \mathcal{V}$, and choose $\mathfrak{q} \notin (\alpha^*)^{-1}\mathcal{V}$. Then $M_{\mathfrak{q}} \cong (M_{\alpha^*(\mathfrak{q})})_{\mathfrak{q}} = 0$.

Assume now that $M_{\mathfrak{p}} \neq 0$ for some $\mathfrak{p} \notin \mathcal{V}$. We view $M_{\mathfrak{p}}$ as an $S_{\mathfrak{p}}$ -module and find therefore a prime ideal \mathfrak{q} in $\text{Spec } S_{\mathfrak{p}} \subseteq \text{Spec } S$ such that $(M_{\mathfrak{p}})_{\mathfrak{q}} \cong M_{\mathfrak{q}}$ is non-zero. It remains to observe that $\alpha^*(\mathfrak{q}) \subseteq \mathfrak{p}$ and hence that $\alpha^*(\mathfrak{q})$ is not in \mathcal{V} . \square

Change of rings and categories. Henceforth, we say $(F; \alpha): (\mathcal{T}; R) \rightarrow (\mathcal{U}; S)$ is an exact functor to mean that \mathcal{T} and \mathcal{U} are compactly generated R -linear and S -linear triangulated categories, respectively; $\alpha: R \rightarrow S$ is a homomorphism of graded rings; and F is an exact functor that is R -linear with respect to the induced R -linear structure on \mathcal{U} ; in other words, that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & S \\ \phi_X \downarrow & & \downarrow \phi_{FX} \\ \text{End}_{\mathcal{T}}^*(X) & \xrightarrow{F} & \text{End}_{\mathcal{U}}^*(FX) \end{array}$$

is commutative for each $X \in \mathcal{T}$.

Theorem 7.7. *Let $(F; \alpha): (\mathcal{T}; R) \rightarrow (\mathcal{U}; S)$ be an exact functor which preserves set-indexed coproducts and products. Let E be a left adjoint and G a right adjoint of F . Let $\mathcal{V} \subseteq \text{Spec } R$ be a specialization closed subset and set $\mathcal{W} = (\alpha^*)^{-1}\mathcal{V}$. Then there are natural isomorphisms:*

- (1) $F\Gamma_{\mathcal{V}} \cong \Gamma_{\mathcal{W}}F, \quad FL_{\mathcal{V}} \cong L_{\mathcal{W}}F, \quad \Gamma_{\mathcal{V}}E \cong E\Gamma_{\mathcal{W}}, \quad \text{and} \quad L_{\mathcal{V}}E \cong EL_{\mathcal{W}}$
- (2) $F\Lambda^{\mathcal{V}} \cong \Lambda^{\mathcal{W}}F, \quad FV^{\mathcal{V}} \cong V^{\mathcal{W}}F, \quad \Lambda^{\mathcal{V}}G \cong G\Lambda^{\mathcal{W}}, \quad \text{and} \quad V^{\mathcal{V}}G \cong GV^{\mathcal{W}}.$

This result contains Propositions 7.2 and 7.5: to recover the first, set $\alpha = \text{id}_R$; for the second set $F = \text{Id}_{\mathcal{T}}$. On the other hand, it is proved using the latter results.

Proof. Since F is linear with respect to the induced R -linear structure on \mathbf{U} , Propositions 7.2 and 7.5 yield the following isomorphism:

$$FG_{\mathcal{V}} \cong \Gamma_{\mathcal{V}}F \cong \Gamma_{\mathcal{W}}F.$$

The other isomorphisms in (1) can be obtained in the same way.

The isomorphisms in (2) are obtained by taking right adjoints of those in (1). \square

As applications, we establish results which track the change in support along linear functors; this is one reason we have had to introduce these notions.

Corollary 7.8. *Let $(F; \alpha): (\mathbf{T}; R) \rightarrow (\mathbf{U}; S)$ be an exact functor which preserves set-indexed coproducts and products. Let E be a left adjoint and G a right adjoint of F . Then for $X \in \mathbf{T}$ and $Y \in \mathbf{U}$ there are inclusions:*

- (1) $\alpha^*(\text{supp}_S FX) \subseteq \text{supp}_R X$ and $\text{supp}_R EY \subseteq \alpha^*(\text{supp}_S Y)$
- (2) $\alpha^*(\text{cosupp}_S FX) \subseteq \text{cosupp}_R X$ and $\text{cosupp}_R GY \subseteq \alpha^*(\text{cosupp}_S Y)$.

Each inclusion is an equality when the corresponding functor is faithful on objects.

Proof. Let \mathfrak{p} be a point in $\text{Spec } R$, and pick specialization closed subsets \mathcal{V} and \mathcal{W} of $\text{Spec } R$ such that $\{\mathfrak{p}\} = \mathcal{V} \setminus \mathcal{W}$. For example, set $\mathcal{V} = \mathcal{V}(\mathfrak{p})$ and $\mathcal{W} = \mathcal{V} \setminus \{\mathfrak{p}\}$.

Setting $\tilde{\mathcal{V}} = (\alpha^*)^{-1}\mathcal{V}$ and $\tilde{\mathcal{W}} = (\alpha^*)^{-1}\mathcal{W}$, one gets isomorphisms

$$F(\Gamma_{\mathfrak{p}}X) = FL_{\mathcal{W}}\Gamma_{\mathcal{V}}X = L_{\tilde{\mathcal{W}}}\Gamma_{\tilde{\mathcal{V}}}FX.$$

Observing that $\tilde{\mathcal{V}} \setminus \tilde{\mathcal{W}} = (\alpha^*)^{-1}\{\mathfrak{p}\}$, this yields equalities

$$\text{supp}_S F(\Gamma_{\mathfrak{p}}X) = \text{supp}_S FX \cap (\tilde{\mathcal{V}} \setminus \tilde{\mathcal{W}}) = \text{supp}_S FX \cap (\alpha^*)^{-1}\{\mathfrak{p}\}.$$

Thus, $\alpha^*(\text{supp}_S FX) \subseteq \text{supp}_R X$, and equality holds if F is faithful on objects.

The other inclusions are obtained in the same way. \square

In the preceding result, the stronger conclusion $\text{supp}_S FX = (\alpha^*)^{-1}\text{supp}_R X$ need not hold, even when F is an equivalence of categories.

Example 7.9. Let R be a field, set $S = R[a]/(a^2 - a)$, and let $\mathbf{U} = \mathbf{D}(S)$ denote the derived category of S -modules, with canonical S -linear structure. Let $\mathbf{T} = \mathbf{U}$ and view this as an R -linear triangulated category via the inclusion $\alpha: R \rightarrow S$.

Let $F: \mathbf{T} \rightarrow \mathbf{U}$ be the identity functor; it is evidently compatible with α and faithful. Observe however that for the module $X = S/(a)$ in \mathbf{T} one has

$$\text{supp}_R X = \text{Spec } R \quad \text{and} \quad \text{Spec}_S FX = \{(a)\}.$$

On the other hand, $(\alpha^*)^{-1}\text{supp}_R X = \text{Spec } S$.

Corollary 7.10. *Let $(F; \alpha): (\mathbf{T}; R) \rightarrow (\mathbf{U}; S)$ be an exact functor which preserves set-indexed coproducts and products. Let E be a left adjoint and G a right adjoint of F . Let $\mathfrak{p} \in \text{Spec } R$ and suppose that $\mathcal{U} = (\alpha^*)^{-1}\{\mathfrak{p}\}$ is a discrete subset of $\text{Spec } S$. Then there are isomorphisms:*

$$(1) \quad F\Gamma_{\mathfrak{p}} \cong \prod_{\mathfrak{q} \in \mathcal{U}} \Gamma_{\mathfrak{q}}F \quad \text{and} \quad \Gamma_{\mathfrak{p}}E \cong \prod_{\mathfrak{q} \in \mathcal{U}} E\Gamma_{\mathfrak{q}}$$

$$(2) \quad F\Lambda^{\mathfrak{p}} \cong \prod_{\mathfrak{q} \in \mathcal{U}} \Lambda^{\mathfrak{q}}F \quad \text{and} \quad \Lambda^{\mathfrak{p}}G \cong \prod_{\mathfrak{q} \in \mathcal{U}} G\Lambda^{\mathfrak{q}}.$$

Proof. Choose specialization closed subsets \mathcal{V} and \mathcal{W} of $\text{Spec } R$ with $\mathcal{V} \setminus \mathcal{W} = \{\mathfrak{p}\}$. Setting $\tilde{\mathcal{V}} = (\alpha^*)^{-1}\mathcal{V}$ and $\tilde{\mathcal{W}} = (\alpha^*)^{-1}\mathcal{W}$, one gets isomorphisms

$$F\Gamma_{\mathfrak{p}} = FL_{\mathcal{W}}\Gamma_{\mathcal{V}} \cong L_{\tilde{\mathcal{W}}}\Gamma_{\tilde{\mathcal{V}}}F \cong \prod_{\mathfrak{q} \in \mathcal{U}} \Gamma_{\mathfrak{q}}L_{\mathcal{W}}\Gamma_{\mathcal{V}}F \cong \prod_{\mathfrak{q} \in \mathcal{U}} \Gamma_{\mathfrak{q}}F,$$

where the first one follows Theorem 7.7. The second is by Proposition 4.15, which applies since $\widetilde{\mathcal{V}} \setminus \widetilde{\mathcal{W}} = \mathcal{U}$ and \mathcal{U} is discrete. The last isomorphism holds because one has $\Gamma_{\mathfrak{q}} L_{\widetilde{\mathcal{W}}} \Gamma_{\widetilde{\mathcal{V}}} \cong \Gamma_{\mathfrak{q}}$ for all $\mathfrak{q} \in \mathcal{U}$; see [4, Proposition 6.1].

The other isomorphisms can be obtained in the same way. \square

The next result involves a notion of costratification of R -linear triangulated categories. This has been introduced in Remark 5.7, and the analogous notion of stratification is from [6, §4].

Theorem 7.11. *Let $(F; \alpha): (\mathbb{T}; R) \rightarrow (\mathbb{U}; S)$ be an exact functor which preserves set-indexed coproducts and products, and fix an object X in \mathbb{T} .*

- (1) *If \mathbb{T} is costratified by R and the right adjoint of F is faithful on objects, then*

$$\mathrm{supp}_S FX = (\alpha^*)^{-1}(\mathrm{supp}_R X) \cap \mathrm{supp}_S \mathbb{U}.$$

- (2) *If \mathbb{T} is stratified by R and the left adjoint of F is faithful on objects, then*

$$\mathrm{cosupp}_S FX = (\alpha^*)^{-1}(\mathrm{cosupp}_R X) \cap \mathrm{supp}_S \mathbb{U}.$$

Proof. We prove the statement concerning cosupports; the argument for the one for supports is exactly analogous.

To begin with, from Corollary 7.8 one gets an inclusion

$$\mathrm{cosupp}_S FX \subseteq (\alpha^*)^{-1}(\mathrm{cosupp}_R X) \cap \mathrm{cosupp}_S \mathbb{U}$$

Now fix a $\mathfrak{q} \in \mathrm{supp}_S \mathbb{U}$ with $\mathfrak{q} \notin \mathrm{cosupp}_S FX$. We need to show that $\mathfrak{p} = \alpha^*(\mathfrak{q})$ is not in $\mathrm{cosupp}_R X$. Let E be a left adjoint of F . Using adjunction, one has

$$\mathrm{Hom}_{\mathbb{T}}(E\Gamma_{\mathfrak{q}}-, X) \cong \mathrm{Hom}_{\mathbb{U}}(-, A^{\mathfrak{q}}FX) = 0.$$

There exists some object U in \mathbb{U} such that $E\Gamma_{\mathfrak{q}}U \neq 0$, since $\mathfrak{q} \in \mathrm{supp}_S \mathbb{U}$ and E is faithful on objects. Moreover, $E\Gamma_{\mathfrak{q}}U$ belongs to $\Gamma_{\mathfrak{p}}\mathbb{T}$, by Corollary 7.8. Since R stratifies \mathbb{T} , the subcategory $\Gamma_{\mathfrak{p}}\mathbb{T}$ contains no non-trivial localizing subcategories, and hence coincides with $\mathrm{Loc}_{\mathbb{T}}(E\Gamma_{\mathfrak{q}}U)$. Thus

$$0 = \mathrm{Hom}_{\mathbb{T}}(\Gamma_{\mathfrak{p}}-, X) \cong \mathrm{Hom}_{\mathbb{T}}(-, A^{\mathfrak{p}}X),$$

and therefore $\mathfrak{p} \notin \mathrm{cosupp}_R X$. \square

Perfect generators and cogenerators. For any subset \mathcal{U} of $\mathrm{Spec} R$ we consider the full subcategories

$$\mathbb{T}_{\mathcal{U}} = \{X \in \mathbb{T} \mid \mathrm{supp}_R X \subseteq \mathcal{U}\} \quad \text{and} \quad \mathbb{T}^{\mathcal{U}} = \{X \in \mathbb{T} \mid \mathrm{cosupp}_R X \subseteq \mathcal{U}\}.$$

The notion of a set of perfect cogenerators was recalled in Section 5. The notion of a set of perfect generators is analogous; see [17].

Lemma 7.12. *Let $(F; \alpha): (\mathbb{T}; R) \rightarrow (\mathbb{U}; S)$ be an exact functor which preserves set-indexed coproducts and products. Let \mathcal{U} be a subset of $\mathrm{Spec} R$ and set $\widetilde{\mathcal{U}} = (\alpha^*)^{-1}\mathcal{U}$. Then the following statements hold:*

- (1) *When F is faithful on objects from $\mathbb{T}_{\mathcal{U}}$, its left adjoint maps any set of perfect generators of $\mathbb{U}_{\widetilde{\mathcal{U}}}$ to a set of perfect generators of $\mathbb{T}_{\mathcal{U}}$.*
- (2) *When F is faithful on objects from $\mathbb{T}^{\mathcal{U}}$, its right adjoint maps any set of perfect cogenerators of $\mathbb{U}^{\widetilde{\mathcal{U}}}$ to a set of perfect cogenerators of $\mathbb{T}^{\mathcal{U}}$,*

Proof. Let E denote a left adjoint of F . It follows from Corollary 7.8 that F and E restrict to functors between $\mathbb{T}_{\mathcal{U}}$ and $\mathbb{U}_{\widetilde{\mathcal{U}}}$. Now use adjunction to prove (1). The proof of (2) is analogous. \square

8. TENSOR TRIANGULATED CATEGORIES

In this section we discuss special properties of triangulated categories which hold when they have a tensor structure.

Let $(\mathbb{T}, \otimes, \mathbb{1})$ be a tensor triangulated category as defined in [4, §8]. In particular, \mathbb{T} is a compactly generated triangulated category with a symmetric monoidal structure; \otimes is its tensor product and $\mathbb{1}$ the unit of the tensor product. The tensor product is exact in each variable and preserves coproducts.

By Brown representability there are function objects $\mathcal{H}om(X, Y)$ satisfying

$$(8.1) \quad \mathrm{Hom}_{\mathbb{T}}(X \otimes Z, Y) \cong \mathrm{Hom}_{\mathbb{T}}(Z, \mathcal{H}om(X, Y)),$$

and we write X^\vee for the *Spanier–Whitehead dual* $\mathcal{H}om(X, \mathbb{1})$. Note that the adjunction extends to function objects, in the sense that there are natural isomorphisms

$$\mathcal{H}om(X \otimes Z, Y) \cong \mathcal{H}om(Z, \mathcal{H}om(X, Y)).$$

This is an easy consequence of Yoneda’s lemma.

We shall assume that the tensor unit $\mathbb{1}$ is compact and that all compact objects C are strongly dualizable in the sense that the canonical morphism

$$C^\vee \otimes X \rightarrow \mathcal{H}om(C, X)$$

is an isomorphism for all X in \mathbb{T} . We also assume that $\mathcal{H}om(-, Y)$ is exact for each object Y in \mathbb{T} .²

Canonical actions. The symmetric monoidal structure of \mathbb{T} ensures that the endomorphism ring $\mathrm{End}_{\mathbb{T}}^*(\mathbb{1})$ is graded commutative. It acts on \mathbb{T} via homomorphisms

$$\mathrm{End}_{\mathbb{T}}^*(\mathbb{1}) \xrightarrow{X \otimes -} \mathrm{End}_{\mathbb{T}}^*(X).$$

In particular, any homomorphism $R \rightarrow \mathrm{End}_{\mathbb{T}}^*(\mathbb{1})$ of rings with R graded commutative induces an action of R on \mathbb{T} . We say that an R action on \mathbb{T} is *canonical* if it arises from such a homomorphism. In that case there are for each specialization closed subset \mathcal{V} and point \mathfrak{p} of $\mathrm{Spec} R$ natural isomorphisms

$$(8.2) \quad \Gamma_{\mathcal{V}} X \cong X \otimes \Gamma_{\mathcal{V}} \mathbb{1}, \quad L_{\mathcal{V}} X \cong X \otimes L_{\mathcal{V}} \mathbb{1}, \quad \text{and} \quad \Gamma_{\mathfrak{p}} X \cong X \otimes \Gamma_{\mathfrak{p}} \mathbb{1}.$$

These isomorphisms are from [4, Theorem 8.2, Corollary 8.3].³

Proposition 8.3. *Let $\mathcal{V} \subseteq \mathrm{Spec} R$ be a specialization closed subset and $\mathfrak{p} \in \mathrm{Spec} R$. Given objects X and Y in \mathbb{T} , there are natural isomorphisms*

$$\begin{aligned} \mathcal{H}om(\Gamma_{\mathcal{V}} X, Y) &\cong \mathcal{H}om(X, \Lambda^{\mathcal{V}} Y) \cong \Lambda^{\mathcal{V}} \mathcal{H}om(X, Y), \\ \mathcal{H}om(L_{\mathcal{V}} X, Y) &\cong \mathcal{H}om(X, V^{\mathcal{V}} Y) \cong V^{\mathcal{V}} \mathcal{H}om(X, Y), \\ \mathcal{H}om(\Gamma_{\mathfrak{p}} X, Y) &\cong \mathcal{H}om(X, \Lambda^{\mathfrak{p}} Y) \cong \Lambda^{\mathfrak{p}} \mathcal{H}om(X, Y). \end{aligned}$$

In particular, there are natural isomorphisms

$$\Lambda^{\mathcal{V}} X \cong \mathcal{H}om(\Gamma_{\mathcal{V}} \mathbb{1}, X), \quad V^{\mathcal{V}} X \cong \mathcal{H}om(L_{\mathcal{V}} \mathbb{1}, X), \quad \Lambda^{\mathfrak{p}} X \cong \mathcal{H}om(\Gamma_{\mathfrak{p}} \mathbb{1}, X).$$

Proof. Combine the isomorphisms in (8.2) with the adjunction defining $\mathcal{H}om$. \square

²The exactness of $\mathcal{H}om(-, Y)$ was omitted from [4, §8], since it was not used there, but it is important in Lemma 8.4 below.

³For these results to hold, the R action should be canonical, for the R -linearity of the adjunction isomorphism (8.1) is used in the arguments.

Colocalizing subcategories. Function objects turn localizing subcategories into colocalizing subcategories in the following sense.

Lemma 8.4. *Let \mathcal{C} be a class of objects in \mathcal{T} and $X, Y \in \mathcal{T}$. If X belongs to $\text{Loc}(\mathcal{C})$, then $\mathcal{H}om(X, Y)$ belongs to $\text{Coloc}(\{\mathcal{H}om(C, Y) \mid C \in \mathcal{C}\})$.*

Proof. This holds as $\mathcal{H}om(-, Y)$ is exact and turns coproducts into products. \square

We focus attention on colocalizing subcategories satisfying the equivalent conditions of the following lemma.

Lemma 8.5. *Let \mathcal{S} be a colocalizing subcategory of \mathcal{T} . Then the following conditions on \mathcal{S} are equivalent:*

- (1) *For all compact objects X in \mathcal{T} and all Y in \mathcal{S} , $X \otimes Y$ is also in \mathcal{S} .*
- (2) *For all compact objects X in \mathcal{T} and all Y in \mathcal{S} , $\mathcal{H}om(X, Y)$ is in \mathcal{S} .*
- (3) *For all objects X in \mathcal{T} and all Y in \mathcal{S} , $\mathcal{H}om(X, Y)$ is in \mathcal{S} .*

Proof. The equivalence of (1) and (2) follows from the isomorphisms $X^\vee \otimes Y \cong \mathcal{H}om(X, Y)$ and $X^{\vee\vee} \cong X$. The equivalence of (2) and (3) follows from Lemma 8.4 and the fact that \mathcal{T} is compactly generated. \square

We say that a colocalizing subcategory is *Hom closed* if the equivalent conditions of the lemma hold, and write $\text{Coloc}^{\mathcal{H}om}(\mathcal{C})$ for the smallest Hom closed colocalizing subcategory containing a class \mathcal{C} of objects in \mathcal{T} .

A localizing subcategory \mathcal{S} of \mathcal{T} is *tensor ideal* or *tensor closed* if $X \in \mathcal{T}$ and $Y \in \mathcal{S}$ imply that $X \otimes Y$ is in \mathcal{S} . Given a class \mathcal{C} of objects in \mathcal{T} , we denote by $\text{Loc}^\otimes(\mathcal{C})$ the smallest tensor ideal localizing subcategory containing \mathcal{C} .

The gist of the next result is that (an appropriate version of) the local-global principle holds for tensor triangulated categories. The first part, about localizing subcategories, is from [5, Theorem 3.6].

Theorem 8.6. *For each $X \in \mathcal{T}$, there are equalities*

$$\begin{aligned} \text{Loc}_\mathcal{T}^\otimes(X) &= \text{Loc}_\mathcal{T}^\otimes(\{\Gamma_\mathfrak{p} X \mid \mathfrak{p} \in \text{Spec } R\}) \\ \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(X) &= \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(\{\Lambda^\mathfrak{p} X \mid \mathfrak{p} \in \text{Spec } R\}). \end{aligned}$$

Proof. The first equality is [5, Theorem 3.6].

In particular, $\text{Loc}_\mathcal{T}^\otimes(\mathbb{1}) = \text{Loc}_\mathcal{T}^\otimes(\{\Gamma_\mathfrak{p} \mathbb{1} \mid \mathfrak{p} \in \text{Spec } R\})$, which in conjunction with Lemma 8.4 gives the second equality below:

$$\begin{aligned} \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(X) &= \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(\mathcal{H}om(\mathbb{1}, X)) \\ &= \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(\{\mathcal{H}om(\Gamma_\mathfrak{p} \mathbb{1}, X) \mid \mathfrak{p} \in \text{Spec } R\}) \\ &= \text{Coloc}_\mathcal{T}^{\mathcal{H}om}(\{\Lambda^\mathfrak{p} X \mid \mathfrak{p} \in \text{Spec } R\}) \end{aligned}$$

The first one holds because $X = \mathcal{H}om(\mathbb{1}, X)$ and last one is by Proposition 8.3. \square

Lemma 8.7. *If \mathcal{S} is a colocalizing subcategory of \mathcal{T} then $\mathcal{H}om(X, Y)$ is in \mathcal{S} for all X in $\text{Loc}_\mathcal{T}(\mathbb{1})$ and Y in \mathcal{S} . In particular, if $\mathbb{1}$ generates \mathcal{T} then all its colocalizing subcategories are Hom closed.*

Proof. This follows from Lemma 8.4. \square

Remark 8.8. Let \mathcal{T} be a tensor triangulated category with a canonical R -action. If \mathcal{T} is generated by its unit $\mathbb{1}$, then the local global principle for (co)localizing subcategories holds. This follows from Theorem 8.6 and Lemma 8.7.

9. COSTRATIFICATION

Let $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ be a tensor triangulated category as in Section 8, endowed with a canonical R -action. In this section, we introduce a variant of the notion of costratification, see Remark 5.7, suitable for this context and explain some consequences, including a classification of the Hom closed colocalizing subcategories.

Recall from Proposition 5.6 that there are maps σ and τ which yield a classification of colocalizing subcategories. Proposition 8.3 implies that each $\Lambda^{\mathfrak{p}}\mathcal{T}$ is Hom closed, so these maps restrict to the following maps on Hom closed subcategories:

$$\left\{ \begin{array}{c} \text{Hom closed colocalizing} \\ \text{subcategories of } \mathcal{T} \end{array} \right\} \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} \left\{ \begin{array}{c} \text{families } (S(\mathfrak{p}))_{\mathfrak{p} \in \text{Spec } R} \text{ with} \\ S(\mathfrak{p}) \subseteq \Lambda^{\mathfrak{p}}\mathcal{T} \text{ a Hom closed} \\ \text{colocalizing subcategory} \end{array} \right\}$$

where $\sigma(S) = (S \cap \Lambda^{\mathfrak{p}}\mathcal{T})$ and $\tau(S(\mathfrak{p}))$ is the colocalizing subcategory of \mathcal{T} cogenerated by all the $S(\mathfrak{p})$. The following result is the analogue of [6, Proposition 3.6].

Proposition 9.1. *The maps σ and τ are mutually inverse bijections.*

Proof. The proof is exactly analogous to that of Proposition 5.6, using the local-global principle from Theorem 8.6. \square

We say that *the tensor triangulated category \mathcal{T} is costratified by R* if for each $\mathfrak{p} \in \text{Spec } R$, the colocalizing subcategory $\Lambda^{\mathfrak{p}}\mathcal{T}$ contains no proper non-zero Hom closed colocalizing subcategories. Compare this definition with the one in Remark 5.7 for general triangulated categories. One does not have to impose the analogue of the local-global principle (C1), for it always holds; see Theorem 8.6.

If \mathcal{T} is costratified by R then for each $\mathfrak{p} \in \text{Spec } R$ there are only two possibilities for $S(\mathfrak{p})$, namely $S(\mathfrak{p}) = \Lambda^{\mathfrak{p}}\mathcal{T}$ or $S(\mathfrak{p}) = 0$. So the maps σ and τ reduce to

$$\left\{ \begin{array}{c} \text{Hom closed colocalizing} \\ \text{subcategories of } \mathcal{T} \end{array} \right\} \begin{array}{c} \xrightarrow{\text{cosupp}_R} \\ \xleftarrow{\text{cosupp}_R^{-1}} \end{array} \{ \text{subsets of } \text{supp}_R \mathcal{T} \}.$$

The next result is now immediate from the definition of costratification.

Corollary 9.2. *If the tensor triangulated category \mathcal{T} is costratified by R then the above maps cosupp_R and cosupp_R^{-1} are mutually inverse bijections.* \square

Stratification. In analogy with the notion of costratification, *the tensor triangulated category \mathcal{T} is stratified by R* if for each $\mathfrak{p} \in \text{Spec } R$ the localizing subcategory $\Gamma_{\mathfrak{p}}\mathcal{T}$ is zero or minimal among tensor ideal localizing subcategories; see [6, §7].

Next we establish a formula relating support and cosupport when \mathcal{T} is stratified.

Lemma 9.3. *An inclusion $\text{cosupp}_R \mathcal{H}om(X, Y) \subseteq \text{supp}_R X \cap \text{cosupp}_R Y$ holds for all objects X and Y in \mathcal{T} .*

Proof. Fix $\mathfrak{p} \in \text{Spec } R$; recall that \mathfrak{p} is in $\text{cosupp}_R X$ if and only if $\mathcal{H}om(\Gamma_{\mathfrak{p}}\mathbb{1}, X) \neq 0$. Using (8.1) and (8.2), and Proposition 8.3, one gets isomorphisms

$$(9.4) \quad \mathcal{H}om(\Gamma_{\mathfrak{p}}\mathbb{1}, \mathcal{H}om(X, Y)) \cong \mathcal{H}om(\Gamma_{\mathfrak{p}}X, Y) \cong \mathcal{H}om(X, \Lambda^{\mathfrak{p}}Y).$$

It follows that $\Gamma_{\mathfrak{p}}X \neq 0$ and $\Lambda^{\mathfrak{p}}Y \neq 0$ when $\Lambda^{\mathfrak{p}}\mathcal{H}om(X, Y) \neq 0$. \square

In [6, Theorem 7.3] we proved that $\text{supp}_R(X \otimes Y) = \text{supp}_R X \cap \text{supp}_R Y$ holds if \mathcal{T} is stratified. An analogue for function objects is contained in the next result.

Theorem 9.5. *The following conditions on \mathcal{T} are equivalent:*

- (1) *The tensor triangulated category \mathcal{T} is stratified by R .*

- (2) $\text{cosupp}_R \mathcal{H}om(X, Y) = \text{supp}_R X \cap \text{cosupp}_R Y$ for all $X, Y \in \mathcal{T}$.
 (3) $\mathcal{H}om(X, Y) = 0$ implies $\text{supp}_R X \cap \text{cosupp}_R Y = \emptyset$ for all $X, Y \in \mathcal{T}$.

Proof. (1) \Rightarrow (2): One inclusion follows from Lemma 9.3. For the other inclusion one uses that \mathcal{T} is stratified by R . Fix $\mathfrak{p} \in \text{supp}_R X \cap \text{cosupp}_R Y$. The minimality of the tensor ideal localizing subcategory $\Gamma_{\mathfrak{p}} \mathcal{T}$ implies $\Gamma_{\mathfrak{p}} \mathbb{1} \in \text{Loc}^{\otimes}(\Gamma_{\mathfrak{p}} X)$, since $\Gamma_{\mathfrak{p}} X \neq 0$. Applying Lemma 8.4, one obtains

$$0 \neq \mathcal{H}om(\Gamma_{\mathfrak{p}} \mathbb{1}, Y) \in \text{Coloc}^{\mathcal{H}om}(\mathcal{H}om(\Gamma_{\mathfrak{p}} X, Y))$$

and therefore $\mathcal{H}om(\Gamma_{\mathfrak{p}} X, Y) \neq 0$. Using the first isomorphism in (9.4), it follows that \mathfrak{p} is in the cosupport of $\mathcal{H}om(X, Y)$.

(2) \Rightarrow (3): Clearly, $\mathcal{H}om(X, Y) = 0$ implies $\text{cosupp}_R \mathcal{H}om(X, Y) = \emptyset$.

(3) \Rightarrow (1): To prove that the tensor triangulated category \mathcal{T} is stratified by R , it suffices to show that given non-zero objects X and Y in $\Gamma_{\mathfrak{p}} \mathcal{T}$ for some \mathfrak{p} in $\text{Spec } R$, there exists a Z such that $\text{Hom}_{\mathcal{T}}^*(X \otimes Z, Y) \neq 0$; see [4, Lemma 3.9].

Since $\text{supp}_R Y = \{\mathfrak{p}\}$, it follows from Theorem 4.13 that $\mathfrak{p} \in \text{cosupp}_R Y$ holds, and hence from our assumption that $\mathcal{H}om(X, Y) \neq 0$. In particular, there exists a $Z \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}^*(Z, \mathcal{H}om(X, Y)) \neq 0$. The adjunction isomorphism (8.1) then yields $\text{Hom}_{\mathcal{T}}^*(X \otimes Z, Y) \neq 0$. \square

The preceding result has the following immediate consequence.

Corollary 9.6. *Suppose the tensor triangulated category \mathcal{T} is generated by its unit. Then \mathcal{T} is stratified by R if and only if for all objects X and Y in \mathcal{T} one has*

$$\text{Hom}_{\mathcal{T}}^*(X, Y) = 0 \iff \text{supp}_R X \cap \text{cosupp}_R Y = \emptyset. \quad \square$$

There is the following connection between stratification and costratification.

Theorem 9.7. *When the tensor triangulated category \mathcal{T} is costratified by R , it is also stratified by R , and then there is an equality*

$$\text{cosupp}_R \mathcal{H}om(X, Y) = \text{supp}_R X \cap \text{cosupp}_R Y \quad \text{for all } X, Y \in \mathcal{T}.$$

Proof. It suffices to prove that given non-zero objects X and Y in $\Gamma_{\mathfrak{p}} \mathcal{T}$ for some \mathfrak{p} in $\text{Spec } R$, there exists a Z such that $\text{Hom}_{\mathcal{T}}^*(X \otimes Z, Y) \neq 0$; see [4, Lemma 3.9].

Assume \mathcal{T} is costratified by R . As $\Gamma_{\mathfrak{p}} X \neq 0$ there exist an object $C \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{T}}^*(X, C(\mathfrak{p})) \neq 0$, by Proposition 4.14. It is easy to verify using the adjunction isomorphism (8.1) that the subcategory

$$\mathcal{S} = \{W \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}^*(X \otimes Z, W) = 0 \text{ for all } Z \in \mathcal{T}\}$$

of \mathcal{T} is colocalizing and Hom closed.

Now observe that $\mathfrak{p} \in \text{cosupp}_R Y$ holds by Theorem 4.13, since $\text{supp}_R Y = \{\mathfrak{p}\}$. Thus $\Lambda^{\mathfrak{p}} \mathcal{T} = \text{Coloc}^{\mathcal{H}om}(\Lambda^{\mathfrak{p}} Y)$, by the costratification hypothesis. This implies $\Lambda^{\mathfrak{p}} Y \notin \mathcal{S}$, since $C(\mathfrak{p}) \in \Lambda^{\mathfrak{p}} \mathcal{T}$ by Theorem 4.13, and $C(\mathfrak{p}) \notin \mathcal{S}$. Thus one obtains that

$$\text{Hom}_{\mathcal{T}}^*(X \otimes Z, Y) \cong \text{Hom}_{\mathcal{T}}^*(\Gamma_{\mathfrak{p}}(X \otimes Z), Y) \cong \text{Hom}_{\mathcal{T}}^*(X \otimes Z, \Lambda^{\mathfrak{p}} Y) \neq 0$$

for some Z in \mathcal{T} , and hence that \mathcal{T} is stratified.

The formula for $\text{cosupp}_R \mathcal{H}om(X, Y)$ follows from Theorem 9.5. \square

Remark 9.8. It is an open question whether stratification implies costratification. The proof of Theorem 9.7 uses the fact that every localizing subcategory generated by a set of objects arises as the kernel of a localization functor; see the proof of [4, Lemma 3.9]. It is not known whether the analogous statement for colocalizing subcategories is true or not. This reflects the fact that products are usually more complicated than coproducts.

The following corollary combines the classification of colocalizing subcategories, Corollary 9.2, with the classification of localizing subcategories in [5, Theorem 3.8].

Corollary 9.9. *If the tensor triangulated category \mathcal{T} is costratified by R then the map sending a subcategory \mathcal{S} to \mathcal{S}^\perp induces a bijection*

$$\left\{ \begin{array}{c} \text{tensor closed localizing} \\ \text{subcategories of } \mathcal{T} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{Hom closed colocalizing} \\ \text{subcategories of } \mathcal{T} \end{array} \right\}.$$

The inverse map sends a Hom closed colocalizing subcategory \mathcal{U} to ${}^\perp\mathcal{U}$.

Proof. Assume \mathcal{T} is costratified by R ; it is then stratified by R , by Theorem 9.7. In particular, both the tensor closed localizing subcategories and the Hom closed colocalizing subcategories of \mathcal{T} are in bijection with the subsets of $\text{supp}_R \mathcal{T}$, via the maps $\text{supp}_R(-)$ and $\text{cosupp}_R(-)$, respectively; see [4, Theorem 3.8] and Corollary 9.2. Now, for any tensor closed localizing subcategory \mathcal{S} of \mathcal{T} one has equalities

$$\begin{aligned} \mathcal{S}^\perp &= \{Y \in \mathcal{T} \mid \mathcal{H}om(X, Y) = 0 \text{ for all } X \in \mathcal{S}\} \\ &= \{Y \in \mathcal{T} \mid \text{cosupp}_R Y \cap \text{supp}_R \mathcal{S} = \emptyset\} \\ &= \text{cosupp}_R^{-1}(\text{supp}_R \mathcal{T} \setminus \text{supp}_R \mathcal{S}) \end{aligned}$$

where the first one is a routine verification, and the second one is by Theorem 9.7. In the same vein, for any Hom closed localizing subcategory \mathcal{U} one has

$${}^\perp\mathcal{U} = \text{supp}_R^{-1}(\text{supp}_R \mathcal{T} \setminus \text{cosupp}_R \mathcal{U}).$$

It thus follows that, under the identification above, both maps $\mathcal{S} \mapsto \mathcal{S}^\perp$ and $\mathcal{U} \mapsto {}^\perp\mathcal{U}$ correspond to the map on $\text{supp}_R \mathcal{T}$ sending a subset to its complement, and are thus inverse to each other. \square

Brown–Comenetz duality. Let k be a commutative ring and suppose that the category \mathcal{T} is k -linear. We denote by $D = \text{Hom}_k(-, E)$ the duality for the category of k -modules with respect to a fixed injective cogenerator E .

The *Brown–Comenetz dual* X^* of an object X is defined by the isomorphism

$$D \text{Hom}_{\mathcal{T}}(\mathbb{1}, - \otimes X) \cong \text{Hom}_{\mathcal{T}}(-, X^*).$$

Note that there is a natural isomorphism $X^* \cong \mathcal{H}om(X, \mathbb{1}^*)$.

Proposition 9.10. *One has $\text{cosupp}_R \mathbb{1}^* = \text{supp}_R \mathcal{T}$ and $\text{cosupp}_R X^* \subseteq \text{supp}_R X$ for any $X \in \mathcal{T}$; equality holds if \mathcal{T} is stratified by R as a tensor triangulated category.*

Proof. The first equality holds because $X^* \cong \mathcal{H}om(X, \mathbb{1}^*)$ and $X^* = 0$ if and only if $X = 0$. The inclusion follows from Lemma 9.3, since

$$\text{cosupp}_R X^* = \text{cosupp}_R \mathcal{H}om(X, \mathbb{1}^*) \subseteq \text{supp}_R X \cap \text{cosupp}_R \mathbb{1}^* = \text{supp}_R X.$$

When \mathcal{T} is stratified, Theorem 9.5 gives equality. \square

Remark 9.11. Let \mathcal{T} be a tensor triangulated category generated by its unit $\mathbb{1}$, and equipped with a canonical R action. Given Remark 8.8, it is clear that \mathcal{T} is (co)stratified by R as a triangulated category if, and only if, it is (co)stratified by R as a tensor triangulated category. And to verify that \mathcal{T} is stratified (respectively, costratified) it suffices to check that each $\Gamma_{\mathfrak{p}} \mathcal{T}$ is a minimal localizing subcategory (respectively, each $\Lambda^{\mathfrak{p}} \mathcal{T}$ is a minimal colocalizing subcategory).

The results in this section thus yield interesting information about all localizing and colocalizing subcategories of \mathcal{T} . For instance, Corollary 9.2 coincides with the bijection in Remark 5.7, while Theorem 9.7 says that if \mathcal{T} is costratified then it is also stratified.

10. FORMAL DG ALGEBRAS

In this section we prove that the derived category of dg (short for “differential graded”) modules over a formal dg algebra is costratified by its cohomology algebra, when that algebra is graded commutative and noetherian.

Let A be a dg algebra and $D(A)$ its derived category of (left) dg modules. It is a triangulated category, generated by the compact object A ; see [16]. A morphism $A \rightarrow B$ of dg algebras is a *quasi-isomorphism* if the induced map $H^*A \rightarrow H^*B$ is an isomorphism. Then restriction induces an equivalence of triangulated categories $D(B) \xrightarrow{\sim} D(A)$, with quasi-inverse the functor $B \otimes_A^L -$. Dg algebras A and B are called *quasi-isomorphic* if there is a finite chain of quasi-isomorphisms linking them.

The multiplication on A induces one on its cohomology H^*A . We say A is *formal* if it is quasi-isomorphic to H^*A , viewed as a dg algebra with zero differential.

Remark 10.1. Suppose that A is formal and that H^*A is graded commutative. Fix a chain of quasi-isomorphisms linking A and H^*A ; it induces an equivalence of categories $D(A) \simeq D(H^*A)$.

The derived tensor product of dg modules endows $D(H^*A)$ with a structure of a tensor triangulated category, with unit H^*A . Thus, $D(A)$ also acquires such a structure, via the equivalence $D(A) \simeq D(H^*A)$; denote \otimes this tensor product on $D(A)$. The object A is a tensor unit, and one gets an action of H^*A on $D(A)$, defined by taking for each object X the composite map

$$H^*A \xrightarrow{\sim} \text{End}_{D(A)}^*(A) \xrightarrow{X \otimes -} \text{End}_{D(A)}^*(X).$$

We refer to this as the action induced by the given chain of quasi-isomorphisms linking A and H^*A .

The action of H^*A on $D(A)$ depends on the choice of quasi-isomorphisms linking A and H^*A . One has however the following independence statement.

Lemma 10.2. *Let A be a formal dg algebra with H^*A graded commutative and noetherian. For any specialization closed set $\mathcal{V} \subseteq \text{Spec } H^*A$ the functors $\Gamma_{\mathcal{V}}$, $L_{\mathcal{V}}$, $\Lambda^{\mathcal{V}}$ and $V^{\mathcal{V}}$ are independent of a chain of quasi-isomorphisms linking A to H^*A .*

Proof. For any chain of quasi-isomorphisms, it is clear from the construction that the homomorphism $H^*A \rightarrow \text{End}_{D(A)}^*(A)$ is the canonical one, and hence it is independent of the action. It thus remains to note that the local cohomology functors on a compactly generated R -linear triangulated category are determined by the action of R on a compact generator, by [5, Corollary 3.3]. \square

In the case when A is a ring, which may be viewed as a dg algebra concentrated in degree zero, the result below is contained in recent work of Neeman [21]. We note that the cosupport, and hence the costratification, are independent of a choice of a canonical action, by Lemma 10.2.

Theorem 10.3. *Let A be a formal dg algebra such that H^*A is graded commutative and noetherian. The category $D(A)$ is costratified by any H^*A -action induced by a chain of quasi-isomorphisms linking A and H^*A .*

Proof. We may replace A by H^*A and assume $d^A = 0$ and A is graded commutative. Set $D = D(A)$. Since A is a unit and a generator of this tensor triangulated category, its colocalizing subcategories are $\mathbf{R}\text{Hom}$ closed; see Lemma 8.7. It remains to verify that $A^{\mathfrak{p}}D$ is a minimal colocalizing subcategory for each homogeneous prime ideal \mathfrak{p} in A ; see Remark 9.11.

Let $k(\mathfrak{p})$ be the graded residue field at \mathfrak{p} . The object $k(\mathfrak{p})$ is in $A^{\mathfrak{p}}D$, so to verify the minimality of $A^{\mathfrak{p}}D$ it suffices to note that for each non-zero M in $A^{\mathfrak{p}}D$, the following equalities hold:

$$\begin{aligned} \text{Coloc}_D(M) &= \text{Coloc}_D(\mathbf{R}\text{Hom}_A(\Gamma_{\mathfrak{p}}A, M)) \\ &= \text{Coloc}_D(\mathbf{R}\text{Hom}_A(k(\mathfrak{p}), M)) \\ &= \text{Coloc}_D(k(\mathfrak{p})) \end{aligned}$$

The first equality holds by Proposition 8.3 since $M \cong A^{\mathfrak{p}}M$. As to the second one, since A is stratified by H^*A , by [6, Theorem 8.1], one has

$$\text{Loc}_D(\Gamma_{\mathfrak{p}}A) = \Gamma_{\mathfrak{p}}D = \text{Loc}_D(k(\mathfrak{p})).$$

Now apply Lemma 8.4. The last equality follows from the fact that $k(\mathfrak{p})$ is a graded field and the action of A on $\mathbf{R}\text{Hom}_A(k(\mathfrak{p}), M)$ factors through $k(\mathfrak{p})$. \square

Exterior algebras. Let Λ be a graded exterior algebra over a field k on indeterminates ξ_1, \dots, ξ_c in negative odd degree, regarded as a dg algebra with zero differential. We give Λ a structure of Hopf algebra via $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i$. In [5, §4], we introduced the homotopy category of graded-injective dg modules over Λ , denoted $\mathbf{K}(\text{Inj } \Lambda)$, and proved the following statements: $\mathbf{K}(\text{Inj } \Lambda)$ is a compactly generated tensor triangulated category, in the sense of Section 8; its unit is an injective resolution of the trivial module k , and this generates $\mathbf{K}(\text{Inj } \Lambda)$; the graded endomorphism algebra of the unit is $\text{Ext}_{\Lambda}^*(k, k)$, which is isomorphic to the graded polynomial k -algebra $S = k[x_1, \dots, x_c]$ with $|x_i| = -|\xi_i| + 1$.

Theorem 10.4. *The category $\mathbf{K}(\text{Inj } \Lambda)$ is costratified by the action of $\text{Ext}_{\Lambda}^*(k, k)$.*

Proof. It is proved in [5, Theorem 6.2] that a suitable dg module J over $\Lambda \otimes_k S$ yields an equivalence $\text{Hom}_{\Lambda}(J, -): \mathbf{K}(\text{Inj } \Lambda) \xrightarrow{\sim} D(S)$. The theorem now follows from Theorem 10.3. \square

11. FINITE GROUPS

Throughout this section, G will be a finite group and k a field whose characteristic divides the order of G . The associated group algebra is denoted kG , and $H^*(G, k)$ denotes its cohomology k -algebra, $\text{Ext}_{kG}^*(k, k)$. This algebra is graded commutative, because kG is a Hopf algebra, and finitely generated and hence noetherian, by a result of Evens and Venkov [10, 24]; see also Golod [13].

Let $\mathbf{K}(\text{Inj } kG)$ be the homotopy category of complexes of injective kG -modules. This is a compactly generated tensor triangulated category, in the sense of Section 8, where the tensor product $X \otimes_k Y$ and the function object $\text{Hom}_k(X, Y)$ are induced by those on kG -modules via the diagonal action of G . The injective resolution ik of the trivial module k is the identity for the tensor product, and it yields a canonical action of $H^*(G, k)$ on $\mathbf{K}(\text{Inj } kG)$; see [7] for details. To simplify notation, we write \mathcal{V}_G for $\text{Spec } H^*(G, k)$, and set for each object X in $\mathbf{K}(\text{Inj } kG)$

$$\text{supp}_G X = \text{supp}_{H^*(G, k)} X \quad \text{and} \quad \text{cosupp}_G X = \text{cosupp}_{H^*(G, k)} X.$$

Example 11.1. We write \mathfrak{m} for the maximal ideal $H^{\geq 1}(G, k)$.

- (1) One has $\text{cosupp}_G X = \{\mathfrak{m}\}$ for any non-zero compact object $X \in \mathbf{K}(\text{Inj } kG)$.
- (2) For each $\mathfrak{p} \in \mathcal{V}_G$ the object $T_{ik}(I(\mathfrak{p}))$, from (5.3), satisfies

$$\text{supp}_G T_{ik}(I(\mathfrak{p})) = \{\mathfrak{p}\} \quad \text{and} \quad \text{cosupp}_G T_{ik}(I(\mathfrak{p})) = \{\mathfrak{q} \in \mathcal{V}_G \mid \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Indeed, for (1) observe that the natural morphism $X \rightarrow X^{**}$ is an isomorphism for each compact object X by [5, Lemma 11.5], where $(-)^*$ denotes Brown–Comenetz duality. Proposition 9.10 therefore gives the first inclusion below

$$\operatorname{cosupp}_G X \subseteq \operatorname{supp}_G X^* \subseteq \{\mathfrak{m}\}.$$

For the second inclusion one uses that the $H^*(G, k)$ -module $\operatorname{Hom}_{\mathbf{K}(kG)}^*(C, X^*)$ is \mathfrak{m} -torsion for each compact $C \in \mathbf{K}(\operatorname{Inj} kG)$, since it is of the form $\operatorname{Hom}_k(M, k)$ for some finitely generated $H^*(G, k)$ -module M by the defining isomorphism of the Brown–Comenetz dual X^* . It remains to observe that $\operatorname{cosupp}_G X \neq \emptyset$ since $X \neq 0$.

To prove (2), note that $\operatorname{Hom}_{\mathbf{K}(kG)}^*(C, T_{ik}(I(\mathfrak{p})))$ is \mathfrak{p} -torsion and \mathfrak{p} -local for each compact object $C \in \mathbf{K}(\operatorname{Inj} kG)$, by the isomorphism (5.3) defining $T_{ik}(I(\mathfrak{p}))$. Thus $\operatorname{supp}_G T_{ik}(I(\mathfrak{p})) = \{\mathfrak{p}\}$, while the cosupport is given by Proposition 5.4.

Restriction and induction. For each subgroup H of G restriction and induction yield exact functors

$$\begin{aligned} (-)\downarrow_H &= \operatorname{Hom}_{kG}(kG, -): \mathbf{K}(\operatorname{Inj} kG) \rightarrow \mathbf{K}(\operatorname{Inj} kH) \quad \text{and} \\ (-)\uparrow^G &= - \otimes_{kH} kG: \mathbf{K}(\operatorname{Inj} kH) \rightarrow \mathbf{K}(\operatorname{Inj} kG). \end{aligned}$$

Restriction yields also a homomorphism $\operatorname{res}_{G,H}: H^*(G, k) \rightarrow H^*(H, k)$ of graded rings, and hence a map:

$$\operatorname{res}_{G,H}^*: \mathcal{V}_H \rightarrow \mathcal{V}_G.$$

It is easy to verify that these functors fit in the framework of Section 7:

Lemma 11.2. *The functor $(-)\downarrow_H$ is $\operatorname{res}_{G,H}$ -linear and induces an exact functor*

$$((-)\downarrow_H, \operatorname{res}_{G,H}): (\mathbf{K}(\operatorname{Inj} kG), H^*(G, k)) \rightarrow (\mathbf{K}(\operatorname{Inj} kH), H^*(H, k))$$

which preserves coproducts and products. Its right adjoint and left adjoint is $(-)\uparrow^G$, and the latter is faithful on objects. \square

The following result is an analogue of [5, Lemma 9.3].

Lemma 11.3. *Let H be a subgroup of G . Fix $\mathfrak{p} \in \mathcal{V}_G$ and set $\mathcal{U} = (\operatorname{res}_{G,H}^*)^{-1}\{\mathfrak{p}\}$. The set \mathcal{U} is finite and discrete, and for any $X \in \mathbf{K}(\operatorname{Inj} kG)$ and any $Y \in \mathbf{K}(\operatorname{Inj} kH)$ there are natural isomorphisms:*

$$\begin{aligned} (\Gamma_{\mathfrak{p}} X)\downarrow_H &\cong \prod_{\mathfrak{q} \in \mathcal{U}} \Gamma_{\mathfrak{q}}(X\downarrow_H) & \Gamma_{\mathfrak{p}}(Y\uparrow^G) &\cong \prod_{\mathfrak{q} \in \mathcal{U}} (\Gamma_{\mathfrak{q}} Y)\uparrow^G \\ (\Lambda^{\mathfrak{p}} X)\downarrow_H &\cong \prod_{\mathfrak{q} \in \mathcal{U}} \Lambda^{\mathfrak{q}}(X\downarrow_H) & \Lambda^{\mathfrak{p}}(Y\uparrow^G) &\cong \prod_{\mathfrak{q} \in \mathcal{U}} (\Lambda^{\mathfrak{q}} Y)\uparrow^G \end{aligned}$$

Proof. The set \mathcal{U} is finite and discrete as $H^*(H, k)$ is finitely generated as a module over $H^*(G, k)$. The result now follows from Corollary 7.10, given Lemma 11.2. \square

The next result is an analogue of [5, Proposition 9.4].

Proposition 11.4. *Let H be a subgroup of G . The following statements hold:*

- (1) *For any object Y in $\mathbf{K}(\operatorname{Inj} kH)$, there is an equality*

$$\operatorname{cosupp}_G(Y\uparrow^G) = \operatorname{res}_{G,H}^*(\operatorname{cosupp}_H Y).$$

- (2) *Any object X in $\mathbf{K}(\operatorname{Inj} kG)$ satisfies $X\downarrow_H\uparrow^G \in \operatorname{Coloc}^{\mathcal{H}om}(X)$, and hence*

$$\operatorname{cosupp}_G(X\downarrow_H\uparrow^G) \subseteq \operatorname{cosupp}_G X.$$

Proof. Part (1) is a special case of Corollary 7.8, which applies by Lemma 11.2.

(2) One has $X \downarrow_H \uparrow^G \cong X \otimes_k k(G/H)$, where $k(G/H)$ is the permutation module on the cosets of H in G . Let W be an injective resolution over kG of $k(G/H)$. Arguing as in the proof of [7, Proposition 5.3], one can prove that the natural map $X \otimes_k k(G/H) \rightarrow X \otimes_k W$ is an isomorphism in $\mathbf{K}(\text{Inj } kG)$. As W is compact in $\mathbf{K}(\text{Inj } kG)$, see [18, Lemma 2.1], it remains to apply Lemma 8.5. \square

Elementary abelian groups. Let E be an elementary abelian p -group

$$E = \langle g_1, \dots, g_r \rangle$$

and k a field of characteristic p . We set $z_i = g_i - 1$, so that the group algebra may be described as

$$kE = k[z_1, \dots, z_r] / (z_1^p, \dots, z_r^p).$$

Let A be the Koszul complex on z_1, \dots, z_r , viewed as a dg algebra: The graded algebra underlying it is generated by kE in degree zero together with exterior generators y_1, \dots, y_r , each of degree -1 . The differential on A is given by $d(y_i) = z_i$ and $d(z_i) = 0$. We write $\mathbf{K}(\text{Inj } A)$ for the homotopy category of graded-injective dg A -modules. It is a compactly generated tensor triangulated category, with a canonical action of $\text{Ext}_A^*(k, k)$; see [5, §8].

Proposition 11.5. *The category $\mathbf{K}(\text{Inj } A)$ is costratified by the action of $\text{Ext}_A^*(k, k)$.*

Proof. Let Λ be an exterior algebra over k on indeterminates ξ_1, \dots, ξ_r of degree -1 , viewed as a dg algebra with $d^\Lambda = 0$. Let $\mathbf{K}(\text{Inj } \Lambda)$ be the homotopy category of graded-injective dg Λ -modules, with tensor triangulated structure described in the paragraph preceding Theorem 10.4.

By [5, Lemma 7.1] there is a quasi-isomorphism of dg k -algebras $\phi: \Lambda \rightarrow A$ defined by $\phi(\xi_i) = z_i^{p-1} y_i$, and by [5, Proposition 4.6] this induces an equivalence of triangulated categories

$$\text{Hom}_\Lambda(A, -): \mathbf{K}(\text{Inj } \Lambda) \xrightarrow{\sim} \mathbf{K}(\text{Inj } A).$$

The desired result is now a consequence of Theorem 10.4. \square

The next result complements [5, Theorem 8.1], concerning stratification.

Theorem 11.6. *Let E be an elementary abelian p -group and k a field of characteristic p . The category $\mathbf{K}(\text{Inj } kE)$ is costratified by the canonical action of $H^*(E, k)$.*

Proof. Write $kE \cong k[z_1, \dots, z_r] / (z_1^p, \dots, z_r^p)$, and let A be the Koszul dg algebra described above. Note that $kE = A_0$ so the inclusion $kE \rightarrow A$ is a morphism of dg algebras; restriction along it gives an exact functor $\mathbf{K}(\text{Inj } A) \rightarrow \mathbf{K}(\text{Inj } kE)$ which preserves coproducts and products, and is compatible with the induced homomorphism $\alpha: \text{Ext}_A^*(k, k) \rightarrow \text{Ext}_{kE}^*(k, k) = H^*(E, k)$. The functor $\text{Hom}_{kE}(A, -)$ is a right adjoint of restriction. We claim that each $X \in \mathbf{K}(\text{Inj } kE)$ satisfies an equality:

$$(*) \quad \text{Thick}(\text{Hom}_{kE}(A, X)) = \text{Thick}(X) \quad \text{in } \mathbf{K}(\text{Inj } kE),$$

and, in particular, that $\text{Hom}_{kE}(A, -)$ is faithful on objects.

Indeed, recall that A is the Koszul complex over kE on $\mathbf{z} = z_1, \dots, z_r$, and in particular, a finite free complex of kE -modules. This yields the first isomorphism below of complexes of kE -modules:

$$\text{Hom}_{kE}(A, X) \cong \text{Hom}_{kE}(A, kE) \otimes_{kE} X \cong \Sigma^r A \otimes_{kE} X$$

The second one is by self-duality of the Koszul complex; see [8, Proposition 1.6.10]. The radical of the ideal (\mathbf{z}) coincides with that of (0) , so the complexes A and kE generate the same thick subcategory in $\mathbf{K}(\text{Inj } kE)$; see [15, Lemma 6.0.9]. Since

$- \otimes_{kE} X$ is an exact functor on $K(\text{Inj } kE)$, it follows from the isomorphism above that $\text{Hom}_{kE}(A, X)$ and X generate the same thick subcategory, as desired.

We are thus in a position to apply Corollary 7.8. Fix a prime $\mathfrak{p} \in \mathcal{V}_E$ and let $\mathfrak{q} = \alpha^{-1}(\mathfrak{p})$. The functor $\text{Hom}_{kE}(A, -)$ then takes $\Lambda^{\mathfrak{p}}K(\text{Inj } kE)$ to $\Lambda^{\mathfrak{q}}K(\text{Inj } A)$ and is faithful. Hence, for any non-zero objects X and Y in $\Lambda^{\mathfrak{p}}K(\text{Inj } kE)$, one has

$$\text{Hom}_{kE}(A, Y) \in \text{Coloc}(\text{Hom}_{kE}(A, X)) \quad \text{in } K(\text{Inj } A),$$

since $K(\text{Inj } A)$ is costratified by the action of $\text{Ext}_A^*(k, k)$; see Proposition 11.5. It implies, in view of the equality (*) above, that in $K(\text{Inj } kE)$ one has

$$Y \in \text{Thick}(\text{Hom}_{kE}(A, Y)) \subseteq \text{Coloc}(\text{Hom}_{kE}(A, X)) = \text{Coloc}(X).$$

Thus, $\Lambda^{\mathfrak{p}}K(\text{Inj } kE)$ is a minimal colocalizing subcategory. Thus, $K(\text{Inj } kE)$ is costratified by the action of $H^*(E, k)$; see Remark 9.11. \square

Next we prepare for a version of the preceding theorem for arbitrary finite groups.

Quillen's stratification. We consider pairs (H, \mathfrak{q}) of subgroups H of G and primes $\mathfrak{q} \in \mathcal{V}_H$, and say that (H, \mathfrak{q}) and (H', \mathfrak{q}') are *G-conjugate* if conjugation with some element in G takes H to H' and \mathfrak{q} to \mathfrak{q}' .

Lemma 11.7. *Let (H, \mathfrak{q}) and (H', \mathfrak{q}') be G-conjugate pairs and X in $K(\text{Inj } kG)$. Then $\Lambda^{\mathfrak{q}}(X \downarrow_H) \neq 0$ if and only if $\Lambda^{\mathfrak{q}'}(X \downarrow_{H'}) \neq 0$.*

Proof. Conjugation with any element $g \in G$ induces an automorphism of $K(\text{Inj } kG)$ that takes an object X to X^g . Note that multiplication with g induces an isomorphism $X \xrightarrow{\sim} X^g$. The assertion follows, since conjugation commutes with restriction and local homology. \square

Quillen has proved that for each \mathfrak{p} in \mathcal{V}_G there exists a pair (E, \mathfrak{q}) such that E is an elementary abelian subgroup of G and $\text{res}_{G,E}^*(\mathfrak{q}) = \mathfrak{p}$; see the discussion after [22, Proposition 11.2]. We say \mathfrak{p} *originates* in such a pair (E, \mathfrak{q}) if there does not exist another such pair (E', \mathfrak{q}') with E' a proper subgroup of E . In this language, [22, Theorem 10.2] reads:

Theorem 11.8. *For any $\mathfrak{p} \in \mathcal{V}_G$ all pairs (E, \mathfrak{q}) where \mathfrak{p} originates are G-conjugate. This sets up a one to one correspondence between primes \mathfrak{p} in \mathcal{V}_G and G-conjugacy classes of such pairs (E, \mathfrak{q}) .* \square

The proof of the next result can be shortened considerably by invoking the subgroup theorem for supports, [4, Theorem 11.2], which is deduced from the stratification theorem for $K(\text{Inj } kG)$, [4, Theorem 9.7]. We give a direct proof, by extracting an argument from the proof of the latter result.

Proposition 11.9. *Fix $\mathfrak{p} \in \mathcal{V}_G$ and suppose \mathfrak{p} originates in E . Given a non-zero object X in $\Lambda^{\mathfrak{p}}K(\text{Inj } kG)$, one has*

$$X \downarrow_E = \prod_{\text{res}_{G,E}^*(\mathfrak{q})=\mathfrak{p}} \Lambda^{\mathfrak{q}}(X \downarrow_E)$$

and $\Lambda^{\mathfrak{q}}(X \downarrow_E) \neq 0$ for each such \mathfrak{q} .

Proof. The decomposition of $X \downarrow_E$ is by Lemma 11.3. For the remaining statements, it suffices to find *one* pair (E', \mathfrak{q}') where \mathfrak{p} originates and such that $\Lambda^{\mathfrak{q}'}(X \downarrow_{E'}) \neq 0$. Then Theorem 11.8 yields that each pair (E, \mathfrak{q}) as in the statement is G-conjugate to (E', \mathfrak{q}') , and hence $\Lambda^{\mathfrak{q}}(X \downarrow_E) \neq 0$, by Lemma 11.7.

Since $X \neq 0$ holds, Chouinard's theorem for $\mathbf{K}(\operatorname{Inj} kG)$, see [4, Proposition 9.6(3)], provides an elementary abelian subgroup E'' of G such that $X \downarrow_{E''} \neq 0$. Fix a \mathfrak{q}'' in $\operatorname{cosupp}_{E''}(X \downarrow_{E''})$. Proposition 11.4 then yields

$$\operatorname{res}_{G,E''}^*(\mathfrak{q}'') \in \operatorname{cosupp}_G(X \downarrow_{E''} \uparrow^G) \subseteq \operatorname{cosupp}_G X = \{\mathfrak{p}\}.$$

Applying Theorem 11.8 to E'' one gets a pair (E', \mathfrak{q}') , with E' a subgroup of E'' , where \mathfrak{q}'' originates. Observe that \mathfrak{p} then originates in (E', \mathfrak{q}') , by functoriality of restrictions and the computation above. It remains to note that

$$\mathfrak{q}' \in (\operatorname{res}_{E'',E'}^*)^{-1}(\operatorname{cosupp}_{E''}(X \downarrow_{E''})) = \operatorname{cosupp}_{E'}(X \downarrow_{E'}),$$

where the inclusion holds by the choice of \mathfrak{q}' and the equality is by Theorem 7.11, applied to the functor

$$((-) \downarrow_{E'}, \operatorname{res}_{E'',E'}^*): (\mathbf{K}(\operatorname{Inj} kE''), H^*(E'', k)) \rightarrow (\mathbf{K}(\operatorname{Inj} kE'), H^*(E', k));$$

noting that the hypotheses are satisfied by Lemma 11.2 and Theorem 11.6. \square

Finite groups. The following theorem is an analogue of [5, Theorem 9.7], which establishes the stratification of $\mathbf{K}(\operatorname{Inj} kG)$; it contains Theorem 11.6, but the latter statement is used in the proof, both directly and by way of Proposition 11.9.

Theorem 11.10. *The tensor triangulated category $\mathbf{K}(\operatorname{Inj} kG)$ is costratified by the canonical action of the cohomology algebra $H^*(G, k)$.*

Proof. Fix $\mathfrak{p} \in \mathcal{V}_G$ and suppose it originates in E . For each compact object C in $\mathbf{K}(\operatorname{Inj} kE)$ and each $\mathfrak{q} \in \mathcal{V}_E$ with $\operatorname{res}_{G,E}^*(\mathfrak{q}) = \mathfrak{p}$, Proposition 5.4 and (3.1) imply

$$\operatorname{cosupp}_E(T_{C//\mathfrak{q}}(I(\mathfrak{q}))) \subseteq \{\mathfrak{q}\}.$$

It follows from Theorem 11.6 and Proposition 11.9 that each non-zero object X in $\Lambda^{\mathfrak{p}}\mathbf{K}(\operatorname{Inj} kG)$ satisfies

$$T_{C//\mathfrak{q}}(I(\mathfrak{q})) \in \operatorname{Coloc}(\Lambda^{\mathfrak{q}}(X \downarrow_E)) \subseteq \operatorname{Coloc}(X \downarrow_E).$$

Together with Proposition 11.4, one thus obtains:

$$T_{C//\mathfrak{q}}(I(\mathfrak{q})) \uparrow^G \in \operatorname{Coloc}(X \downarrow_E \uparrow^G) \subseteq \operatorname{Coloc}^{\mathcal{H}om}(X).$$

It remains to observe that the collection of objects $T_{C//\mathfrak{q}}(I(\mathfrak{q})) \uparrow^G$ cogenerates the triangulated category $\Lambda^{\mathfrak{p}}\mathbf{K}(\operatorname{Inj} kG)$. This is a consequence of Proposition 5.4 and Lemma 7.12, which can be applied, thanks to Lemma 11.2 and Proposition 11.9. \square

Applications. The consequences of costratification described in Section 9 apply to $\mathbf{K}(\operatorname{Inj} kG)$. In particular, Theorem 9.7 implies that $\mathbf{K}(\operatorname{Inj} kG)$ is stratified as a tensor triangulated category by $H^*(G, k)$, which is [5, Theorem 9.7]. A modification of Theorem 7.11 yields the following subgroup theorem for cosupport, which is analogous to the one for support; see [5, Theorem 11.2].

Theorem 11.11. *Let H be a subgroup of G . Each object X in $\mathbf{K}(\operatorname{Inj} kG)$ satisfies*

$$\operatorname{cosupp}_H(X \downarrow_H) = (\operatorname{res}_{G,H}^*)^{-1}(\operatorname{cosupp}_G X).$$

Proof. Restriction and induction form an adjoint pair of functors satisfying the assumptions of Theorem 7.11. Thus one gets

$$\operatorname{cosupp}_H(X \downarrow_H) \subseteq (\operatorname{res}_{G,H}^*)^{-1}(\operatorname{cosupp}_G X).$$

For the other inclusion, one uses stratification of $\mathbf{K}(\operatorname{Inj} kG)$ as follows.

Fix a $\mathfrak{q} \in \mathcal{V}_H$ with $\mathfrak{q} \notin \text{cosupp}_H(X \downarrow_H)$. We need to show that $\mathfrak{p} = \text{res}_{G,H}^*(\mathfrak{q})$ is not in $\text{cosupp}_G X$. Using the adjunction formula for function objects from Proposition 8.3, one has

$$\text{Hom}_k(\Gamma_{\mathfrak{q}} -, X \downarrow_H) \cong \text{Hom}_k(-, \Lambda^{\mathfrak{q}} X \downarrow_H) = 0.$$

This implies $\text{Hom}_k((\Gamma_{\mathfrak{q}} -) \uparrow^G, X) = 0$, since

$$\text{Hom}_k(U, V \downarrow_H) \uparrow^G \cong \text{Hom}_k(U \uparrow^G, V)$$

for all U in $\mathbf{K}(\text{Inj } kH)$ and V in $\mathbf{K}(\text{Inj } kG)$, by [1, Proposition 3.3]. There exists some U in $\mathbf{K}(\text{Inj } H)$ such that $(\Gamma_{\mathfrak{q}} U) \uparrow^G \neq 0$ since induction is faithful on objects. Moreover, $(\Gamma_{\mathfrak{q}} U) \uparrow^G$ belongs to $\Gamma_{\mathfrak{p}} \mathbf{K}(\text{Inj } kG)$, by Corollary 7.8. Since $\mathbf{K}(\text{Inj } kG)$ is stratified as a tensor triangulated category by $H^*(G, k)$, the subcategory $\Gamma_{\mathfrak{p}} \mathbf{K}(\text{Inj } kG)$ contains no non-trivial tensor ideal localizing subcategories, and hence coincides with $\text{Loc}^{\otimes}((\Gamma_{\mathfrak{q}} U) \uparrow^G)$. Thus

$$0 = \text{Hom}_k(\Gamma_{\mathfrak{p}} -, X) \cong \text{Hom}_k(-, \Lambda^{\mathfrak{p}} X),$$

again by Proposition 8.3, and therefore $\mathfrak{p} \notin \text{cosupp}_R X$. \square

Stable module category. Let $\text{Mod}(kG)$ denote the (abelian) category of all kG -modules. The stable module category, $\text{StMod}(kG)$, has the same objects as the module category $\text{Mod}(kG)$, but the morphisms in $\text{StMod}(kG)$ are given by quotienting out those morphisms in $\text{Mod}(kG)$ that factor through a projective module. The category $\text{StMod}(kG)$ is tensor triangulated, with triangles induced from short exact sequences of kG -modules, and product the tensor product over k with diagonal action. The trivial module k is the unit of the product, and its graded endomorphism ring in $\text{StMod}(kG)$ is the Tate cohomology algebra $\hat{H}^*(G, k)$. There is thus an action of $H^*(G, k)$ on $\text{StMod}(kG)$, via the natural map $H^*(G, k) \rightarrow \hat{H}^*(G, k)$.

We regard $\text{StMod}(kG)$ as a triangulated subcategory of $\mathbf{K}(\text{Inj } kG)$, as in [7, §6].

Proposition 11.12. *The functor $\text{StMod}(kG) \rightarrow \mathbf{K}(\text{Inj } kG)$ that takes a kG -module to its Tate resolution identifies the tensor triangulated category $\text{StMod}(kG)$ with the (co)localizing subcategory consisting of all acyclic complexes.* \square

The action of $H^*(G, k)$ on $\text{StMod}(kG)$ is compatible with this identification. Moreover the acyclic complexes form a localizing and colocalizing subcategory of $\mathbf{K}(\text{Inj } kG)$ that is Hom and tensor closed. Note that a complex is acyclic if and only if its (co)support does not contain the maximal ideal of $H^*(G, k)$. This follows from [5, Proposition 9.6] for the support, and then from Corollary 4.9 for the cosupport.

Theorem 11.13. *The tensor triangulated category $\text{StMod}(kG)$ is costratified by the canonical action of $H^*(G, k)$.*

Proof. Given Proposition 11.12, the desired result follows from Theorem 11.10, since $\Lambda^{\mathfrak{p}} \text{StMod}(kG) = \Lambda^{\mathfrak{p}} \mathbf{K}(\text{Inj } kG)$ for each non-maximal \mathfrak{p} in $\text{Spec } H^*(G, k)$. \square

We can now justify the results stated in the introduction.

Proof of Theorem 1.1. From Theorems 11.13 and 9.7 it follows that the tensor triangulated category $\text{StMod}(kG)$ is stratified and costratified by the canonical action of $H^*(G, k)$. Thus the map sending a subset \mathcal{U} of \mathcal{V}_G to the subcategory of kG -modules X satisfying $\text{supp}_G X \subseteq \mathcal{U}$ yields a bijection between subsets of \mathcal{V}_G and tensor ideal localizing subcategories of $\text{StMod}(kG)$; see [5, Theorem 3.8]. Composing this bijection with the one between localizing and colocalizing subcategories from Corollary 9.9 gives the desired result. \square

Proof of Corollary 1.2. Given Theorem 11.13, this follows from Corollary 9.9. \square

We close this discussion on the stable module category with the following example. There is no analogue for arbitrary tensor triangulated categories; see Example 11.1 and Proposition 4.19.

Example 11.14. There is an equality $\text{cosupp}_G M = \text{supp}_G M$ for any finite dimensional module M in $\text{StMod}(kG)$.

Indeed, denote by $(-)^*$ the Brown–Comenetz duality on $\text{StMod}(kG)$ which equals $\Omega \text{Hom}_k(-, k)$ by [5, Proposition 11.6], where ΩN denotes the kernel of a projective cover of a kG -module N . Then one gets the first equality below because M is finite dimensional:

$$\text{cosupp}_G M = \text{cosupp}_G M^{**} = \text{supp}_G M^* = \text{supp}_G M.$$

The second equality is Proposition 9.10, and the last one is well-known; see [2, Theorem 5.1.1].

Modules. Although $\text{Mod}(kG)$ is not a triangulated category, we define colocalizing subcategories in an analogous way. A full subcategory of $\text{Mod}(kG)$ is said to be *thick* if whenever two modules in a short exact sequence are in, then so is the third. A *colocalizing subcategory* \mathcal{S} of $\text{Mod}(kG)$ is a thick subcategory closed under all products, that is, for any family of modules M_i ($i \in I$) in \mathcal{S} the product $\prod_i M_i$ is in \mathcal{S} . The next lemma describes an extra tensor condition for colocalizing subcategories; it parallels Lemma 8.5.

Lemma 11.15. *Let \mathcal{S} be a colocalizing subcategory of $\text{Mod}(kG)$. Then the following conditions are equivalent:*

- (1) *If N is a simple kG -module and M is in \mathcal{S} then $N \otimes_k M$ is in \mathcal{S} .*
- (2) *If N is a finitely generated kG -module and M is in \mathcal{S} then $N \otimes_k M$ is in \mathcal{S} .*
- (3) *If N is a kG -module and M is in \mathcal{S} then $\text{Hom}_k(N, M)$ is in \mathcal{S} .* \square

A colocalizing subcategory of $\text{Mod}(kG)$ is said to be *Hom closed* if the equivalent conditions of the lemma hold. The next result is analogous to [5, Proposition 2.1].

Proposition 11.16. *The canonical functor from $\text{Mod}(kG)$ to $\text{StMod}(kG)$ induces a one to one correspondence between non-zero Hom closed colocalizing subcategories of $\text{Mod}(kG)$ and Hom closed colocalizing subcategories of $\text{StMod}(kG)$.* \square

Thus the classification of Hom closed colocalizing subcategories of $\text{Mod}(kG)$ is a consequence of the costratification of $\text{StMod}(kG)$ formulated in Theorem 11.13.

APPENDIX A. LOCALIZATION FUNCTORS AND THEIR ADJOINTS

Localization functors. A functor $L: \mathcal{C} \rightarrow \mathcal{C}$ is called *localization functor* if there exists a morphism $\eta: \text{Id}_{\mathcal{C}} \rightarrow L$ such that the morphism $L\eta: L \rightarrow L^2$ is invertible and $L\eta = \eta L$. The morphism η is called *adjunction*. Recall that a morphism $\eta: F \rightarrow F'$ between functors is *invertible* if $\eta X: FX \rightarrow F'X$ is invertible for each object X . A functor $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ is by definition a *colocalization functor* if its opposite functor $\Gamma^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ is a localization functor. The corresponding morphism $\Gamma \rightarrow \text{Id}_{\mathcal{C}}$ is called *coadjunction*.

Any localization functor $L: \mathcal{C} \rightarrow \mathcal{C}$ can be written as the composite of two functors that form an adjoint pair. In order to define these functors let us denote by $\text{Im } L$ the *essential image* of L , that is, the full subcategory of \mathcal{C} that is formed by all objects isomorphic to one of the form LX for some X in \mathcal{C} .

Lemma A.1. *Let $L: \mathcal{C} \rightarrow \mathcal{C}$ be a localization functor and write L as a composite*

$$\mathcal{C} \xrightarrow{F} \operatorname{Im} L \xrightarrow{G} \mathcal{C}$$

where G denotes the inclusion functor. Then G is a right adjoint of F .

Proof. See [4, Lemma 3.1] □

Next we characterize the adjoint pairs of functors (F, G) such that its composite $L = GF$ is a localization functor. Given any category \mathcal{C} and a class Σ of morphisms in \mathcal{C} , we denote by $Q: \mathcal{C} \rightarrow \mathcal{C}[\Sigma^{-1}]$ the universal functor that inverts all morphisms in Σ ; see [12, §I.1]. Thus $Q\phi$ is invertible for all ϕ in Σ , and any functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F\phi$ is invertible for all ϕ in Σ factors uniquely through Q .

Lemma A.2. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be a pair of functors such that G is a right adjoint of F . Let $L = GF$ and $\eta: \operatorname{Id}_{\mathcal{C}} \rightarrow L$ be the adjunction morphism. Then the following are equivalent:*

- (1) *The morphism $L\eta: L \rightarrow L^2$ is invertible and $L\eta = \eta L$ (that is, L is a localization functor).*
- (2) *The functor F induces an equivalence $\mathcal{C}[\Sigma^{-1}] \xrightarrow{\sim} \mathcal{D}$, where Σ denotes the class of morphisms ϕ in \mathcal{C} such that $F\phi$ is invertible.*
- (3) *The functor G is fully faithful.*

Proof. For (1) \Leftrightarrow (3) see [4, Lemma 3.1]; for (2) \Leftrightarrow (3) see [12, Proposition I.1.3]. □

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between triangulated categories. Then the full subcategory $\operatorname{Ker} F$ consisting of all objects that are annihilated by F is a thick subcategory of \mathcal{C} . Denote by Σ the class of morphisms ϕ in \mathcal{C} such that $F\phi$ is invertible. Then a morphism in \mathcal{C} belongs to Σ if and only if its cone belongs to $\operatorname{Ker} F$. Thus $\mathcal{C}[\Sigma^{-1}] = \mathcal{C}/\operatorname{Ker} F$; see [25].

The right adjoint of a localization functor. We discuss the existence of a right adjoint of a localization functor.

Lemma A.3. *Let $L: \mathcal{C} \rightarrow \mathcal{C}$ be a localization functor with a right adjoint Γ . Then Γ is a colocalization functor satisfying $\operatorname{Im} \Gamma = \operatorname{Im} L$. If $\eta: \operatorname{Id}_{\mathcal{C}} \rightarrow L$ denotes the adjunction of L , then the coadjunction $\theta: \Gamma \rightarrow \operatorname{Id}_{\mathcal{C}}$ of Γ is given by the composite*

$$\operatorname{Hom}_{\mathcal{C}}(X, \Gamma Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(LX, Y) \xrightarrow{(\eta X, Y)} \operatorname{Hom}_{\mathcal{C}}(X, Y), \quad X, Y \in \mathcal{C}.$$

Proof. Fix an object Y in \mathcal{C} . We need to show that $\Gamma(\theta Y)$ is an isomorphism and that $\Gamma(\theta Y) = \theta(\Gamma Y)$. Denote by

$$\phi_{X, Y}: \operatorname{Hom}_{\mathcal{C}}(X, \Gamma Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(LX, Y)$$

the natural bijection given by the adjunction between L and Γ . We use that $L(\eta X)$ is an isomorphism and that $L(\eta X) = \eta(LX)$. The following commutative diagram shows that $\Gamma(\theta Y)$ is an isomorphism since all horizontal maps are bijections.

$$\begin{array}{ccccc} \operatorname{Hom}_{\mathcal{C}}(X, \Gamma^2 Y) & \xrightarrow{\phi_{X, \Gamma Y}} & \operatorname{Hom}_{\mathcal{C}}(LX, \Gamma Y) & \xrightarrow{\phi_{LX, Y}} & \operatorname{Hom}_{\mathcal{C}}(L^2 X, Y) \\ \downarrow (X, \Gamma \theta Y) & & \downarrow (LX, \theta Y) & & \downarrow (\eta LX, Y) \\ \operatorname{Hom}_{\mathcal{C}}(X, \Gamma Y) & \xrightarrow{\phi_{X, Y}} & \operatorname{Hom}_{\mathcal{C}}(LX, Y) & \xrightarrow{\operatorname{id}} & \operatorname{Hom}_{\mathcal{C}}(LX, Y) \end{array}$$

Combining the previous diagram with the next one shows that $\Gamma(\theta Y) = \theta(\Gamma Y)$.

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(X, \Gamma^2 Y) & \xrightarrow{\phi_{X, \Gamma Y}} & \mathrm{Hom}_{\mathcal{C}}(LX, \Gamma Y) & \xrightarrow{\phi_{LX, Y}} & \mathrm{Hom}_{\mathcal{C}}(L^2 X, Y) \\
(X, \theta \Gamma Y) \downarrow & & \downarrow (\eta X, \Gamma Y) & & \downarrow (L\eta X, Y) \\
\mathrm{Hom}_{\mathcal{C}}(X, \Gamma Y) & \xrightarrow{\mathrm{id}} & \mathrm{Hom}_{\mathcal{C}}(X, \Gamma Y) & \xrightarrow{\phi_{X, Y}} & \mathrm{Hom}_{\mathcal{C}}(LX, Y)
\end{array}$$

It remains to show that $\mathrm{Im} L = \mathrm{Im} \Gamma$. Suppose that X belongs to $\mathrm{Im} L$. Then ηX is invertible, and therefore the map $\mathrm{Hom}_{\mathcal{C}}(X, \theta Y)$ is bijective for all Y . Using this fact for $Y = X$ and $Y = \Gamma X$ shows that θX is invertible. Thus $\mathrm{Im} L \subseteq \mathrm{Im} \Gamma$. The proof of the other inclusion is similar. \square

Proposition A.4. *For a localization functor $L: \mathcal{C} \rightarrow \mathcal{C}$ are equivalent:*

- (1) *The functor L admits a right adjoint.*
- (2) *The inclusion functor $G: \mathrm{Im} L \rightarrow \mathcal{C}$ admits a right adjoint G_ρ .*
- (3) *There exists a colocalization functor $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ such that $\mathrm{Im} \Gamma = \mathrm{Im} L$.*

In that case $\Gamma \cong GG_\rho$ and Γ is a right adjoint of L .

Proof. (1) \Rightarrow (3): See Lemma A.3.

(3) \Rightarrow (2): See Lemma A.1.

(2) \Rightarrow (1): Write L as composite $L = GF$ as in Lemma A.1. Then the composite GG_ρ is a right adjoint of L , since G is a right adjoint of F . \square

The left adjoint of a localization functor. We discuss the existence of a left adjoint of a localization functor.

Proposition A.5. *For a localization functor $L: \mathcal{C} \rightarrow \mathcal{C}$ are equivalent:*

- (1) *The functor L admits a left adjoint.*
- (2) *The functor $F: \mathcal{C} \rightarrow \mathrm{Im} L$ sending X in \mathcal{C} to LX admits a left adjoint F_λ .*
- (3) *There exists a colocalization functor $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ such that $\Gamma\phi$ is invertible if and only if $L\phi$ is invertible for each morphism ϕ in \mathcal{C} .*

In that case $\Gamma \cong F_\lambda F$ and Γ is a left adjoint of L .

Proof. Denote by Σ be the class of morphisms in \mathcal{C} that are inverted by L .

(1) \Rightarrow (2): Let L_λ be a left adjoint of L and write $L = GF$ as in Lemma A.1. Then we have for X, Y in \mathcal{C}

$$\mathrm{Hom}_{\mathcal{C}}(X, FY) \cong \mathrm{Hom}_{\mathcal{C}}(GX, GFY) \cong \mathrm{Hom}_{\mathcal{C}}(L_\lambda GX, Y).$$

Thus $L_\lambda G$ is a left adjoint of F .

(2) \Rightarrow (3): It follows from Lemma A.2 that F induces an equivalence $\mathcal{C}[\Sigma^{-1}] \xrightarrow{\sim} \mathrm{Im} L$. Applying the dual assertion of this lemma shows that F_λ is fully faithful and that $\Gamma = F_\lambda F$ is a colocalization functor. Moreover, the class of morphisms that are inverted by Γ coincides with the corresponding class for F , which equals Σ .

(3) \Rightarrow (1): Denote by $Q: \mathcal{C} \rightarrow \mathcal{C}[\Sigma^{-1}]$ the universal functor inverting Σ and let $\bar{L}, \bar{\Gamma}: \mathcal{C}[\Sigma^{-1}] \rightarrow \mathcal{C}$ be the induced functors satisfying $L = \bar{L}Q$ and $\Gamma = \bar{\Gamma}Q$. It follows from Lemma A.2 that \bar{L} is a right adjoint of Q and that $\bar{\Gamma}$ is a left adjoint of Q . Thus for X, Y in \mathcal{C} we have

$$\mathrm{Hom}_{\mathcal{C}}(\bar{\Gamma}QX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(QX, QY) \cong \mathrm{Hom}_{\mathcal{C}}(X, \bar{L}QY).$$

It follows that Γ is a left adjoint of L . \square

Remark A.6. Let \mathcal{C} be a triangulated category and $L: \mathcal{C} \rightarrow \mathcal{C}$ an exact localization functor. Then statements (2) and (3) in Proposition A.5 admits the following equivalent reformulations:

- (2') The quotient functor $\mathcal{C} \rightarrow \mathcal{C}/\text{Ker } L$ admits a left adjoint.
- (3') There exists an exact colocalization functor $\Gamma: \mathcal{C} \rightarrow \mathcal{C}$ such that $\text{Ker } \Gamma = \text{Ker } L$.

The equivalence $\mathcal{C}/\text{Ker } L \xrightarrow{\sim} \text{Im } L$ has already been mentioned. The reformulation of (3) relies on the same argument: an exact functor inverts a morphism ϕ in \mathcal{C} if and only if it annihilates the cone of ϕ .

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