A LOCAL-GLOBAL PRINCIPLE FOR SMALL TRIANGULATED CATEGORIES

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Abstract. Local cohomology functors are constructed for the category of cohomological functors on an essentially small triangulated category T equipped with an action of a commutative noetherian ring. This is used to establish a local-global principle and to develop a notion of stratification, for T and the cohomological functors on it, analogous to such concepts for compactly generated triangulated categories.

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1. Introduction

In this paper we establish an analogue of the local-global principle from commutative algebra [8, Chap. II, §3] for an essentially small triangulated category, using the central action of a graded commutative ring. This has applications to the theory of support varieties in representation theory and in commutative algebra, that started with the work of Quillen [23] and Carlson [9].

Our paradigms for the local-global principle are the ones from [5] and from Stevenson's work [24] for compactly generated (so "big") triangulated categories. These play a crucial role in the classification of localising subcategories of the stable module category of the group algebra of a finite group [5] and of the singularity category of a locally complete intersection ring [25]. The local-global principle from [5] does yield an analogue (see Proposition 8.3) for the "small" category of compact objects, which is useful in studying its thick subcategories, for example. The work presented here arose from a search for a more direct proof of this result. Our reason

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for doing so, besides the obvious aesthetic one, is that there are small categories for which there is no canonical choice of a big category. And even if there were one, it is not clear that an action of a ring of operators on the small category extends to an action on the big one.

Following an idea of Grothendieck–Verdier [13], in this work we propose a different model for such constructions. Namely, given an essentially small triangulated category T, we consider the category $\mathsf{Coh}\,\mathsf{T}$ of cohomological functors $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups. Up to an equivalence, this is the category of ind-objects of T in the sense of [13]. The functor category contains a copy of T, because Yoneda's lemma allows to identify an object $X \in \mathsf{T}$ with the representable functor $\mathrm{Hom}_{\mathsf{T}}(-, X)$. While $\mathsf{Coh}\,\mathsf{T}$ is no longer triangulated, it does carry an exact structure that is sufficient for our purposes; moreover, it admits filtered colimits. These are the basic ingredients we use for setting up our machinery.

Let us explain the main results in this work. Fix a noetherian graded commutative ring R acting centrally on T; the principal examples are listed in Example 3.1. This gives for each pair of objects X, Y in T an R-action on the graded abelian group

$$\mathrm{Hom}^*_{\mathsf{T}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{T}}(X, \Sigma^n Y).$$

Let $\operatorname{Spec} R$ denote the set of homogeneous prime ideals. For each $\mathfrak{p} \in \operatorname{Spec} R$ there is a localisation functor $\mathsf{T} \to \mathsf{T}_{\mathfrak{p}}$ taking an object X to $X_{\mathfrak{p}}$. The category $\mathsf{T}_{\mathfrak{p}}$ has the same objects as T and there is a natural isomorphism

$$\mathrm{Hom}^*_{\mathsf{T}}(X,Y)_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Hom}^*_{\mathsf{T}_{\mathfrak{p}}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

The formulation of the following local-global principle involves *Koszul objects.* Given an object $X \in \mathsf{T}$ and a homogeneous ideal $\mathfrak{a}$ of R, an iterated cone construction yields an object $X/\!\!/\mathfrak{a}$. While this object depends on a choice of a sequence of generators of $\mathfrak{a}$, the thick subcategory generated by it does not; see Lemma 3.9. For $\mathfrak{p} \in \operatorname{Spec} R$ set $X(\mathfrak{p}) = X_{\mathfrak{p}}/\!\!/\mathfrak{p}$.

Theorem 5.10 (Local-global principle)**.** *Let S be a thick subcategory of T. Then the following conditions are equivalent for an object X in T:*

(1) *X belongs to S.*
(2) *$X_{\mathfrak{p}}$ belongs to $\mathsf{Thick}(\mathsf{S}_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*
(3) *$X(\mathfrak{p})$ belongs to $\mathsf{Thick}(\mathsf{S}_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*

Motivated by this result, we define the *support* of an object $X \in \mathsf{T}$ to be the set

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X(\mathfrak{p}) \neq 0\}.$$

When the R-module $\mathrm{End}^*_{\mathsf{T}}(X)$ is finitely generated, this coincides with the support, in the usual sense in commutative algebra, of the R-module $\mathrm{End}^*_{\mathsf{T}}(X)$; see Proposition 4.2. For each $\mathfrak{p}$ in $\operatorname{Spec} R$, set

$$\varGamma_{\mathfrak{p}}\mathsf{T} = \{X \in \mathsf{T}_{\mathfrak{p}} \mid \mathrm{End}^*_{\mathsf{T}_{\mathfrak{p}}}(X)_{\mathfrak{q}} = 0 \text{ for all } \mathfrak{q} \not\supseteq \mathfrak{p}\}.$$

This is a thick subcategory of $\mathsf{T}_{\mathfrak{p}}$. We say that T is *stratified* by the action of R if for each $\mathfrak{p}$ in $\operatorname{Spec} R$, the category $\varGamma_{\mathfrak{p}}\mathsf{T}$ admits no proper thick subcategory.

Theorem 7.4. *Suppose that T is stratified by the action of R. For any pair of objects X, Y in T one has*

$$\begin{aligned} X \in \mathsf{Thick}(Y) \quad &\iff \quad \operatorname{supp}_R X \subseteq \operatorname{supp}_R Y, \\ \mathrm{Hom}^*_{\mathsf{T}}(X,Y) = 0 \quad &\iff \quad (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y) = \varnothing. \end{aligned}$$

There is a partial converse when the endomorphism rings of objects in T are finitely generated R-modules: If T is not stratified, then there are objects in T having the same support but generating different thick subcategories; see Proposition 7.5.

The proofs of Theorems 5.10 and 7.4 involve the category of cohomological functors. Indeed, the R-action on T has an obvious extension to $\mathsf{Coh}\,\mathsf{T}$, and based on this we develop a theory of local cohomology and support for objects in $\mathsf{Coh}\,\mathsf{T}$, analogous to the one in [4] for compactly generated triangulated categories. This forms the foundation for much of this work.

In order to illustrate these results and techniques, it is shown that the category of perfect complexes over a commutative noetherian ring is stratified; this amounts to a classical theorem of Hopkins [14] and Neeman [20]. Applications of the local-global principle, Theorem 5.10, to the study of modules over locally complete intersections and over integral group rings, will appear elsewhere.

Most ideas in this paper are taken from our previous work [4, 5, 6]; see also the references given there for inspiration by other authors. However, the categorical setting in this work is fundamentally different and the systematic use of the category of cohomological functors in this context seems to be new. We do not work with a compactly generated triangulated category having arbitrary coproducts, but instead with an essentially small triangulated category. A brief comparison between these two approaches can be found in the final section. Otherwise, references to previous work are kept to a minimum.

2. Cohomological functors

In this section we introduce the category of cohomological functors on a triangulated category and study its basic properties. For instance, we discuss base change and a long exact sequence corresponding to a Verdier quotient.

Cohomological functors. Let T be an essentially small triangulated category with suspension $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$. Recall that a functor $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups is *cohomological* if it takes exact triangles to exact sequences. We denote by $\mathsf{Coh}\,\mathsf{T}$ the category of cohomological functors. Morphisms in $\mathsf{Coh}\,\mathsf{T}$ are natural transformation and the Yoneda functor $\mathsf{T} \to \mathsf{Coh}\,\mathsf{T}$ sending $X \in \mathsf{T}$ to

$$H_X = \mathrm{Hom}_{\mathsf{T}}(-, X)$$

is fully faithful. The suspension Σ extends to a functor $\mathsf{Coh}\,\mathsf{T} \xrightarrow{\sim} \mathsf{Coh}\,\mathsf{T}$ by taking F in $\mathsf{Coh}\,\mathsf{T}$ to $F \circ \Sigma^{-1}$; we denote this again by Σ.

It is convenient to view $\mathsf{Coh}\,\mathsf{T}$ as a full subcategory of the category $\mathsf{Mod}\,\mathsf{T}$ of all additive functors $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$. Note that (co)limits in $\mathsf{Mod}\,\mathsf{T}$ are computed pointwise. For E and F in $\mathsf{Mod}\,\mathsf{T}$ we write $\mathrm{Hom}(E, F)$ for the set of morphisms from E to F. Thus $\mathrm{Hom}(H_X, F) \cong F(X)$ for X in T, by Yoneda's lemma.

Any additive functor $F\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ can be written canonically as a colimit of representable functors

$$(\operatorname{colim}_{H_X \to F} H_X) \xrightarrow{\sim} F \tag{2.1}$$

where the colimit is taken over the slice category T/F; see [13, Proposition 3.4]. Objects in T/F are morphisms $H_X \to F$ where X runs through the objects of T. A morphism in T/F from $H_X \xrightarrow{\phi} F$ to $H_{X'} \xrightarrow{\phi'} F$ is a morphism $\alpha\colon X \to X'$ in T such that $\phi' H_\alpha = \phi$.

A theorem of Lazard says that a module is flat if and only if it is a filtered colimit of finitely generated free modules; this has been generalised to functor categories by Oberst and Röhrl [22]. The following lemma shows that cohomological and flat functors agree; this is well-known, for instance from [19, Lemma 2.1].

Lemma 2.2. *The cohomological functors* $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$ *are precisely the filtered colimits of representable functors (in the category of additive functors* $\mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}$*). In particular, the category* $\mathsf{Coh}\,\mathsf{T}$ *has filtered colimits.*

Sketch of proof. A filtered colimit of exact sequences is again exact. Thus a filtered colimit of representable functors is cohomological. Conversely, given a cohomological functor F, one easily checks that the slice category T/F is filtered. Thus (2.1) gives a presentation of F as a filtered colimit of representable functors. □

We say that a sequence of morphisms in $\mathsf{Coh}\,\mathsf{T}$ is *exact* provided that evaluation at each object in T yields an exact sequence in Ab.

Lemma 2.3. *The category* $\mathsf{Coh}\,\mathsf{T}$ *is an exact category in the sense of Quillen; it admits enough projective and enough injective objects.*

Proof. The cohomological functors form an extension closed subcategory of $\mathsf{Mod}\,\mathsf{T}$ containing all projective objects and all injective objects. This is clear for the projective objects and follows easily from Yoneda's lemma for the injectives. □

Exact functors. Let T and U be essentially small triangulated categories. A functor $P\colon \mathsf{Coh}\,\mathsf{T} \to \mathsf{Coh}\,\mathsf{U}$ is said to be *exact* if it takes exact sequences to exact sequences and if there is a natural isomorphism $P \circ \Sigma \xrightarrow{\sim} \Sigma \circ P$.

An exact functor $f\colon \mathsf{T} \to \mathsf{U}$ induces a pair of functors

$$f^*\colon \mathsf{Coh}\,\mathsf{T} \longrightarrow \mathsf{Coh}\,\mathsf{U} \qquad \text{and} \qquad f_*\colon \mathsf{Coh}\,\mathsf{U} \longrightarrow \mathsf{Coh}\,\mathsf{T}$$

where $f^*(F) = \operatorname{colim}_{H_X \to F} H_{f(X)}$ and $f_*(G) = G \circ f$. The next lemma collects some of their basic properties. Recall that a *triangulated subcategory* is a full additive subcategory closed under forming cones and suspensions. The functor f is a *quotient functor* when it is equivalent to the canonical functor $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ given by a triangulated subcategory $\mathsf{S} \subseteq \mathsf{T}$.

Lemma 2.4. *Let* $f\colon \mathsf{T} \to \mathsf{U}$ *be an exact functor between essentially small triangulated categories.*

(1) *The functor* f^* *is a left adjoint of* f_*.
(2) *The functors* f^* *and* f_* *are exact and preserve filtered colimits.*
(3) *If* f *is fully faithful, then* f^* *is fully faithful and* $\mathrm{Id} \xrightarrow{\sim} f_* \circ f^*$.
(4) *If* f *is a quotient functor, then* f_* *is fully faithful and* $f^* \circ f_* \xrightarrow{\sim} \mathrm{Id}$.

Proof. (1) Given $F \in \mathsf{Coh}\,\mathsf{T}$ and $G \in \mathsf{Coh}\,\mathsf{U}$, we claim that

$$\operatorname{Hom}(f^*F, G) \cong \operatorname{Hom}(F, f_*G).$$

When F is representable this is immediate from Yoneda's lemma. The general case then follows since F can be written as a colimit of representable functors.

(2) Clearly, f_* is exact and preserves filtered colimits. A left adjoint, in particular, f^*, automatically preserves colimits. The exactness of f^* follows from the fact that f is exact; see [16, Lemma 2.2].

(3) We use the fact that for any pair (S, T) of adjoint functors, the left adjoint S is fully faithful iff the unit $\mathrm{Id} \to T \circ S$ is invertible; see [12, Proposition 1.3]. If f is fully faithful, then $F \cong (f_* \circ f^*)(F)$ for any representable functor F, and the general case follows by taking filtered colimits.

(4) If f is a quotient functor, then f_* is fully faithful; see [12, Lemma 1.2]. Thus the counit $f^* \circ f_* \to \mathrm{Id}$ is invertible, by the argument dual to the one in (3). □

Remark 2.5. Observe that when f^* is fully faithful so is f, since the Yoneda embedding of T into $\operatorname{Coh}\mathsf{T}$ is fully faithful. However, there are examples where f_* is fully faithful but f is not a quotient functor; see [17, Example 7.4].

Triangulated subcategories. Let $\mathsf{S}\subseteq\mathsf{T}$ be a triangulated subcategory. We write $i\colon\mathsf{S}\to\mathsf{T}$ for the inclusion and $q\colon\mathsf{T}\to\mathsf{T}/\mathsf{S}$ for the corresponding quotient functor. Henceforth we view $\operatorname{Coh}\mathsf{S}$ and $\operatorname{Coh}\mathsf{T}/\mathsf{S}$ as full subcategories of $\operatorname{Coh}\mathsf{T}$, via i^* and q_* respectively. More specifically, there are identifications

$$\operatorname{Coh}\mathsf{S}=\{F\in\operatorname{Coh}\mathsf{T}\mid F=\operatorname{colim}_\alpha H_{X_\alpha}\text{ with all }X_\alpha\in\mathsf{S}\},$$
$$\operatorname{Coh}\mathsf{T}/\mathsf{S}=\{F\in\operatorname{Coh}\mathsf{T}\mid F|_{\mathsf{S}}=0\}.$$

The second identification follows from the universal property of the quotient T/S [26, Chap. II, Cor. 2.2.11]. Here is a useful recognition criterion for objects in $\operatorname{Coh}\mathsf{S}$.

Lemma 2.6. *Let $\mathsf{S}\subseteq\mathsf{T}$ be a triangulated subcategory. Then $F\in\operatorname{Coh}\mathsf{T}$ belongs to $\operatorname{Coh}\mathsf{S}$ iff each morphism $H_X\to F$ with $X\in\mathsf{T}$ factors through H_Y for some $Y\in\mathsf{S}$.*

Proof. As in (2.1), we write F as a filtered colimit

$$F=\operatorname{colim}_{H_X\to F}H_X.$$

Then the assertion is an immediate consequence of the following lemma, applied with C' and C the slice categories S/F and T/F, respectively. □

Lemma 2.7. *Let $i\colon\mathsf{C}'\to\mathsf{C}$ be a fully faithful functor with C a small filtered category. Suppose that for any $X\in\mathsf{C}$ there is an object $Y\in\mathsf{C}'$ and a morphism $X\to iY$. Then C' is a small filtered category, and for any functor $F\colon\mathsf{C}\to\mathsf{D}$ into a category which admits filtered colimits, the natural morphism*

$$\operatorname{colim}_{Y\in\mathsf{C}'}F(iY)\longrightarrow\operatorname{colim}_{X\in\mathsf{C}}F(X)$$

is an isomorphism.

Proof. See [13, Proposition 8.1.3]. □

Localisation. Let $\mathsf{S}\subseteq\mathsf{T}$ be a triangulated subcategory. With $i\colon\mathsf{S}\to\mathsf{T}$ and $q\colon\mathsf{T}\to\mathsf{T}/\mathsf{S}$ the canonical functors, set

$$\varGamma=i^*\circ i_*\qquad\text{and}\qquad L=q_*\circ q^*. \tag{2.8}$$

These are exact functors on $\operatorname{Coh}\mathsf{T}$. By definition, for any $F\in\operatorname{Coh}\mathsf{T}$ one has

$$\varGamma F=\operatorname{colim}_{H_S\to F}\operatorname{Hom}_{\mathsf{T}}(-,S) \tag{2.9}$$

where the colimit is taken over the slice category S/F, which is filtered because S is a triangulated subcategory.

Proposition 2.10. *In $\operatorname{Coh}\mathsf{T}$ each object F fits into a functorial exact sequence*

$$\cdots\longrightarrow\Sigma^{-1}(LF)\longrightarrow\varGamma F\longrightarrow F\longrightarrow LF\longrightarrow\Sigma(\varGamma F)\longrightarrow\Sigma F\longrightarrow\cdots \tag{2.11}$$

Moreover, any exact sequence

$$\cdots\longrightarrow\Sigma^{-1}F''\longrightarrow F'\longrightarrow F\longrightarrow F''\longrightarrow\Sigma F'\longrightarrow\Sigma F\longrightarrow\cdots \tag{2.12}$$

in $\operatorname{Coh}\mathsf{T}$ with $F'\in\operatorname{Coh}\mathsf{S}$ and $F''\in\operatorname{Coh}\mathsf{T}/\mathsf{S}$ is isomorphic to (2.11).

For $F=\operatorname{Hom}_{\mathsf{T}}(-,X)$, the sequence (2.11) *specialises to*

$$\begin{aligned}\cdots\longrightarrow\operatorname{Hom}_{\mathsf{T}/\mathsf{S}}(-,\Sigma^{-1}X)\longrightarrow\operatorname{colim}_{S\to X}\operatorname{Hom}_{\mathsf{T}}(-,S)\longrightarrow&\\ \longrightarrow\operatorname{Hom}_{\mathsf{T}}(-,X)\longrightarrow\operatorname{Hom}_{\mathsf{T}/\mathsf{S}}(-,X)\longrightarrow\cdots&\end{aligned} \tag{2.13}$$

Proof. Fix an object X in T. Specialising (2.9) one gets that

$$\varGamma\operatorname{Hom}_{\mathsf{T}}(-,X) = \operatorname{colim}_{S\to X}\operatorname{Hom}_{\mathsf{T}}(-,S)$$

This functor fits into an exact sequence of cohomological functors of the form (2.13) since one has by definition

$$\operatorname{Hom}_{\mathsf{T}/\mathsf{S}}(-,X) = \operatorname{colim}_{X\to Y}\operatorname{Hom}_{\mathsf{T}}(-,Y)$$

where $X\to Y$ runs through all morphisms with cone in S. The sequence is functorial and hence yields an exact sequence for each filtered colimit of representable functors. This justifies (2.11).

Given another sequence (2.12), we apply the exact functor L and obtain the following commuting diagram.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & \Sigma F' & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & LF' & \longrightarrow & LF & \longrightarrow & LF'' & \longrightarrow & L(\Sigma F') & \longrightarrow \cdots
\end{array}$$

The morphism $F\to F''$ is isomorphic to $F\to LF$, since $LF'=0=\Sigma(LF')$ and $F''\xrightarrow{\sim} LF''$. Analogously, an application of $\varGamma$ shows that $F'\to F$ is isomorphic to $\varGamma F\to F$. This yields an isomorphism between (2.11) and (2.12). □

The functor L from (2.8) is a *localisation functor*[1], while $\varGamma$ is a *colocalisation functor*[2], and the functorial exact sequence (2.11) is called the *localisation sequence* for $\mathsf{S}\subseteq\mathsf{T}$. In the following we consider the case that one of the natural morphisms $\varGamma F\to F$ and $F\to LF$ is an isomorphism.

Corollary 2.14. *Let* $\mathsf{S}\subseteq\mathsf{T}$ *be a triangulated subcategory and* $F\in\operatorname{\mathsf{Coh}}\mathsf{T}$. *Then*

$$\begin{aligned}
F\in\operatorname{\mathsf{Coh}}\mathsf{S} \quad&\Longleftrightarrow\quad \varGamma F\cong F \quad\Longleftrightarrow\quad LF=0,\\
F\in\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S} \quad&\Longleftrightarrow\quad F\cong LF \quad\Longleftrightarrow\quad \varGamma F=0.
\end{aligned}$$ □

A pair (U,V) of full subcategories of an additive category forms a *torsion pair* provided that the inclusion of U admits a right adjoint, the inclusion of V admits a left adjoint, $\mathsf{U}=\{X\mid \operatorname{Hom}(X,Y)=0 \text{ for all } Y\in\mathsf{V}\}$, and $\mathsf{V}=\{Y\mid \operatorname{Hom}(X,Y)=0 \text{ for all } X\in\mathsf{U}\}$.

Corollary 2.15. *Let* $\mathsf{S}\subseteq\mathsf{T}$ *be a triangulated subcategory. Then* $\operatorname{\mathsf{Coh}}\mathsf{S}$ *and* $\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S}$ *form a torsion pair in* $\operatorname{\mathsf{Coh}}\mathsf{T}$. *Thus*

$$\begin{aligned}
\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S} &= \{F\in\operatorname{\mathsf{Coh}}\mathsf{T}\mid \operatorname{Hom}(E,F)=0 \textit{ for all } E\in\operatorname{\mathsf{Coh}}\mathsf{S}\},\\
\operatorname{\mathsf{Coh}}\mathsf{S} &= \{F\in\operatorname{\mathsf{Coh}}\mathsf{T}\mid \operatorname{Hom}(F,G)=0 \textit{ for all } G\in\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S}\}.
\end{aligned}$$

Proof. We have already seen in Lemma 2.4 that the inclusion of $\operatorname{\mathsf{Coh}}\mathsf{S}$ admits a right adjoint while the inclusion of $\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S}$ admits a left adjoint. For the first equality, observe that $\operatorname{Hom}(E,F)=0$ for all $E\in\operatorname{\mathsf{Coh}}\mathsf{S}$ means that $F(X)=0$ for all $X\in\mathsf{S}$. This is equivalent to $\varGamma F=0$, and therefore to $F\in\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S}$. For the second equality, it remains to show that $\operatorname{Hom}(F,G)=0$ for all $G\in\operatorname{\mathsf{Coh}}\mathsf{T}/\mathsf{S}$ implies $F\in\operatorname{\mathsf{Coh}}\mathsf{S}$. The assumption on F implies $LF=0$ since $L=q_*\circ q^*$ and

$$\operatorname{Hom}(q^*F,q^*F)\cong\operatorname{Hom}(F,(q_*\circ q^*)F)=0.$$

Thus F belongs to $\operatorname{\mathsf{Coh}}\mathsf{S}$. □

[1] The natural morphism $\eta\colon \operatorname{Id}\to L$ has the property that $L\eta$ is invertible and $L\eta=\eta L$.

[2] The natural morphism $\theta\colon \varGamma\to\operatorname{Id}$ has the property that $\varGamma\theta$ is invertible and $\varGamma\theta=\theta\varGamma$.

In general, (co)localisation functors do note commute; see [4, Example 3.5]. The following lemma identifies some conditions under which they do.

Lemma 2.16. *Let* $\mathsf{S}_1 \subseteq \mathsf{S}_2 \subseteq \mathsf{T}$ *be triangulated subcategories and let* $(\varGamma_1, L_1)$ *and* $(\varGamma_2, L_2)$ *be the corresponding pairs of (co)localisation functors on* $\operatorname{\mathsf{Coh}} \mathsf{T}$*. The morphisms in* (2.11) *induce isomorphisms*

$$\varGamma_1\varGamma_2 \cong \varGamma_1 \cong \varGamma_2\varGamma_1, \quad L_1L_2 \cong L_2 \cong L_2L_1, \quad \varGamma_1L_2 = 0 = L_2\varGamma_1, \quad \varGamma_2L_1 \cong L_1\varGamma_2.$$

Proof. Apply the localisation sequence (2.11). □

This has the following useful consequence.

Corollary 2.17. *Given thick subcategories* S_1 *and* S_2 *of* T*, one has*

$$\mathsf{S}_1 \subseteq \mathsf{S}_2 \qquad \iff \qquad \operatorname{\mathsf{Coh}} \mathsf{S}_1 \subseteq \operatorname{\mathsf{Coh}} \mathsf{S}_2. \qquad \square$$

3. Cohomological localisation

In this section we introduce cohomological localisation functors for categories of cohomological functors and explain how to compute these functors in terms of Koszul objects. These are analogues of results in [4, §§4–6].

Let T be an essentially small triangulated category. For objects X, Y in T set

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(\Sigma^{-n}X, Y).$$

More generally, each F in $\operatorname{\mathsf{Mod}} \mathsf{T}$ induces a functor $F^*\colon \mathsf{T}^{\mathrm{op}} \to \mathsf{Ab}^*$ into the category of graded abelian groups, with

$$F^n(X) = F(\Sigma^{-n}X) \quad \text{for each } n \in \mathbb{Z}.$$

Central ring actions. Let R be a graded commutative ring; thus R is $\mathbb{Z}$-graded and satisfies $rs = (-1)^{|r||s|}sr$ for each pair of homogeneous elements r, s in R. We say that T is *R-linear*, or that *R acts* on T, if there is a homomorphism $\phi\colon R \to Z^*(\mathsf{T})$ of graded rings, where

$$Z^*(\mathsf{T}) = \bigoplus_{n\in\mathbb{Z}} \{\eta\colon \operatorname{Id}_{\mathsf{T}} \to \Sigma^n \mid \eta\Sigma = (-1)^n\Sigma\eta\}$$

is the graded centre of T. For each object X in T this yields a homomorphism $\phi_X\colon R \to \operatorname{End}^*_{\mathsf{T}}(X)$ of graded rings such that for all objects $X, Y \in \mathsf{T}$ the R-module structures on $\operatorname{Hom}^*_{\mathsf{T}}(X,Y)$ induced by ϕ_X and ϕ_Y agree up to a sign. Namely, for any homogeneous elements $r \in R$ and $\alpha \in \operatorname{Hom}^*_{\mathsf{T}}(X,Y)$, one has

$$\phi_Y(r)\alpha = (-1)^{|r||\alpha|}\alpha\phi_X(r).$$

Here are some examples.

Example 3.1. (1) Any triangulated category admits a canonical action of $\mathbb{Z}$.

(2) The derived category of a ring A has a canonical action of the centre of A.

(3) If A is an algebra over a field k, the derived category of A has a canonical action of the Hochschild cohomology of A over k.

(4) Given a finite dimensional Hopf algebra H over a field k (for example, the group algebra of a finite group), the derived category of H (and hence also the stable module category) has a canonical action of the k-algebra $\operatorname{Ext}^*_H(k,k)$.

We fix an action of R on T. The following observations will be used repeatedly.

Remark 3.2. The R-action on T induces an action on any triangulated subcategory $\mathsf{S} \subseteq \mathsf{T}$ and on the quotient T/S, compatible with the inclusion and quotient functors, respectively. It also extends to an action on $\operatorname{\mathsf{Mod}} \mathsf{T}$.

Torsion objects. The set of homogeneous prime ideals of R is denoted $\operatorname{Spec} R$. For a homogeneous ideal $\mathfrak{a}$ of R we set

$$\mathcal{V}(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Let $\mathcal{V}$ be a *specialisation closed* subset of $\operatorname{Spec} R$; this condition means that if $\mathcal{V}$ contains a prime $\mathfrak{p}$, then it contains everything in $\mathcal{V}(\mathfrak{p})$. An R-module M is $\mathcal{V}$-*torsion* if $M_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \operatorname{Spec} R \setminus \mathcal{V}$. Note that M is $\mathcal{V}(\mathfrak{a})$-torsion if and only if each $r \in \mathfrak{a}$ and $x \in M$ satisfy $r^n x = 0$ for $n \gg 0$.

A functor $F \in \operatorname{Coh} \mathsf{T}$ is $\mathcal{V}$-*torsion* if $F^*(X)$ is $\mathcal{V}$-torsion for all $X \in \mathsf{T}$. The full subcategory of all $\mathcal{V}$-torsion functors is denoted by $(\operatorname{Coh} \mathsf{T})_{\mathcal{V}}$. Analogously, an object $Y \in \mathsf{T}$ is $\mathcal{V}$-*torsion* if $\operatorname{End}^*_{\mathsf{T}}(Y)$ is $\mathcal{V}$-torsion. This means the functor $\operatorname{Hom}_{\mathsf{T}}(-, Y)$ is $\mathcal{V}$-torsion, since for each $X \in \mathsf{T}$ the R-action on $\operatorname{Hom}^*_{\mathsf{T}}(X, Y)$ factors through $\operatorname{End}^*_{\mathsf{T}}(Y)$. Set

$$\mathsf{T}_{\mathcal{V}} = \{X \in \mathsf{T} \mid \operatorname{End}^*_{\mathsf{T}}(X)_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \setminus \mathcal{V}\}.$$

Note that $\mathsf{T}_{\mathcal{V}}$ is a thick subcategory of T. Recall that we view $\operatorname{Coh}(\mathsf{T}_{\mathcal{V}})$ as a full subcategory of $\operatorname{Coh} \mathsf{T}$. It follows from the definitions that there is an inclusion

$$\operatorname{Coh}(\mathsf{T}_{\mathcal{V}}) \subseteq (\operatorname{Coh} \mathsf{T})_{\mathcal{V}}. \tag{3.3}$$

Equality holds when R is noetherian; see Corollary 4.4, and also Propositions 3.6 and 3.10. Following the definition in (2.8), the inclusion $\mathsf{T}_{\mathcal{V}} \subseteq \mathsf{T}$ induces functors

$$\varGamma_{\mathcal{V}}, L_{\mathcal{V}} \colon \operatorname{Coh} \mathsf{T} \longrightarrow \operatorname{Coh} \mathsf{T},$$

where $\varGamma_{\mathcal{V}}$ is a colocalisation functor and $L_{\mathcal{V}}$ is a localisation functor. Note that these functors are exact and preserve filtered colimits.

Inverting central elements. Given a homogeneous element $r \in R$ of degree d and $X \in \mathsf{T}$, we write $X/\!\!/r$ for the cone of the morphisms $X \xrightarrow{r} \Sigma^d X$. This definition yields the following exact sequence.

$$\cdots \longrightarrow H_X \xrightarrow{\pm r} H_{\Sigma^d X} \longrightarrow H_{X/\!\!/r} \longrightarrow H_{\Sigma X} \xrightarrow{\pm r} H_{\Sigma^{d+1} X} \longrightarrow \cdots \tag{3.4}$$

In particular, inverting r in T is equivalent to annihilating $X/\!\!/r$ for all $X \in \mathsf{T}$.

Let Φ be a multiplicatively closed set of homogeneous elements in R. The following lemma describes the quotient functor for T that inverts the elements of Φ. We consider the specialisation closed set

$$\mathcal{Z}(\Phi) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap \Phi \neq \varnothing\}.$$

Given an R-module M, we write $M[\Phi^{-1}]$ for the localisation of M with respect to Φ. Note that M is $\mathcal{Z}(\Phi)$-torsion iff $M[\Phi^{-1}] = 0$.

The following lemma is a variation of known results; see for instance [2, Theorem 3.6] or [15, Theorem 3.3.7].

Lemma 3.5. *Let Φ be a multiplicatively closed set of homogeneous elements in R and $\mathsf{T}' \subseteq \mathsf{T}$ a subcategory satisfying $\operatorname{Thick}(\mathsf{T}') = \mathsf{T}$. Then there is an equality*

$$\mathsf{T}_{\mathcal{Z}(\Phi)} = \operatorname{Thick}(\{X/\!\!/r \mid X \in \mathsf{T}',\, r \in \Phi\}),$$

and the quotient functor $\mathsf{T} \to \mathsf{T}/\mathsf{T}_{\mathcal{Z}(\Phi)}$ induces a natural isomorphism

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y)[\Phi^{-1}] \xrightarrow{\sim} \operatorname{Hom}^*_{\mathsf{T}/\mathsf{T}_{\mathcal{Z}(\Phi)}}(X, Y)$$

for all objects X, Y in T.

Proof. Set $\mathsf{S} = \operatorname{Thick}(\{X/\!\!/r \mid X \in \mathsf{T}',\, r \in \Phi\})$ and $\mathsf{U} = \mathsf{T}/\mathsf{S}$. We claim:

(1) If X or Y is in S, then $\operatorname{Hom}^*_{\mathsf{T}}(X, Y)[\Phi^{-1}] = 0$.

(2) For any X, Y in T, the natural morphism $\operatorname{Hom}^*_{\mathsf{T}}(X,Y) \to \operatorname{Hom}^*_{\mathsf{U}}(X,Y)$ induces an isomorphism

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y)[\Phi^{-1}] \xrightarrow{\sim} \operatorname{Hom}^*_{\mathsf{U}}(X,Y).$$

Indeed, (1) follows from (3.4). Given this, it follows from the exact sequence (2.13) that the morphism in (2) induces an isomorphism

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y)[\Phi^{-1}] \xrightarrow{\sim} \operatorname{Hom}^*_{\mathsf{U}}(X,Y)[\Phi^{-1}].$$

On the other hand, Φ acts invertibly on $\operatorname{Hom}^*_{\mathsf{U}}(X,Y)$, that is to say,

$$\operatorname{Hom}^*_{\mathsf{U}}(X,Y) \xrightarrow{\sim} \operatorname{Hom}^*_{\mathsf{U}}(X,Y)[\Phi^{-1}].$$

It suffices to check this claim for all $Y \in \mathsf{T}'$, and then it is clear from (3.4). Combining both isomorphisms yields (2), and completes the proof of the claims.

It remains to observe that $X \in \mathsf{T}$ is $\mathcal{Z}(\Phi)$-torsion iff $\operatorname{End}^*_{\mathsf{T}}(X)[\Phi^{-1}] = 0$; given (2) above, the latter condition translates to $X = 0$ in U. Thus $\mathsf{T}_{\mathcal{Z}(\Phi)} = \mathsf{S}$. □

Let Φ be a multiplicatively closed set. We define a functor

$$L_\Phi \colon \operatorname{\mathsf{Coh}} \mathsf{T} \longrightarrow \operatorname{\mathsf{Coh}} \mathsf{T}$$

by taking F in $\operatorname{\mathsf{Coh}} \mathsf{T}$ to $F[\Phi^{-1}]$ given by $F[\Phi^{-1}]^*(X) = F^*(X)[\Phi^{-1}]$ for $X \in \mathsf{T}$. It is easy to verify that this is an exact localisation functor; the corresponding colocalisation functor is denoted $\varGamma_\Phi$.

Proposition 3.6. *There is a natural isomorphism $L_\Phi \xrightarrow{\sim} L_{\mathcal{Z}(\Phi)}$ and hence*

$$\operatorname{\mathsf{Coh}}(\mathsf{T}_{\mathcal{Z}(\Phi)}) = (\operatorname{\mathsf{Coh}} \mathsf{T})_{\mathcal{Z}(\Phi)}.$$

Proof. Lemma 3.5 yields the isomorphism for representable functors, and the general case follows since L_Φ and $L_{\mathcal{Z}(\Phi)}$ preserve filtered colimits. □

Localisation at a prime ideal. Let $\mathfrak{p}$ be a homogeneous prime ideal of R. Thus $R \setminus \mathfrak{p}$ is a multiplicatively closed subset and

$$\mathcal{Z}(R \setminus \mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

Set

$$\mathsf{T}_\mathfrak{p} = \mathsf{T}/\mathsf{T}_{\mathcal{Z}(R\setminus\mathfrak{p})}$$

and let $X_\mathfrak{p}$ denote the image of an object X in T under the natural functor $\mathsf{T} \to \mathsf{T}_\mathfrak{p}$. This quotient category is described in Lemma 3.5. Thus for all $X, Y \in \mathsf{T}$ there is a natural isomorphism

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y)_\mathfrak{p} \cong \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}).$$

It follows from Proposition 3.6 that the localisation functor

$$\operatorname{\mathsf{Coh}} \mathsf{T} \longrightarrow \operatorname{\mathsf{Coh}} \mathsf{T}, \quad F \mapsto F_\mathfrak{p},$$

defined by $(F_\mathfrak{p})^*(X) = F^*(X)_\mathfrak{p}$ is isomorphic to $L_{\mathcal{Z}(R\setminus\mathfrak{p})}$. A functor $F \in \operatorname{\mathsf{Coh}} \mathsf{T}$ is called $\mathfrak{p}$-*local* if $F \cong F_\mathfrak{p}$, and $(\operatorname{\mathsf{Coh}} \mathsf{T})_\mathfrak{p}$ denotes the full subcategory formed by all $\mathfrak{p}$-local functors. Proposition 3.6 yields an identification

(3.7) $$\operatorname{\mathsf{Coh}}(\mathsf{T}_\mathfrak{p}) = (\operatorname{\mathsf{Coh}} \mathsf{T})_\mathfrak{p}.$$

Koszul objects. Fix a homogeneous element $r \in R$ of degree d. For each X in T and each integer n set $X_n = \Sigma^{nd} X$ and consider the commuting diagram

$$\begin{array}{ccccccc}
X & = & X & = & X & = & \cdots \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r^2} & & \downarrow{\scriptstyle r^3} & & \\
X_1 & \xrightarrow{r} & X_2 & \xrightarrow{r} & X_3 & \xrightarrow{r} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X/\!\!/r & \longrightarrow & X/\!\!/r^2 & \longrightarrow & X/\!\!/r^3 & \longrightarrow & \cdots
\end{array}$$

where each vertical sequence is given by the exact triangle defining $X/\!\!/r^n$, and the morphisms in the last row are the (non-canonical) ones induced by the commutativity of the upper squares.

Lemma 3.8. *Let $r \in R$ be a homogeneous element of degree d.*

(1) *For $F \in \mathsf{Coh}\,\mathsf{T}$, the colimit of the sequence*

$$F \xrightarrow{r} \Sigma^d F \xrightarrow{r} \Sigma^{2d} F \xrightarrow{r} \Sigma^{3d} F \xrightarrow{r} \cdots$$

is naturally isomorphic to $L_{\mathcal{V}(r)} F$.

(2) *For $X \in \mathsf{T}$, the colimit of the sequence*

$$H_{\Sigma^{-1} X/\!\!/r^1} \longrightarrow H_{\Sigma^{-1} X/\!\!/r^2} \longrightarrow H_{\Sigma^{-1} X/\!\!/r^3} \longrightarrow \cdots$$

is naturally isomorphic to $\Gamma_{\mathcal{V}(r)} H_X$.

Proof. The colimit construction in (1) yields a functor $\mathsf{Coh}\,\mathsf{T} \to \mathsf{Coh}\,\mathsf{T}$; we claim that it is isomorphic to $L_{\mathcal{V}(r)}$. It suffices to prove this for representable functors as both functors preserve filtered colimits. When $F = H_X$ one has an exact sequence

$$\cdots \longrightarrow \operatorname{colim} H_{\Sigma^{-1} X/\!\!/r^n} \longrightarrow H_X \longrightarrow \operatorname{colim} H_{X_n} \longrightarrow \operatorname{colim} H_{X/\!\!/r^n} \longrightarrow \cdots$$

where r acts invertibly on $\operatorname{colim} H_{X_n}$ while $\operatorname{colim} H_{X/\!\!/r^n}$ is $\mathcal{V}(r)$-torsion. Thus the sequence is isomorphic to the localisation sequence for $\mathsf{T}_{\mathcal{V}(r)} \subseteq \mathsf{T}$, by Propositions 2.10 and 3.6. $\square$

Let $\mathfrak{a}$ be a finitely generated homogeneous ideal of R and $X \in \mathsf{T}$. Pick a sequence of elements $r_1, \ldots, r_n$ in R that generate the ideal $\mathfrak{a}$ and define inductively

$$X_0 = X \qquad \text{and} \qquad X_i = X_{i-1}/\!\!/r_i \text{ for } 1 \le i \le n.$$

We call X_n a *Koszul object* of X with respect to $\mathfrak{a}$, and denote it $X/\!\!/\mathfrak{a}$. This depends on a choice of a sequence of generators for $\mathfrak{a}$, so our notation is ambiguous. However, there is the following uniqueness result.

Lemma 3.9. *There is an equality*

$$\mathsf{Thick}(X/\!\!/\mathfrak{a}) = \{Y \in \mathsf{Thick}(X) \mid \operatorname{End}^*_{\mathsf{T}}(Y)_{\mathfrak{p}} = 0 \textit{ for all } \mathfrak{p} \not\supseteq \mathfrak{a}\}.$$

Proof. When $\mathfrak{a}$ is generated by a single element the desired statement follows from Lemma 3.5, applied to $\mathsf{T} = \mathsf{Thick}(X)$. An iteration settles the general case. $\square$

Proposition 3.10. *Let $\mathfrak{a}$ be a finitely generated homogeneous ideal of R. Then*

$$\mathsf{T}_{\mathcal{V}(\mathfrak{a})} = \mathsf{Thick}(\{X/\!\!/\mathfrak{a} \mid X \in \mathsf{T}\}) \qquad \textit{and} \qquad \mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{a})}) = (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(\mathfrak{a})}.$$

Moreover, the objects of $\mathsf{T}_{\mathcal{V}(\mathfrak{a})}$ are precisely the direct summands of Koszul objects $X/\!\!/\mathfrak{b}$ with $X \in \mathsf{T}$ and $\mathfrak{b}$ an ideal of R satisfying $\sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}}$.

Proof. Set $\mathsf{S} = \mathsf{Thick}(\{X/\!\!/\mathfrak{a} \mid X \in \mathsf{T}\})$. It suffices to show that

$$\mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{a})}) \subseteq (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(\mathfrak{a})} \subseteq \mathsf{Coh}\,\mathsf{S} \subseteq \mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{a})}).$$

From this the first part of the assertion follows. The fact that all objects in $\mathsf{T}_{\mathcal{V}(\mathfrak{a})}$ are direct summands of Koszul objects follows from the proof.

The first inclusion is by definition and the last follows from Lemma 3.9. To verify the inclusion in the middle, it suffices to show that for any $F \in (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(\mathfrak{a})}$ each morphism $\phi\colon H_X \to F$ with X in T factors through a morphism $H_X \to H_Y$ with Y in S; see Lemma 2.6. To this end, let $r_1, \dots, r_n$ be a sequence of elements that generate the ideal $\mathfrak{a}$. Starting with $X_0 = X$ and $\phi_0 = \phi$, we construct factorisations

$$\phi_{i-1}\colon H_{X_{i-1}} \longrightarrow H_{X_i} \xrightarrow{\phi_i} F$$

for $i = 1, 2, \dots, n$. The assumption on F implies that each ϕ_{i-1} is annihilated by $r_i^{\alpha_i}$, for some $\alpha_i \geq 1$. Thus we set

$$X_i = \Sigma^{-\alpha_i |r_i|} X_{i-1}/\!\!/r_i^{\alpha_i}.$$

The object $Y = X_n$ is the desired object; it belongs to S by Lemma 3.9. □

Composition laws. We show that cohomological localisation and colocalisation functors commute; see Lemma 2.16 for related commutation rules.

Lemma 3.11. *Let Φ and Ψ be multiplicatively closed sets of homogeneous elements in R. Then*

$$L_\Phi \circ L_\Psi \cong L_\Psi \circ L_\Phi, \qquad L_\Phi \circ \Gamma_\Psi \cong \Gamma_\Psi \circ L_\Phi, \qquad \Gamma_\Phi \circ \Gamma_\Psi \cong \Gamma_\Psi \circ \Gamma_\Phi.$$

Proof. The first isomorphism is clear since localising R-modules with respect to Φ and Ψ commutes. We consider the exact localisation sequence

$$\mathbb{E}_\Psi\colon \qquad \cdots \longrightarrow \Gamma_\Psi \longrightarrow \mathrm{Id} \longrightarrow L_\Psi \longrightarrow \cdots$$

and the pair of morphisms

$$(L_\Phi \mathbb{E}_\Psi) \longrightarrow (L_\Phi \mathbb{E}_\Psi) L_\Phi = L_\Phi(\mathbb{E}_\Psi L_\Phi) \longleftarrow (\mathbb{E}_\Psi L_\Phi).$$

This yields the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & L_\Phi \Gamma_\Psi & \longrightarrow & L_\Phi & \longrightarrow & L_\Phi L_\Psi & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & L_\Phi \Gamma_\Psi L_\Phi & \longrightarrow & L_\Phi L_\Phi & \longrightarrow & L_\Phi L_\Psi L_\Phi & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \Gamma_\Psi L_\Phi & \longrightarrow & L_\Phi & \longrightarrow & L_\Psi L_\Phi & \longrightarrow & \cdots
\end{array}$$

In two of three columns the vertical morphisms are isomorphisms. Thus the five lemma implies that $L_\Phi \Gamma_\Psi \cong \Gamma_\Psi L_\Phi$. A similar argument based on the pair of morphisms $\mathbb{E}_\Psi \Gamma_\Phi \leftarrow \Gamma_\Phi \mathbb{E}_\Psi \Gamma_\Phi \to \Gamma_\Phi \mathbb{E}_\Psi$ is used to deduce the third isomorphism $\Gamma_\Phi \Gamma_\Psi \cong \Gamma_\Psi \Gamma_\Phi$ from the second. □

Next observe for a multiplicatively closed set $\Phi = \{r^i \mid i \in \mathbb{N}\}$ that $\mathcal{Z}(\Phi) = \mathcal{V}(r)$ and so $\Gamma_\Phi \cong \Gamma_{\mathcal{V}(r)}$ by Proposition 3.6.

Lemma 3.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be finitely generated homogeneous ideals of R. Then*

$$\Gamma_{\mathcal{V}(\mathfrak{a})} \circ \Gamma_{\mathcal{V}(\mathfrak{b})} \cong \Gamma_{\mathcal{V}(\mathfrak{a}) \cap \mathcal{V}(\mathfrak{b})}.$$

Proof. It suffices to show for $\mathfrak{a}$ generated by homogeneous elements $r_1, \ldots, r_n$ that

$$\varGamma_{\mathcal{V}(\mathfrak{a})} = \varGamma_{\mathcal{V}(r_n)} \circ \ldots \circ \varGamma_{\mathcal{V}(r_1)}.$$

We prove this assertion by induction on n. Let $\mathfrak{a}' = (r_1, \ldots, r_{n-1})$. We know from Lemma 3.11 that the $\varGamma_{\mathcal{V}(r_i)}$ commute. Thus $\varGamma_{\mathcal{V}(\mathfrak{a}')}\varGamma_{\mathcal{V}(r_n)} \cong \varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}$ by the induction hypothesis. Using Proposition 3.10, it follows that the image of $\varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}$ belongs to

$$\begin{aligned} \mathsf{Coh}(\mathsf{T}_{\mathcal{V}(r_n)}) \cap \mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{a}')}) &= (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(r_n)} \cap (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(\mathfrak{a}')} \\ &= (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}(\mathfrak{a})} \\ &= \mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{a})}). \end{aligned}$$

Therefore $L_{\mathcal{V}(\mathfrak{a})}(\varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}) = 0$. On the other hand, $\varGamma_{\mathcal{V}(\mathfrak{a})}(\varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}) = \varGamma_{\mathcal{V}(\mathfrak{a})}$. The exact localisation sequence (2.11) yields the following exact sequence

$$\cdots \to \varGamma_{\mathcal{V}(\mathfrak{a})}(\varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}) \to \varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')} \to L_{\mathcal{V}(\mathfrak{a})}(\varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}) \to \cdots$$

and therefore $\varGamma_{\mathcal{V}(\mathfrak{a})} \cong \varGamma_{\mathcal{V}(r_n)}\varGamma_{\mathcal{V}(\mathfrak{a}')}$. □

Corollary 3.13. *Let $\mathfrak{a}$ be a finitely generated homogeneous ideal, and $\mathfrak{p}$ a homogeneous prime ideal, of R. For each $F \in \mathsf{Coh}\,\mathsf{T}$, there is a natural isomorphism*

$$(\varGamma_{\mathcal{V}(\mathfrak{a})}F)_{\mathfrak{p}} \cong \varGamma_{\mathcal{V}(\mathfrak{a})}(F_{\mathfrak{p}}).$$

Proof. Apply Lemmas 3.11 and 3.12. □

4. Support

In this section, we define the support of a cohomological functor and establish some useful rules for computing it; the development parallels the one in [4, §5].

Let R be a graded commutative ring and T be an essentially small R-linear triangulated category. From now on we assume R to be noetherian.

Support. For each F in $\mathsf{Coh}\,\mathsf{T}$ and $\mathfrak{p}$ in $\operatorname{Spec} R$ set

$$\varGamma_{\mathfrak{p}}F = \varGamma_{\mathcal{V}(\mathfrak{p})}(F_{\mathfrak{p}}).$$

Then $\varGamma_{\mathfrak{p}}$ is an exact functor on $\mathsf{Coh}\,\mathsf{T}$ that preserves filtered colimits. The subset

$$\operatorname{supp}_R F = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}}F \neq 0\}$$

is called the *support* of F.

Proposition 4.1. *Let $F \in \mathsf{Coh}\,\mathsf{T}$. Then $\operatorname{supp}_R F = \varnothing$ if and only if $F = 0$.*

Proof. Clearly, $\operatorname{supp}_R F = \varnothing$ when $F = 0$. Suppose F is non-zero. Recall that if an R-module M is non-zero, then there exists a $\mathfrak{p} \in \operatorname{Spec} R$ such that $M_{\mathfrak{p}} \neq 0$. Choose a prime $\mathfrak{p}$ that is minimal subject to the condition that $F_{\mathfrak{p}} \neq 0$. Then for all primes $\mathfrak{q}$ properly contained in $\mathfrak{p}$ and all $X \in \mathsf{T}$, one has

$$F_{\mathfrak{p}}(X)_{\mathfrak{q}} \cong F(X)_{\mathfrak{q}} = 0.$$

Hence $F_{\mathfrak{p}}(X)$ is $\mathcal{V}(\mathfrak{p})$-torsion, by [18, Theorem 6.5]; that is to say, $F_{\mathfrak{p}}$ is $\mathcal{V}(\mathfrak{p})$-torsion. It then follows from Proposition 3.10 that $F_{\mathfrak{p}}$ is in $\mathsf{Coh}(\mathsf{T}_{\mathcal{V}(\mathfrak{p})})$, so the natural map $\varGamma_{\mathfrak{p}}F \to F_{\mathfrak{p}}$ is an isomorphism. As $F_{\mathfrak{p}} \neq 0$, one gets that $\mathfrak{p}$ is in $\operatorname{supp}_R F$. □

We can compute the support of the representable functors as follows.

Proposition 4.2. *Let X be an object in T. Then*

$$\operatorname{supp}_R H_X \subseteq \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{End}^*_{\mathsf{T}}(X)_{\mathfrak{p}} \neq 0\},$$

and equality holds when $\operatorname{End}^*_{\mathsf{T}}(X)$ is finitely generated over R.

Proof. The inclusion holds because $(H_X)_{\mathfrak{p}} = 0$ iff $\operatorname{End}^*_{\mathsf{T}}(X)_{\mathfrak{p}} = 0$. Now suppose that $\operatorname{End}^*_{\mathsf{T}}(X)_{\mathfrak{p}} \neq 0$ and that $\operatorname{End}^*_{\mathsf{T}}(X)$ is finitely generated. Then $X_{\mathfrak{p}} \neq 0$, and an application of Nakayama's lemma gives $X_{\mathfrak{p}}/\!\!/\mathfrak{p} \neq 0$; see [4, Lemma 5.11]. The functor $H_{X_{\mathfrak{p}}/\!\!/\mathfrak{p}}$ is $\mathcal{V}(\mathfrak{p})$-torsion by Lemma 3.9. It is also $\mathfrak{p}$-local, so one gets

$$\varGamma_{\mathfrak{p}} H_{X/\!\!/\mathfrak{p}} = \varGamma_{\mathcal{V}(\mathfrak{p})} H_{X_{\mathfrak{p}}/\!\!/\mathfrak{p}} = H_{X_{\mathfrak{p}}/\!\!/\mathfrak{p}} \neq 0.$$

Hence $\mathfrak{p}$ is in $\operatorname{supp}_R H_X$. □

Composition laws. Computing support is compatible with cohomological localisation and colocalisation.

Proposition 4.3. *Let* $\mathcal{V} \subseteq \operatorname{Spec} R$ *be a specialisation closed subset. For each* $F \in \mathsf{Coh}\,\mathsf{T}$ *the following equalities hold*

$$\operatorname{supp}_R \varGamma_{\mathcal{V}} F = \mathcal{V} \cap \operatorname{supp}_R F,$$
$$\operatorname{supp}_R L_{\mathcal{V}} F = (\operatorname{Spec} R \setminus \mathcal{V}) \cap \operatorname{supp}_R F.$$

Proof. If $\mathfrak{p} \notin \mathcal{V}$ then $(\varGamma_{\mathcal{V}} F)_{\mathfrak{p}} = 0$, since $\varGamma_{\mathcal{V}} F$ is $\mathcal{V}$-torsion. Thus $\operatorname{supp}_R \varGamma_{\mathcal{V}} F \subseteq \mathcal{V}$. If $\mathfrak{p} \in \mathcal{V}$ then $\mathsf{T}_{\mathcal{V}(\mathfrak{p})} \subseteq \mathsf{T}_{\mathcal{V}}$, hence $\varGamma_{\mathcal{V}(\mathfrak{p})}\varGamma_{\mathcal{V}} = \varGamma_{\mathcal{V}(\mathfrak{p})}$; see Lemma 2.16. This gives the second equality below:

$$\varGamma_{\mathfrak{p}}(\varGamma_{\mathcal{V}} F) = (\varGamma_{\mathcal{V}(\mathfrak{p})}\varGamma_{\mathcal{V}} F)_{\mathfrak{p}} = (\varGamma_{\mathcal{V}(\mathfrak{p})} F)_{\mathfrak{p}} = \varGamma_{\mathfrak{p}} F,$$

while the other two are by Corollary 3.13. Thus

$$\operatorname{supp}_R \varGamma_{\mathcal{V}} F = \mathcal{V} \cap \operatorname{supp}_R \varGamma_{\mathcal{V}} F = \mathcal{V} \cap \operatorname{supp}_R F.$$

This proves the first equality; the proof of the second is similar. □

The following result says that a $\mathcal{V}$-torsion functor is a colimit of representable functors defined by $\mathcal{V}$-torsion objects.

Corollary 4.4. *Let* $\mathcal{V} \subseteq \operatorname{Spec} R$ *be specialisation closed. Then* $\mathsf{Coh}(\mathsf{T}_{\mathcal{V}}) = \mathsf{Coh}(\mathsf{T})_{\mathcal{V}}$.

Proof. It suffices to prove that $\mathsf{Coh}(\mathsf{T})_{\mathcal{V}} \subseteq \mathsf{Coh}(\mathsf{T}_{\mathcal{V}})$; confer (3.3). Fix $F \in (\mathsf{Coh}\,\mathsf{T})_{\mathcal{V}}$. For any $\mathfrak{p} \notin \mathcal{V}$, one has $F_{\mathfrak{p}} = 0$, and hence $\varGamma_{\mathfrak{p}} F = 0$, that is to say, $\operatorname{supp}_R F \subseteq \mathcal{V}$. Proposition 4.3 then implies that $L_{\mathcal{V}} F = 0$ and it follows from (2.11) that the natural map $\varGamma_{\mathcal{V}} F \to F$ is an isomorphism. This is the desired result. □

Corollary 4.5. *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be specialisation closed subsets of* $\operatorname{Spec} R$. *Then*

$$\varGamma_{\mathcal{V}}\varGamma_{\mathcal{W}} \cong \varGamma_{\mathcal{V}\cap\mathcal{W}} \cong \varGamma_{\mathcal{W}}\varGamma_{\mathcal{V}}, \qquad L_{\mathcal{V}}L_{\mathcal{W}} \cong L_{\mathcal{V}\cup\mathcal{W}} \cong L_{\mathcal{W}}L_{\mathcal{V}}, \qquad \varGamma_{\mathcal{V}}L_{\mathcal{W}} \cong L_{\mathcal{W}}\varGamma_{\mathcal{V}}.$$

Proof. Given Proposition 4.3, one can argue as for [4, Proposition 6.1]. □

Corollary 4.6. *Let* $\mathfrak{p} \in \operatorname{Spec} R$, *and let* $\mathcal{V}$ *and* $\mathcal{W}$ *be specialisation closed subsets of* $\operatorname{Spec} R$ *such that* $\mathcal{V} \setminus \mathcal{W} = \{\mathfrak{p}\}$. *Then*

$$L_{\mathcal{W}}\varGamma_{\mathcal{V}} \cong \varGamma_{\mathfrak{p}} \cong \varGamma_{\mathcal{V}}L_{\mathcal{W}}.$$

Proof. Given Proposition 4.3, one can argue as in the proof of [4, Theorem 6.2]. □

Corollary 4.7. *Let* $\mathfrak{p} \in \operatorname{Spec} R$ *and* $F \in \mathsf{Coh}\,\mathsf{T}$. *Then* $\operatorname{supp}_R \varGamma_{\mathfrak{p}} F \subseteq \{\mathfrak{p}\}$.

Proof. Combine Proposition 4.3 and Corollary 4.6. □

Colimits. The following result can be used to reduce computations involving specialisation closed subsets to those involving closed sets; it is an analogue of [24, Lemma 6.6] in the compactly generated context.

Lemma 4.8. *Let $\mathcal{V} = \bigcup_\alpha \mathcal{V}_\alpha$ be a directed union of specialisation closed subsets of* $\operatorname{Spec} R$. *Then*

$$\operatorname{colim} \varGamma_{\mathcal{V}_\alpha} \xrightarrow{\sim} \varGamma_{\mathcal{V}} \qquad \text{and} \qquad \operatorname{colim} L_{\mathcal{V}_\alpha} \xrightarrow{\sim} L_{\mathcal{V}}.$$

Proof. We make repeated use of Proposition 4.1. Since $\varGamma_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} = \varGamma_{\mathcal{V}_\alpha}$, it follows from the localisation sequence (2.11) that the natural morphism $\operatorname{colim} \varGamma_{\mathcal{V}_\alpha} \to \varGamma_{\mathcal{V}}$ fits into an exact sequence

$$\cdots \longrightarrow \operatorname{colim} \varGamma_{\mathcal{V}_\alpha} \longrightarrow \varGamma_{\mathcal{V}} \longrightarrow \operatorname{colim} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} \longrightarrow \cdots$$

We claim that $\operatorname{colim} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} = 0$. Indeed, $\varGamma_{\mathfrak{p}}$ commutes with filtered colimits so the claim is that $\operatorname{colim}(\varGamma_{\mathfrak{p}} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}}) = 0$ for each $\mathfrak{p}$ in $\operatorname{Spec} R$. Proposition 4.3 and its corollaries yield

$$\operatorname{supp}_R(\varGamma_{\mathfrak{p}} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} F) \subseteq \{\mathfrak{p}\} \cap (\operatorname{Spec} R \setminus \mathcal{V}_\alpha) \cap \mathcal{V}$$

for each F in $\mathsf{Coh}\,\mathsf{T}$. Thus, if $\mathfrak{p} \notin \mathcal{V}$, then evidently $\varGamma_{\mathfrak{p}} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} = 0$. Assume $\mathfrak{p} \in \mathcal{V}$. When $\mathfrak{p} \in \mathcal{V}_\alpha$ as well, it again follows from the equality above that $\varGamma_{\mathfrak{p}} L_{\mathcal{V}_\alpha}\varGamma_{\mathcal{V}} = 0$. Since $\mathcal{V}$ is directed union of the $\mathcal{V}_\alpha$, the desired vanishing follows.

This proves the claim. The exact sequence above then yields the isomorphism involving $\varGamma_{\mathcal{V}}$. The assertion for $L_{\mathcal{V}}$ follows, using again (2.11). □

5. The local-global principle

Let R be a noetherian graded commutative ring and T be an essentially small R-linear triangulated category. In this section we establish a local-global principle for $\mathsf{Coh}\,\mathsf{T}$, analogous to the one in [5, §3]. A local-global principle for T then follows.

Localising subcategories. We call a full subcategory of $\mathsf{Coh}\,\mathsf{T}$ *localising* if it is closed under forming coproducts, extensions, and suspensions. Here, F is an *extension* of F' and F'' if there is an exact sequence $F' \to F \to F''$ in $\mathsf{Coh}\,\mathsf{T}$. Any localising subcategory is closed under subobjects and quotient objects; this follows by specialising $F' = 0$ or $F'' = 0$. In particular, a localising subcategory is closed under filtered colimits. The smallest localising subcategory containing a subcategory C of T is denoted $\mathsf{Loc}(\mathsf{C})$.

The following lemma provides some basic properties of localising subcategories; they will be used without further mention. The argument is straightforward.

Lemma 5.1. *Let $P\colon \mathsf{Coh}\,\mathsf{T} \to \mathsf{Coh}\,\mathsf{U}$ be an exact functor that preserves coproducts.*

(1) *If $\mathsf{C} \subseteq \mathsf{Coh}\,\mathsf{U}$ is localising, then $\{F \in \mathsf{Coh}\,\mathsf{T} \mid P(F) \in \mathsf{C}\}$ is a localising subcategory of $\mathsf{Coh}\,\mathsf{T}$. In particular, the kernel*

$$\operatorname{Ker}(P) = \{F \in \mathsf{Coh}\,\mathsf{T} \mid P(F) = 0\}$$

is a localising subcategory of $\mathsf{Coh}\,\mathsf{T}$.

(2) *Any subcategory $\mathsf{C} \subseteq \mathsf{Coh}\,\mathsf{T}$ satisfies $P(\mathsf{Loc}(\mathsf{C})) \subseteq \mathsf{Loc}(P(\mathsf{C}))$.* □

We provide important examples of localising subcategories.

Lemma 5.2. *Let $\mathsf{S} \subseteq \mathsf{T}$ be a triangulated subcategory. Then $\mathsf{Coh}\,\mathsf{S}$ and $\mathsf{Coh}\,\mathsf{T}/\mathsf{S}$ are localising when viewed as subcategories of $\mathsf{Coh}\,\mathsf{T}$.*

Proof. From Corollary 2.14 it follows that $\mathsf{Coh}\,\mathsf{S}$ equals the kernel of the functor L, while $\mathsf{Coh}\,\mathsf{T}/\mathsf{S}$ equals the kernel of the functor $\varGamma$. It remains to observe that both functors are exact and preserve coproducts, by Lemma 2.4. □

Proposition 5.3. *Let $\mathcal{V}$ be a specialisation closed subset of* $\operatorname{Spec} R$ *and* $F \in \mathsf{Coh}\,\mathsf{T}$. *Then $\varGamma_{\mathcal{V}}F$ and $L_{\mathcal{V}}F$ belong to* $\mathsf{Loc}(F)$.

Proof. From the localisation sequence (2.11) it follows that it suffices to prove this for $\varGamma_{\mathcal{V}}$. The assertion follows from Lemmas 3.8 and 3.12 when $\mathcal{V} = \mathcal{V}(\mathfrak{a})$ for some ideal $\mathfrak{a}$. The general case then follows from Lemma 4.8. □

Lemma 5.4. *Let $X \in \mathsf{T}$ and $\mathfrak{a}$ be a homogeneous ideal of R. Then*

$$\mathsf{Loc}(H_X) = \mathsf{Coh}\,\mathsf{Thick}(X) \qquad \textit{and} \qquad \mathsf{Loc}(\varGamma_{\mathcal{V}(\mathfrak{a})}H_X) = \mathsf{Loc}(H_{X/\!\!/\mathfrak{a}}).$$

Proof. Both assertions use the following observation:

$$Y \in \mathsf{Thick}(X) \quad \Longrightarrow \quad H_Y \in \mathsf{Loc}(H_X).$$

From this it follows that $\mathsf{Coh}\,\mathsf{Thick}(X) \subseteq \mathsf{Loc}(H_X)$, while the reverse inclusion is by Lemma 5.2. This settles the first of the desired equalities.

For the second one, note that $\varGamma_{\mathcal{V}(r)}H_X \in \mathsf{Loc}(H_{X/\!\!/r})$ for any homogeneous element $r \in R$, by Lemma 3.8. Thus an induction on the number of generators of $\mathfrak{a}$ shows that $\varGamma_{\mathcal{V}(\mathfrak{a})}H_X \in \mathsf{Loc}(H_{X/\!\!/\mathfrak{a}})$. Conversely, $H_{X/\!\!/\mathfrak{a}}$ belongs to $\mathsf{Loc}(H_X)$, by the observation above, so $\varGamma_{\mathcal{V}(\mathfrak{a})}H_{X/\!\!/\mathfrak{a}} \cong H_{X/\!\!/\mathfrak{a}}$ belongs to $\mathsf{Loc}(\varGamma_{\mathcal{V}(\mathfrak{a})}H_X)$. □

The local-global principle. The following result is the analogue of a local-global principle for compactly generated triangulated categories [5, Theorem 3.1].

Theorem 5.5 (Local-global principle)**.** *Let $F \in \mathsf{Coh}\,\mathsf{T}$. Then*

$$\mathsf{Loc}(F) = \mathsf{Loc}(\{\varGamma_{\mathfrak{p}}F \mid \mathfrak{p} \in \operatorname{Spec} R\}) = \mathsf{Loc}(\{F_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}).$$

Proof. The idea for this proof is taken from [20, Lemma 2.10].

It follows from Proposition 5.3 that $\varGamma_{\mathfrak{p}}F$ and $F_{\mathfrak{p}}$ belong to $\mathsf{Loc}(F)$.

Now we set $\mathsf{C} = \mathsf{Loc}(\{\varGamma_{\mathfrak{p}}F \mid \mathfrak{p} \in \operatorname{Spec} R\})$ and prove that $F \in \mathsf{C}$. Consider the set of specialisation closed subsets $\mathcal{W}$ of $\operatorname{Spec} R$ such that $\varGamma_{\mathcal{W}}F \in \mathsf{C}$. This set is non-empty, for it contains the empty set, and it is closed under directed unions by Lemma 4.8. Thus it has a maximal element, say $\mathcal{V}$, by Zorn's lemma. We claim that $\mathcal{V} = \operatorname{Spec} R$. To this end assume $\mathcal{V} \neq \operatorname{Spec} R$ and choose a prime ideal $\mathfrak{p}$ maximal in the subset $\operatorname{Spec} R \setminus \mathcal{V}$. The subset $\mathcal{V} \cup \{\mathfrak{p}\}$ is then specialisation closed. Consider the localisation sequence with respect to $\mathcal{V}$:

$$\cdots \longrightarrow \varGamma_{\mathcal{V}}\varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}F \longrightarrow \varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}F \longrightarrow L_{\mathcal{V}}\varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}F \longrightarrow \cdots$$

Corollaries 4.5 and 4.6 yield that $\varGamma_{\mathcal{V}} \cong \varGamma_{\mathcal{V}}\varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}$ and $\varGamma_{\mathfrak{p}} \cong L_{\mathcal{V}}\varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}$. Hence the terms on the left and on the right of the localisation sequence are in C, and hence so is the one in the middle, $\varGamma_{\mathcal{V}\cup\{\mathfrak{p}\}}F$. This contradicts the maximality of $\mathcal{V}$ and yields the first equality.

The second equality follows from the first since $\varGamma_{\mathfrak{p}}F = \varGamma_{\mathcal{V}(\mathfrak{p})}(F_{\mathfrak{p}}) \in \mathsf{Loc}(F_{\mathfrak{p}})$ for each prime $\mathfrak{p}$, by Proposition 5.3. □

Recall that the R-action on T induces an action on any triangulated subcategory $\mathsf{S} \subseteq \mathsf{T}$ and on T/S. It is not hard to see that the induced functor $\mathsf{S}_{\mathfrak{p}} \to \mathsf{T}_{\mathfrak{p}}$ is exact, full, and faithful. For this reason, we view $\mathsf{S}_{\mathfrak{p}}$ as a triangulated subcategory of $\mathsf{T}_{\mathfrak{p}}$.

Lemma 5.6. *Let $\mathsf{S} \subseteq \mathsf{T}$ be a triangulated subcategory and $\mathfrak{p} \in \operatorname{Spec} R$. Then the canonical functor $\mathsf{T}_{\mathfrak{p}} \to (\mathsf{T}/\mathsf{S})_{\mathfrak{p}}$ induces an equivalence of triangulated categories*

$$\mathsf{T}_{\mathfrak{p}}/\mathsf{S}_{\mathfrak{p}} \xrightarrow{\sim} (\mathsf{T}/\mathsf{S})_{\mathfrak{p}}.$$

Proof. Consider the commutative diagrams below. The one on the left is clear from the constructions and induces the one on the right.

$$\begin{array}{ccc} \mathsf{T} & \xrightarrow{f} & \mathsf{T}/\mathsf{S} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ \mathsf{T}_{\mathfrak{p}} & \xrightarrow{f_{\mathfrak{p}}} & (\mathsf{T}/\mathsf{S})_{\mathfrak{p}} \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{\mathsf{Coh}} \mathsf{T} & \xleftarrow{f_*} & \operatorname{\mathsf{Coh}} \mathsf{T}/\mathsf{S} \\ \big\uparrow{\scriptstyle p_*} & & \big\uparrow{\scriptstyle q_*} \\ \operatorname{\mathsf{Coh}}(\mathsf{T}_{\mathfrak{p}}) & \xleftarrow{(f_{\mathfrak{p}})_*} & \operatorname{\mathsf{Coh}}(\mathsf{T}/\mathsf{S})_{\mathfrak{p}} \end{array}$$

Since f and q are quotient functors, so is their composition qf. Then $f_{\mathfrak{p}}p$ and p are quotient functors, and from this it is not hard to verify that so is $f_{\mathfrak{p}}$. We claim that its kernel consists precisely of direct summands of objects of $\mathsf{S}_{\mathfrak{p}}$, and the statement would then follow.

As to the claim: In the display above, the functors in the square on the right are all fully faithful, since they are induced by quotient functors. Hence we view all the categories in the diagram as subcategories of $\operatorname{\mathsf{Coh}} \mathsf{T}$. Recall from Corollary 2.14 that $\operatorname{Ker}(f^*) = \operatorname{\mathsf{Coh}} \mathsf{S}$. Using (3.7) it follows that $\operatorname{Ker}(f^*_{\mathfrak{p}}) = \operatorname{\mathsf{Coh}}(\mathsf{S}_{\mathfrak{p}})$. In particular, for an object $X \in \mathsf{T}_{\mathfrak{p}}$, one has $f_{\mathfrak{p}}(X) = 0$ iff $f^*_{\mathfrak{p}}(H_X) = 0$ iff X is a direct summand of an object in $\mathsf{S}_{\mathfrak{p}}$; the second equivalence is by Lemma 2.6. This justifies the claim. □

For $\mathfrak{p} \in \operatorname{Spec} R$ we set $\varGamma_{\mathfrak{p}}\mathsf{T} = (\mathsf{T}_{\mathfrak{p}})_{\mathcal{V}(\mathfrak{p})}$. This yields the following diagram

$$\mathsf{T} \twoheadrightarrow \mathsf{T}_{\mathfrak{p}} \hookleftarrow \varGamma_{\mathfrak{p}}\mathsf{T}$$

and henceforth we make the identification

$$\operatorname{\mathsf{Coh}} \varGamma_{\mathfrak{p}}\mathsf{T} = (\operatorname{\mathsf{Coh}} \mathsf{T})_{\mathfrak{p}} \cap (\operatorname{\mathsf{Coh}} \mathsf{T})_{\mathcal{V}(\mathfrak{p})}.$$

Corollary 5.7. *Taking a localising subcategory $\mathsf{C} \subseteq \operatorname{\mathsf{Coh}} \mathsf{T}$ to the family*

$$(\mathsf{C} \cap \operatorname{\mathsf{Coh}} \varGamma_{\mathfrak{p}}\mathsf{T})_{\mathfrak{p} \in \operatorname{Spec} R}$$

induces a bijection between

- *the localising subcategories of $\operatorname{\mathsf{Coh}} \mathsf{T}$, and*
- *the families $(\mathsf{C}(\mathfrak{p}))_{\mathfrak{p} \in \operatorname{Spec} R}$ with $\mathsf{C}(\mathfrak{p})$ a localising subcategory of $\operatorname{\mathsf{Coh}} \varGamma_{\mathfrak{p}}\mathsf{T}$.*

Proof. The inverse map takes a family $(\mathsf{C}(\mathfrak{p}))_{\mathfrak{p} \in \operatorname{Spec} R}$ to the smallest localising subcategory of $\operatorname{\mathsf{Coh}} \mathsf{T}$ containing all $\mathsf{C}(\mathfrak{p})$. □

Remark 5.8. There is an analogue of Corollary 5.7 for thick subcategories of T since each thick subcategory $\mathsf{S} \subseteq \mathsf{T}$ is determined by the localising subcategory $\operatorname{\mathsf{Coh}} \mathsf{S} \subseteq \operatorname{\mathsf{Coh}} \mathsf{T}$; see Lemma 5.2.

Consequences of the local-global principle. For $X \in \mathsf{T}$ and $\mathfrak{p} \in \operatorname{Spec} R$, we set $X(\mathfrak{p}) = (X /\!\!/ \mathfrak{p})_{\mathfrak{p}}$ and identify this with $(X_{\mathfrak{p}}) /\!\!/ \mathfrak{p}$. Note that Lemma 5.4 implies

$$\operatorname{\mathsf{Loc}}(\varGamma_{\mathfrak{p}} H_X) = \operatorname{\mathsf{Loc}}(H_{X(\mathfrak{p})}). \tag{5.9}$$

The following is the local-global principle for T announced in the introduction.

Theorem 5.10 (Local-global principle)**.** *Let S be a thick subcategory of T. Then the following conditions are equivalent for an object X in T:*

(1) *X belongs to S.*
(2) *$X_{\mathfrak{p}}$ belongs to $\operatorname{\mathsf{Thick}}(\mathsf{S}_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*
(3) *$X(\mathfrak{p})$ belongs to $\operatorname{\mathsf{Thick}}(\mathsf{S}_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Evidently (1) ⇒ (2) and (2) ⇒ (3).

Assume (3) holds. We work in $\mathsf{Coh}\,\mathsf{T}$. For each $\mathfrak{p} \in \operatorname{Spec} R$, the hypothesis implies the first inclusion below:

$$H_{X(\mathfrak{p})} \in \mathsf{Coh}(\mathsf{S}_\mathfrak{p}) = (\mathsf{Coh}\,\mathsf{S})_\mathfrak{p} \subseteq \mathsf{Coh}\,\mathsf{S}.$$

The equality is by Proposition 3.6. Thus $\varGamma_\mathfrak{p} H_X \in \mathsf{Coh}\,\mathsf{S}$ for all $\mathfrak{p}$, by (5.9). It follows from Theorem 5.5 that H_X belongs to $\mathsf{Coh}\,\mathsf{S}$, and hence that $X \in \mathsf{S}$. □

Theorem 5.11. *For any pair of objects X, Y in T the following holds:*

$$\begin{aligned} \operatorname{Hom}^*_\mathsf{T}(X,Y) = 0 \quad &\iff \quad \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \\ &\iff \quad \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X(\mathfrak{p}), Y(\mathfrak{p})) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R. \end{aligned}$$

Proof. Let $\mathsf{S} = \mathsf{Thick}(X)$. Then Theorem 5.5 yields the following equivalences:

$$\begin{aligned} \operatorname{Hom}^*_\mathsf{T}(X,Y) = 0 \quad &\iff \quad H_Y \in \mathsf{Coh}\,\mathsf{T}/\mathsf{S} \\ &\iff \quad H_{Y_\mathfrak{p}} \in (\mathsf{Coh}\,\mathsf{T}/\mathsf{S})_\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \\ &\iff \quad \varGamma_\mathfrak{p} H_Y \in (\mathsf{Coh}\,\mathsf{T}/\mathsf{S})_\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R. \end{aligned}$$

Using the identification $\mathsf{Coh}(\mathsf{T}_\mathfrak{p}/\mathsf{S}_\mathfrak{p}) = \mathsf{Coh}(\mathsf{T}/\mathsf{S})_\mathfrak{p} = (\mathsf{Coh}\,\mathsf{T}/\mathsf{S})_\mathfrak{p}$ from Lemma 5.6 and the identity (5.9), we obtain

$$\begin{aligned} \operatorname{Hom}^*_\mathsf{T}(X,Y) = 0 \quad &\iff \quad \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \\ &\iff \quad \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y(\mathfrak{p})) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R. \end{aligned}$$

In the last condition, $X_\mathfrak{p}$ can be replaced by $X(\mathfrak{p})$. This follows from the general fact that for any homogenous ideal $\mathfrak{a}$ of R and any pair of objects U, V in T

$$\operatorname{Hom}^*_\mathsf{T}(U,V) = 0 \quad \iff \quad \operatorname{Hom}^*_\mathsf{T}(U /\!\!/ \mathfrak{a}, V) = 0$$

when $\operatorname{Hom}^*_\mathsf{T}(U,V)$ is $\mathcal{V}(\mathfrak{a})$-torsion; for a proof use (3.4) or see [4, Lemma 5.11]. □

6. Tensor triangulated categories

Let $(\mathsf{T}, \otimes, \mathbf{1})$ be a tensor triangulated category that is essentially small. The tensor product $\otimes\colon \mathsf{T} \times \mathsf{T} \to \mathsf{T}$ is then symmetric monoidal, exact in each variable, and admits a unit $\mathbf{1}$. The tensor product on T extends to a tensor product $\mathsf{Coh}\,\mathsf{T} \times \mathsf{Coh}\,\mathsf{T} \to \mathsf{Coh}\,\mathsf{T}$ that we denote again by $\otimes$. We list the basic (and defining) properties. For objects $X, Y \in \mathsf{T}$ and $F, G \in \mathsf{Coh}\,\mathsf{T}$ we have:

(1) $H_X \otimes H_Y \cong H_{X \otimes Y}$.
(2) $F \otimes -$ and $- \otimes G$ are exact and preserve filtered colimits.
(3) $F \otimes G \cong G \otimes F$.

Lemma 6.1. $(\mathsf{Coh}\,\mathsf{T}, \otimes, H_\mathbf{1})$ *is a symmetric monoidal category.* □

Strongly monoidal functors. A functor f between symmetric monoidal categories is called *strongly monoidal* if there are isomorphisms

$$\mathbf{1} \xrightarrow{\sim} f(\mathbf{1}) \qquad \text{and} \qquad f(X) \otimes f(Y) \xrightarrow{\sim} f(X \otimes Y)$$

that are natural and compatible with the monoidal structures. We have the following projection formula.

Lemma 6.2. *Let $f\colon \mathsf{T} \to \mathsf{U}$ be a strongly monoidal exact functor between tensor triangulated categories. For $F \in \mathsf{Coh}\,\mathsf{T}$ and $G \in \mathsf{Coh}\,\mathsf{U}$, there is a natural morphism*

$$\alpha_{F,G}\colon F \otimes f_*(G) \longrightarrow f_*(f^*(F) \otimes G).$$

This is an isomorphism when T is generated as a triangulated category by $\mathbf{1}$.

Proof. Observe that $f^*\colon \mathsf{Coh}\,\mathsf{T} \to \mathsf{Coh}\,\mathsf{U}$ is strongly monoidal. This is clear for representable functors; the general case follows by taking filtered colimits in one argument and then in the other. The morphism $\alpha_{F,G}$ is the adjoint of the composition

$$f^*(F \otimes f_*(G)) \xrightarrow{\sim} f^*(F) \otimes f^* f_*(G) \to f^*(F) \otimes G.$$

Observe that the objects F such that $\alpha_{F,G}$ is an isomorphism for all G form a localising subcategory of $\mathsf{Coh}\,\mathsf{T}$ containing $H_{\mathbf{1}}$. If $\mathbf{1}$ generates T, then $\mathsf{Loc}(H_{\mathbf{1}}) = \mathsf{Coh}\,\mathsf{T}$, by Lemma 5.4. Thus $\alpha_{F,G}$ is an isomorphism for all F and G. □

Cohomological localisation. Let R be a noetherian graded commutative ring acting on T. The cohomological (co)localisation functors arising from this action can be expressed as tensor functors.

Proposition 6.3. *Let $\mathcal{V}$ be a specialisation closed subset of* $\operatorname{Spec} R$. *Then*

$$\varGamma_{\mathcal{V}} \cong \varGamma_{\mathcal{V}} H_{\mathbf{1}} \otimes - \qquad \textit{and} \qquad L_{\mathcal{V}} \cong L_{\mathcal{V}} H_{\mathbf{1}} \otimes -.$$

Proof. A simple calculation shows that one isomorphism implies the other. For instance, when $L_{\mathcal{V}} \cong L_{\mathcal{V}} H_{\mathbf{1}} \otimes -$ then $L_{\mathcal{V}}(\varGamma_{\mathcal{V}} H_{\mathbf{1}} \otimes -) = 0$. This yields a morphism $\varGamma_{\mathcal{V}} H_{\mathbf{1}} \otimes - \to \varGamma_{\mathcal{V}}$ making the following diagram commutative.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \varGamma_{\mathcal{V}} H_{\mathbf{1}} \otimes - & \longrightarrow & \mathrm{Id} & \longrightarrow & L_{\mathcal{V}} H_{\mathbf{1}} \otimes - & \longrightarrow & \cdots \\
 & & \downarrow & & \Vert & & \downarrow & & \\
\cdots & \longrightarrow & \varGamma_{\mathcal{V}} & \longrightarrow & \mathrm{Id} & \longrightarrow & L_{\mathcal{V}} & \longrightarrow & \cdots
\end{array}$$

The five lemma then shows that this is an isomorphism.

It follows from the description of $L_{\mathcal{V}(r)}$ in Lemma 3.8 that the assertion holds for a closed set $\mathcal{V}(r)$ given by some $r \in R$. Lemma 3.12 then implies the assertion for a closed set $\mathcal{V}(\mathfrak{a})$ given by a finitely generated ideal $\mathfrak{a}$ of R, and Lemma 4.8 implies the assertion for an arbitrary specialisation closed subset. □

7. Stratification

Let R be a noetherian graded commutative ring and T be an essentially small R-linear triangulated category. In this section we study the stratification of T and $\mathsf{Coh}\,\mathsf{T}$; this is the analogue of stratification for compactly generated triangulated categories introduced in [5, §4] and inspired by [15, §6].

Stratification. The triangulated category T is called *minimal* if T admits no proper thick subcategory. This means if $\mathsf{S} \subseteq \mathsf{T}$ is a thick subcategory then $\mathsf{S} = 0$ or $\mathsf{S} = \mathsf{T}$. Analogously, $\mathsf{Coh}\,\mathsf{T}$ is said to be minimal if $\mathsf{Coh}\,\mathsf{T}$ admits no proper localising subcategory. Clearly, T is minimal when $\mathsf{Coh}\,\mathsf{T}$ is minimal.

Definition 7.1. We say that T is *stratified* by the action of R if $\varGamma_{\mathfrak{p}}\mathsf{T}$ is minimal for each $\mathfrak{p} \in \operatorname{Spec} R$. In the same vein, $\mathsf{Coh}\,\mathsf{T}$ is *stratified* by the action of R if $\mathsf{Coh}\,\varGamma_{\mathfrak{p}}\mathsf{T}$ is minimal for each $\mathfrak{p} \in \operatorname{Spec} R$.

In each case stratification yields a classification of thick or localising subcategories in terms of subsets of $\operatorname{Spec} R$; see Corollary 5.7.

Remark 7.2. Suppose that T is minimal. Then T is stratified by any R-action, and in particular, by the canonical action of $\mathbb{Z}$. Moreover, there is a unique prime $\mathfrak{p} \in \operatorname{Spec} R$ such that $\varGamma_{\mathfrak{p}}\mathsf{T} \neq 0$. Clearly, this implies $\varGamma_{\mathfrak{q}}\mathsf{T} = 0$ for all $\mathfrak{q} \neq \mathfrak{p}$ in $\operatorname{Spec} R$.

Consequences of stratification. It is convenient to set $\operatorname{supp}_R X = \operatorname{supp}_R H_X$ for each object $X \in \mathsf{T}$. Observe that (5.9) implies

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X(\mathfrak{p}) \neq 0\},$$

and this can be reformulated in terms of the following identity which is an immediate consequence of Lemma 3.9

$$\mathsf{Thick}(X(\mathfrak{p})) = \mathsf{Thick}(X_\mathfrak{p}) \cap \varGamma_\mathfrak{p}\mathsf{T} = \varGamma_\mathfrak{p}\,\mathsf{Thick}(X). \tag{7.3}$$

Theorem 7.4. *Suppose that* T *is stratified by the action of* R*. Given objects* X, Y *in* T*, we have*

$$\begin{aligned} X \in \mathsf{Thick}(Y) \quad &\iff \quad \operatorname{supp}_R X \subseteq \operatorname{supp}_R Y, \\ \operatorname{Hom}^*_\mathsf{T}(X,Y) = 0 \quad &\iff \quad (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y) = \varnothing. \end{aligned}$$

Proof. For the first assertion set $\mathsf{S} = \mathsf{Thick}(Y)$. The local-global principle from Theorem 5.10 gives the first equivalence:

$$\begin{aligned} X \in \mathsf{Thick}(Y) \quad &\iff \quad X(\mathfrak{p}) \in \varGamma_\mathfrak{p}\mathsf{S} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \\ &\iff \quad \operatorname{supp}_R X \subseteq \operatorname{supp}_R Y. \end{aligned}$$

The second equivalence uses the minimality of $\varGamma_\mathfrak{p}\mathsf{T}$ and the identity (7.3).

For the second assertion recall from Theorem 5.11 that

$$\operatorname{Hom}^*_\mathsf{T}(X,Y) = 0 \quad \iff \quad \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X(\mathfrak{p}), Y(\mathfrak{p})) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R.$$

The minimality of $\varGamma_\mathfrak{p}\mathsf{T}$ implies for objects U, V in $\varGamma_\mathfrak{p}\mathsf{T}$ that $\operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(U,V) \neq 0$ iff $U \neq 0 \neq V$. Thus $\operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X(\mathfrak{p}), Y(\mathfrak{p})) = 0$ iff $\mathfrak{p} \notin (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y)$. □

Theorem 7.4 has a converse when endomorphism rings are finitely generated.

Proposition 7.5. *Suppose that for each object* X *in* T *the endomorphism ring* $\operatorname{End}^*_\mathsf{T}(X)$ *is finitely generated over* R*. If* T *is not stratified by* R*, then there are objects* $X, Y \in \mathsf{T}$ *such that* $\operatorname{supp}_R X = \operatorname{supp}_R Y$ *but* $\mathsf{Thick}(X) \neq \mathsf{Thick}(Y)$*.*

Proof. Assume $\varGamma_\mathfrak{p}\mathsf{T}$ is not minimal. Thus there are non-zero objects $X_\mathfrak{p}, Y_\mathfrak{p}$ in $\varGamma_\mathfrak{p}\mathsf{T}$ such that $\mathsf{Thick}(X_\mathfrak{p}) \neq \mathsf{Thick}(Y_\mathfrak{p})$. In T consider the objects $X' = X/\!\!/\mathfrak{p}$ and $Y' = Y/\!\!/\mathfrak{p}$. Then $\operatorname{supp}_R X' = \mathcal{V}(\mathfrak{p}) = \operatorname{supp}_R Y'$ by Proposition 4.2. On the other hand, Lemma 3.9 gives the equalities below:

$$\mathsf{Thick}(X'_\mathfrak{p}) = \mathsf{Thick}(X_\mathfrak{p}) \neq \mathsf{Thick}(Y_\mathfrak{p}) = \mathsf{Thick}(Y'_\mathfrak{p}),$$

so that $\mathsf{Thick}(X') \neq \mathsf{Thick}(Y')$. □

The Hom vanishing statement in Theorem 7.4 can be strengthened when morphism spaces are finitely generated over R. For an R-module M, we write

$$\operatorname{Supp}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq 0\}.$$

The following theorem can be used to explain results on the symmetry of Hom vanishing, as studied in [1, 7].

Theorem 7.6. *Let* X *and* Y *be objects in* T*.*

(1) *If* $\operatorname{Hom}^*_\mathsf{T}(X,Y)$ *is finitely generated over* R*, then*

$$\operatorname{Supp}_R \operatorname{Hom}^*_\mathsf{T}(X,Y) \subseteq (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y).$$

(2) *If* T *is stratified by the action of* R*, then*

$$\operatorname{Supp}_R \operatorname{Hom}^*_\mathsf{T}(X,Y) \supseteq (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y).$$

Proof. (1) Let $\mathfrak{p} \in \operatorname{Supp}_R \operatorname{Hom}^*_{\mathsf{T}}(X, Y)$. Thus $\operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) \neq 0$. The assumption implies that this is finitely generated, and an application of Nakayama's lemma gives

$$\operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X(\mathfrak{p}), Y(\mathfrak{p})) \neq 0;$$

use (3.4) or see [4, Lemma 5.11] for a proof. Thus $\mathfrak{p} \in (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y)$.

(2) Let $\mathfrak{p} \in (\operatorname{supp}_R X) \cap (\operatorname{supp}_R Y)$. Then stratification implies

$$\operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X(\mathfrak{p}), Y(\mathfrak{p})) \neq 0,$$

and therefore

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y)_\mathfrak{p} \cong \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) \neq 0.$$

Thus $\mathfrak{p} \in \operatorname{Supp}_R \operatorname{Hom}^*_{\mathsf{T}}(X, Y)$. □

Perfect complexes. Let A be a noetherian commutative ring. We denote by $\mathsf{D}(A)$ the derived category of the category of A-modules. An object in $\mathsf{D}(A)$ is called *perfect* if it is isomorphic to a bounded complex of finitely generated projective A-modules; these form a thick subcategory denoted by $\mathsf{D}^{\mathrm{per}}(A)$. For $X \in \mathsf{D}(A)$ set

$$H_X = \operatorname{Hom}_A(-, X)|_{\mathsf{D}^{\mathrm{per}}(A)}.$$

The ring $R = A$ acts canonically on $\mathsf{T} = \mathsf{D}^{\mathrm{per}}(A)$ and we show that $\operatorname{\mathsf{Coh}} \mathsf{T}$ is stratified by this action. The residue fields play a special role. For $\mathfrak{p} \in \operatorname{Spec} R$ let $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$, viewed as a complex concentrated in degree zero.

Lemma 7.7. *Let* $\mathfrak{p} \in \operatorname{Spec} R$. *Then* $\operatorname{\mathsf{Coh}} \varGamma_\mathfrak{p} \mathsf{T} = \operatorname{\mathsf{Loc}}(H_{k(\mathfrak{p})})$.

Proof. Since $H_{k(\mathfrak{p})}$ is $\mathfrak{p}$-local and $\mathcal{V}(\mathfrak{p})$-torsion when evaluated at any object from T, it belongs to $\operatorname{\mathsf{Coh}} \varGamma_\mathfrak{p} \mathsf{T}$; this justfies one inclusion. For the other one, it is convenient to identify $\mathsf{T}_\mathfrak{p}$ and $\mathsf{D}^{\mathrm{per}}(A_\mathfrak{p})$. Thus an object X in $\varGamma_\mathfrak{p}\mathsf{T}$ is a perfect complex over $A_\mathfrak{p}$ such that its cohomology is of finite length over $A_\mathfrak{p}$. It follows that

$$X \in \operatorname{\mathsf{Thick}}(k(\mathfrak{p})) \subseteq \mathsf{D}(A_\mathfrak{p}).$$

This is easily shown by an induction on the number of integers n such $H^n(X) \neq 0$; see for example [11, Example 3.5]. Thus $H_X \in \operatorname{\mathsf{Loc}}(H_{k(\mathfrak{p})})$. □

Theorem 7.8. *Let A be a commutative noetherian ring. Then* $\operatorname{\mathsf{Coh}} \mathsf{D}^{\mathrm{per}}(A)$ *is stratified by the canonical action of A.*

Proof. Fix $\mathfrak{p} \in \operatorname{Spec} A$. We need to show that $\operatorname{\mathsf{Coh}} \varGamma_\mathfrak{p} \mathsf{T}$ is minimal, which is equivalent to $\operatorname{\mathsf{Loc}}(F) = \operatorname{\mathsf{Loc}}(H_{k(\mathfrak{p})})$ for each non-zero $F \in \operatorname{\mathsf{Coh}} \varGamma_\mathfrak{p}\mathsf{T}$, by Lemma 7.7. This is clear for $F = \varGamma_\mathfrak{p} H_A$. This gives the second of the following equalities:

$$\operatorname{\mathsf{Loc}}(F) = \operatorname{\mathsf{Loc}}(F \otimes \varGamma_\mathfrak{p} H_A) = \operatorname{\mathsf{Loc}}(F \otimes H_{k(\mathfrak{p})}).$$

The first one is by Proposition 6.3. Let f denote the functor

$$- \otimes_A k(\mathfrak{p}) \colon \mathsf{D}^{\mathrm{per}}(A) \longrightarrow \mathsf{D}^{\mathrm{per}}(k(\mathfrak{p})).$$

Then $F \otimes H_{k(\mathfrak{p})} \cong f_* f^*(F)$ by Lemma 6.2. Thus $F \otimes H_{k(\mathfrak{p})}$ is a direct sum of suspensions of $H_{k(\mathfrak{p})}$, since every object in $\mathsf{D}^{\mathrm{per}}(k(\mathfrak{p}))$ is a direct sum of suspensions of $k(\mathfrak{p})$. It follows that $\operatorname{\mathsf{Loc}}(F) = \operatorname{\mathsf{Loc}}(H_{k(\mathfrak{p})})$. □

The following result is due to Hopkins [14] and Neeman [20].

Corollary 7.9. *Let* $\mathsf{S} \subseteq \mathsf{D}^{\mathrm{per}}(A)$ *be a thick subcategory. Then*

$$\mathsf{S} = \{X \in \mathsf{D}^{\mathrm{per}}(A) \mid \operatorname{supp}_A X \subseteq \mathcal{V}\}$$

for some specialisation closed subset $\mathcal{V} \subseteq \operatorname{Spec} A$.

Proof. The assertion follows from Theorems 7.4 and 7.8, using the fact that $\operatorname{supp}_A X$ is specialisation closed for each $X \in \mathsf{D}^{\mathrm{per}}(A)$ by Proposition 4.2. □

Remark 7.10. Theorem 7.8 generalises with same proof in two directions as follows.

(1) Let A be a commutative differential graded algebra such that the ring $H^*(A)$ is noetherian. If A is formal, then $\mathsf{Coh}\,\mathsf{D}^{\mathrm{per}}(A)$ is stratified by the canonical $H^*(A)$-action. This is an analogue of Theorem 8.1 in [5] that asserts that $\mathsf{D}(A)$ is stratified by $H^*(A)$.

(2) Let A be a graded commutative noetherian ring. More precisely, we fix an abelian grading group G endowed with a symmetric bilinear form

$$(-,-)\colon G \times G \longrightarrow \mathbb{Z}/2,$$

and A admits a decomposition

$$A = \bigoplus_{g\in G} A_g$$

such that the multiplication satisfies $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$ and $xy = (-1)^{(g,h)} yx$ for all homogeneous elements $x \in A_g$, $y \in A_h$. We consider G-graded A-modules with degree zero morphisms and let $\mathsf{D}(A)$ denote its derived category. Localising subcategories of $\mathsf{Coh}\,\mathsf{D}^{\mathrm{per}}(A)$ and $\mathsf{D}(A)$ are supposed to be closed under twists, where the *twist* of a module or complex X by $g \in G$ is given by $X(g)_h = X_{g+h}$. Suitably adapting definitions and constructions to take into account twists, one can establish that $\mathsf{Coh}\,\mathsf{D}^{\mathrm{per}}(A)$ is stratified by the canonical A-action. This is an analogue of Corollary 5.7 in [10] that asserts that $\mathsf{D}(A)$ is stratified by A.

8. Compactly generated triangulated categories

Let R be a noetherian graded commutative ring and T be a compactly generated R-linear triangulated category. The subcategory of compacts, T^{c}, is an essentially small triangulated category and has an induced R-action. In [4] we developed a theory of local cohomology and support for T. In this section, we use the *restricted Yoneda functor*

$$\mathsf{T} \longrightarrow \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}, \quad X \mapsto H_X = \operatorname{Hom}_{\mathsf{T}}(-,X)|_{\mathsf{T}^{\mathsf{c}}},$$

to compare it with the one for $\mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ introduced in this article.

Remark 8.1. The morphisms annihilated by the functor $\mathsf{T} \to \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ are called *phantom maps*. In the context of the stable module category $\mathsf{T} = \mathsf{StMod}\,kG$ of a finite group G, these were studied by Benson and Gnacadja. In particular, in [3, §4] it is shown that there are filtered systems in $\mathsf{T}^{\mathsf{c}} = \mathsf{stmod}\,kG$ that do not lift to $\mathsf{mod}\,kG$. As a consequence, there are objects in $\mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ that are not in the image of $\mathsf{T} \to \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$, namely the filtered colimit of the corresponding representable functors. In the context of the derived category of a commutative noetherian ring, examples of filtered systems that do not lift can be found in Neeman [21].

Cohomological localisation. Given a specialisation closed subset $\mathcal{V}$ of $\operatorname{Spec} R$, there is an exact localisation functor $\bar{L}_{\mathcal{V}}\colon \mathsf{T} \to \mathsf{T}$ such that $\bar{L}_{\mathcal{V}} X = 0$ iff H_X is $\mathcal{V}$-torsion; see [4, §4]. The corresponding colocalisation functor is denoted by $\bar{\varGamma}_{\mathcal{V}}$. Thus each $X \in \mathsf{T}$ fits into an exact *localisation triangle*

$$\bar{\varGamma}_{\mathcal{V}} X \longrightarrow X \longrightarrow \bar{L}_{\mathcal{V}} X \longrightarrow .$$

The following proposition says that notions developed in [4] for compactly generated triangulated categories are determined by analogous concepts for the category of cohomological functors which are discussed in this work. This applies, for instance, to the notion of support.

Proposition 8.2. *Let $\mathcal{V} \subseteq \operatorname{Spec} R$ be specialisation closed and $X \in \mathsf{T}$. Then*

$$H_{\bar{\Gamma}_{\mathcal{V}}X} \cong \Gamma_{\mathcal{V}} H_X \qquad \text{and} \qquad H_{\bar{L}_{\mathcal{V}}X} \cong L_{\mathcal{V}} H_X.$$

Proof. It follows from Proposition 2.10 that the long exact sequence

$$\cdots \longrightarrow H_{\Sigma^{-1}(\bar{L}_{\mathcal{V}}X)} \longrightarrow H_{\bar{\Gamma}_{\mathcal{V}}X} \longrightarrow H_X \longrightarrow H_{\bar{L}_{\mathcal{V}}X} \longrightarrow H_{\Sigma(\bar{\Gamma}_{\mathcal{V}}X)} \longrightarrow \cdots$$

is isomorphic to the localisation sequence (2.11) for $\mathsf{T}^{\mathsf{c}}_{\mathcal{V}} \subseteq \mathsf{T}^{\mathsf{c}}$, applied to H_X. □

Localising subcategories. A full triangulated subcategory of T is *localising* if it is closed under forming coproducts. Following [5, §3], we say that the *local-global principle* holds for T, if for each object $X \in \mathsf{T}$ we have

$$\mathsf{Loc}(X) = \mathsf{Loc}(\{\bar{\Gamma}_{\mathfrak{p}}X \mid \mathfrak{p} \in \operatorname{Spec} R\}).$$

This local-global principle has been established in a number of relevant cases. For instance, it holds when R has finite Krull dimension [5, Corollary 3.5], or when T admits a model [24, Theorem 6.9].

We obtain an alternative proof of Theorem 5.10, provided the local-global principle holds for T.

Proposition 8.3. *The local-global principle for T implies the principle for T^{c}.*

Proof. We verify that conditions (1)–(3) of Theorem 5.10 are equivalent. Evidently (1) ⇒ (2) and (2) ⇒ (3).

Assume (3) holds. Given $\mathfrak{p} \in \operatorname{Spec} R$, it follows from [5, Theorem 3.1] that $\mathsf{S}_{\mathfrak{p}}$ is contained in $\mathsf{Loc}(\mathsf{S})$. Thus

$$\mathsf{Loc}_{\mathsf{T}}(\bar{\Gamma}_{\mathfrak{p}}X) = \mathsf{Loc}(X(\mathfrak{p})) \subseteq \mathsf{Loc}(\mathsf{S}_{\mathfrak{p}}) \subseteq \mathsf{Loc}(\mathsf{S}),$$

where the equality holds by [5, Lemma 3.8]. Now the local-global principle for T yields that X belongs to $\mathsf{Loc}(\mathsf{S})$. It remains to observe that this implies $X \in \mathsf{S}$, because X is compact and S is a subcategory of compact objects. □

Cohomological localising subcategories. Any localising subcategory of $\mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ induces one of T via the restricted Yoneda functor. However, we do not know if this is a bijection between the corresponding localising subcategories. This changes when one restricts to localising subcategories that are defined cohomologically.

We call a localising subcategory $\mathsf{S} \subseteq \mathsf{T}$ *cohomological* if there is a cohomological functor $F\colon \mathsf{T} \to \mathsf{A}$ such that

(1) A is an abelian category with exact filtered colimits,
(2) F preserves coproducts, and
(3) S equals the full subcategory of objects in T annihilated by F.

Analogously, a localising subcategory $\mathsf{C} \subseteq \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ is called *cohomological* if there is an exact functor $F\colon \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}} \to \mathsf{A}$ such that

(1) A is an abelian category with exact filtered colimits,
(2) F preserves coproducts, and
(3) C equals the full subcategory of objects in $\mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ annihilated by F.

Example 8.4. An intersection of cohomological localising subcategories is cohomological. Given a subset $\mathcal{U} \subseteq \operatorname{Spec} R$, the localising subcategories

$$\{X \in \mathsf{T} \mid \operatorname{supp}_R X \subseteq \mathcal{U}\} \qquad \text{and} \qquad \{X \in \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}} \mid \operatorname{supp}_R X \subseteq \mathcal{U}\}$$

are cohomological.

Proposition 8.5. *Taking a localising subcategory $\mathsf{C} \subseteq \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$ to $\{X \in \mathsf{T} \mid H_X \in \mathsf{C}\}$ induces a bijection between*

– *the cohomological localising subcategories of* $\mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}$*, and*
– *the cohomological localising subcategories of* T*.*

Proof. To describe the inverse map, let $F\colon \mathsf{T}\to\mathsf{A}$ be a cohomological functor which preserves coproducts. This extends essentially uniquely to an exact and coproduct preserving functor $\bar{F}\colon \mathsf{Coh}\,\mathsf{T}^{\mathsf{c}}\to\mathsf{A}$ by sending a filtered colimit of representable functors $\operatorname{colim}_\alpha H_{X_\alpha}$ to $\operatorname{colim}_\alpha F(X_\alpha)$. It remains to observe that for each $X\in\mathsf{T}$ we have $F(X)=0$ iff $\bar{F}(H_X)=0$. Thus the localising subcategories determined by F and $\bar{F}$ correspond to each other under the above assignment. □

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References

[1] L. L. Avramov and R.-O. Buchweitz, Support varieties and cohomology over complete intersections, Invent. Math. **142** (2000), 285–318.
[2] P. Balmer, Spectra, spectra, spectra – Tensor triangular spectra versus Zariski spectra of endomorphism rings, Algebr. Geom. Topol. **10** (2010), no. 3, 1521–1563.
[3] D. J. Benson and G. Ph. Gnacadja, Phantom maps and purity in modular representation theory. II, Algebr. Represent. Theory **4** (2001), no. 4, 395–404.
[4] D. J. Benson, S. B. Iyengar, and H. Krause, Local cohomology and support for triangulated categories, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.
[5] D. Benson, S. B. Iyengar and H. Krause, Stratifying triangulated categories, J. Topol. **4** (2011), no. 3, 641–666.
[6] D. J. Benson, S. B. Iyengar, and H. Krause, Stratifying modular representations of finite groups, Ann. of Math. **174** (2011), 1643–1684.
[7] P. A. Bergh and S. Oppermann, Cohomological symmetry in triangulated categories, J. Algebra **347** (2011), 143–152.
[8] N. Bourbaki, *Éléments de mathématique. Fascicule XXVII. Algèbre commutative. Chapitre 1: Modules plats. Chapitre 2: Localisation*, Actualités Scientifiques et Industrielles, No. 1290 Herman, Paris, 1961.
[9] J. F. Carlson, The varieties and the cohomology ring of a module, J. Algebra **85** (1983), 104–143.
[10] I. Dell'Ambrogio and G. Stevenson, On the derived category of a graded commutative Noetherian ring, J. Algebra **373** (2013), 356–376.
[11] W. Dwyer, J. P. C. Greenlees and S. Iyengar, Finiteness in derived categories of local rings, Comment. Math. Helv. **81** (2006), no. 2, 383–432.
[12] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Springer-Verlag New York, Inc., New York, 1967.
[13] A. Grothendieck and J. L. Verdier, Préfaisceaux, in *SGA 4, Théorie des Topos et Cohomologie Etale des Schémas, Tome 1. Théorie des Topos*, Lect. Notes in Math., vol. 269, Springer, Heidelberg, 1972-1973, pp. 1-184.
[14] M. J. Hopkins, Global methods in homotopy theory, in *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser., 117 Cambridge Univ. Press, Cambridge, 1987.
[15] M. Hovey, J. H. Palmieri, and N. P. Strickland, Axiomatic stable homotopy theory, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114.
[16] H. Krause, Smashing subcategories and the telescope conjecture—an algebraic approach, Invent. Math. **139** (2000), no. 1, 99–133.
[17] H. Krause, Cohomological quotients and smashing localizations, Amer. J. Math. **127** (2005), no. 6, 1191–1246.
[18] H. Matsumura, *Commutative ring theory*, translated from the Japanese by M. Reid, Cambridge Studies in Advanced Mathematics, 8, Cambridge Univ. Press, Cambridge, 1986.
[19] A. Neeman, The Brown representability theorem and phantomless triangulated categories, J. Algebra **151** (1992), no. 1, 118–155.
[20] A. Neeman, The chromatic tower for $D(R)$, Topology **31** (1992), no. 3, 519–532.
[21] A. Neeman, On a theorem of Brown and Adams, Topology **36** (1997), no. 3, 619–645.
[22] U. Oberst and H. Röhrl, Flat and coherent functors, J. Algebra **14** (1970), 91–105.
[23] D. Quillen, The spectrum of an equivariant cohomology ring. I, II, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) **94** (1971), 573–602.

[24] G. Stevenson, Support theory via actions of tensor triangulated categories, J. Reine Angew. Math. **681** (2013), 219–254.
[25] G. Stevenson, Subcategories of singularity categories via tensor actions, arXiv:1105.4698 (to appear in Compositio Math.).
[26] J.-L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque No. 239 (1996), xii+253 pp. (1997).

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