LOCALISING SUBCATEGORIES FOR COCHAINS ON THE CLASSIFYING SPACE OF A FINITE GROUP

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Abstract. The localising subcategories of the derived category of the cochains on the classifying space of a finite group are classified. They are in one to one correspondence with the subsets of the set of homogeneous prime ideals of the cohomology ring $H^*(G,k)$.

1. Introduction

Let G be a finite group and k a field of characteristic p. Let $C^*(BG;k)$ be the cochains on the classifying space BG. Using the machinery of Elmendorf, Kříž, Mandell and May [8], one can regard $C^*(BG;k)$ as a strictly commutative S-algebra over the field k. The derived category $\mathsf{D}(C^*(BG;k))$ has thus a structure of a tensor triangulated category via the left derived tensor product $-\otimes^{\mathbf{L}}_{C^*(BG;k)}-$. The unit for the tensor product is $C^*(BG;k)$.

In this paper we apply techniques and results from [3, 4, 5, 6] to classify the localising subcategories of $\mathsf{D}(C^*(BG;k))$. More precisely, there is a notion of stratification for triangulated categories via the action of a graded commutative ring which implies that the localising subcategories are parameterised by sets of homogeneous prime ideals [4]. For $\mathsf{D}(C^*(BG;k))$ we use the natural action of the endomorphism ring of the tensor identity which is isomorphic to the cohomology algebra $H^*(G,k)$ of the group G.

Theorem 1.1. *The derived category $\mathsf{D}(C^*(BG;k))$ is stratified by the ring $H^*(G,k)$. This yields a one to one correspondence between the localising subcategories of $\mathsf{D}(C^*(BG;k))$ and subsets of the set of homogeneous prime ideals of $H^*(G,k)$.*

It is proved in [6] that there is an equivalence of tensor triangulated categories between $\mathsf{D}(C^*(BG;k))$ and the localising subcategory of $\mathsf{K}(\mathsf{Inj}\, kG)$ generated by the tensor identity. Here, $\mathsf{K}(\mathsf{Inj}\, kG)$ is the homotopy category of complexes of injective (= projective) kG-modules, studied in [6, 9].

The main theorem of [5] states that $\mathsf{K}(\mathsf{Inj}\, kG)$ is stratified as a tensor triangulated category by $H^*(G,k)$. Theorem 1.1 is a consequence of a more general result concerning tensor triangulated categories, which is described below.

Let $(\mathsf{T},\otimes,\mathbb{1})$ be a compactly generated tensor triangulated category, as described in [3, §8], and R a graded commutative noetherian ring acting on T via a homomorphism $R\to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. In this case, for each homogeneous prime ideal $\mathfrak{p}$ of R there exists a *local cohomology functor* $\varGamma_{\mathfrak{p}}\colon \mathsf{T}\to\mathsf{T}$; see [3]. The *support* of an object X in T is then defined to be

$$\operatorname{supp}_R X = \{\mathfrak{p}\in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}}X\neq 0\}\,.$$

The research of the second author was undertaken during a visit to the University of Bielefeld, supported by a research prize from the Humboldt Foundation, and by NSF grant DMS 0903493.

The condition that T is *stratified* by the action of R means that assigning a subcategory S of T to its support

$$\operatorname{supp}_R \mathsf{S} = \bigcup_{X \in \mathsf{S}} \operatorname{supp}_R X$$

yields a bijection between *tensor ideal* localising subcategories of T and subsets of the homogeneous prime ideal spectrum $\operatorname{Spec} R$ contained in $\operatorname{supp}_R \mathsf{T}$; see [4, Theorem 4.2]. Theorem 1.1 is thus a special case of the result below that relates tensor ideal localising subcategories of T and the localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, the localising subcategory of T generated by the tensor unit. We note that $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is a compactly generated tensor triangulated category in its own right and that R acts on it as well.

Theorem 1.2. *Suppose that the Krull dimension of R is finite. If T is stratified by R as a tensor triangulated category, then so is $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, and there is a bijection*

$$\left\{ \begin{matrix} \textit{Tensor ideal localising} \\ \textit{subcategories of } \mathsf{T} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{Localising subcategories} \\ \textit{of } \operatorname{Loc}_{\mathsf{T}}(\mathbb{1}) \end{matrix} \right\}.$$

It assigns each tensor ideal localising subcategory S of T to $\mathsf{S} \cap \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$.

Remark 1.3. The theorem is not true without the assumption that T is stratified by R. For example, let T be the derived category of quasi-coherent sheaves on the projective line $\mathbb{P}^1_k$. The tensor unit is $\mathcal{O}$. In this example there are no proper localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathcal{O})$ since $\operatorname{End}^*_{\mathsf{T}}(\mathcal{O}) = k$, while there are many tensor ideal localising subcategories of T.

Remark 1.4. The assumption that the Krull dimension of R is finite is artificial, and is used only to ensure that for each $X \in \mathsf{T}$ and $\mathfrak{p} \in \operatorname{Spec} R$ the object $\varGamma_{\mathfrak{p}} X$ belongs to $\operatorname{Loc}_{\mathsf{T}}(X)$. One can replace this condition by, for instance, the assumption that T arises as the homotopy category of a Quillen model category [10, §6].

2. Localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$

In this section T is a triangulated category with set-indexed coproducts and the tensor product $\otimes$ provides a symmetric monoidal structure with unit $\mathbb{1}$ on T, which is exact in each variable and preserves set-indexed coproducts.

The proof of Theorem 1.2 is based on a sequence of elementary lemmas. The first one describes the tensor ideal localising subcategory of T which is generated by a class C of objects; we denote this by $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{C})$.

Lemma 2.1. *Let C be a class of objects of T. Then*

$$\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{C}) = \operatorname{Loc}_{\mathsf{T}}(\{X \otimes Y \mid X \in \mathsf{C}, Y \in \mathsf{T}\}).$$

Proof. Set $\mathsf{S} = \operatorname{Loc}_{\mathsf{T}}(\{X \otimes Y \mid X \in \mathsf{C}, Y \in \mathsf{T}\})$. It suffices to show that S is tensor ideal. This means that $F\mathsf{S} \subseteq \mathsf{S}$ for each tensor functor $F = - \otimes Y$, which is an immediate consequence of Lemma 2.2 below. □

Lemma 2.2. *Let $F \colon \mathsf{U} \to \mathsf{V}$ be an exact functor between triangulated categories that preserves set-indexed coproducts. If C is a class of objects of U, then*

$$F \operatorname{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}).$$

Proof. The preimage $F^{-1} \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C})$ is a localising subcategory of U containing C. Thus it contains $\operatorname{Loc}_{\mathsf{U}}(\mathsf{C})$, and one gets

$$F \operatorname{Loc}_{\mathsf{U}}(\mathsf{C}) \subseteq F F^{-1} \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}) \subseteq \operatorname{Loc}_{\mathsf{V}}(F\mathsf{C}).$$ □

Lemma 2.3. *Let $\Gamma\colon \mathsf{T}\to\mathsf{T}$ be a colocalization functor that preserves set-indexed coproducts. Then for any $X\in\mathsf{T}$ and $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, there is a natural isomorphism*

$$\Gamma X\otimes Y \xrightarrow{\sim} \Gamma(X\otimes Y).$$

Remark 2.4. There is an analogous result for a localization functor $L\colon\mathsf{T}\to\mathsf{T}$ that preserves set-indexed coproducts: For any $X\in\mathsf{T}$ and $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, there is a natural isomorphism $L(X\otimes Y)\xrightarrow{\sim} LX\otimes Y$.

Proof. A colocalisation functor Γ comes with a natural morphism $\Gamma X\to X$. Tensoring this with an object $Y\in\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ gives a morphism $\Gamma X\otimes Y\to X\otimes Y$ that factors through the natural morphism $\Gamma(X\otimes Y)\to X\otimes Y$. Here, one uses that $\Gamma X\otimes Y$ belongs to $\Gamma\mathsf{T}$, since the objects $Y'\in\mathsf{T}$ with $\Gamma X\otimes Y'\in\Gamma\mathsf{T}$ form a localising subcategory containing $\mathbb{1}$. The induced morphism $\phi_Y\colon \Gamma X\otimes Y\to\Gamma(X\otimes Y)$ is an isomorphism. To see this, observe that the objects $Y'\in\mathsf{T}$ such that $\phi_{Y'}$ is an isomorphism form a localising subcategory containing $\mathbb{1}$. □

Proposition 2.5. *Suppose that the unit $\mathbb{1}$ is compact in T and let $\Gamma\colon\mathsf{T}\to\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ denote the right adjoint of the inclusion $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\to\mathsf{T}$. If S is a localising subcategory of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, then*

$$\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})=\Gamma(\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}))=\mathsf{S}.$$

Proof. We verify each of the following inclusions

$$\mathsf{S}\subseteq\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\subseteq\Gamma(\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}))\subseteq\mathsf{S}.$$

The first one is clear. Composing the functor Γ with the inclusion $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\to\mathsf{T}$ yields a colocalisation functor that preserves set-indexed coproducts, since $\mathbb{1}$ is compact. For an object X in $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$, we have $\Gamma X\cong X$. This gives the second inclusion. Applying Lemma 2.3 together with the description of $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$ from Lemma 2.1 yields the third inclusion. □

Corollary 2.6. *Suppose that the unit $\mathbb{1}$ is a compact object in T. Assigning each localising subcategory S of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ to $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$ gives a bijection*

$$\left\{\begin{matrix}\textit{Localising subcategories}\\ \textit{of } \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\end{matrix}\right\}\xrightarrow{\sim}\left\{\begin{matrix}\textit{Tensor ideal localising subcategories of}\\ \mathsf{T}\ \textit{generated by objects from}\ \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})\end{matrix}\right\}.$$

Proof. The inverse map sends $\mathsf{U}\subseteq\mathsf{T}$ to $\mathsf{U}\cap\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. □

We are now ready to prove Theorem 1.2. Note that in this T is a compactly generated tensor triangulated category, which entails a host of additional requirements; see [3, §8] for a list.

Proof of Theorem 1.2. It follows from Proposition 2.5 that the assignment

$$\mathsf{S}\longmapsto\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S})$$

is an injective map from the localising subcategories of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ to the tensor ideal localising subcategories of T. In general, it is not bijective, as the example of Remark 1.3 shows. However, since T is stratified by R as a tensor triangulated category, it follows from [4, §7] that each tensor ideal localising subcategory is generated by a set of objects of the form $\Gamma_{\mathfrak{p}}\mathbb{1}$. Since R has finite Krull dimension, [4, Theorem 3.4] yields that $\Gamma_{\mathfrak{p}}\mathbb{1}$ is in $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. Therefore, given a tensor ideal localising subcategory U of T, the localising subcategory

$$\mathsf{U}'=\operatorname{Loc}_{\mathsf{T}}(\{\Gamma_{\mathfrak{p}}\mathbb{1}\mid\mathfrak{p}\in\operatorname{Supp}_R\mathsf{U}\})\subseteq\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$$

satisfies $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{U}') = \mathsf{U}$. This proves the surjectivity of the assignment. Moreover, we have shown that each localising subcategory of $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is generated by objects of the form $\varGamma_{\mathfrak{p}}\mathbb{1}$, so $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ is stratified by the action of R; see [4, Theorem 4.2]. □

3. The cohomological nucleus

Let $(\mathsf{T}, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category and let R be a graded commutative noetherian ring acting on T via a homomorphism $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. Suppose in addition that R has finite Krull dimension.

We define the *cohomological nucleus* of T as the set of homogeneous prime ideals $\mathfrak{p}$ of R such that there exists an object $X \in \mathsf{T}$ satisfying $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$ and $\varGamma_{\mathfrak{p}}X \neq 0$. This definition is motivated by work of Benson, Carlson, and Robinson in the context of modular group representations [2].

For $\mathfrak{p}$ in $\operatorname{Spec} R$ consider the tensor ideal localising subcategory

$$\varGamma_{\mathfrak{p}}\mathsf{T} = \{Y \in \mathsf{T} \mid Y \cong \varGamma_{\mathfrak{p}}X \text{ for some } X \in \mathsf{T}\}.$$

Note that an object $X \in \mathsf{T}$ belongs to $\varGamma_{\mathfrak{p}}\mathsf{T}$ if and only if $\operatorname{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for every compact $C \in \mathsf{T}$, by [3, Corollary 4.10]. The result below gives a local description of the cohomological nucleus.

Proposition 3.1. *Let $\mathfrak{p}$ be a homogeneous prime ideal of R. The following conditions are equivalent:*

(1) *Every object X in T with $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$ satisfies $\varGamma_{\mathfrak{p}}X = 0$.*
(2) *One has $\operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}\mathbb{1}) = \varGamma_{\mathfrak{p}}\mathsf{T}$.*
(3) *Every localising subcategory of $\varGamma_{\mathfrak{p}}\mathsf{T}$ is a tensor ideal of T.*

Proof. The Krull dimension of R is finite, so $\varGamma_{\mathfrak{p}}X$ is in $\operatorname{Loc}_{\mathsf{T}}(X)$ for each X in T, by [4, Theorem 3.4]. This fact is used without further comment.

(1) ⇒ (2): Set $\mathsf{S} = \operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}\mathbb{1})$. Note that $\mathsf{S} \subseteq \varGamma_{\mathfrak{p}}\mathsf{T}$; we claim that equality holds. Indeed, $\mathsf{S} \subseteq \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$ and also $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}) = \varGamma_{\mathfrak{p}}\mathsf{T}$, since $\varGamma_{\mathfrak{p}} = \varGamma_{\mathfrak{p}}\mathbb{1} \otimes -$. Thus, for any X in $\varGamma_{\mathfrak{p}}\mathsf{T}$ from Proposition 2.5 one gets an exact triangle $\varGamma X \to X \to X' \to$ with $\varGamma X \in \mathsf{S}$ and $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$. Then (1) implies $X' = 0$ and hence $X \in \mathsf{S}$.

(2) ⇒ (3): Let S be a localising subcategory of $\varGamma_{\mathfrak{p}}\mathsf{T}$. Using (2) and the fact that $\varGamma_{\mathfrak{p}}\mathsf{T}$ is a tensor ideal of T, one has $\operatorname{Loc}^{\otimes}_{\mathsf{T}}(\mathsf{S}) \subseteq \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. Then it follows, again from Proposition 2.5, that S is a tensor ideal of T.

(3) ⇒ (1): Assume $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1}, X) = 0$; then $\operatorname{Hom}^*_{\mathsf{T}}(\mathbb{1}, \varGamma_{\mathfrak{p}}X) = 0$, as $\mathbb{1}$ is compact. Condition (3) implies that $\operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}\mathbb{1}) = \varGamma_{\mathfrak{p}}\mathsf{T}$. Thus $\varGamma_{\mathfrak{p}}X$ belongs to $\operatorname{Loc}_{\mathsf{T}}(\varGamma_{\mathfrak{p}}\mathbb{1})$ and therefore also to $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. So one obtains $\operatorname{Hom}^*_{\mathsf{T}}(\varGamma_{\mathfrak{p}}X, \varGamma_{\mathfrak{p}}X) = 0$, which implies $\varGamma_{\mathfrak{p}}X = 0$. □

Consider as an example for T the stable module category $\mathsf{StMod}\, kG$ of a finite group G with the canonical action of $R = H^*(G, k)$. We refer to [1, 2] for the discussion of two variations of the nucleus, namely the *group theoretic* and the *representation theoretic* nucleus. There it is shown that $\operatorname{Loc}_{\mathsf{T}}(\mathbb{1}) = \mathsf{T}$ if and only if the centraliser of every element of order p in G is p-nilpotent and every block is either principal or semisimple, where p denotes the characteristic of the field k.

It is convenient to define for any class C of objects of T

$$\mathsf{C}^{\perp} = \{Y \in \mathsf{T} \mid \operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \text{ for all } X \in \mathsf{C}\},$$

$${}^{\perp}\mathsf{C} = \{X \in \mathsf{T} \mid \operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \text{ for all } Y \in \mathsf{C}\}.$$

Now let $\mathsf{S} = \operatorname{Loc}_{\mathsf{T}}(\mathbb{1})$. The *representation theoretic nucleus* is by definition

$$\bigcup_{X \in \mathsf{S}^{\perp} \cap \mathsf{T}^c} \operatorname{supp}_R X.$$

Clearly, this is contained in the cohomological nucleus. It is a remarkable fact that the representation theoretic nucleus is non-empty if $\mathsf{S}^\perp \neq 0$; this is proved in [1, 2]. Moreover, Question 13 of [7] asks whether $\mathsf{S} = {}^\perp(\mathsf{S}^\perp \cap \mathsf{T}^c)$. Note that $\mathsf{S} = {}^\perp(\mathsf{S}^\perp)$ follows from general principles.

Acknowledgments. It is a pleasure to thank Greg Stevenson for helpful comments on this work.

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