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EXISTENCE OF WEAK SOLUTIONS TO AN EVOLUTIONARY MODEL FOR MAGNETOELASTICITY*

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Abstract. We prove existence of weak solutions to an evolutionary model derived for magnetoelastic materials. The model is phrased in Eulerian coordinates and consists in particular of (i) a Navier–Stokes equation that involves magnetic and elastic terms in the stress tensor, of (ii) a regularized transport equation for the deformation gradient, and of (iii) the Landau–Lifshitz–Gilbert equation for the dynamics of the magnetization. The proof is built on a Galerkin method and a fixed-point argument. It is based on ideas from Lin and the third author for systems modeling the flow of liquid crystals as well as on methods by Carbou and Fabrie for solutions of the Landau–Lifshitz equation.

Key words. magnetoelasticity, Eulerian coordinates, existence of weak solutions, Landau–Lifshitz–Gilbert equation

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1. Introduction.

Magnetoelastic (or magnetostrictive) materials respond mechanically to applied magnetic fields (magnetostriction) and/or react with a change of magnetization to mechanical stresses (magnetoelastic effect). Because of their remarkable response to external stimuli, they are attractive not only from the point of view of mathematical modeling but also for applications. Magnetoelastic materials are, among others, used in sensors to measure force or torque (cf., e.g., [6, 7, 30]) as well as magnetic actuators (cf., e.g., [48]) or generators for ultrasonic sound (cf., e.g., [9]).

Modeling of magnetoelastic materials goes back to Brown [8] as well as Tiersten [51, 52]. Later, many works appeared studying magnetoelasticity particularly in the static case relying on energy minimization; see, e.g., [17, 18, 33]. Let us point out that the magnetoelastic models investigated there can be seen as generalizations of models for micromagnetics that are also studied for their own right; cf., e.g., the reviews [19, 26, 35]. Based on the analysis in the static case, rate-independent evolution models were studied in [34] using the concept of energetic solutions; cf. [43]. However, in micromagnetics the dynamics is usually governed by the Landau–Lifshitz–Gilbert (LLG) equation [27, 28, 36], which has been extensively studied analytically; see, e.g., [1, 11, 41, 42]. Nevertheless, if the LLG equation is coupled with elasticity, the available works confine themselves to the small strain setting; cf., e.g., [10, 14].

The prominent difficulty in analyzing magnetoelastic models lies in the fact that elasticity is commonly formulated in the reference configuration, while micromag-

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netics is modeled in the current or deformed configuration. To overcome this issue, one might transform the magnetic part back into the reference configuration as in [17, 18, 34]. However, this is only possible if one can assure, by suitable modelling assumptions, that the deformation is invertible. In the static case, this can be enforced by suitable coercivity of the elastic energy; in particular, the energy has to blow up as the determinant of the deformation gradient tends to 0. The dynamic case is more involved, though, because the balance law for the deformation also features the inertia term. We refer to [45], where a magneto-elastic model is formulated in the fully Lagrangian setting; however, mathematical analysis of such a model could only be performed under several simplifying assumptions (cf. [45] for details). Another possibility is to work in the small-strain setting in which the difference between the actual and the reference configuration is neglected; cf., e.g., [10, 14].

In this article, we shall take a different approach and formulate the fully nonlinear problem of magnetoelasticity completely in Eulerian coordinates in the current configuration. In the Eulerian setting, the main state variable is the velocity and not the deformation. This poses an obstacle from the point of view of elasticity since then the deformation gradient is not readily available. To overcome this difficulty, we follow the approach of Liu and Walkington [40], where this issue has been resolved by finding a differential equation—a transport equation for the deformation gradient—that allows one to obtain the deformation gradient (in the current configuration) from the velocity gradient. Therefore, we will not need to care about the invertibility of the deformation. Moreover, the model is perfectly fitted to be used in modeling of so-called magnetorheological fluids; cf., e.g., [53]. Those are so-called smart fluids containing magnetoelastic particles in a carrier fluid. Indeed, it seems feasible that the system of partial differential equations under consideration (1)–(4) can be extended to fluid models via a phase field approach (cf. also [40]).

As for the magnetic part, we model the evolution of magnetization by the Landau–Lifshitz–Gilbert equation with, however, the time derivative replaced by the convective one. This is in order to take into account that changes of the magnetization also occur due to transport by the underlying viscoelastic material. We refer to section 2 for a detailed description of the model; see also [24]. In this work, we prove the existence of weak solutions in the case where we regularize the evolution equation for the deformation gradient. Our proof is based on a Galerkin method discretizing the velocity in the balance of momentum and a fixed point argument. It borrows ideas from the work of Lin and Liu [38], beyond which our system is further coupled to the evolution of the deformation gradient and the LLG equation. For the treatment of the LLG equation we further utilize methods from Carbou and Fabrie [11] in order to pass to the limit in the Galerkin approximation; see also [4] for a sketch of the proof and an announcement of this work, and cf. [24].

We point out that the system we are considering in this article indeed has the Navier–Stokes equations as a subsystem. The Galerkin method utilized here is also applied in the Navier–Stokes context; see, e.g., [20, 50]. For a broad insight into the Navier–Stokes equations, we further refer to, e.g., [16], and to [25] for a steady-state analysis; see also, e.g., [37, 13] for recent contributions in the context of the Navier–Stokes equations coupled to Maxwell's equations for charged systems, which yields the magnetohydrodynamic equations. Further, we would like to mention that also viscoelastic flows, even without magnetism as considered here, are of ongoing interest; cf., e.g, [2, 21, 31, 32].

The paper is structured as follows: we start with a presentation of the considered model for magnetoelastic materials in section 2. There, we list the model equations and give a brief derivation. In section 3, we state the main result of this article, viz., the existence of weak solutions to the evolutionary model for magnetoelasticity in Theorem 3.2. The proof of this theorem is presented in section 4. In section 5, we prove two auxiliary lemmas used in the proof of Theorem 3.2.

2. Presentation of the model.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, represent the current configuration. Then we consider the following model for magnetoelastic solids:

$$\partial_t v + (v \cdot \nabla)v - \operatorname{div}\mathcal{T} = f \qquad \text{(balance of momentum)}, \tag{1}$$

$$\nabla \cdot v = 0 \qquad \text{(incompressibility)}, \tag{2}$$

$$\partial_t F + (v \cdot \nabla)F - \nabla v F = \kappa \Delta F \qquad \text{(evolution of deformation gradient)}, \tag{3}$$

$$\partial_t M + (v \cdot \nabla)M = -\gamma M \times H_{\text{eff}} - \lambda M \times M \times H_{\text{eff}} \qquad \text{(LLG equation)}, \tag{4}$$

closed by boundary conditions (14)–(16) and initial conditions (17)–(19) below. Here, (1) is the balance of momentum in Eulerian coordinates with $v : \Omega \times (0, T) \to \mathbb{R}^d$ being the velocity mapping, $\mathcal{T}$ the stress tensor, and f the applied body forces. Similarly, (4) is a variant of the LLG evolution equation for the magnetization $M : \Omega \times (0, T) \to \mathbb{R}^3$, in which we replaced the time-derivatives in the LLG equation by the convective one in order to take changes of the magnetization through transport into account—see also Remark 2.1 for a detailed discussion regrading the transport dynamics of the magnetization. In this equation, H_{eff} is the effective magnetic field (cf. (8) below), $\gamma > 0$ is the electron gyromagnetic ratio, and $\lambda > 0$ is a phenomenological damping parameter. Here and in the following, we impose the standard constraint

$$|M| = 1 \text{ almost everywhere in } \Omega \times (0, T), \tag{5}$$

which corresponds to the existence of a saturation magnetization. Equation (3) is an evolution equation for F which, in our modeling, is an approximation for the deformation gradient in Eulerian coordinates. Indeed, if $\kappa = 0$, (3) is obtained by taking a time derivative of the deformation gradient and rephrasing it in Eulerian coordinates; cf. [40, equation (5)]. In this case, (3) is an evolution equation for the deformation gradient, but taking $\kappa = 0$ would not allow us to pass to the limit in the stress tensor in the presented Galerkin approximation. Therefore, we include a regularization term (cf., e.g., [39, p. 1461]) with κ presumably small.

The stress-tensor $\mathcal{T}$ as well as the effective field H_{eff} are constitutive quantities. In this work, we assume the decomposition

$$\mathcal{T} = -p\mathbb{I} + \nu\left(\nabla v + (\nabla v)^\top\right) + \mathcal{T}_{\text{rev}},$$

where $-p\mathbb{I}$ represents the pressure (which, however, shall not appear in our work since we will consider weak solutions only) and $\nu(\nabla v + (\nabla v)^\top)$ is the viscous stress corresponding to the frame indifferent dissipation functional $\int_\Omega \nu |\frac{1}{2}\left(\nabla v + (\nabla v)^\top\right)|^2 dx$. Due to the assumed incompressibility and the zero boundary condition (15), this yields the term $\nu \Delta v$ in the balance of momentum (11). Finally, $\mathcal{T}_{\text{rev}}$ is the magnetoelastic part of the stress tensor that, similarly as the effective magnetic field H_{eff}, will be deduced from the Helmholtz free energy.

For the Helmholtz free energy in magnetoelasticity we have the following general form

$$\psi(F,M) = \underbrace{A\int_\Omega |\nabla M|^2\,dx}_{\text{exchange energy}} + \underbrace{\int_\Omega \phi(F,M)\,dx}_{\text{anisotropy energy}} + \underbrace{\int_\Omega \psi^{(d)}_{\text{stray}}\,dx}_{\text{stray field energy}} \tag{6}$$
$$+ \underbrace{\int_\Omega W(F)\,dx}_{\text{elastic energy}} \underbrace{-\mu_0\int_\Omega M\cdot H_{\text{ext}}dx}_{\text{Zeeman energy}},$$

where $\psi^{(d)}_{\text{stray}}$ equals $\frac{\mu_0}{2}(M\cdot e_3)^2$ with e_3 being a unit vector orthogonal to Ω if $d=2$, and $-\frac{\mu_0}{2}M\cdot H$ if $d=3$. The stray field $H:\mathbb{R}^3\to\mathbb{R}^3$ is obtained from (possibly a reduced set) of the Maxwell equations; cf., e.g., [18, 17] or [35] for the micromagnetic part and [29] for the case $d=2$. Moreover, H_{ext} denotes the external magnetic field. Notice that the whole energy, including its elastic part, is formulated in the current configuration. From the Helmholtz free energy we obtain the effective field H_{eff} by taking the negative variational derivative of ψ with respect to M. In order to obtain $\mathcal{T}_{\text{rev}}$ we use that the elastic stress is a variational derivative of the Helmholtz free energy with respect to the deformation gradient F; similarly, in order to determine the applied force f we need to take a variational derivative with respect to the deformation. However, care is needed during this procedure since the free energy has to be transferred back to the reference configuration and then the derivative with respect to the deformation gradient as well as the deformation is taken in order to obtain the Piola–Kirchhoff stress tensor and the Langrangian volume force, respectively. This stress tensor and this force are subsequently again transformed into the current configuration to obtain the Cauchy stress tensor and the Eulerian volume force. We present the derivation only for the simplified case presented below considered in this article and refer to [24] for a detailed derivation of the simplified as well as the general model, which is based on taking variations of the action functional while carefully taking into account changes between the Eulerian and Lagrangian coordinates.

Here, we study a simplified situation of isotropic magnetic particles (which allows us to set the anisotropy energy to zero), with the stray field energy neglected; cf. Remark 3.9. Thus, we are left with

$$\psi(F,M) = A\int_\Omega |\nabla M|^2\,dx + \int_\Omega W(F)\,dx - \mu_0\int_\Omega M\cdot H_{\text{ext}}dx, \tag{7}$$

and so the effective magnetic field, which equals the negative variational derivative of ψ with respect to M, is given by

$$H_{\text{eff}} = 2A\Delta M + \mu_0 H_{\text{ext}}. \tag{8}$$

To obtain $\mathcal{T}_{\text{rev}}$, we need to transform ψ from (7) to the reference configuration $\widetilde{\Omega}$. To this end, we denote by $X\in\widetilde{\Omega}$ material points in the reference configuration (Lagrangian coordinates) and by $x\in\Omega$ spatial points in the current configuration (Eulerian coordinates). Further, we assume that the deformation from the reference to the current configuration is governed by the smooth bijective flow map

$$x:\widetilde{\Omega}\times[0,T]\to\Omega \quad \text{that is} \quad (X,t)\mapsto x(X,t).$$

With the flow map, the velocity is defined by $\widetilde{v}(X,t) = \frac{\partial}{\partial t}x(X,t)$, which, in Eulerian coordinates, is denoted by $v:\Omega\times[0,T]\to\mathbb{R}^d$ chosen such that $v(x(X,t),t) = \widetilde{v}(X,t)$.

Moreover, we define $\widetilde{M} : \widetilde{\Omega} \times [0,T] \to \mathbb{R}^3$ to be the magnetization in the reference configuration by setting

$$\widetilde{M}(X,t) = M(x(X,t),t), \tag{9}$$

and $\widetilde{F} : \widetilde{\Omega} \times [0,T] \to \mathbb{R}^{d\times d}$ to be the deformation gradient in the reference configuration satisfying

$$\widetilde{F}(X,t) = F(x(X,t),t).$$

Remark 2.1. Let us remark that by defining the magnetization in the reference configuration by (9), we actually assume that, by the movement of the flow, the magnetization vector in each particular material point can neither be stretched nor rotated but it can be transported to a different position. In other words, we treat the magnetization as a 3-scalar function with respect to the transformation from the reference to the deformed configuration. This approach is, for example, used in [17]. Let us note that taking the material time derivative in (9) translates to taking the convective derivative in the deformed configuration, i.e., the time derivative that we used in the LLG equation (4).

However, more complicated transformation rules for the magnetization vector could be assumed. For example, even in the incompressible case treated here, the modeling assumption could be that the magnetization may rotate or stretch by the flow of the underlying medium, which would lead to the idea that

$$\widetilde{M}(X,t) = F^{-1}(X,t)M(x(X,t),t);$$

in other words, the magnetization would transform as a one-form. Under such more general kinematic assumptions also the unit vector constraint (5) is relaxed; on the other hand, under the rule (9) it can be maintained. Finally, let us stress that the material derivative under the more general kinematic assumptions is more complicated than the convective one, so that, in this case, also the form of the LLG had to be changed for consistency.

Next, we obtain for the Helmholtz free energy in Lagrangian coordinates, denoted by $\widetilde{\psi}(x, \widetilde{F}, \widetilde{M})$,

$$\widetilde{\psi}(x, \widetilde{F}, \widetilde{M}) = \int_{\widetilde{\Omega}} A|\nabla_X \widetilde{M}(X,t)\widetilde{F}^{-1}(X,t)|^2 - \mu_0 \widetilde{M}(X,t)\cdot H_{\text{ext}}(x(X,t),t) \\ + W(\widetilde{F}(X,t)) \,\mathrm{d}X.$$

Notice that, due to incompressibility (2), the Jacobian of the transformation is one. Moreover, notice that through the external magnetic field, the Helmholtz free energy in the reference configuration also depends on the deformation $x(X,t)$ itself. Thus, the term $\widetilde{\mathcal{F}}(x) = -\int_{\widetilde{\Omega}} \mu_0 \widetilde{M}(X) \cdot H_{\text{ext}}(x(X,t),t) \,\mathrm{d}X$ can be understood as the potential of an applied volume force to the mechanical system (cf. forces with generalized potentials in, e.g., [15]), whence the volume force $\widetilde{f}$ is obtained as the negative variational derivative of $\widetilde{\mathcal{F}}$ with respect to x. Transforming back to the current configuration, we have that

$$f = \mu_0 \nabla H_{\text{ext}}^\top M.$$

Moreover, taking the variational derivative of $\widetilde{\psi}$ with respect to $\widetilde{F}$ and transforming back to the current configuration, we obtain for the elastic stress tensor

$$\mathcal{T}_{\text{rev}} = -2A\nabla M \odot \nabla M + W'(F)F^\top \qquad \text{with} \quad (\nabla M \odot \nabla M)_{ij} = \sum_k \nabla_i M_k \nabla_j M_k.$$

Altogether, we are left with the following system of partial differential equations,

$$\partial_t v + (v\cdot\nabla)v + \nabla p + \nabla\cdot(2A\nabla M \odot \nabla M - W'(F)F^\top) - \nu\Delta v = \mu_0(\nabla H_{\mathrm{ext}})^\top M, \tag{10}$$

$$\nabla\cdot v = 0, \tag{11}$$

$$\partial_t F + (v\cdot\nabla)F - \nabla v F = \kappa\Delta F, \tag{12}$$

$$\partial_t M + (v\cdot\nabla)M = -\gamma M \times (2A\Delta M + \mu_0 H_{\mathrm{ext}}) - \lambda M \times M \times (2A\Delta M + \mu_0 H_{\mathrm{ext}}) \tag{13}$$

in $\Omega \times (0,T)$, accompanied by the following boundary/initial conditions,

$$v = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{14}$$

$$F = F_{\min} \qquad \text{on } \partial\Omega \times (0,T), \tag{15}$$

$$\frac{\partial M}{\partial n} = (\nabla M)n = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{16}$$

$$v(x,0) = v_0(x), \quad \nabla\cdot v_0(x) = 0, \tag{17}$$

$$F(x,0) = F_0(x), \tag{18}$$

$$M(x,0) = M_0(x), \quad |M_0| \equiv 1, \tag{19}$$

where n denotes the outer normal to the boundary of Ω. Here, $F_{\min} \subset \mathbb{R}^{d\times d}$ is a matrix for which $W'(F_{\min}) = 0$. Moreover, let us note that the boundary condition (15) is needed only due to the regularizing term $\kappa\Delta F$ in (13), which brings in higher derivatives of F.

3. Main result. As the main result of this contribution, we prove existence of weak solutions to the system (10)–(13). We start by defining the notion of weak solutions we shall work with. Here and in the following we set $A = \frac{1}{2}$, $\mu_0 = 1$, and $\gamma = \lambda = 1$ since constants are irrelevant for this mathematical analysis.

Moreover, we shall restrict our scope to $\Omega \subset \mathbb{R}^2$; in this setting we may obtain a weak solution globally in time. If $\Omega \subset \mathbb{R}^3$, the presented proof remains valid up to small modifications but only to obtain short-time existence of solutions; cf. Remark 3.6 below.

Let us start the discussion by giving the notion of weak solution that we shall use in this work.

DEFINITION 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a C^∞-domain and let $T > 0$ be the final time of the evolution. Then, we call (v, F, M) enjoying the regularity*

$$\begin{aligned} v &\in L^\infty\left(0,T; L^2_{\mathrm{div}}\left(\Omega;\mathbb{R}^2\right)\right) \cap L^2\left(0,T; W^{1,2}_{0,\mathrm{div}}\left(\Omega;\mathbb{R}^2\right)\right), \\ F &\in L^\infty\left(0,T; L^2\left(\Omega;\mathbb{R}^{2\times 2}\right)\right) \cap L^2\left(0,T; W^{1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right), \\ M &\in L^\infty\left(0,T; W^{1,2}\left(\Omega;\mathbb{R}^3\right)\right) \cap L^2\left(0,T; W^{2,2}\left(\Omega;\mathbb{R}^3\right)\right) \end{aligned}$$

a weak solution of the system (10)–(13) *accompanied with initial/boundary conditions* (14)–(19) *if it satisfies* (15)–(16) *in the sense of traces as well as the initial conditions* (17)–(19) *in the sense*

$$v(\cdot,t) \xrightarrow{L^2(\Omega)} v_0(\cdot), \qquad F(\cdot,t) \xrightarrow{L^2(\Omega)} F_0(\cdot), \qquad M(\cdot,t) \xrightarrow{W^{1,2}(\Omega)} M_0(\cdot) \qquad \text{as } t \to 0^+,$$

and if it fulfills the system

$$\begin{aligned}\int_0^T\int_\Omega &- v\cdot\partial_t\phi + (v\cdot\nabla)v\cdot\phi - \big(\nabla M\odot\nabla M - W'(F)F^\top - \nu\nabla v\big)\cdot\nabla\phi\\ &- (\nabla H_{\text{ext}}^\top M)\cdot\phi\;dx\;dt = \int_\Omega v_0(x)\phi(x,0)dx,\end{aligned}\tag{20}$$

$$\int_0^T\int_\Omega -F\cdot\partial_t\xi + (v\cdot\nabla)F\cdot\xi - (\nabla vF)\cdot\xi + \kappa\nabla F\cdot\nabla\xi\;dx\;dt = \int_\Omega F_0(x)\cdot\xi(x,0)\;dx,\tag{21}$$

$$\begin{aligned}&\int_0^T\int_\Omega -M\cdot\partial_t\zeta + (v\cdot\nabla)M\cdot\zeta + (M\times(\Delta M + H_{\text{ext}}))\cdot\zeta - |\nabla M|^2M\cdot\zeta\\ &\quad - \Delta M\cdot\zeta\;dx\;dt = \int_0^T\int_\Omega\big(-M\cdot H_{\text{ext}}M + H_{\text{ext}}\big)\zeta dxdt + \int_\Omega M_0(x)\cdot\zeta(x,0)\;dx\end{aligned}\tag{22}$$

for all $\phi(x,t)=\phi_1(t)\phi_2(x)$ *with* $\phi_1\in W^{1,\infty}(0,T)$ *satisfying* $\phi_1(T)=0$ *and* $\phi_2\in W^{1,2}_{0,\text{div}}(\Omega;\mathbb{R}^2)$*, all* $\xi(x,t)=\xi_1(t)\xi_2(x)$ *with* $\xi_1\in W^{1,\infty}(0,T)$ *satisfying* $\xi_1(T)=0$ *and* $\xi_2\in W^{1,2}_0(\Omega;\mathbb{R}^{2\times 2})$*, and all* $\zeta(x,t)=\zeta_1(t)\zeta_2(x)$ *with* $\zeta_1\in W^{1,\infty}(0,T)$ *satisfying* $\zeta_1(T)=0$ *and* $\zeta_2\in L^2(\Omega;\mathbb{R}^3)$.

Above and throughout the paper, Bochner spaces are denoted by $L^p(\mathrm{O};V)$ and $W^{k,p}(\mathrm{O};V)$ for functions mapping $\mathrm{O}\subset\mathbb{R}^m$ to a Banach space V whose norm in V belongs to the appropriate Lebesgue or Sobolev space. In the special case in which V is $\mathbb{R}^n$, we denote by $L^p_{\text{div}}(\mathrm{O};\mathbb{R}^n)$, $W^{1,p}_{0,\text{div}}(\mathrm{O};\mathbb{R}^n)$ those subsets of the appropriate Lebesgue or Sobolev space on which the distributional divergence vanishes; in the Sobolev space, also the boundary values (in the sense of traces) are 0. We will use the notation $W^{-1,2}(\mathrm{O};\mathbb{R}^n)$ for the dual space of $W^{1,2}_0(\mathrm{O};\mathbb{R}^n)$; moreover, we shall denote the duality pairing between $W^{-1,2}(\mathrm{O};\mathbb{R}^n)$ and $W^{1,2}_0(\mathrm{O};\mathbb{R}^n)$ by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$.

In the weak formulation of (10) and (12) we used integration by parts to transfer the highest derivatives in the Laplacian to the test function, which is standard. Moreover, we used that, as long as $|M|=1$, (13) is equivalent to (see, e.g., [5, 11])

$$\partial_t M + (v\cdot\nabla)M = -M\times(\Delta M + H_{\text{ext}}) + |\nabla M|^2M + \Delta M - M(M\cdot H_{\text{ext}}) + H_{\text{ext}}.\tag{23}$$

Before formulating our main result, let us summarize the assumptions on the data in the model that we shall need: Let us start with the elastic energy W, which must be independent of the observer; that is, it has to satisfy $W(\mathcal{R}A)=W(A)$ for all $\mathcal{R}\in SO(2)$ (and thus $W'(\mathcal{R}A)=\mathcal{R}W'(A)$; see also [40]) for all $A\in\mathbb{R}^{2\times 2}$. We assume that $W\in C^2(\mathbb{R}^{2\times 2})$ is of 2-growth, i.e., there exists a constant $C_1>0$ such that

$$C_1|A|^2\le W(A)\le C_1\left(|A|^2+1\right)\qquad\forall A\in\mathbb{R}^{2\times 2}.\tag{24}$$

Further, notice that due to the differentiability of W this implies that $W'(\cdot)$ is of 1-growth, that is,

$$|W'(A)|\le C_2(|A|+1)\qquad\forall A\in\mathbb{R}^{2\times 2}\tag{25}$$

and likewise $W''(\cdot)$ is bounded, i.e.,

$$|W''(A)|\le C_3\qquad\forall A\in\mathbb{R}^{2\times 2}.\tag{26}$$

Finally, we assume that W is strictly convex, that is,

$$\exists a > 0 \qquad (W''(\Xi)A) \cdot A \geq a|A|^2 \qquad \forall \Xi, A \in \mathbb{R}^{2\times 2}, \tag{27}$$

and let the matrix $F_{\min} \in \mathbb{R}^{2\times 2}$ be such that $W'(F_{\min}) = 0$.

Our main result is the existence of weak solutions to (10)–(13) in the sense of Definition 3.1.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a C^∞-domain and let $T > 0$ be the final time of the evolution. Let $W \in C^2(\mathbb{R}^{2\times 2}; \mathbb{R})$ satisfy* (24)–(27). *In addition, assume that*

$$H_{\text{ext}} \in C^0\left(0,T;L^2\left(\Omega;\mathbb{R}^3\right)\right) \cap L^2\left(0,T;L^\infty\left(\Omega;\mathbb{R}^3\right)\right) \cap L^3\left(0,T;W^{1,4}\left(\Omega;\mathbb{R}^3\right)\right), \tag{28}$$

$$\partial_t H_{\text{ext}} \in L^1\left(0,T;L^1\left(\Omega;\mathbb{R}^3\right)\right) \tag{29}$$

and $v_0 \in L^2_{\text{div}}(\Omega;\mathbb{R}^2)$, $F_0 \in L^2(\Omega;\mathbb{R}^{2\times 2})$ and $M_0 \in W^{2,2}(\Omega;\mathbb{R}^3)$. Moreover, let the initial data and the external field satisfy the smallness condition

$$\begin{aligned} \text{IED} := \int_\Omega \frac{1}{2}|v_0|^2 + \frac{1}{2}|\nabla M_0|^2 + W(F_0)\,dx + 2\|H_{\text{ext}}\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^3))} \\ + \|\partial_t H_{\text{ext}}\|_{L^1(0,T;L^1(\Omega;\mathbb{R}^3))} < \frac{1}{\widetilde{C}} \end{aligned} \tag{30}$$

for a suitably large constant $\widetilde{C} > 0$ depending just on Ω. Then there exists a weak solution of the system (10)–(13) *accompanied with initial/boundary conditions* (14)–(19) *in the sense of Definition* 3.1.

Let us note that we can specify the $\widetilde{C}$ in (30) explicitly. Namely, it corresponds to the constant in the well-known Ladyzhenskaya inequality

$$\|f\|^4_{L^4(\Omega;\mathbb{R}^{\tilde{d}})} \leq \tilde{C}\left(\|f\|^4_{L^2(\Omega;\mathbb{R}^{\tilde{d}})} + \|\nabla f\|^2_{L^2(\Omega;\mathbb{R}^{\tilde{d}\times 2})}\|f\|^2_{L^2(\Omega;\mathbb{R}^{\tilde{d}})}\right),$$

which holds for all functions in $L^4(\Omega;\mathbb{R}^{\tilde{d}})$ since $\Omega \subset \mathbb{R}^2$; this inequality is applied in (57) below.

It is interesting to note that the smallness constant in (30) would also depend on λ, appearing in (13), if we did not set λ to be one. Nevertheless, $\widetilde{C}$ depends neither on ν nor on κ, appearing in (10) and (12), respectively. This shows that the smallness of the initial data is required because of the complicated rheology and not of the Navier–Stokes equations.

We prove Theorem 3.2 in section 4 below. The proof is based on a Galerkin approximation of the system (10)–(13). As is standard in the context of the Navier–Stokes equation, we approximate the velocity in terms of basis functions of the Stokes operator. We leave (12) as well as the LLG equation (13) undiscretized but insert the discretized velocity into these equations. A similar approach has already been used in [38], [49] but here the partial discretization of the system is crucial also in order to keep the constraint $|M| = 1$ satisfied in the Galerkin scheme.

We deduce energy estimates that are, in turn, used for the convergence of the Galerkin scheme. However, the energetic a priori estimates do not yield enough regularity of M for proving convergence of the solutions to a solution of the system because we get ∇M bounded only in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{3\times 2}))$. Thus, we need to adapt parts of the regularity analysis for the LLG equation (cf., e.g., [11, 41, 42]) to the case of our system. Our argument here is based on the technique from [11].

A further peculiarity is brought into the proof by the fact that an adaptation of the technique of [11] to our case is fully possible only on the level of the Galerkin approximation since then v is smooth. Nevertheless, we obtain a bound on ΔM in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ that is uniform in the Galerkin index. This is all that we need to make the limiting process of the nonlinear terms in the stress tensor involving the magnetization work.

Before embarking onto the proof of Theorem 3.2, let us consider some remarks about the assumptions of this theorem as well as possible extensions.

Remark 3.3 (Heisenberg constraint). Let us note that the Heisenberg constraint (5) is automatically included in the LLG equation, i.e., any weak solution in the sense of Definition 3.1 will fulfill this constraint. This is the most standard setting in micromagnetics, even though there exist also variants of the LLG that do not enforce this constraint and allow to relax it; see, e.g., [44]. From the mathematical point of view, such variants could allow for stronger results than the one presented in Theorem 3.2. Indeed, enforcing the constraint $|M| = 1$ relates the LLG to the harmonic-map heat-flow for which finite time blow-up under large initial data has been proved. (See also Remark 3.8 below.)

Remark 3.4 (weak formulation of the LLG equation). Let us note that our weak formulation of the LLG equation (22) is actually stronger than the standardly used weak formulation as proposed in [1]. Notice that we keep the highest derivatives (i.e., the Laplacian) in (22) and, in fact, since no partial integration in space has been used, we can deduce from (22) that the LLG equation actually holds a.e. in Ω. We can afford to require this stronger formulation since we anyway need to prove a bound on ΔM in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ in order to be able to pass to the limit in the Galerkin approximation in the stress tensor.

Remark 3.5 (convexity of W). The convexity assumption (27) makes sure that the energy is lower semicontinuous, which we will need in order to pass to the limit in the energy inequality. Nevertheless, this assumption is not optimal from the physical point of view since elastic energies in the large strain setting are not convex. However, if the material is incompressible, that is, we assure that $\det F = 1$, even a convex energy is acceptable since interpenetration of matter is not possible under this assumption. Yet, even if we assume incompressibility here, we cannot make sure that $\det F = 1$ due to the regularization in (3). In order to relax the convexity assumption, it would be necessary to change the used mathematical methods; in particular the assumption (27) enters in Step 2 of the proof of Theorem 3.2.

Remark 3.6 ($\Omega \subset \mathbb{R}^2$). The fact that $\Omega \subset \mathbb{R}^2$ enters at several places in the proof of Theorem 3.2 but most crucially in Step 2, where higher order a priori estimates for the magnetization are derived and the Ladyzhenskaya inequality is used. Nevertheless, the proof could be easily adapted by using techniques from [11], to hold also for $\Omega \subset \mathbb{R}^3$ but with a sufficiently short final time of the evolution.

Remark 3.7 (positive κ). In the system studied in this contribution, it is essential that we keep $\kappa > 0$. In fact, the possibility of sending $\kappa \to 0$ in related system is discussed in [39], where it is shown that while the limit passage in the weak formulation of (12) seems feasible a limit passage in the stress tensor in (10) leads to a defect measure (cf. also [22] in the context of the Navier–Stokes equation) that can be proved to vanish if the velocity is Lipschitz.

Remark 3.8 (smallness of the initial data). The smallness condition (30) on the initial data is quite limiting but a condition of this type seems to be necessary in order to prove existence of weak solutions to (10)–(13). In fact, in order to pass to the limit in the stress tensor in the balance of momentum, we need sufficient integrability of ∇M from which we employ the higher regularity of M. However, if the initial data are not small, higher regularity cannot be expected. Indeed, blowup in finite time for the LLG equation from smooth but not small initial data has been numerically reported in [3]. An analytical proof of this phenomenon seems to be missing for the LLG equation in two dimensions but has been given in the related harmonic-map heat-flow equation in [12].

Remark 3.9 (stray-field energy). In the analysis of this work we neglect the stray-field energy. However, recall that in $d = 2$ the stray-field energy reads $\frac{\mu_0}{2}(M \cdot e_3)^2$, which is a lower order term compared to the exchange energy. Therefore, it seems feasible to include the stray-field energy in the presented analysis in $d = 2$.

Remark 3.10 (gradient flow). We remark that a system corresponding to (10)–(13) but equipped with a gradient flow for the magnetization M instead of the LLG equation is studied in [24]. Moreover, in [47] weak-strong uniqueness is shown under the assumption that strong solutions exist. A proof of higher regularity of weak solutions is an open topic in the case of the gradient flow dynamics as well as in the LLG setting. The system with gradient flow has the advantage of being closer to the system studied in [38] in terms of the magnetization. The gradient flow type dynamics are less involved than the LLG equation, which makes the treatment of the equation for the magnetization M a lot easier. However, in the context of micromagnetics, the LLG equation is the established description of the dynamics of the magnetization. For the gradient flow case, existence of weak solutions is proved by a Galerkin approximation and a fixed-point argument similar to the proofs of this paper; however, for that proof there is not as much regularity needed for the magnetization as in the LLG case.

4. Proof of Theorem 3.2.

Let us now give a detailed proof of Theorem 3.2. Everywhere in the proof, we use C as a generic constant that may change from expression to expression. It may only depend on the problem parameters that are fixed throughout the proof such as Ω, but dependence on other data, in particular on the initial conditions or the Galerkin index, is specified explicitly. Moreover, note that we do not always display the dependence of v on x and t; instead of $v(x,t)$ we may also write $v(t)$ if we want to stress the dependence on time, or just v, and correspondingly for F and M.

Proof of Theorem 3.2. We start by constructing suitable approximate solutions:

Step 1: *Discrete formulation and existence of discrete solutions.* Let us construct Galerkin approximations of the velocity via eigenfunctions of the Stokes operator, i.e., let $\{\xi_i\}_{i=1}^{\infty} \subset C^\infty(\bar{\Omega};\mathbb{R}^2)$ be an orthogonal basis of $W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)$ and an orthonormal basis of $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^2)$ satisfying

$$\Delta\xi_i + \nabla p_i = -\lambda_i \xi_i \tag{31}$$

in Ω and vanishing on the boundary. Here, $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m \le \cdots$ with $\lambda_m \xrightarrow{m\to\infty} \infty$. Notice that Ω is a C^∞-domain so the assumed regularity of the eigenfunctions can indeed be guaranteed. Further, let us denote

$$P_m : W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2) \to \mathbf{H}_m := \mathrm{span}\{\xi_1,\xi_2,\ldots,\xi_m\}.$$

We start by defining the notion of a weak solution to the approximate problem.

Definition 4.1. *We call* (v_m, F_m, M_m) *a weak discrete solution of the system* (10)–(13) *on some time interval* $(0,t) \subset (0,T)$ *provided that the pair* (F_m, M_m) *enjoys the regularity*

(32)
$$F_m \in W^{1,2}\left(0,t;W^{-1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right) \cap L^\infty\left(0,t;L^2(\Omega;\mathbb{R}^{2\times 2})\right) \cap L^2\left(0,t;W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2})\right),$$

(33)
$$M_m \in W^{1,\infty}\left(0,t;L^2\left(\Omega;\mathbb{R}^3\right)\right) \cap L^\infty\left(0,t;W^{2,2}\left(\Omega;\mathbb{R}^3\right)\right) \cap L^2\left(0,t;W^{3,2}\left(\Omega;\mathbb{R}^3\right)\right)$$

and solves

(34)
$$\langle\!\langle \partial_t F_m, \Xi \rangle\!\rangle + \int_\Omega (v_m \cdot \nabla) F_m \cdot \Xi - (\nabla v_m F_m) \cdot \Xi + \kappa \nabla F_m \cdot \nabla \Xi \; dx = 0 \qquad \text{in } (0,t),$$

(35)
$$\begin{aligned}\partial_t M_m + (v_m \cdot \nabla) M_m = {} & |\nabla M_m|^2 M_m + \Delta M_m - M_m \times (\Delta M_m + H_{\text{ext}}) \\ & - M_m (M_m \cdot H_{\text{ext}}) + H_{\text{ext}} \qquad \text{in } \Omega \times (0,t)\end{aligned}$$

for all $\Xi \in W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2})$, *together with the initial conditions* (18)–(19) *and boundary conditions* (15)–(16).

Moreover, $v_m(x,s) = \sum_{i=1}^m g_m^i(s)\xi_i(x)$ *with* $g_m^i : (0,t) \to \mathbb{R}$ *being the Lipschitz continuous solution of*

(36)
$$\frac{d}{dt} g_m^i(s) = -\nu \lambda_i g_m^i(s) + \sum_{j,k=1}^m g_m^j(s) g_m^k(s) A_{jk}^i + D_m^i(s, F_m, M_m), \quad i = 1, \dots, m,$$

with the initial condition $g_m^i(0) = \int_\Omega v_0 \cdot \xi_i \; dx$ *and*

(37)
$$\begin{aligned} A_{jk}^i &:= -\int_\Omega (\xi_j \cdot \nabla)\xi_k \cdot \xi_i \; dx, \\ D_m^i(s, F_m, M_m) &:= \int_\Omega \left(\nabla M_m(s) \odot \nabla M_m(s) - W'(F_m(s)) F_m(s)^\top\right) \cdot \nabla \xi_i \\ &\qquad + \left(\nabla H_{\text{ext}}^\top(s) M_m(s)\right) \cdot \xi_i dx \end{aligned}$$

for any $s \in (0,t)$, $i,j,k = 1,\dots,m$ *and any* (F,M) *in the function spaces mentioned in* (32) *and* (33).

For further convenience, let us denote

$$\text{IN} := \left(\|W(F_0)\|_{L^1(\Omega)}, \|M_0\|_{W^{2,2}(\Omega;\mathbb{R}^3)}\right).$$

We prove existence of discrete solutions to (10)–(13) in the sense of Definition 4.1 by a fixed point argument. To this end, we define for all $0 < t_0 \le T$ and for $L := \|v_0\|_{L^2(\Omega;\mathbb{R}^2)} + 1$ the set

$$V_m(t_0) = \left\{ v(x,t) = \sum_{i=1}^m g_m^i(t)\xi_i(x) \text{ in } \Omega \times [0,t_0) : \sup_{t \in [0,t_0)} \left(\sum_{i=1}^m \left|g_m^i(t)\right|^2 \right)^{\frac{1}{2}} \le L, \right.$$
$$\left. g_m^i(0) = \int_\Omega v_0(x) \cdot \xi_i(x) \; dx \right\}.$$

Notice that $V_m(t_0)$ is a closed and convex subset of $C([0,t_0];\mathbf{H}_m)$, which itself is a subset of $C([0,t_0];L^2(\Omega;\mathbb{R}^2))$. With some $v \in V_m(t_0)$ fixed we may find weak solutions to (12)–(13) by means of the following lemma.

LEMMA 4.2. *For $v \in V_m(t_0)$ fixed and H_{ext} satisfying* (28) *there is $0 < t_1 \le t_0$ that only depends on L, m, IN, and the external field H_{ext}, such that we can find unique (F,M) with*

$$F \in W^{1,2}\left(0,t_1;W^{-1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right) \cap L^2\left(0,t_1;W_0^{1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right), \tag{38}$$

$$M \in W^{1,\infty}\left(0,t_1;L^2\left(\Omega;\mathbb{R}^3\right)\right) \cap L^2\left(0,t_1;W^{3,2}\left(\Omega;\mathbb{R}^3\right)\right) \tag{39}$$

satisfying

$$\langle\!\langle \partial_t F, \Xi \rangle\!\rangle + \int_\Omega (v\cdot\nabla)F\cdot\Xi - (\nabla v F)\cdot \Xi + \kappa \nabla F \cdot \nabla \Xi \, dx = 0 \qquad \text{a.e. in } (0,t_1), \tag{40}$$

$$\partial_t M + (v\cdot\nabla)M = |\nabla M|^2 M + \Delta M - M\times(\Delta M + H_{\text{ext}}) - M(M\cdot H_{\text{ext}}) + H_{\text{ext}} \quad \text{a.e. in } \Omega\times(0,t_1) \tag{41}$$

for all $\Xi \in W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2})$, together with the initial conditions (18)–(19) *and boundary conditions* (15)–(16). *Moreover, the pair (F,M) satisfies the following bounds:*

$$\|F\|_{L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^{2\times 2}))} \le C(L,m,\text{IN}), \quad \|M\|_{L^\infty(0,t_1;W^{2,2}(\Omega;\mathbb{R}^3))} \le C(L,m,\text{IN},H_{\text{ext}}). \tag{42}$$

In addition, we have that $|M| = 1$ a.e. in $\Omega\times(0,t)$ and the estimate

$$\begin{aligned} &\|\Delta M(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad \le \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ &\qquad + C(L,m,H_{\text{ext}}) \int_0^t \left(1 + \|\nabla M\|^6_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} \|\Delta M\|^4_{L^2(\Omega;\mathbb{R}^3)}\right) ds \end{aligned} \tag{43}$$

for any t for which (41) *is satisfied.*

The proof of Lemma 4.2 is based on a Galerkin approximation within which the estimates (42) and (43) can be obtained by following the reasoning of [11]. We postpone it, for the sake of clarity, to section 5 and rather continue with the proof of Theorem 3.2 at this point.

By Lemma 4.2, we have now found, for some fixed $v \in V_m(t_0)$, functions (F,M) that solve (40)–(41) and are such that

$$D_m^i(t;F,M) \in L^\infty(0,t_1)$$

with the L^∞-norm of $D_m^i(t;F,M)$ depending only on L and m, the initial data through IN and the external magnetic field.

Thus, we can apply Carathéodory's existence theorem to obtain existence of unique Lipschitz continuous solutions $\tilde{g}_m^i(t)$ of

$$\frac{d}{dt}\tilde{g}_m^i(t) = -\nu\lambda_i \tilde{g}_m^i(t) + \sum_{j,k=1}^m \tilde{g}_m^j(t)\tilde{g}_m^k(t) A_{jk}^i + D_m^i(t;F,M), \qquad i = 1,\dots,m, \tag{44}$$

with the initial condition $\tilde{g}_m^i(0) = \int_\Omega v_0 \cdot \xi_i \, dx = g_m^i(0)$, at least on a time interval

$(0, t_2)$ with $t_2 \le t_1$. Notice that, for $t \in [0, t_1]$ and for $|\tilde{g}_m - g_m(0)| \le b$, $b > 0$, where $\tilde{g}_m = (\tilde{g}_m^1, \dots, \tilde{g}_m^m)$, we can bound the right-hand side of (44) by the constant

$$R = -\nu\lambda_i(2b + |g_m(0)|) + (2b + |g_m(0)|)^2 \sum_{j,k=1}^{m} \left|A_{jk}^i\right| + \|D^i(t; F, M)\|_{L^\infty(0,t_1)}.$$

Thus, it follows from [23, Chapter 1, Theorem 1] that t_2 has to be chosen in such a way that $Rt_2 \le b$; in other words t_2 depends just on the L^∞-norm of $D_m^i(t; F, M)$ (that in turn only depends on L and m, the initial data through IN and the external magnetic field).

Choosing $0 < t^* \le t_2$ small enough (but, as we shall see, only dependent on L and m, the initial data through IN and the external magnetic field), we can assure that

$$\tilde{v}(x,t) = \sum_{i=1}^{m} \tilde{g}_m^i(t)\xi_i(x) \tag{45}$$

is in $V_m(t^*)$. To prove this, note that we can deduce from (44) and (37) that $\tilde{v}$ satisfies

$$\int_\Omega \partial_t\tilde{v}\cdot\zeta + (\tilde{v}\cdot\nabla)\tilde{v}\cdot\zeta - \big(\nabla M \odot \nabla M - W'(F)F^\top - \nu\nabla\tilde{v}\big)\cdot\nabla\zeta - (\nabla H_{\text{ext}}^\top M)\cdot\zeta \, dx = 0 \tag{46}$$

for all $\zeta \in \mathbf{H}_m$. Choosing $\zeta = \tilde{v}$ in (46) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^2)}
&= -\underbrace{\int_\Omega (\tilde{v}\cdot\nabla)\tilde{v}\cdot\tilde{v}\, dx}_{=0} + \nu\|\nabla\tilde{v}\|^2_{L^2(\Omega;\mathbb{R}^2)} + \int_\Omega (\nabla M \odot \nabla M - W'(F)F^\top)\cdot\nabla\tilde{v}\, dx \\
&\quad + \int_\Omega (\nabla H_{\text{ext}}^\top M)\cdot\tilde{v}\, dx \\
&\le C(m, H_{\text{ext}})\|\tilde{v}(t)\|_{L^2(\Omega;\mathbb{R}^2)}\left(\left\|\int_\Omega |\nabla M \odot \nabla M - W'(F)F^\top|\, dx\right\|_{L^\infty(0,t_2)} + 1\right) \\
&\quad + \nu\|\tilde{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^2)}.
\end{aligned}$$

Further, we have that

$$\|\tilde{v}(t)\|_{L^2(\Omega;\mathbb{R}^2)} \le \sum_{i=1}^{m} |\tilde{g}_m^i(t)|\|\xi_i\|_{L^2(\Omega;\mathbb{R}^2)} = \sum_{i=1}^{m} |\tilde{g}_m^i(t)| \le C(m, L),$$

because everywhere on the interval $[0, t_2]$ we have the bound $|\tilde{g}_m - g_m(0))| \le b$ from Carathéodory's existence theorem. Thus, we obtain that $\frac{1}{2}\frac{d}{dt}\|\tilde{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^2)}$ is bounded by $C(m, L, \text{IN}, H_{\text{ext}})$. Hence

$$\|\tilde{v}(t)\|^2_{L^2(\Omega;\mathbb{R}^2)} \le \|\tilde{v}_0\|^2_{L^2(\Omega;\mathbb{R}^2)} + C(m, L, \text{IN}, H_{\text{ext}})t.$$

Now, we can define an operator

$$\mathcal{L} : V_m(t^*) \to V_m(t^*), \quad v \mapsto \tilde{v}, \tag{47}$$

with $\tilde{v}$ defined as in (45). Notice that the range of $\mathcal{L}$ is precompact in $C([0, t^*]; \mathbf{H}_m)$. This can be seen from the Arzelà–Ascoli theorem since any $\tilde{v}$ in the range of $\mathcal{L}$ is obtained from (44) and thus is uniformly Lipschitz continuous in time with a Lipschitz constant depending just on L and m, the initial data through IN and the external magnetic field.

Moreover, we will prove in the following lemma (the proof of which is technical but straightforward and thus postponed to section 5) that $\mathcal{L}$ is continuous.

LEMMA 4.3. *The operator $\mathcal{L}$ defined in* (47) *is continuous on $V_m(t^*)$ in the topology of $C(0,t^*;\mathbf{H}_m)$.*

Thus, Schauder's fixed point theorem assures the existence of a

$$v_m \in V_m(t^*)$$

such that $\mathcal{L}(v_m) = v_m$. In turn, v_m together with the associated pair (F_m, M_m) is a discrete weak solution in the sense of Definition 4.1 of the system (10)–(13) on the time interval $[0,t^*]$.

Step 2: *A priori estimates.* Let us now deduce the a priori estimates, i.e., in particular (1) and (58) below. To this end, let us first multiply (4.1) by $-H_{\text{eff}} = -\Delta M_m - H_{\text{ext}}$ to get that

(48)
$$\begin{aligned}(\partial_t M_m + (v_m\cdot\nabla)M_m)\cdot(-H_{\text{eff}}) &= (M_m \times H_{\text{eff}})\cdot H_{\text{eff}} + (M_m\times M_m\times H_{\text{eff}})\cdot H_{\text{eff}}\\ &= |M_m\cdot H_{\text{eff}}|^2 - |H_{\text{eff}}|^2 \le 0,\end{aligned}$$

since $|M_m| = 1$ by Lemma 4.2. After plugging the definition of the effective field into this equation, we obtain

(49)
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla M_m|^2 - M_m\cdot H_{\text{ext}}dx + \int_\Omega M_m\cdot\partial_t H_{\text{ext}}dx - \int_\Omega((v_m\cdot\nabla)M_m)\cdot(\Delta M_m + H_{\text{ext}}) \le 0.$$

Note that for any smooth M the following identity holds, $\nabla\cdot(\nabla M\otimes\nabla M) = \nabla\frac{|\nabla M|^2}{2} + (\nabla M)^\top\Delta M$, and thus $\nabla\cdot(\nabla M\otimes\nabla M)\cdot v_m = \nabla\frac{|\nabla M|^2}{2}\cdot v_m + (v_m\cdot\nabla)M\Delta M$. Therefore, using integration by parts and the fact that v_m is divergence free together with the vanishing boundary conditions, we obtain the identity

$$-\int_\Omega (\nabla M\otimes\nabla M)\cdot\nabla v_m\,dx = \int_\Omega (v_m\cdot\nabla)M\Delta M\,dx,$$

which holds by approximation also for M_m for almost all $t\in[0,t^*)$. Moreover, by integration by parts we get that

$$-\int_\Omega (v_m\cdot\nabla M_m)\cdot H_{\text{ext}} = \int_\Omega \nabla\cdot v_m M_m\cdot H_{\text{ext}} + \left(\nabla H_{\text{ext}}^\top M_m\right)\cdot v_m\,dx = \int_\Omega\left(\nabla H_{\text{ext}}^\top M_m\right)\cdot v_m\,dx.$$

Plugging this into (49) leads to

(50)
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla M_m|^2 - M_m\cdot H_{\text{ext}}dx + \int_\Omega M_m\cdot\partial_t H_{\text{ext}}dx + \int_\Omega (\nabla M_m\odot\nabla M_m)\cdot\nabla v_m\\ + \left(\nabla H_{\text{ext}}^\top M_m\right)\cdot v_m\,dx \le 0.\end{aligned}$$

Let us now test (34) with $W'(F_m)$. Notice that this is an admissible test function since for almost all $t\in[0,t^*)$ we have that $W'(F_m)$ is in $W^{1,2}(\Omega;\mathbb{R}^{2\times 2})$. Indeed, due to growth condition (25), $W'(F_m)$ is in $L^2(\Omega;\mathbb{R}^{2\times 2})$ if $F_m\in L^2(\Omega;\mathbb{R}^{2\times 2})$, which is guaranteed by Lemma 4.2. Moreover, since $W''(\cdot)$ is bounded by (26),

$\nabla W'(F_m) = W''(F_m)\nabla F_m$ is in $L^2(\Omega;\mathbb{R}^{2\times2\times2})$ for almost all $t \in [0,t^*)$ if $\nabla F_m \in L^2(\Omega;\mathbb{R}^{2\times2\times2})$, which is again guaranteed by Lemma 4.2, where a bound on F_m in $L^2(0,t^*;W^{1,2}(\Omega;\mathbb{R}^{2\times2}))$ is obtained. Finally, due to the continuity of the trace operator and $W'(F_{\min}) = 0$, we know that $W'(F_m) = 0$ on $\partial\Omega$. Plugging in the test, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega W(F_m)dx + \int_\Omega \kappa\nabla F_m \cdot \nabla W'(F_m) + ((v_m\cdot\nabla)F_m - \nabla v_m F_m)W'(F_m)\,dx \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega W(F_m)dx + \int_\Omega \kappa\nabla F_m \cdot (W''(F_m)\nabla F_m) + (v_m\cdot\nabla)W(F_m) \\
&\qquad - (W'(F_m)F_m^\top)\cdot\nabla v_m\,dx = 0.
\end{aligned}$$

Using that v_m is divergence free and exploiting condition (27), we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega W(F_m)dx + \int_\Omega \kappa a|\nabla F_m|^2 - (W'(F_m)F_m^\top)\cdot\nabla v_m\ dx \le 0.$$

Last, we deduce from (36) and (37) that $v_m = \sum_{i=1}^m g_m^i(t)\xi_i(x)$ satisfies

$$\begin{aligned}
\int_\Omega \partial_t v_m\cdot\zeta + (v_m\cdot\nabla)v_m\cdot\zeta - \big(\nabla M_m \odot \nabla M_m - W'(F_m)F_m^\top - \nu\nabla v_m\big)\cdot\nabla\zeta& \\
- \big(\nabla H_{\mathrm{ext}}^\top M_m\big)\cdot\zeta\ dx = 0&
\end{aligned} \tag{51}$$

for all $\zeta \in \mathbf{H}_m$. Testing this equality with v_m itself yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |v_m|^2dx + \frac{1}{2}\int_\Omega \nu|\nabla v_m|^2 - (\nabla M_m\odot\nabla M_m)\cdot\nabla v_m + (W'(F_m)F_m^\top)\cdot\nabla v_m \\
&\quad - \big(\nabla H_{\mathrm{ext}}^\top M_m\big)\cdot v_m\ dx = 0,
\end{aligned}$$

because $\int_\Omega (v_m\cdot\nabla v_m)\cdot v_m dx = 0$. Summing the three expressions above, we get the overall energy inequality for any $t \in [0,t^*)$ as follows:

$$\begin{aligned}
&\underbrace{\int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 - M_m(t)\cdot H_{\mathrm{ext}}(t) + W(F_m(t))\,dx}_{\text{energy at time } t} + \underbrace{\int_0^t\!\!\int_\Omega \kappa a|\nabla F_m|^2 dxds}_{\text{regularization}} \\
&+ \underbrace{\int_0^t\!\!\int_\Omega \nu|\nabla v_m|^2\,dx\,ds}_{\text{dissipation}} \\
&\quad\le \underbrace{\int_\Omega \frac{1}{2}|v_m(0)|^2 + \frac{1}{2}|\nabla M_m(0)|^2 - M_m(0)\cdot H_{\mathrm{ext}}(0) + W(F_m(0))\,dx}_{\text{approximate initial energy}} \\
&\qquad \underbrace{- \int_0^t\!\!\int_\Omega M_m\cdot\partial_t H_{\mathrm{ext}}\,dx\,ds}_{\text{work of external forces}} \\
&\quad\le \underbrace{\int_\Omega \frac{1}{2}|v_0|^2 + \frac{1}{2}|\nabla M_0|^2 - M_0\cdot H_{\mathrm{ext}}(0) + W(F_0)\,dx}_{\text{initial energy}} \underbrace{- \int_0^t\!\!\int_\Omega M_m\cdot\partial_t H_{\mathrm{ext}}\,dx\,ds}_{\text{work of external forces}},
\end{aligned} \tag{52}$$

where in the last line we exploited that F_m and M_m already satisfy the initial conditions exactly. From (1), we obtain the following estimate for any $t \in [0, t^*)$:

$$\begin{aligned}(53)\quad &\int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 + W(F_m(t))\,dx + \int_0^t\!\!\int_\Omega \kappa a|\nabla F_m|^2 + \nu|\nabla v_m|^2\,dx\,ds \\ &\quad\le \int_\Omega \frac{1}{2}|v_0|^2 + \frac{1}{2}|\nabla M_0|^2 + W(F_0)\,dx + 2\|H_{\mathrm{ext}}\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^3))} \\ &\qquad + \|\partial_t H_{\mathrm{ext}}\|_{L^1(0,T;L^1(\Omega;\mathbb{R}^3))} = \mathrm{IED};\end{aligned}$$

plugging in additionally (24) we obtain that

$$\begin{aligned}(54)\quad &\sup_{t\in[0,t^*)} \int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 + |F_m(t)|^2\,dx \\ &\quad + \int_0^t\!\!\int_\Omega \kappa a|\nabla F_m|^2 + \nu|\nabla v_m|^2\,dx\,ds \\ &\quad \le C(\mathrm{IED}).\end{aligned}$$

The above estimate is based on the inequality in (48), i.e., on $|M\cdot H_{\mathrm{eff}}|^2 - |H_{\mathrm{eff}}|^2 \le 0$, cf. (49). We can refine the a priori estimate in (50) by working with the following expression obtained with $H_{\mathrm{eff}} = \Delta M_m + H_{\mathrm{ext}}$:

$$\begin{aligned}|M_m\cdot H_{\mathrm{eff}}|^2 - |H_{\mathrm{eff}}|^2 &= (M_m\cdot\Delta M_m)^2 + 2(M_m\cdot\Delta M_m)(M_m\cdot H_{\mathrm{ext}}) + (M_m\cdot H_{\mathrm{ext}})^2 \\ &\quad - |\Delta M_m|^2 + 2\Delta M_m\cdot H_{\mathrm{ext}} + |H_{\mathrm{ext}}|^2.\end{aligned}$$

For any $t \in [0, t^*)$ we get by the same procedure as above that

$$\begin{aligned}&\int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 + W(F_m(t))\,dx \\ &\quad + \int_0^t\!\!\int_\Omega \kappa a|\nabla F_m|^2 + \nu|\nabla v_m|^2 + |\Delta M_m|^2 + |H_{\mathrm{ext}}|^2\,dx\,ds \\ &\le \int_\Omega \frac{1}{2}|v_0|^2 + \frac{1}{2}|\nabla M_0|^2 - M_0\cdot H_{\mathrm{ext}}(0) + M_m(t)\cdot H_{\mathrm{ext}}(t) + W(F_0)\,dx \\ &\quad - \int_0^t\!\!\int_\Omega M_m\cdot\partial_t H_{\mathrm{ext}}\,dx\,ds + \int_0^t\!\!\int_\Omega |\nabla M_m|^4 + 2(M_m\cdot\Delta M_m)(M_m\cdot H_{\mathrm{ext}}) \\ &\quad + (M_m\cdot H_{\mathrm{ext}})^2 - 2\Delta M_m\cdot H_{\mathrm{ext}}\,dxds,\end{aligned}$$

where in the last term, we used that $-M_m\cdot\Delta M_m = |\nabla M_m|^2$ since the modulus of M_m is equal to one. By Young's and Hölder's inequalities, this leads to

$$\begin{aligned}&\int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 + W(F_m(t))\,dx \\ &\quad + \int_0^t\!\!\int_\Omega \kappa a|\nabla F_m|^2 + \nu|\nabla v_m|^2 + \left(1-\varepsilon^2\right)|\Delta M_m|^2\,dx\,ds \\ &\quad \le \mathrm{IED} + \int_0^t\!\!\int_\Omega |\nabla M_m|^4 + \frac{1}{\varepsilon^2}|H_{\mathrm{ext}}|^2\,dxds,\end{aligned}$$

where $\epsilon > 0$ can be arbitrarily small.

Now, we exploit an observation from [11]: the term $\int_0^t\int_\Omega |\nabla M_m|^4\,dxds$ on the right-hand side of the above expression can actually be absorbed into the Laplacian on the left-hand side, which yields a bound on the second gradient of M_m. To this end, observe that for any $t \in [0, t^*)$

$$\|\Delta M_m(t)\|_{L^2(\Omega;\mathbb{R}^3)} = \left\|\nabla^2 M_m(t)\right\|_{L^2(\Omega;\mathbb{R}^{3\times2\times2})} \tag{55}$$

due to the Neumann boundary conditions for M_m. Further, by Ladyzhenskaya's inequality, it holds for any $f \in W^{1,2}(\Omega;\mathbb{R}^{\tilde{d}})$ that

$$\|f\|_{L^4(\Omega;\mathbb{R}^{\tilde{d}})} \leq C\left(\|f\|_{L^2(\Omega;\mathbb{R}^{\tilde{d}})} + \|\nabla f\|^{1/2}_{L^2(\Omega;\mathbb{R}^{\tilde{d}\times2})}\|f\|^{1/2}_{L^2(\Omega;\mathbb{R}^{\tilde{d}})}\right) \tag{56}$$

for some $C > 0$ depending on Ω only. Hence,

$$\begin{aligned}\|\nabla M_m\|^4_{L^4(\Omega;\mathbb{R}^{3\times2})} \\ \leq \widetilde{C}\left(\|\nabla M_m\|^4_{L^2(\Omega;\mathbb{R}^{3\times2})} + \|\nabla^2 M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times2\times2})}\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}\right)\end{aligned} \tag{57}$$

for some $\widetilde{C} > 0$ depending on Ω only. Thus, we have for any $t \in [0, t^*)$ that

$$\begin{aligned}&\int_\Omega \frac{1}{2}|v_m(t)|^2 + \frac{1}{2}|\nabla M_m(t)|^2 + W(F_m(t))\,dx \\ &\quad + \int_0^t\!\!\int_\Omega \kappa a|\nabla F_m(t)|^2 + \nu|\nabla v_m|^2 + \left(1-\varepsilon^2\right)|\nabla^2 M_m|^2\,dx\,ds \\ &\quad \leq \mathrm{IED} + \widetilde{C}\int_0^t \|\nabla M_m\|^4_{L^2(\Omega;\mathbb{R}^{3\times2})} + \left\|\nabla^2 M_m\right\|^2_{L^2(\Omega;\mathbb{R}^{3\times2\times2}}\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}\,ds \\ &\qquad + \int_0^t\int_\Omega \frac{1}{\varepsilon^2}|H_{\mathrm{ext}}|^2\,dxds \\ &\quad \leq (1+\widetilde{C}T)\mathrm{IED} + \widetilde{C}\,\mathrm{IED}\int_0^t \left\|\nabla^2 M_m\right\|^2_{L^2(\Omega;\mathbb{R}^{3\times2\times2})}\,ds + \frac{1}{\varepsilon^2}\|H_{\mathrm{ext}}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))},\end{aligned}$$

where we applied that $\|\nabla M_m\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})} \leq \mathrm{IED}$ uniformly in the time by (1). Therefore, if $\widetilde{C}\,\mathrm{IED} < 1-\varepsilon^2$, we get, additionally to (1), the a priori estimate:

$$\left\|\nabla^2 M_m\right\|^2_{L^2(0,t^*;L^2(\Omega;\mathbb{R}^{3\times2\times2}))} \leq C(T, \mathrm{IED}, H_{\mathrm{ext}}). \tag{58}$$

Notice that, owing to estimate (43), we can strengthen (58) albeit not uniformly in the Galerkin variable m. Indeed, since $\|\nabla M_m(t)\|_{L^2(\Omega;\mathbb{R}^{3\times2})}$ is bounded uniformly by IED on $(0, t^*)$, we may rewrite (43) as

$$\begin{aligned}&\|\Delta M_m(t)\|^2{}_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad \leq \|\Delta M_0\|^2{}_{L^2(\Omega;\mathbb{R}^3)} + C(L, m, \mathrm{IED}, H_{\mathrm{ext}})\int_0^t \left(1 + \|\Delta M_m(s)\|^4_{L^2(\Omega;\mathbb{R}^3)}\right)ds,\end{aligned}$$

whence we obtain by the Gronwall lemma that for all $t \in [0, t^*)$

$$\|\Delta M_m(t)\|^2{}_{L^2(\Omega;\mathbb{R}^3)} \leq C(L, m, \mathrm{IED}, H_{\mathrm{ext}})(\|\Delta M_0\|_{L^2(\Omega;\mathbb{R}^3)} + T), \tag{59}$$

where we also used that $\int_0^{t^*}\|\Delta M_m(s)\|^2_{L^2(\Omega;\mathbb{R}^3)}ds \leq \mathrm{IED}$.

Step 3: *Dual a priori estimates.* Notice that the a priori estimates obtained in Step 2 do not give any information on the time derivatives of the quantities v_m, F_m, M_m. However, these will be needed since without a uniform bound on time derivatives we cannot expect strong convergence in Bochner spaces, which in turn is crucial to pass to the limit in the nonlinearities in the system. We deduce these estimates directly from the discrete system (34), (4.1), and (51) itself on a time interval $(0,t) \subset (0,T)$ such that (v_m, F_m, M_m) is a weak solution on $(0,t)$ in the sense of Definition 4.1.

Indeed, for the velocity and for any $\xi \in L^2(0,t)$ and any $\zeta \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)$ with

$$\|\xi\|_{L^2(0,t)} \le 1 \qquad \text{and} \qquad \|\zeta\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \le 1$$

we get that

$$\begin{aligned}
&\int_0^t\int_\Omega \partial_t v_m \cdot (\xi\zeta)\, dx\, ds \\
&\quad = \int_0^t\int_\Omega \partial_t v_m \cdot (\xi P_m\zeta)\, dx\, ds \\
&\quad = \int_0^t\int_\Omega -(v_m\cdot\nabla)v_m(\xi P_m\zeta) + \nu\nabla v_m\cdot(\xi\nabla P_m\zeta) \\
&\qquad + \left(\nabla M_m \odot \nabla M_m - W'(F_m)F_m^\top\right)\cdot(\xi\nabla P_m\zeta) + (\nabla H_{\mathrm{ext}}^\top M_m)\cdot(\xi P_m\zeta)\, dx\, ds \\
&\quad = \int_0^t\int_\Omega (v_m\otimes v_m)\cdot(\xi\nabla P_m\zeta) + \nu\nabla v_m\cdot(\xi\nabla P_m\zeta) \\
&\qquad + \left(\nabla M_m \odot \nabla M_m + W'(F_m)F_m^\top\right)\cdot(\xi\nabla P_m\zeta) + (\nabla H_{\mathrm{ext}}^\top M_m)\cdot(\xi P_m\zeta)\, dx\, ds \\
&\quad \le \int_0^t\int_\Omega |v_m|^2|\xi||\nabla P_m\zeta| + \nu|\nabla v_m||\xi||\nabla P_m\zeta| \\
&\qquad + \left(|\nabla M_m\odot\nabla M_m| + |W'(F_m)F_m^\top|\right)|\xi||\nabla P_m\zeta| + |\nabla H_{\mathrm{ext}}^\top M_m||\xi||P_m\zeta|\, dx\, ds \\
&\quad \le \int_0^t \|v_m\|^2_{L^4(\Omega;\mathbb{R}^2)}|\xi|\|\nabla P_m\zeta\|_{L^2(\Omega,\mathbb{R}^{2\times2})} + \nu\|\nabla v_m\|_{L^2(\Omega;\mathbb{R}^{2\times2})}|\xi|\|\nabla P_m\zeta\|_{L^2(\Omega;\mathbb{R}^{2\times2})}\, ds \\
&\qquad + \int_0^t \big(\|\nabla M_m\odot\nabla M_m\|_{L^2(\Omega;\mathbb{R}^{2\times2})} \\
&\qquad\qquad + \|W'(F_m)F_m^\top\|_{L^2(\Omega;\mathbb{R}^{2\times2})}\big)|\xi|\|\nabla P_m\zeta\|_{L^2(\Omega;\mathbb{R}^{2\times2})}\, ds \\
&\qquad + \int_0^t \|\nabla H_{\mathrm{ext}}\|_{L^2(\Omega;\mathbb{R}^{3\times2})}\|M_m\|_{L^\infty(\Omega;\mathbb{R}^3)}|\xi|\|P_m\zeta\|_{L^2(\Omega;\mathbb{R}^2)}\, ds \\
&\quad \le c\|v_m\|^2_{L^4(0,t;L^4(\Omega,\mathbb{R}^2))} + \nu\|\nabla v_m\|_{L^2(0,t;L^2(\Omega))} + \|\nabla M_m\|^2_{L^4(0,t;L^4(\Omega;\mathbb{R}^{3\times2}))} \\
&\qquad + c\left(1 + \|F_m\|^2_{L^4(0,t;L^4(\Omega;\mathbb{R}^{2\times2}))}\right) \\
&\qquad + \|\nabla H_{\mathrm{ext}}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times2}))}\|M_m\|_{L^\infty(0,t;L^\infty(\Omega;\mathbb{R}^3))},
\end{aligned}$$

where we used that $\|P_m\zeta\|_{W_0^{1,2}(\Omega;\mathbb{R}^2)} \le \|\zeta\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \le 1$ and exploited the growth condition (25); we recall also that the velocity is, albeit not uniformly in m, Lipschitz-continuous in time, which makes the time integration feasible. Notice that the terms appearing on the right-hand side of this expression are bounded by interpolation of the energetic estimate that we already obtained in the previous step. Indeed, applying Ladyzhenskaya's inequality (56) to v_m, ∇M_m, and F_m, we get the asserted Bochner-space regularities.

Thus, taking a supremum over all ξ and ζ as above, we see that

$$\|\partial_t v_m\|_{L^2(0,t;W^{-1,2}(\Omega;\mathbb{R}^2))} \le C(T, \mathrm{IED}, H_{\mathrm{ext}}). \tag{60}$$

In the same spirit, we deduce also estimates on the time derivatives of the magnetization M_m and the deformation gradient F_m. Let us start with the magnetization, multiply (4.1) by some arbitrary $\xi \in L^2(0,t)$ and any $\zeta \in L^2(\Omega;\mathbb{R}^3)$ satisfying

$$\|\xi\|_{L^2(0,t)} \leq 1 \qquad \text{and} \qquad \|\zeta\|_{L^2(\Omega;\mathbb{R}^3)} \leq 1,$$

and integrate over Ω and $(0,t)$. We obtain

$$\begin{aligned}
&\int_0^t\!\!\int_\Omega \partial_t M_m \cdot (\xi\zeta)\, dx\, ds \\
&\quad \leq \int_0^t\!\!\int_\Omega |(v_m\cdot\nabla)M_m\cdot(\xi\zeta)| \\
&\qquad + |(M_m \times (\Delta M_m + H_{\text{ext}}))\cdot(\xi\zeta)| + |(M_m\times M_m\times(\Delta M_m + H_{\text{ext}}))\cdot(\xi\zeta)| dxds \\
&\quad \leq \int_0^t \|v_m\|_{L^4(\Omega;\mathbb{R}^2)}\|\nabla M_m\|_{L^4(\Omega;\mathbb{R}^{3\times 2})}|\xi|\|\zeta\|_{L^2(\Omega;\mathbb{R}^2)} \\
&\qquad + 2\big(\|\Delta M_m\|_{L^2(\Omega;\mathbb{R}^3)} + \|H_{\text{ext}}\|_{L^2(\Omega;\mathbb{R}^3)}\big)|\xi|\|\zeta\|_{L^2(\Omega;\mathbb{R}^3)}\, ds \\
&\quad \leq \|v_m\|_{L^4(0,t;L^4(\Omega;\mathbb{R}^2))}\|\nabla M_m\|_{L^4(0,t;L^4(\Omega;\mathbb{R}^{3\times 2}))} \\
&\qquad + 2(\|\Delta M_m\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^3))} + \|H_{\text{ext}}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}).
\end{aligned}$$

We again employ (55) and the Ladyzhenskaya inequality (56) and obtain

$$\|\partial_t M_m\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^3))} \leq C(T, \text{IED}, H_{\text{ext}}), \tag{61}$$

which implies that

$$\|\partial_t \nabla M_m\|_{L^2(0,t;W^{-1,2}(\Omega;\mathbb{R}^{3\times 2}))} \leq C(T, \text{IED}, H_{\text{ext}}). \tag{62}$$

Let us make one more observation on $\partial_t \nabla M_m$. To this end set

$$W^{1,2}_n\left(\Omega;\mathbb{R}^{3\times 2}\right) := \left\{X \in W^{1,2}\left(\Omega;\mathbb{R}^{3\times 2}\right)) : Xn = 0 \text{ a.e. on } \partial\Omega\right\};$$

here, recall that n denotes the outer normal to the boundary of Ω. Notice that $\nabla M_m \in W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2}))$, and hence, in order to form the Gelfand triple

$$W^{1,2}_n\left(\Omega;\mathbb{R}^{3\times 2}\right) \hookrightarrow L^2\left(\Omega;\mathbb{R}^{3\times 2}\right) \hookrightarrow \left(W^{1,2}_n\left(\Omega;\mathbb{R}^{3\times 2}\right)\right)',$$

we would like to assert that $\partial_t \nabla M_m \in L^2(0,t;(W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2}))')$; this can be seen using (61). Indeed, for any $g \in L^2(\Omega;\mathbb{R}^2)$ we have that $\nabla g \in (W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2}))'$ because for any smooth g and any arbitrary $\Phi \in W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2})$ we obtain

$$\int_\Omega \nabla g \cdot \Phi\, dx = -\int_\Omega g(\nabla\cdot\Phi)\, dx \leq \|g\|_{L^2(\Omega;\mathbb{R}^2)}\|\Phi\|_{W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2})},$$

so that the claim follows by approximation. This calculation also shows that

$$\|\partial_t \nabla M_m\|_{L^2(0,t;(W^{1,2}_n(\Omega;\mathbb{R}^{3\times 2}))')} \leq C(T, \text{IED}, H_{\text{ext}}). \tag{63}$$

Finally, we consider $\partial_t F_m$. To this end, let us take any arbitrary $\xi \in L^4(0,t)$ and $\zeta \in W^{1,2}(\Omega;\mathbb{R}^{2\times 2})$ such that

$$\|\xi\|_{L^4(0,t)} \leq 1 \qquad \text{and} \qquad \|\zeta\|_{W^{1,2}(\Omega;\mathbb{R}^{2\times 2})} \leq 1$$

and estimate

$$\begin{aligned}
&\int_0^t \langle\!\langle \partial_t F_m, \zeta \rangle\!\rangle \xi \, ds \\
&\quad \leq \int_0^t \int_\Omega |(v_m \cdot \nabla) F_m \cdot (\xi\zeta)| + |(\nabla v_m F_m) \cdot (\xi\zeta)| + \kappa \, |\nabla F_m \cdot (\xi \nabla \zeta)| \;\, dx \, ds \\
&\quad \leq \int_0^t \|v_m\|_{L^3(\Omega;\mathbb{R}^2)} \|\nabla F_m\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2})} |\xi| \|\zeta\|_{L^6(\Omega;\mathbb{R}^{2\times2})} \\
&\qquad + \|\nabla v_m\|_{L^2(\Omega;\mathbb{R}^{2\times2})} \|F_m\|_{L^3(\Omega;\mathbb{R}^{2\times2})} |\xi| \|\zeta\|_{L^6(\Omega;\mathbb{R}^{2\times2})} \\
&\qquad + \kappa \|\nabla F_m\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2})} |\xi| \|\nabla\zeta\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2})} \, ds \\
&\quad \leq \|v_m\|_{L^4(0,t;L^3(\Omega;\mathbb{R}^2))} \|\nabla F_m\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{2\times2\times2}))} \\
&\qquad + \|\nabla v_m\|_{L^2(0,t;L^2(\Omega;\mathbb{R}^{2\times2}))} \|F_m\|_{L^4(0,t;L^3(\Omega;\mathbb{R}^{2\times2}))} + \kappa \|\nabla F_m\|_{L^{\frac{4}{3}}(0,t;L^2(\Omega;\mathbb{R}^{2\times2\times2}))},
\end{aligned}$$

where again all terms are bounded by the energetic estimates (1) when taking also interpolation inequalities, analogous to those that we used in the balance of momentum, into account. In total, we obtain that

$$\|\partial_t F_m\|_{L^{\frac{4}{3}}(0,t;W^{-1,2}(\Omega;\mathbb{R}^{2\times2}))} \leq C(\text{IED}). \tag{64}$$

Notice that the dual estimate (64) that we obtain for F_m is slightly worse than those that we got for M_m and v_m in (63) and (60), respectively, hence, proving that F_m attains the right initial data will be slightly more difficult; see Step 6.

Step 4: *Extending the approximate solution.* The approximate solution and the a priori estimates that we obtained so far only hold on a short interval $[0, t^*)$. Nevertheless, they can be extended to the interval $[0, T)$ with T as in Theorem 3.2. Indeed, we may find a time instant t_* such that t_* is arbitrarily close to t^* and $(v_m(t_*), F_m(t_*), M_m(t_*))$ are well defined and bounded in $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^{2\times2}) \times W^{1,2}(\Omega;\mathbb{R}^3)$ by IED; cf. (1)–(1). Moreover, due to (59), we can assure that $M_m(t_*)$ is bounded in the $W^{2,2}$-norm by a constant that only depends on m, L (which in fact is only dependent on IED) and IED. Thus, as long as m is fixed, and in particular in this step, we can assure that the magnetization is bounded in the $W^{2,2}$-norm uniformly in time, which is needed in Step 1.

Thus, we may regard $(v_m(t_*), F_m(t_*), M_m(t_*))$ as new initial datum and repeat the procedure from Step 1. This allows us to find a solution (v_m, F_m, M_m) of the system (34)–(36) on $\Omega \times [t_*, t_* + \delta)$ coinciding with the earlier solution $(v_m, F_m, M_m)(t_*)$ in t_*. Notice also that the procedure in Step 1 allows us to choose the length of the solution interval δ only depending on m, L (which is controlled by IED), the initial data through IN and global-in-time properties of the external magnetic field. In particular, we see that $\delta > |t_* - t^*|$ since the latter can be made arbitrarily small.

Gluing the two solutions together, we thus obtain a solution on a time interval $[0, t_* + \delta)$. Repeating the procedure in Steps 2 and 3 then gives that the same a priori estimates hold for the prolonged solution on the solution interval $[0, t_* + \delta)$. Notice that repeating this procedure on the whole interval $[0, t_* + \delta)$ and not just on $[t_*, t_* + \delta)$ allows us to bound $(v_m(t), F_m(t), M_m(t))$ in $L^2(\Omega;\mathbb{R}^2) \times L^2(\Omega;\mathbb{R}^{2\times2}) \times W^{1,2}(\Omega;\mathbb{R}^3)$ for almost all $t \in [0, t_* + \delta)$ by IED, i.e., by the initial data and the external field, and not just by the norms of $(v_m, F_m, M_m)(t_*)$ and the external field.

Thus, we can continue the extension on another time instant of length δ which is the same as above. This is due to the fact that the initial data for this extension will again be bounded by IED. Finally, we obtain a solution (v_m, F_m, M_m) on $\Omega \times (0, T)$.

Step 5: *Convergence of the approximate system.* From the a priori estimates (1) and (58) obtained in Step 2 as well as the dual a priori estimates (60), (63), and (64) from Step 3, we conclude by the Aubin–Lions lemma (cf., e.g., [46]) that, up to a nonrelabeled subsequence, there exist $(v,F,M) \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)) \times L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^{2\times 2})) \times L^2(0,T;W^{2,2}(\Omega;\mathbb{R}^3))$ such that

$$v_m \to v \quad \text{in } L^2\left(0,T;L^4\left(\Omega;\mathbb{R}^2\right)\right), \tag{65}$$
$$\nabla v_m \rightharpoonup \nabla v \quad \text{in } L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{2\times 2}\right)\right), \tag{66}$$
$$F_m \to F \quad \text{in } L^2\left(0,T;L^4\left(\Omega;\mathbb{R}^{2\times 2}\right)\right), \tag{67}$$
$$\nabla F_m \rightharpoonup \nabla F \quad \text{in } L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{2\times 2\times 2}\right)\right) \tag{68}$$
$$\nabla M_m \to \nabla M \quad \text{in } L^2\left(0,T;L^4\left(\Omega;\mathbb{R}^{3\times 2}\right)\right), \tag{69}$$
$$\Delta M_m \rightharpoonup \Delta M \quad \text{in } L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{3}\right)\right). \tag{70}$$

Moreover, due to the continuous embedding of $W^{1,4}(\Omega;\mathbb{R}^3) \hookrightarrow L^\infty(\Omega;\mathbb{R}^3)$, we also have that

$$M_m \to M \quad \text{in } L^2\left(0,T;L^\infty\left(\Omega;\mathbb{R}^3\right)\right). \tag{71}$$

At this point, we are ready to pass to the limit in (51), (34), and (4.1) that together form the discrete system. Let us start with the balance of momentum (51). To this end, let us choose some arbitrary $\zeta \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)$ and use $\zeta_m := P_m(\zeta) \in \mathbf{H}_m$ as a test function in (51). Moreover, multiply this equation by $\xi \in W^{1,\infty}(0,T)$ with $\xi(T) = 0$ and integrate over $[0,T)$ to obtain

$$\begin{aligned} &\int_0^T\!\!\int_\Omega -v_m\cdot\zeta_m\xi' + (v_m\cdot\nabla)v_m\cdot(\xi\zeta_m) + \nu\nabla v_m\cdot(\xi\nabla\zeta_m) \\ &\qquad - \left(\nabla M_m \odot \nabla M_m - W'(F_m)F_m^\top\right)\cdot(\xi\nabla\zeta_m) - ((\nabla H_{\mathrm{ext}})^\top M_m)\cdot(\xi\zeta_m)\,dx\,dt \\ &\quad = \int_\Omega v_m(0)\cdot(\xi(0)\zeta_m)\,dx, \end{aligned} \tag{72}$$

where we used integration by parts with respect to time.

Due to the continuity of the Nemytskiĭ mapping induced by $W'(\cdot)$ (cf., e.g., [46]), we get that $W'(F_m) \to W'(F)$ in $L^2(0,T;L^4(\Omega;\mathbb{R}^{2\times 2}))$. Therefore, by standard weak-strong convergence arguments we get that (72) converges to

$$\begin{aligned} &\int_0^T\!\!\int_\Omega -v\cdot\zeta\xi' + (v\cdot\nabla)v\cdot(\xi\zeta) + \nu\nabla v\cdot(\xi\nabla\zeta) - \left(\nabla M \odot \nabla M - W'(F)F^\top\right)\cdot(\xi\nabla\zeta)\,dx\,dt \\ &\qquad - ((\nabla H_{\mathrm{ext}})^\top M)\cdot(\xi\zeta)\,dx\,dt = \int_\Omega v_0\cdot(\xi(0)\zeta)\,dx \end{aligned} \tag{73}$$

as $m \to \infty$.

Further, multiply (34) by $\xi \in W^{1,\infty}(0,T)$ with $\xi(T) = 0$ and integrate over $[0,T)$ to get

$$\begin{aligned} &\int_0^T\!\!\int_\Omega -F_m\cdot(\xi'\Xi) + ((v_m\cdot\nabla)F_m - \nabla v_m F_m)\cdot(\xi\Xi) + \kappa\nabla F_m\cdot(\xi\nabla\Xi)\,dx\,dt \\ &\qquad = \int_\Omega F_0\cdot(\xi(0)\Xi)\,dx, \end{aligned} \tag{74}$$

where we used that $F_m(0) = F_0$ and that, due to Lemma 4.2, we have $\partial_t F_m \in L^2(0,T;W^{-1,2}(\Omega;\mathbb{R}^{2\times 2}))$ and simultaneously $F_m \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2}))$ whence $F_m \in C(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})$ and the integration by-parts formula holds; cf., e.g., [46, Lemma 7.3]. Then, after integrating by parts in time, the duality pairing between $W^{-1,2}(\Omega;\mathbb{R}^{2\times 2})$ and $W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2})$ reduces to a scalar product on $L^2(\Omega;\mathbb{R}^{2\times 2})$.

Standard weak-strong convergence arguments allow us to identify the limit as

(75)
$$\int_0^T\int_\Omega -F\cdot(\xi'\Xi) + ((v\cdot\nabla)F - \nabla v F)\cdot(\xi\Xi) + \kappa\nabla F\cdot(\xi\nabla\Xi)\;dx\;dt = \int_\Omega F_0\cdot(\xi(0)\Xi)\;dx$$

as $m\to\infty$.

Finally, we pass to the limit in the LLG. By multiplying (4.1) by $\tilde\zeta \in L^2(\Omega;\mathbb{R}^3)$ and $\xi \in W^{1,\infty}(0,T)$ with $\xi(T) = 0$ and integrating over space and time, we obtain with $M_m(0) = M_0$ that

(76)
$$\begin{aligned}\int_0^T\int_\Omega & -M_m\cdot(\xi'\tilde\zeta) + \big((v_m\cdot\nabla)M_m + (M_m\times(\Delta M_m + H_{\text{ext}}) - \Delta M_m\big)\cdot(\xi\tilde\zeta)\;dx\;dt\\ &= \int_0^T\int_\Omega \big(|\nabla M_m|^2 M_m - M_m(M_m\cdot H_{\text{ext}}) + H_{\text{ext}}\big)\cdot(\xi\tilde\zeta)\,dxdt + \int_\Omega M_0\cdot(\xi(0)\tilde\zeta)\;dx.\end{aligned}$$

As $m\to\infty$, this equation converges to

(77)
$$\begin{aligned}\int_0^T\int_\Omega & -M\cdot(\xi'\tilde\zeta) + \big((v_m\cdot\nabla)M + (M\times(\Delta M + H_{\text{ext}}) - \Delta M\big)\cdot(\xi\tilde\zeta)\;dx\;dt\\ &= \int_0^T\int_\Omega \big(|\nabla M|^2 M - M(M\cdot H_{\text{ext}}) + H_{\text{ext}}\big)\cdot(\xi\tilde\zeta)\,dxdt + \int_\Omega M_0\cdot(\xi(0)\tilde\zeta)\;dx.\end{aligned}$$

Indeed, for the term $\int_0^T\int_\Omega |\nabla M_m|^2 M_m\cdot(\xi\tilde\zeta)\,dxdt$ this is obtained by the following calculation:

$$\begin{aligned}&\left|\int_0^T\int_\Omega \big(|\nabla M_m|^2 M_m - |\nabla M|^2 M\big)\cdot(\xi\tilde\zeta)\,dxdt\right|\\ &= \left|\int_0^T\int_\Omega \big(\big(|\nabla M_m|^2 - |\nabla M|^2\big)\,M_m + |\nabla M|^2(M_m - M)\big)\cdot(\xi\tilde\zeta)\;dx\;dt\right|\\ &= \left|\int_0^T\int_\Omega \big((\nabla M_m - \nabla M)\cdot(\nabla M_m + \nabla M)M_m + |\nabla M|^2(M_m - M)\big)\cdot(\xi\tilde\zeta)\;dx\;dt\right|\\ &\le \|\nabla M_m + \nabla M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))}\|\nabla M_m - \nabla M\|_{L^2(0,T;L^4(\Omega;\mathbb{R}^{3\times 2}))}\\ &\quad\times\|M_m\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))}\|\xi\|_{L^\infty(0,T)}\|\tilde\zeta\|_{L^2(\Omega;\mathbb{R}^3))}\\ &\quad+\|\nabla M\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 2}))}\|M_m - M\|_{L^2(0,T;L^\infty(\Omega))}\|\xi\|_{L^\infty(0,T)}\|\tilde\zeta\|_{L^2(\Omega;\mathbb{R}^3))},\end{aligned}$$

where the second term on the right-hand side tends to zero owing to (71) while the first term on the right-hand side vanishes thanks to (69).

All other terms converge by a combination of weak and strong convergences in (65)–(71). Hence, the discrete solution that we constructed in Step 1 and extended in Step 4 converges in the sense of (65)–(71) to a solution of (20)–(22).

Moreover, the L^∞-in-time regularities in Definition 3.1 hold by the lower semicontinuity of norms, and since the estimate (1) is uniform in m and is obtained for the entire time interval $(0,T)$.

Step 6: *Attainment of the initial data.* Finally, we are left to prove that the initial data is actually attained by the solution in the sense of Definition 3.1. As for v and M this is fairly easy because the a priori estimates (1), (58), (60), (63), and (64) translate by weak lower semicontinuity to the limit so that by

$$\begin{aligned} v \in L^2\left(0,T;W_0^{1,2}\left(\Omega;\mathbb{R}^2\right)\right) \quad &\text{and} \quad \partial_t v \in L^2\left(0,T;W^{-1,2}\left(\Omega;\mathbb{R}^2\right)\right), \\ M \in L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^3\right)\right) \quad &\text{and} \quad \partial_t M \in L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^3\right)\right), \\ \nabla M \in L^2\left(0,T;W_n^{1,2}\left(\Omega;\mathbb{R}^{3\times 2}\right)\right) \quad &\text{and} \quad \partial_t \nabla M \in L^2\left(0,T;\left(W_n^{1,2}\left(\Omega;\mathbb{R}^{3\times 2}\right)\right)'\right), \end{aligned}$$

and by, e.g., [46, Lemma 7.3] we have that $v \in C(0,T;L^2(\Omega;\mathbb{R}^2))$ and that $M \in C(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. Moreover, we can see directly from (20) that $v(0) = v_0$ a.e. in Ω. Indeed, for some $\varepsilon > 0$ take $\phi(x,t) = \phi_1(t)\phi_2(x)$ in such a way that $\phi_1(0) = 1$, $\phi_1(t)$ linear on $(0,\varepsilon)$ and $\phi_1(t) = 0$ for all $t \in [\varepsilon, T]$ while $\phi_2 \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)$ is arbitrary. Then, as $\varepsilon \to 0$ we have $\phi(\cdot,t) \to 0$ a.e. in Ω while $\partial_t\phi(t) \rightharpoonup -\delta_0$ in measures, where δ_0 denotes the Dirac measure centered at 0. Thus,

$$\int_\Omega (v(0) - v_0)\cdot \phi_2 dx = 0$$

for all $\phi_2 \in W^{1,2}_{0,\mathrm{div}}(\Omega;\mathbb{R}^2)$, which shows the claim. The situation is analogous for M.

For F, the situation is slightly more complicated since the obtained integrability of the time derivative does not allow us to immediately form a Gelfand triple since $L^{4/3}$ (in-time integrability of the time derivative of F) is not dual to L^2 (in-time integrability of ∇F). Nevertheless, we conclude from the a priori estimates (1) and (64) that (notice that we actually get from (1) that $F \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2}))$, which yields the first statement)

$$F \in L^4\left(0,T;W^{-1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right) \quad \text{and} \quad \partial_t F \in L^{\frac{4}{3}}\left(0,T;W^{-1,2}\left(\Omega;\mathbb{R}^{2\times 2}\right)\right),$$

which implies that (see, e.g., [46, Lemma 7.3]) $F \in C(0,T;W^{-1,2}(\Omega;\mathbb{R}^{2\times 2}))$; combining this with the fact that $F \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2}))$, we have (see, e.g., [50, Chapter III, Lemma 1.4]) that

$$F \in C\left(0,T;L^2_w\left(\Omega;\mathbb{R}^{2\times 2}\right)\right),$$

where $L^2_w(\Omega;\mathbb{R}^{2\times 2})$ is the space of L^2-functions whose values are 2×2-matrices and which are equipped with the weak topology. Moreover, by the same procedure as above, we may identify that $F(0) = F_0$ whence

$$F(t) \rightharpoonup F_0 \quad \text{in } L^2\left(\Omega;\mathbb{R}^{2\times 2}\right) \quad \text{as } t \to 0_+. \tag{78}$$

By the convexity of W this translates to

$$\int_\Omega W(F_0)dx \le \liminf_{t\to 0_+} \int_\Omega W(F(t))dx.$$

On the other hand, the energy estimate (1) also translates to the limit by weak* lower semicontinuity of the energy with respect to the convergence of $(v_m, F_m, M_m) \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^2)) \times L^\infty(0,T;L^2(\Omega;\mathbb{R}^{2\times 2})) \times L^\infty(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$. Hence

$$\int_\Omega \frac{1}{2}|v(t)|^2 + \frac{1}{2}|\nabla M(t)|^2 - M(t)\cdot H_{\text{ext}}(t) + W(F)\,dx + \int_0^t\int_\Omega \kappa a|\nabla F|^2 + \nu|\nabla v|^2\,dx\,dt$$
$$\leq \int_\Omega \frac{1}{2}|v_0|^2 + \frac{1}{2}|\nabla M_0|^2 - M_0\cdot H_{\text{ext}}(0) + W(F_0)\,dx - \int_0^t\int_\Omega M\cdot\partial_t H_{\text{ext}}\,dx\,dt$$

for almost all $t \in (0,T)$. By continuity, we may extend the estimate to hold even for all $t \in (0,T)$. Thus, taking the $\limsup_{t\to 0^+}$ and using the already proved attainment of initial data (as well as the continuity of the external field in time) we get that

$$\limsup_{t\to 0^+}\int_\Omega W(F(t))dx \leq \int_\Omega W(F_0)dx,$$

so that altogether $\int_\Omega W(F(t))dx \to \int_\Omega W(F_0)dx$. By the strict convexity and growth of W this means that

$$\lim_{t\to 0^+}\|F(t)\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} = \|F_0\|_{L^2(\Omega;\mathbb{R}^{2\times 2})},$$

which combined with the already obtained weak convergence of $F(t)$ to F_0 in (78) means that the initial data are attained in the strong sense as claimed. ∎

5. Proofs of Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. Recall that for a fixed $v \in V_m(t_0)$, we aim to construct (F,M) satisfying

$$\langle\!\langle \partial_t F, \Xi\rangle\!\rangle + \int_\Omega (v\cdot\nabla)F\cdot\Xi - (\nabla v F)\cdot\Xi + \kappa\nabla F\cdot\nabla\Xi\;dx = 0 \qquad \text{a.e. in } (0,t_1), \tag{79}$$

$$\begin{aligned}\partial_t M + (v\cdot\nabla)M = |\nabla M|^2 M + \Delta M - M\times(\Delta M + H_{\text{ext}})&\\ - M(M\cdot H_{\text{ext}}) + H_{\text{ext}}&\qquad \text{a.e. in } \Omega\times(0,t_1)\end{aligned} \tag{80}$$

for all $\Xi \in W_0^{1,2}(\Omega;\mathbb{R}^{2\times 2})$, together with the initial conditions (18)–(19) and boundary conditions (15)–(16).

Notice that the two equations (79) and (2) are decoupled. Consequently, we can prove existence of solutions to each of the equations separately. To prove the existence, we rely on similar methods as in the proof of Theorem 3.2, i.e., we use a Galerkin approximation and standard ODE theory to prove existence of approximate solutions. Thus, the existence of solutions is proved at first on some short time interval $[0,\tilde{t})$ for some $0 < \tilde{t} \leq t_1$, but we can extend the solution later to the entire interval $[0,t_1]$ due to the a priori estimates obtained.

Existence of weak solution to (79). As for the Galerkin approximation, we project F and (79) on finite dimensional subspaces of the eigenfunctions of the Laplace-operator that form an orthonormal basis of $L^2(\Omega;\mathbb{R}^{2\times 2})$ and an orthogonal basis of $W^{1,2}(\Omega;\mathbb{R}^{2\times 2})$. Let $P_k : L^2(\Omega;\mathbb{R}^{2\times 2}) \to \{\Xi_1,\Xi_2,\ldots,\Xi_k\}$ be this orthonormal projection.

For a fixed $k \in \mathbb{N}$, we look for a function F_k of the form

$$F_k(x,t) = \sum_{i=1}^{k} d_k^i(t)\Xi_i(x) + F_{\min} \tag{81}$$

solving the projection of (79) on the span$\{\Xi_1, \Xi_2, \ldots, \Xi_k\}$, i.e., we solve the ODE

$$\frac{d}{dt} d_k^i(t) = -\kappa\mu_i d_k^i(t) + \sum_{j=1}^{k} d_k^j(t)\tilde{A}_j^i(t) + \tilde{B}^i(t), \qquad i = 1, \ldots, k, \tag{82}$$

where

$$\tilde{A}_j^i(t) = -\int_\Omega (v(x,t)\cdot\nabla)\Xi_j(x)\cdot\Xi_i(x) - (\nabla v(x,t)\Xi_j(x))\cdot\Xi_i(x)\ dx, \tag{83}$$

$$\tilde{B}^i(t) = \int_\Omega \nabla v(x,t)F_{\min}\cdot\Xi_i(x)\ dx. \tag{84}$$

The initial condition becomes

$$d_k^i(0) = \int_\Omega F_0(x)\cdot\Xi_i(x)\ dx \tag{85}$$

for $i = 1, \ldots, k$. We apply Carathéodory's existence theorem (see, e.g., [23, Chapter 1, Theorem 1]) to obtain absolutely continuous solution $d_k^i(t)$ of (82) on the interval $[0, \tilde{t})$. Notice that the solution interval will thus depend only on $F_{\min}$, the initial condition, and the $L^\infty(0, t_1; W^{1,\infty}(\Omega; \mathbb{R}^2))$-norm of v, i.e., on m and L. Notice also that, since the right-hand side of (82) is locally Lipschitz, the obtained solution is unique.

We now prove all the needed a priori estimates. To this end, let us first sum (82) over all $i = 1 \ldots k$ to get

$$\int_\Omega \big(\partial_t F_k + (v\cdot\nabla)F_k - \nabla v F_k\big)\cdot\Xi dx + \int_\Omega \kappa\nabla F_k\cdot\nabla\Xi dx = 0 \tag{86}$$

for all $\Xi \in \text{span}\{\Xi_1, \Xi_2, \ldots, \Xi_k\}$.

Let us now test (86) by $F_k - F_{\min}$ and integrate over $[0,t]$ for $t \le \tilde{t}$ to find

$$\begin{aligned}
\frac{1}{2}\int_\Omega |F_k(t)|^2\ dx &- \int_0^t\!\!\int_\Omega \partial_t F_k\cdot F_{\min}\ dx\ ds + \int_0^t\!\!\int_\Omega (v\cdot\nabla)\frac{|F_k|^2}{2} - (v\cdot\nabla)F_k\cdot F_{\min}\ dx\ ds \\
&- \int_0^t\!\!\int_\Omega \nabla v\cdot(F_k F_k^\top) - \nabla v F_k\cdot F_{\min} - \kappa|\nabla F_k|^2\ dx\ ds \\
&= \frac{1}{2}\int_\Omega |P_k(F_0)|^2 dx \le \frac{1}{2}\int_\Omega |F_0|^2 dx.
\end{aligned}$$

As the third integral term vanishes because v is divergence free, we get, by integrating the second expression by parts with respect to time, and rearranging,

$$\begin{aligned}
\frac{1}{4}\int_\Omega |F_k(t)|^2\ dx &+ \int_0^t\!\!\int_\Omega \kappa|\nabla F_k|^2\ dx\ ds \\
&\le \int_0^t\!\!\int_\Omega |\nabla v\cdot(F_k F_k^\top)| + |\nabla v||F_k||F_{\min}|\ dx\ ds + \int_\Omega \frac{5}{2}|F_{\min}|^2\ dx + \int_\Omega |F_0|^2\ dx \\
&\le \|F_0\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + C(L, m, F_{\min})\left(1 + \int_0^t\!\!\int_\Omega |F_k|^2\ dx\ ds\right),
\end{aligned} \tag{87}$$

where in the last line we used that $\|\nabla v\|_{L^\infty(0,t;L^\infty(\Omega;\mathbb{R}^2))} \leq C(L,m)$ since $v \in V_m(t_0)$. Applying Gronwall's inequality yields that

$$\|F_k\|_{L^\infty(0,\tilde{t};L^2(\Omega;\mathbb{R}^{2\times2}))\cap L^2(0,\tilde{t};W^{1,2}(\Omega;\mathbb{R}^{2\times2}))} \leq C(L,m,T)\big(1+\|F_0\|^2_{L^2(\Omega;\mathbb{R}^{2\times2})}\big). \tag{88}$$

Notice that we drop the dependence of $F_{\min}$ in the notation of the constant here and in the following. Moreover, from this estimate it follows that we may extend the approximate solution onto the interval $[0,t_0)$ by the same procedure as in Step 4 of the proof of Theorem 3.2. Next, we derive an estimate on the time derivative $\partial_t F_k$ in $L^2(0,t_0;W^{-1,2}(\Omega;\mathbb{R}^{2\times2}))$. To this end, let us choose some arbitrary $\zeta \in L^2(0,t_0)$ and $\Xi \in W_0^{1,2}(\Omega;\mathbb{R}^{2\times2})$ satisfying

$$\|\zeta\|_{L^2(0,t_0)} \leq 1 \qquad \text{and} \qquad \|\Xi\|_{W_0^{1,2}(\Omega;\mathbb{R}^{2\times2})} \leq 1$$

and calculate

$$\begin{aligned}
&\int_0^{t_0} \partial_t F_k \cdot (P_k\Xi)\zeta \; dt \\
&\quad= \int_0^{t_0} \partial_t F_k \cdot \Xi\zeta \; dt \\
&\quad= \int_0^{t_0}\!\!\int_\Omega -(v\cdot\nabla)F_k : (\zeta\Xi) + (\nabla v F_k) : (\zeta\Xi) - \kappa\nabla F_k \,\vdots\, (\zeta\nabla\Xi) \; dx\; dt \\
&\quad\leq \int_0^{t_0} \Big(\big(\|v\|_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla F_k\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2})} \\
&\qquad\qquad + \|\nabla v\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\|F_k\|_{L^2(\Omega;\mathbb{R}^{2\times2})}\big)\|\Xi\|_{L^2(\Omega;\mathbb{R}^{2\times2})} \\
&\qquad\qquad + \kappa\|\nabla F_k\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2})}\|\nabla\Xi\|_{L^2(\Omega;\mathbb{R}^{2\times2\times2\times2})}\Big)|\zeta| \; dt \\
&\quad\leq C(L,m)\|F_k\|_{L^2(0,t_0;W^{1,2}(\Omega;\mathbb{R}^{2\times2})};
\end{aligned}$$

and since for $\|F_k\|_{L^2(0,t_0;W^{1,2}(\Omega;\mathbb{R}^{2\times2}))}$ we already got an estimate in (87), we see that

$$\|\partial_t F_k\|_{L^2(0,t_0;W^{-1,2}(\Omega;\mathbb{R}^{2\times2}))} \leq C(L,m) + \|F_0\|_{L^2(\Omega;\mathbb{R}^{2\times2})}. \tag{89}$$

From the preceding estimates, we see that we may extract a subsequence (not relabeled) from $(F_k)_{n\in\mathbb{N}}$ such that

$$F_k \rightharpoonup F \quad \text{in } L^2\left(0,t_0;L^2\left(\Omega;\mathbb{R}^{2\times2}\right)\right), \tag{90}$$

$$\partial_t F_k \rightharpoonup \partial_t F \quad \text{in } L^2\left(0,t_0;W^{-1,2}\left(\Omega;\mathbb{R}^{2\times2}\right)\right), \tag{91}$$

$$\nabla F_k \rightharpoonup \nabla F \quad \text{in } L^2\left(0,t_0;L^2\left(\Omega;\mathbb{R}^{2\times2\times2}\right)\right). \tag{92}$$

As, by fixing v, (86) is a linear equation, we may pass with k to ∞ to get that F solves (79). Owing to the linearity once again, this is the unique solution of (79).

Existence of weak solutions to (2). Let us now prove the existence of solutions as well as suitable a priori estimates for (2). The procedure to obtain these is inspired by [11]. As above, we perform a Galerkin approximation; to this end, let $\{\eta_i\}_{i=1}^\infty \subset C^\infty(\Omega;\mathbb{R}^3)$ be an orthonormal basis of $L^2(\Omega;\mathbb{R}^3)$ and an orthogonal basis of $W_n^{2,2}(\Omega;\mathbb{R}^3)$, where

$$W_n^{2,2}(\Omega;\mathbb{R}^3) = \left\{X \in W_n^{2,2}\left(\Omega;\mathbb{R}^3\right) : \nabla X n = 0 \text{ a.e. on } \partial\Omega\right\}.$$

For example, this basis may be composed of eigenfunctions of the operator $\Delta^2 +$ id subject to vanishing Neumann boundary condition for the eigenfunction and its Laplacian. Let $\widetilde{P}_k : L^2(\Omega;\mathbb{R}^3) \to \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_k\}$ be the orthonormal projection onto finite dimensional subspaces formed by this basis. For a fixed $k \in \mathbb{N}$, we look for a function M_k of the form

$$M_k(x,t) = \sum_{i=1}^{k} h_k^i(t)\eta_i(x) \tag{93}$$

that satisfies the projection of (2) onto $\operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_k\}$; this amounts to solving the following ODE:

$$\frac{d}{dt} h_k^i(t) = \sum_{j=1}^{k} h_k^j(t)\hat{A}_j^i(t) + \sum_{j,l=1}^{k} h_k^j(t)h_k^l(t)\hat{B}_{jl}^i + \sum_{j,l,m=1}^{k} h_k^j(t)h_k^l(t)h_k^m(t)\hat{C}_{jlm}^i \tag{94}$$

for $i = 1, \ldots, k$, where

$$\hat{A}_j^i(t) = -\int_\Omega \big((v(x,t)\cdot\nabla)\eta_j(x) + (\eta_j(x) \times H_{\text{ext}}(x,t)) - \Delta\eta_j(x) - H_{\text{ext}}(x,t)\big) \cdot \eta_i(x)\, dx, \tag{95}$$

$$\hat{B}_{jk}^i = -\int_\Omega (\eta_j(x) \times \Delta\eta_k(x) + (\eta_k(x)\cdot H_{\text{ext}})\eta_j(x)) \cdot \eta_i(x)\, dx, \tag{96}$$

$$\hat{C}_{jkl}^i = \int_\Omega (\nabla\eta_j(x) : \nabla\eta_k(x))(\eta_l(x)\cdot\eta_i(x))\, dx. \tag{97}$$

The initial condition becomes

$$h_k^i(0) = \int_\Omega M_0(x)\cdot\eta_i(x)\, dx, \qquad i = 1, \ldots, n. \tag{98}$$

Existence of unique Lipschitz continuous solutions $h_n^i(t)$ is also here obatined by Carathéodory's existence theorem on a time interval $[0, t^{**})$. Notice that the length of the solution interval depends just on the $L^2(\Omega;\mathbb{R}^3)$ norm of H_{ext} (which is controlled by assumption uniformly on $[0,T]$) and the $L^\infty(\Omega;\mathbb{R}^2)$ norm of v (which is controlled uniformly by $C(m,L)$ on $[0,t_0]$ since $v \in V_m(t_0)$).

In order to deduce suitable a priori estimates, we first rewrite (94) as

$$\begin{aligned} \int_\Omega \big(&\partial_t M_k + (v\cdot\nabla)M_k - |\nabla M_k|^2 M_k - \Delta M_k + M_k \times (\Delta M_k + H_{\text{ext}}) \\ &+ M_k(M_k \cdot H_{\text{ext}}) - H_{\text{ext}}\big)\eta\, dx = 0 \end{aligned} \tag{99}$$

for all $\eta \in \operatorname{span}\{\eta_1, \eta_2, \ldots, \eta_k\}$. Let us first test (99) by M_k to obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Omega |M_k|^2\, dx + \int_\Omega |\nabla M_k|\, dx \\ &= \int_\Omega |\nabla M_k|^2|M_k|^2 - |M_k|^2(M_k\cdot H_{\text{ext}}) + M_k\cdot H_{\text{ext}}\, dx \\ &\le 2\Big(\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}^2 \|\nabla M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}^2 \\ &\qquad + \big(\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}^2 + 1\big)\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|H_{\text{ext}}\|_{L^1(\Omega;\mathbb{R}^3)}\Big). \end{aligned} \tag{100}$$

Next, we test (99) by $\Delta^2 M_k$ and obtain for all $t \in [0, t^{**})$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta M_k|^2\,dx + \int_\Omega |\nabla\Delta M_k|^2\,dx \\
&\quad\le \underbrace{\int_\Omega |(v\cdot\nabla)M_k\cdot\Delta^2 M_k|\,dx}_{:=\mathrm{I}_1} + \underbrace{\int_\Omega |(M_k\times(\Delta M_n + H_{\mathrm{ext}}))\cdot\Delta^2 M_k|\,dx}_{:=\mathrm{I}_2} \\
&\qquad + \underbrace{\int_\Omega \big||\nabla M_k|^2 M_k\cdot\Delta^2 M_k\big|\,dx}_{:=\mathrm{I}_3} + \underbrace{\int_\Omega |(M_k\cdot H_{\mathrm{ext}})M_k\cdot\Delta^2 M_k|\,dx}_{:=\mathrm{I}_4} \\
&\qquad + \underbrace{\int_\Omega |H_{\mathrm{ext}}\cdot\Delta^2 M_k|\;dx}_{:=\mathrm{I}_5}\,.
\end{aligned} \tag{101}
$$

We will estimate the integrals I_1–I_5 separately. To do so, we will utilize the following estimates, which hold for all smooth $M:\Omega\to\mathbb{R}^3$ ($\Omega\subset\mathbb{R}^2$) with zero Neumann boundary conditions:

$$\|M\|_{W^{2,2}(\Omega;\mathbb{R}^3)} \le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \tag{102}$$

$$\|\nabla M\|_{L^4(\Omega;\mathbb{R}^{3\times2})} \le C\|\nabla M\|^{\frac12}_{L^2(\Omega;\mathbb{R}^{3\times2})}\left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac14}, \tag{103}$$

$$\|\nabla M\|_{L^6(\Omega;\mathbb{R}^{3\times2})} \le C\|\nabla M\|^{\frac13}_{L^2(\Omega;\mathbb{R}^{3\times2})}\left(\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac13}, \tag{104}$$

$$
\begin{aligned}
\|\nabla M\|_{L^\infty(\Omega;\mathbb{R}^{3\times2})} \le C\|\nabla M\|^{\frac12}_{L^2(\Omega;\mathbb{R}^{3\times2})}\big(&\|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})} \\
&+ \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}\big)^{\frac14},
\end{aligned} \tag{105}
$$

$$\|\Delta M\|_{L^4(\Omega;\mathbb{R}^3)} \le C\|\Delta M\|^{\frac12}_{L^2(\Omega;\mathbb{R}^3)}\left(\|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}\right)^{\frac14} \tag{106}$$

for some constant $C>0$ depending just on Ω. Indeed, (102) is a variant of Poincaré inequality after realizing that $\|\nabla^2 M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2\times2})} = \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}$ by integration by parts due to the vanishing Neumann boundary conditions. Further, (103) and (106) are variants of the Ladyzhenskaya inequality formulated here for functions the traces of which do not necessarily vanish on $\partial\Omega$ while (104) is a more general interpolation inequality obtained from the Gagliardo–Nirenberg theorem. Finally, (105) is a variant of the Agmon inequality valid in two dimensions.

We start to estimate the term I_1 and get, since $v\in V_m(t_0)$,

$$
\begin{aligned}
\mathrm{I}_1 &\le \int_\Omega |(\nabla v(\nabla M_k)^\top)\cdot\nabla\Delta M_k| + |(v\cdot\nabla)\nabla M_k\cdot\nabla\Delta M_k|\;dx \\
&\le \|\nabla v\|_{L^\infty(\Omega;\mathbb{R}^{2\times2})}\|\nabla M_k\|_{L^2(\Omega;\mathbb{R}^{3\times2})}\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times2})} \\
&\quad + \|v\|_{L^\infty(\Omega;\mathbb{R}^2)}\|\nabla^2 M_k\|_{L^2(\Omega;\mathbb{R}^{3\times2\times2})}\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times2})} \\
&\le C(L,m)\left(\|\nabla M_k\|_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_k\|_{L^2(\Omega;\mathbb{R}^3)}\right)\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times2})},
\end{aligned} \tag{107}
$$

where we used (55) and (102). For the integral term I_2, we obtain

(108)
$$\begin{aligned}
\mathrm{I}_2 &\leq \int_\Omega |(\nabla M_k \times (\Delta M_k + H_{\mathrm{ext}})) \cdot \nabla\Delta M_k| + |(\nabla M_k \times (H_{\mathrm{ext}} + \nabla H_{\mathrm{ext}})) \cdot \nabla \Delta M_k| \; dx \\
&\leq \|\nabla M_k\|_{L^4(\Omega;\mathbb{R}^{3\times 2})}\big(\|\Delta M_k\|_{L^4(\Omega;\mathbb{R}^3)} + 2\|H_{\mathrm{ext}}\|_{W^{1,4}(\Omega;\mathbb{R}^3)}\big)\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\leq C\|\nabla M_k\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})}\left(\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{4}} \\
&\quad \times \Bigg(\|\Delta M_k\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^3)}\left(\|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)^{\frac{1}{4}} \\
&\qquad\qquad + 2\|H_{\mathrm{ext}}\|_{W^{1,4}(\Omega;\mathbb{R}^3)}\Bigg)\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned}$$

We estimate the integral term I_3 and find out that

(109)
$$\begin{aligned}
\mathrm{I}_3 &= \int_\Omega |(2(\nabla^2 M_k \nabla M_k) \otimes M_k) \cdot \nabla\Delta M_k| + |\nabla M_k|^2 |\nabla M_k \cdot \nabla \Delta M_k| \; dx \\
&\leq 2\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla M_k\|_{L^\infty(\Omega;\mathbb{R}^{3\times 2})}\|\nabla^2 M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2\times 2})}\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad + \|\nabla M_k\|^3_{L^6(\Omega;\mathbb{R}^{3\times 2})}\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\leq C\Big(\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\Delta M_k\|_{L^2(\Omega;\mathbb{R}^3)}\|\nabla M_k\|^{\frac{1}{2}}_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\quad \times \left(\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\right)^{\frac{1}{4}} \\
&\quad + \|\nabla M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}\Big(\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big)\Big)\|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})}.
\end{aligned}$$

For the integral term I_4, we estimate

(110)
$$\begin{aligned}
\mathrm{I}_4 &= \int_\Omega |(M_k \cdot H_{\mathrm{ext}})(\nabla M_k \cdot \nabla\Delta M_k)| \\
&\quad + |(\nabla\Delta M_k)^\top M_k) \cdot ((\nabla M_k)^\top H_{\mathrm{ext}} + (\nabla H_{\mathrm{ext}})^\top M_k)| \; dx \\
&\leq \|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|H_{\mathrm{ext}}\|_{W^{1,3}(\Omega;\mathbb{R}^3)}\big(2\|\nabla M_k\|_{L^6(\Omega;\mathbb{R}^{3\times 2})} + \|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\big) \\
&\quad \times \|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} \\
&\leq \|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|H_{\mathrm{ext}}\|_{W^{1,3}(\Omega;\mathbb{R}^3)}\big(2\|\nabla M_n\|^2_{L^2(\Omega)} + 2\|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\big) \\
&\quad \times \|\nabla\Delta M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})},
\end{aligned}$$

where to get the last expression we used (104) combined with the Young inequality. Finally, estimating the integral term I_5 yields

(111)
$$\mathrm{I}_5 \leq \|\nabla H_{\mathrm{ext}}\|_{L^2(\Omega)}\|\nabla\Delta M_n\|_{L^2(\Omega)}.$$

Combining (2)–(111), we obtain from (101) and an iterative application of Young's inequality that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta M_k|^2\,dx + \frac{1}{2}\int_\Omega |\nabla\Delta M_k|^2\,dx \\
&\quad\le C(L,m)\left(\|\nabla M_k\|^4_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_k\|^4_{L^2(\Omega;\mathbb{R}^3)}\right) \\
&\qquad + C(H_{\mathrm{ext}})\big(1+\|M_k\|^4_{L^\infty(\Omega;\mathbb{R}^3)}\big)\big(1+\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\big) \\
&\qquad \times\left(\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + \|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + 1\right)^2 .
\end{aligned} \tag{112}
$$

We shall make use of (112) later to derive (43). In order to derive further a priori estimates, let us use that $W^{2,2}(\Omega;\mathbb{R}^3)$ embeds continuously into $W^{1,2}(\Omega;\mathbb{R}^3)$ as well as $L^\infty(\Omega;\mathbb{R}^3)$ so that with the help of (102) we have that

$$
\begin{aligned}
\|\nabla M_k\|_{L^2(\Omega;\mathbb{R}^{3\times 2})} &\le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}, \\
\|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)} &\le C\left(\|M\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using this in (112) and adding it to (100) lets us deduce that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |M_k|^2 + |\Delta M_k|^2 dx + \int_\Omega |\nabla M_n|^2 + |\nabla\Delta M_n|^2 dx \\
&\qquad \le C(L,m,H_{\mathrm{ext}})\left(1+\left(\|M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\Delta M_n\|^2_{L^2(\Omega;\mathbb{R}^3)}\right)^8\right).
\end{aligned} \tag{113}
$$

In the next step, we make use of the following classical comparison lemma; see, e.g., [11, Lemma 2.4]).

LEMMA 5.1. *Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be C^1 and nondecreasing in its second variable. Assume further that $y: I\subset\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying $y(t)\le y_0+\int_0^t f(s,y(s))\,ds$ for all $t>0$. Let $z: I\to\mathbb{R}$ be the solution of $z'(t)=f(t,z(t))$, $z(0)=y_0$. Then, it holds that $y(t)\le z(t)$ for all $t>0$.*

From (113) and Lemma 5.1 we deduce the existence of a time $0<t_1\le t^{**}$ such that

$$
\|M_k\|_{L^\infty(0,t_1;W^{1,2}(\Omega;\mathbb{R}^3))} + \|M_k\|_{L^2(0,t_1;W^{3,2}(\Omega;H_{\mathrm{ext}}))} \le C(L,m,M_0,H_{\mathrm{ext}}). \tag{114}
$$

In order to be able to pass to the limit as $k\to\infty$ in (99), we need to derive further estimates on the time derivative of M_k as well as of ∇M_k. To this end, let us test (99) by $\partial_t M_k$ to get

$$
\begin{aligned}
\int_\Omega |\partial_t M_k|^2\,dx &= \int_\Omega \big(-(v\cdot\nabla)M_k - (M_k\times\Delta M_k) + |\nabla M_k|^2 M_k + \Delta M_k \\
&\qquad - (M_k\cdot H_{\mathrm{ext}})M_k + H_{\mathrm{ext}}\big)\cdot\partial_t M_k\,dx \\
&\le 3\int_\Omega |(v\cdot\nabla)M_k|^2 + |M_k\times\Delta M_k|^2 + |\nabla M_k|^4|M_k|^2 + |\Delta M_k|^2 \\
&\qquad + (M_k\cdot H_{\mathrm{ext}})^2|M_k|^2 + |H_{\mathrm{ext}}|^2\,dx + \frac{1}{2}\int_\Omega |\partial_t M_k|^2\,dx.
\end{aligned}
$$

From there, we get

$$\begin{aligned}\|\partial_t M_k\|_{L^2(\Omega;\mathbb{R}^3)} \leq 6\Big(&C(L,m)\|\nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|M_k\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &+ \|\nabla M_k\|^4_{L^4(\Omega;\mathbb{R}^{3\times 2})}\|M_k\|^2_{L^\infty(\Omega;\mathbb{R}^3)} + \|\Delta M_k\|^2_{L^2(\Omega;\mathbb{R}^3)}\\ &+ \|H_{\text{ext}}\|^2_{L^2(\Omega;\mathbb{R}^3)}\|M_k\|^4_{L^\infty(\Omega;\mathbb{R}^3)} + \|H_{\text{ext}}\|^2_{L^2(\Omega;\mathbb{R}^3)}\Big),\end{aligned}$$

where we take the supremum over all $t \in [0, t_1)$ to find, using (114) and the fact that $H_{\text{ext}} \in C(0, T; L^2(\Omega; \mathbb{R}^3))$,

$$\|\partial_t M_k\|_{L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^3))} \leq C(L, m, M_0, H_{\text{ext}}). \tag{115}$$

Next, we test (99) by $-\partial_t \Delta M_k$ and integrate over $(0, t)$ for $t \leq t_1$ to find out that

$$\begin{aligned}&\int_0^t \|\partial_t \nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\, ds + \frac{1}{2}\big(\|\Delta M_k(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} - \|\Delta M_k(0)\|^2_{L^2(\Omega;\mathbb{R})}\big)\\ &= \int_0^t\!\!\int_\Omega \big((v\cdot\nabla)M_k + (M_k \times \Delta M_k) - |\nabla M_k|^2 M_k - (M_k \cdot H_{\text{ext}})M_k + H_{\text{ext}}\big)\\ &\quad \cdot \partial_t \Delta M_k\, dx\, ds\\ &= \int_0^t\!\!\int_\Omega -\Big((\nabla v(\nabla M_k)^\top) - (v\cdot\nabla)\nabla M_k - \nabla(M_k \times \Delta M_k)\\ &\qquad + (2(\nabla^2 M_k \nabla M_k) \otimes M_k + |\nabla M_k|^2 \nabla M_k\\ &\qquad + (M_k \cdot H_{\text{ext}})\nabla M_k - \nabla H_{\text{ext}}\big) \cdot \partial_t \nabla M_k\\ &\qquad + ((\partial_t \nabla M_k)^\top M_k) \cdot ((\nabla M_k)^\top H_{\text{ext}} + (\nabla H_{\text{ext}})^\top M_k\Big)\, dx\, ds\\ &\leq \frac{1}{2}\int_0^t \|\partial_t \nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})} + 5\int_0^t\!\!\int_\Omega C(L,m)\big(|\nabla M_k|^2 + |\nabla^2 M_k|^2\big)\\ &\quad + |\nabla M_k|^2|\Delta M_k|^2 + |M_k|^2|\nabla\Delta M_k|^2 + 4|M_k|^2|\nabla M_k|^2|\nabla^2 M_k|^2 + |\nabla M_k|^6\\ &\quad + 2|M_k|^2|\nabla M_k|^2|H_{\text{ext}}|^2 + |\nabla H_{\text{ext}}|^2 + |M_k|^4|\nabla H_{\text{ext}}|^2\, dx\, ds\\ &\leq \int_0^t \|\partial_t \nabla M_k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 2})}\\ &\quad + 5\Big(C(L,m)\big(\|\nabla M_k\|^2_{L^2(0,t_1;L^2(\Omega;\mathbb{R}^{3\times 2}))} + \|\nabla^2 M_k\|^2_{L^2(0,t_1;L^2(\Omega;\mathbb{R}^{3\times 2\times 2}))}\big)\\ &\qquad + (1 + 4\|M_k\|^2_{L^\infty(0,t_1 t;L^\infty(\Omega,\mathbb{R}^3))})\|\nabla M_k\|_{L^\infty(0,t_1;L^4(\Omega;\mathbb{R}^{3\times 2}))}\\ &\qquad\quad \times \|\nabla^2 M_k\|_{L^2(0,t_1;L^4(\Omega;\mathbb{R}^{3\times 2\times 2}))}\\ &\qquad + \|M_k\|^2_{L^\infty(0,t_1;L^\infty(\Omega;\mathbb{R}^3))}\|\nabla\Delta M_k\|^2_{L^2(0,t_1;L^2(\Omega;\mathbb{R}^{3\times 2}))} + \|\nabla M_k\|^6_{L^6(0,t_1;L^6(\Omega;\mathbb{R}^{3\times 2}))}\\ &\qquad\quad + \|\nabla H_{\text{ext}}\|^2_{L^2(0,t_1;L^2(\Omega;\mathbb{R}^{3\times 2}))}\\ &\qquad + 2\|M_k\|^2_{L^\infty(0,t_1;L^\infty(\Omega;\mathbb{R}^3))}2\|H_{\text{ext}}\|^2_{L^\infty(0,t_1;L^2(\Omega;\mathbb{R}^3))}\|\nabla M_k\|^2_{L^2(0,t_1;L^\infty(\Omega;\mathbb{R}^{3\times 2}))}\\ &\qquad + \|M_k\|^4_{L^\infty(0,t_1;L^\infty(\Omega;\mathbb{R}^3))}\|\nabla H_{\text{ext}}\|^2_{L^2(0,t_1;L^2(\Omega;\mathbb{R}^{3\times 2}))}\Big).\end{aligned}$$

Taking the supremum over all $t \in [0, t_1)$ and using (114), we get the bound

$$\|\partial_t \nabla M_k\|_{L^2(0,t_1;L^2(\Omega))} \leq C(v, M_0, H_{\text{ext}}). \tag{116}$$

We now pass to the limit as $k \to \infty$ to obtain a weak solution to (2). By our a priori estimates , we can find $M \in L^\infty(0, t_1, W^{2,2}(\Omega; \mathbb{R}^3)) \cap W^{1,\infty}(0, t_1; L^2(\Omega; \mathbb{R}^3))$

$\cap\, L^2(0,t_1;W^{3,2}(\Omega;\mathbb{R}^3))$ such that for a (nonrelabeled) subsequence of $(M_k)_{k\in\mathbb{N}}$, we have that

$$M_k \to M \quad \text{in } L^p\left(0,t_1;W^{1,2}\left(\Omega;\mathbb{R}^3\right)\right), \quad 1<p<\infty, \tag{117}$$

$$\partial_t M_k \rightharpoonup \partial_t M \quad \text{in } L^2\left(0,t_1;W^{2,2}\left(\Omega;\mathbb{R}^3\right)\right). \tag{118}$$

Indeed, the weak convergence result follows by the Banach–Alaoglu theorem, while the strong convergence (117) is obtained from the Aubin–Lions lemma. In fact, the Aubin–Lions lemma yields at first the strong convergence $M_k \to M$ in $L^2(0,t_1;W^{2,2}(\Omega;\mathbb{R}^3))$ but combining this with the boundedness of $(M_k)_{k\in\mathbb{N}}$ in $L^\infty(0,t_1;W^{2,2}(\Omega;\mathbb{R}^3))$ we obtain (117).

Thus, multiplying (99) with $\zeta \in L^2(0,t_1)$, integrating over $(0,t_1)$, and passing to the limit $k\to\infty$, we get the equation

$$\int_0^{t_1}\int_\Omega \big(\partial_t M + (v\cdot\nabla)M + (M\times\Delta M) - |\nabla M|^2 M - \Delta M \\ + (M\cdot H_{\text{ext}})M + H_{\text{ext}}\big)\cdot\varphi\zeta\, dx\, dt = 0,$$

which holds for all $\varphi\in L^2(\Omega;\mathbb{R}^3)$ and all $\zeta\in L^2(0,t_1)$. From this, we can conclude that M satisfies (2).

Furthermore, notice that

$$M \in L^\infty\left(0,t_1,W^{2,2}\left(\Omega;\mathbb{R}^3\right)\right)\cap W^{1,\infty}\left(0,t_1;L^2\left(\Omega;\mathbb{R}^3\right)\right)\cap L^2\left(0,t_1;W^{3,2}\left(\Omega;\mathbb{R}^3\right)\right)$$

is the unique solution of (2). Indeed, assume that there existed two solutions $M_1, M_2 \in L^\infty(0,t_1,W^{2,2}(\Omega;\mathbb{R}^3))\cap W^{1,\infty}(0,t_1;L^2(\Omega;\mathbb{R}^3))\cap L^2(0,t_1;W^{3,2}(\Omega;\mathbb{R}^3))$ with $M_1\neq M_2$. The difference M_1-M_2 would then fulfill for almost all $x\in\Omega$ and almost all $t\in[0,t_1)$

$$\begin{aligned}
&\partial_t(M_1-M_2) + (v\cdot\nabla)(M_1-M_2)\\
&\quad = \Delta(M_1-M_2) - (M_1-M_2)\times\Delta M_1 + M_2\times(\Delta(M_1-M_2))\\
&\qquad + \left(|\nabla M_1|^2 - |\nabla M_2|^2\right)M_1 + |\nabla M_2|^2(M_1-M_2) - (M_1-M_2)(M_1\cdot H_{\text{ext}})\\
&\qquad - M_2((M_1-M_2)\cdot H_{\text{ext}}).
\end{aligned}$$

We multiply this equation by (M_1-M_2), integrate over Ω, and use the identity $(a\times b)\cdot c = (b\times c)\cdot a$ to find out that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |M_1-M_2|^2 dx + \int_\Omega |\nabla(M_1-M_2)|^2 dx\\
&\quad = \int_\Omega \big((\Delta(M_1-M_2))\times(M_1-M_2)\big)\cdot M_2\\
&\qquad + \left((\nabla M_1-\nabla M_2)\cdot(\nabla M_1+\nabla M_2)\right)M_1\cdot(M_1-M_2) + |\nabla M_2|^2|M_1-M_2|^2\\
&\qquad - |M_1-M_2|^2(M_1\cdot H_{\text{ext}}) - M_2\cdot(M_1-M_2)((M_1-M_2)\cdot H_{\text{ext}})\, dx\\
&\quad \le \frac{1}{2}\int_\Omega |\nabla(M_1-M_2)|^2 dx\\
&\qquad + \int_\Omega \underbrace{\left(|\nabla M_2|^2 + |\nabla(M_1+M_2)|^2|M_1|^2 + |\nabla M_2|^2 + (|M_1|+|M_2|)|H_{\text{ext}}|\right)}_{(\star)}\\
&\qquad\qquad \times |M_1-M_2|^2 dx,
\end{aligned}$$

where we integrated by parts in the first term on the second line. Now, due to the assumed regularity of M_1 and M_2, we know that $(\star)$ is bounded in $L^1(0,t_1;L^\infty(\Omega))$ which allows us to apply the Gronwall lemma. Thus, since $M_1(0) = M_2(0)$, the two solutions coincide.

Moreover, let us show that $M \in L^\infty(0,t_1,W^{2,2}(\Omega;\mathbb{R}^3)) \cap W^{1,\infty}(0,t_1;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,t_1;W^{3,2}(\Omega;\mathbb{R}^3))$, which is the solution of (2), fulfills that $|M(t)| = 1$ a.e. in Ω for a.a. $t \in [0,t_1)$. To this end, let us multiply (2) with M to obtain

$$\begin{aligned}(119)\qquad &\frac{1}{2}\Big(\partial_t|M|^2 + (v\cdot\nabla)|M|^2 - \Delta|M|^2\Big)\\ &\quad= (|M|^2-1)\left(|\nabla M|^2 - M\cdot H_{\text{ext}}\right) \quad \text{a.e. in } \Omega\times[0,t_1).\end{aligned}$$

Notice that (119) is solved by $|M| = 1$ so we just need to show that this is the unique solution. Let us set $\theta := |M|^2$ and since M is fixed being the unique solution of (2), we may denote $f(M) := |\nabla M|^2 - M\cdot H_{\text{ext}}$. Thus (119) transfers to an equation for θ that reads

(120)
$$\frac{1}{2}\Big(\partial_t\theta + (v\cdot\nabla)\theta - \Delta\theta\Big) = (\theta-1)f(M) \qquad \text{a.e. in } \Omega\times[0,t_1) \text{ with } \theta(0) = |M_0|^2 = 1;$$

now if (120) had two solutions $\theta_1,\theta_2 \in L^\infty(0,t_1,W^{2,2}(\Omega)) \cap W^{1,\infty}(0,t_1;L^2(\Omega))$, we could subtract (119) for θ_1 and θ_2, multiply by $\theta_1-\theta_2$, and conclude by the Gronwall lemma that the two solutions have to coincide.

Finally, we pass to the limit in the inequality in (112) integrated over $(0,t_1)$. On the left-hand side we rely on the convexity of the norm, while on the right-hand side it is enough to use the strong convergence (117). Therefore, since $|M| = 1$ a.e. in $\Omega\times[0,t_1)$ in the limit, we obtain for almost all $t\in[0,t_1)$

$$\begin{aligned}\|\Delta M(t)\|^2_{L^2(\Omega;\mathbb{R}^3)} \le \|\Delta M_0\|^2_{L^2(\Omega;\mathbb{R}^3)} + C(L,m,H_{\text{ext}})\int_0^t 1\\ + \|\nabla M\|^6_{L^2(\Omega;\mathbb{R}^{3\times2})} + \|\nabla M\|^2_{L^2(\Omega;\mathbb{R}^{3\times2})}\|\Delta M\|^4_{L^2(\Omega;\mathbb{R}^3)}\,ds.\end{aligned}$$ □

Proof of Lemma 4.3. We show that $\mathcal{L}$ defined in (47) is continuous on $V_m(t^*)$ in the topology of $C(0,t^*;\mathbf{H}_m)$. To this end, let $(v_l)_{l\in\mathbb{N}} \subset V_m(t^*)$ converge to some $v \in V_m(t^*)$ in the sense that $(g^i_m)_l \to g^i_m$ in $C(0,t^*)$ for $i = 1,\dots,m$, where $v_l = \sum_{i=1}^m (g^i_m)_l(t)\xi_i(x)$ and $v = \sum_{i=1}^m g^i_m(t)\xi_i(x)$.

Let us denote by (F_l,M_l) and (F,M) the solutions of (40)–(41) corresponding to v_l and v, respectively. Notice that their existence is guaranteed by Lemma 4.2.

Let us first realize that $F_l \to F$ in $L^\infty(0,t^*;L^2(\Omega))$. To this end, subtract (40) for F from (40) for F_l, test the result by $F_l - F$, and integrate of over $(0,t)$ with some $0 \le t \le t^*$to obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega |F_l-F|^2(t)\,dx + \int_0^t\!\!\int_\Omega \kappa|\nabla(F_l-F)|^2\,dxds\\ &\quad= -\frac{1}{2}\int_0^t\!\!\int_\Omega (v_l\cdot\nabla)|F_l-F|^2dxds + \int_0^t\!\!\int_\Omega (\nabla v_l(F_l-F))\cdot(F_l-F)\\ &\qquad - ((v_l-v)\cdot\nabla)F\cdot(F_l-F) + (\nabla v_l - \nabla v)F\cdot(F_l-F)dxds,\end{aligned}$$

where we used that F_l and F have the same initial data. Realizing that $\int_0^t\int_\Omega \kappa|\nabla(F_l - F)|^2\,dx\,ds = 0$ because v_l is divergence free and employing Young's inequality yields that

$$
\begin{aligned}
(121)\qquad & \|(F_l - F)(t)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} \\
&\le \int_0^t \|(F_l - F)(s)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})}\, ds \\
&\quad + C\int_0^t \|((v_l - v)\cdot\nabla)F\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} + \|(\nabla v_l - \nabla v)F\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2})} ds,
\end{aligned}
$$

where the first integral on the right-hand side vanishes as $l \to \infty$ due to the assumed convergence of $(v_l)_{l\in\mathbb{N}}$. The claim thus follows by the Gronwall inequality.

Next, we check that $M_l \to M$ in $L^2(0, t^*; W^{1,2}(\Omega;\mathbb{R}^3))$. Similarly as above, we subtract (41) for M from (41) for M_l to have that for a.a. $x \in \Omega$ and a.e. $t \in [0, t^*)$

$$
\begin{aligned}
(122)\qquad & \partial_t(M_l - M) - \Delta(M_l - M) + (v_l\cdot\nabla)(M_l - M) + ((v_l - v)\cdot\nabla)M \\
&= -(M_l - M)\times(\Delta M_l + H_{\text{ext}}) + M\times(\Delta(M_l - M)) + \\
&\quad \left(|\nabla M_l|^2 - |\nabla M|^2\right) M_l + \left(|\nabla M|^2 - (M_l\cdot H_{\text{ext}})\right)(M_l - M) \\
&\quad - M((M_l - M)\cdot H_{\text{ext}});
\end{aligned}
$$

further multiply the result by $M_l - M$ and integrate over Ω and $(0, t)$ with some $0 \le t \le t^*$ to get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |M_l - M|^2(t)\, dx + \int_0^t\!\!\int_\Omega |\nabla(M_l - M)|^2\, dx\, ds \\
&\le \int_0^t\!\!\int_\Omega \left(|v_l - v||\nabla M| + |\nabla(M_l - M)|(2|\nabla M| + |\nabla M_l|)\right)|(M_l - M)| \\
&\quad + \left(|\nabla M|^2 + 2|H_{\text{ext}}|\right)|M_l - M|^2\, dx\, ds.
\end{aligned}
$$

Using now Young's inequality, we obtain

$$
\begin{aligned}
(123)\quad & \frac{1}{2}\int_\Omega |M_l - M|^2(t)\, dx + \int_0^t\!\!\int_\Omega |\nabla(M_l - M)|^2\, dx\, ds \\
&\le \int_0^t\!\!\int_\Omega |v_l - v||\nabla M| dx ds \\
&\quad + C\int_0^t \Big(1 + \|\nabla M\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} + \|\nabla M_l\|^2_{L^\infty(\Omega;\mathbb{R}^{3\times 2})} + \|H_{\text{ext}}\|_{L^\infty(\Omega;\mathbb{R}^3)}\Big) \\
&\quad \times \|M_l - M\|^2_{L^2(\Omega;\mathbb{R}^3)}\, ds,
\end{aligned}
$$

from which the claim follows by the Gronwall inequality.

Let us define $(D^i_m)_l(t)$ and $D^i_m(t)$ via (37) by using (F_l, M_l) and (F, M), respectively. Notice that due to the already proved convergence of $(F_l)_{l\in\mathbb{N}}$ to F and $(M_l)_{l\in\mathbb{N}}$ to M, we see that $(D^i_m)_l \to D^i_m$ in $L^1(0, t^*)$.

Further take $\tilde{v}_l = \sum_{i=1}^m (\tilde{g}^i_m)_l(t)\xi_i(x)$ as $\mathcal{L}(v_l)$ and $\tilde{v} = \sum_{i=1}^m (\tilde{g}^i_m)(t)\xi_i(x)$ as $\mathcal{L}(v)$, i.e., the solutions of (94) with $(D^i_m)_l$ and D^i_m as the right-hand side, respectively. The proof is finished if we can show that $(\tilde{g}_m)_l \to \tilde{g}_m$ uniformly on $[0, t^*)$. To this end, subtract (94) for $\tilde{v}$ from the one for $\tilde{v}_l$ and write in matrix notation

$$\begin{aligned}\partial_t\big((\tilde{g}_m)_l(t) - \tilde{g}_m(t)\big) = &- \nu\,\mathrm{diag}(\lambda_1, \dots, \lambda_m)\big((\tilde{g}_m)_l(t) - \tilde{g}_m(t)\big) + (D_m)_l(t) - D_m(t)\\ &+ \big(A^1(\tilde{g}_m)_l(t)\cdot(\tilde{g}_m)_l(t), \dots, A^m(\tilde{g}_m)_l(t)\cdot(\tilde{g}_m)_l(t)\big)\\ &- \big(A^1\tilde{g}_m(t)\cdot\tilde{g}_m(t), \dots, A^m\tilde{g}_m(t)\cdot\tilde{g}_m(t)\big).\end{aligned}$$

Adding and subtracting the vector $\big(A^1\tilde{g}_m(t)\cdot(\tilde{g}_m)_l(t), \dots, A^m\tilde{g}_m(t)\cdot\tilde{g}_m(t)\big)$ and integrating over $(0,t)$ with some $0 \le t \le t^*$ gives

$$|(\tilde{g}_m)_l(t) - \tilde{g}_m(t)| \le C(L,m)\int_0^t |(\tilde{g}_m)_l(s) - \tilde{g}_m(s)|\ ds + \int_0^t |(D_m(s))_l - D_m(s)|\ ds,$$

and, by means of the Gronwall inequality, we obtain

$$|(\tilde{g}_m)_l(t) - \tilde{g}_m(t)| \le \left(\int_0^t \big|(D_m)_l(s) - D_m(s)\big|\ ds\right) \mathrm{e}^{C(L,m)t^*}.$$

□

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