

RESTRICTION OF ODD DEGREE CHARACTERS AND NATURAL CORRESPONDENCES

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ABSTRACT. Let q be an odd prime power, $n > 1$, and let P denote a maximal parabolic subgroup of $GL_n(q)$ with Levi subgroup $GL_{n-1}(q) \times GL_1(q)$. We restrict the odd-degree irreducible characters of $GL_n(q)$ to P to discover a natural correspondence of characters, both for $GL_n(q)$ and $SL_n(q)$. A similar result is established for certain finite groups with self-normalizing Sylow p -subgroups. Next, we construct a canonical bijection between the odd-degree irreducible characters of $G = S_n$, $GL_n(q)$ or $GU_n(q)$ with q odd, and those of $\mathbf{N}_G(P)$, where P is a Sylow 2-subgroup of G . Since our bijections commute with the action of the absolute Galois group over the rationals, we conclude that the fields of values of character correspondents are the same. We use this to answer some questions of R. Gow.

1. INTRODUCTION

It is not often the case that a natural correspondence of characters between a group G and a subgroup H of G is found. Even more rarely this correspondence can be described by inspecting the restriction of characters from G to H . The paradigmatic example of this is the Glauberman correspondence, which is a natural bijection between the P -invariant irreducible characters $\text{Irr}_P(G)$ of a finite group G of order not divisible by a prime p , acted on by the p -group P , and the irreducible characters of the fixed point subgroup $\mathbf{C}_G(P)$. The fact that $\text{Irr}_P(G)$ and $\text{Irr}(\mathbf{C}_G(P))$ have the same number of elements is very important, but that these sets are canonically isomorphic is what lies behind the origin of deep theorems and conjectures in Representation Theory.

If G is a finite group and p is a prime, the *McKay conjecture* (cf. [M₁, M₂]) asserts that $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\mathbf{N}_G(P))|$ for $P \in \text{Syl}_p(G)$, where $\text{Irr}_{p'}(G)$ is the set of the irreducible complex characters of G of degree not divisible by p . Until very recently, this conjecture was known to hold for various classes of finite groups, but remained open in general. A milestone result has been achieved by Malle and Späth in [MS] where they succeeded in proving the McKay conjecture in the case $p = 2$.

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We thank Rod Gow for communicating his observation concerning $GL_n(q)$ in Corollary F to us, as well as for suggesting that the same statement should hold for $GU_n(q)$.

The focus of this paper is, on the other hand, on the existence of *canonical correspondences* between $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(H)$ for certain pairs (G, H) of finite groups with $G > H \geq \mathbf{N}_G(P)$, and specially for $p = 2$. Even in the cases where the McKay conjecture is known to hold for G and H (cf. [O1], [MS]) and thereby the existence of a bijection between $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(H)$ is guaranteed, this resulting bijection usually does not give a canonical correspondence: choices have to be made, and this is what complicates the study of such maps. In the first instance, one expects a canonical correspondence to commute with the action of the absolute Galois group over the rationals, and in this case, the fields of values of character correspondents must be the same. This does not happen often. Also, one expects that canonical correspondences between $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(H)$ will commute with every automorphism of G that stabilizes H , and provide essential information on cohomological character theoretic questions. Furthermore, there is some hope that certain canonical correspondences will play an important role in proving various refinements of the McKay conjecture (eg. [N2]).

In conclusion, when a canonical correspondence is found (and as we say, this does not happen often) it should be possible to understand fundamental properties of certain characters of G by studying the characters of a smaller subgroup H of G . Even more, this correspondence usually affects the behavior of the character theory of convenient overgroups which contain G and H in a certain way. The main purpose of this paper is to prove that for $p = 2$, and for symmetric, general linear, general unitary, and solvable groups, this rare phenomenon does happen. Why these groups and why only for $p = 2$ is a mystery whose explanation we do not see. As we will point out, this is not going to happen for other groups and for other primes, but that is not a surprise. The surprise is that this phenomenon does happen for those selected groups.

For any character χ of G we denote by χ_H its restriction to a subgroup H .

Definition 1.1. An arbitrary subgroup $H \leq G$ is called *p-restriction good* if for every $\chi \in \text{Irr}_{p'}(G)$, there exists a unique $\chi^* \in \text{Irr}_{p'}(H)$ such that $\chi_H = \chi^* + \Delta$ and either $\Delta = 0$ or all irreducible constituents of Δ have degrees divisible by p . A *p-restriction good* subgroup $H \leq G$ is called *p-restriction canonical* if the map $\chi \mapsto \chi^*$ yields a bijection between $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(H)$.

Very recently, the following result* has been proved:

Theorem 1.2. [APS] *Let $n \in \mathbb{Z}_{>1}$. Then S_{n-1} is a 2-restriction good subgroup in S_n . Moreover, if n is odd, then S_{n-1} is a 2-restriction canonical subgroup in S_n .*

In this paper, we prove:

Theorem A. *Let $n \in \mathbb{Z}_{>1}$, q be an odd prime power, and P be a maximal parabolic subgroup of $GL_n(q)$ with Levi subgroup $GL_{n-1}(q) \times GL_1(q)$. Then P is a 2-restriction good subgroup in $GL_n(q)$. Moreover, if n is odd, then P is a 2-restriction canonical subgroup in $GL_n(q)$.*

*In fact, we independently conjectured this statement. But while we were working on the proof of it, we learned of the preprint [APS] in which the conjecture was proved.

Theorem B. *Let $n \in \mathbb{Z}_{>1}$ be odd, q an odd prime power, and Q be a maximal parabolic subgroup of $SL_n(q)$ with Levi subgroup $(GL_{n-1}(q) \times GL_1(q)) \cap SL_n(q)$. Then Q is a 2-restriction canonical subgroup in $SL_n(q)$.*

Theorem C. *Let G be a finite group, p be a prime, and $P \in \text{Syl}_p(G)$. Suppose that $P = \mathbf{N}_G(P)$, and in addition that G is a solvable group if $p = 2$. Let $P \leq H \leq G$. Then H is a p -restriction canonical subgroup in G .*

All these theorems might suggest that further results of this type can hold true for arbitrary finite groups with self-normalizing Sylow 2-subgroups. However, the group $G = SL_3(2)$ has self-normalizing Sylow 2-subgroups, and an irreducible character $\chi \in \text{Irr}(G)$ of degree 7 such that the restriction of χ to *every* odd-index proper subgroup H of G has exactly three irreducible constituents of odd degree. This example also shows that Theorem A does not hold when $2|q$. Other examples also show that analogues of Theorem 1.2 and Theorems A, B do not seem to hold when $p > 2$. However, canonical character correspondences, although not necessarily defined by restriction, can be obtained for symmetric groups and finite general linear and unitary groups, again for $p = 2$.

In this paper we are often speaking of canonical or natural correspondences between characters of a group G and a subgroup $H < G$. We use the word *canonical* or *natural* in the following sense: either the correspondence is obtained in terms of an explicit representation theoretic construction involving restriction from G to H (for example as in Theorems A, B, C), or it is obtained by means of an explicitly described combinatorial bijection on the labels, or a combination of the two. In all cases, we require that the bijection commutes with the action of Galois automorphisms and group automorphisms (of G that stabilize H). We quote I. M. Isaacs in his landmark paper [Is1] that the word *natural* “is intended to mean that an algorithm is given for constructing the correspondence and that the result is independent of any choices made in the application of the algorithm”. We refer the reader to Sections 4, 5 for more details on the following theorems.

Theorem D. *Let $n \in \mathbb{Z}_{>1}$ and let M be a maximal subgroup of S_n of odd index. Then there is a canonical bijection between $\text{Irr}_{2'}(S_n)$ and $\text{Irr}_{2'}(M)$.*

Theorem E. *Let $n \in \mathbb{Z}_{\geq 1}$, q be an odd prime power, $G = GL_n(q)$ or $GU_n(q)$, and $P \in \text{Syl}_2(G)$. Then there is a canonical bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(\mathbf{N}_G(P))$.*

Since our bijection in Theorem E commutes with Galois action (see Theorem 5.3), it follows, for instance, that the fields of values of the odd-degree irreducible characters of $G = GL_n(q)$ and $GU_n(q)$, if q is odd, are in bijection with the fields of values of the odd-degree characters of $\mathbf{N}_G(P)$ for $P \in \text{Syl}_2(G)$. This does not happen in $GL_2(4)$ or $GL_2(8)$. Two other cases where there exists a canonical correspondence for the McKay conjecture for $p = 2$ are in solvable groups [Is1], and symmetric groups [G], see also Theorem 4.3, where the constructed correspondence $\chi \mapsto \chi^\sharp$ has the additional property that χ^\sharp is a constituent of $\chi|_{\mathbf{N}_G(P)}$.

Note that, for some quasisimple groups S and primes p , certain bijections between $\text{Irr}_{p'}(S)$ and $\text{Irr}_{p'}(N)$ for some subgroups $N < S$ containing $\mathbf{N}_S(P)$ with $P \in \text{Syl}_p(S)$ have been constructed which commute with group automorphisms (see e.g. [CS], [MS]). These equivariant bijections play an important role in the recent proof [MS] of the McKay conjecture

for the prime $p = 2$. But it is not clear how these bijections behave with respect to Galois action. In fact, the example of $S = \mathsf{A}_5$ shows that no bijection between $\text{Irr}_{2'}(S)$ and $\text{Irr}_{2'}(\mathbf{N}_S(P))$ (for $p = 2$) can commute with Galois action. Hence, the existence of canonical correspondences in the case of S_n , $GL_n(q)$ and $GU_n(q)$ with q odd, is somewhat a miracle which deserves further investigation for a conceptual explanation.

To illustrate the power of canonical maps, we can answer a question of Gow, which was privately communicated to us.

Corollary F. *The number of real-valued, irreducible characters of odd degree of $G = GL_n(q)$ and $GU_n(q)$, with q any odd prime power, is equal to that of $\mathbf{N}_G(P)$ for $P \in \text{Syl}_2(G)$, which is $2^{n_1+n_2+\dots+n_r+r}$ if $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$ is the 2-adic decomposition of n . Furthermore, all such characters are rational-valued.*

2. RESTRICTION TO A MAXIMAL PARABOLIC SUBGROUP

Unless otherwise stated, we always assume that p is a prime and H is a subgroup of a finite group G . We begin with some simple observations. Note that if the p -restriction good subgroup H of G satisfies $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(H)|$ and $p \nmid |G : H|$, then it is p -restriction canonical, by the following lemma. (The lemma also indicates a possible weakening of the notion of p -restriction good subgroups when one allows a multiplicity > 1 of the p' -degree irreducible constituent).

Lemma 2.1. *Let H have p' -index in G and $|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(H)|$. Suppose that for every $\chi \in \text{Irr}_{p'}(G)$, among the irreducible constituents of χ_H there is only one (but possibly with multiplicity > 1), denoted by χ^* , that has p' -degree. Then the map $* : \chi \mapsto \chi^*$ is a bijection between $\text{Irr}_{p'}(G)$ and $\text{Irr}_{p'}(H)$.*

Proof. For any $\rho \in \text{Irr}_{p'}(H)$, the induced character ρ^G has p' -degree, so it contains a constituent $\chi \in \text{Irr}_{p'}(G)$. By assumption, $\rho = \chi^*$. Thus $*$ is surjective, and so it is injective as well. \square

The following result is well known, see for example [J₁, 22.4].

Lemma 2.2. *Let $a, b \in \mathbb{Z}_{\geq 0}$, $n = a + b$, and consider the decompositions*

$$n = \sum_{i=0}^t 2^i n_i, \quad a = \sum_{i=0}^t 2^i a_i, \quad b = \sum_{i=0}^t 2^i b_i,$$

where $0 \leq a_i, b_i, n_i \leq 1$. Then the following statements are equivalent:

- (a) The binomial coefficient $\binom{n}{a}$ is odd.
- (b) $a_i + b_i = n_i$ for all i .
- (c) $0 \leq a_i \leq n_i$ for all i .

For $r \in \mathbb{Z}_{>0}$ we denote by $[r]_p$ the largest p -power that divides r ; we also set $[0]_p := \infty$.

Corollary 2.3. *Let $a_1, \dots, a_m \in \mathbb{Z}_{>0}$ and $n = \sum_{i=1}^m a_i$. Suppose that $n!/\prod_{i=1}^m a_i!$ is odd. Then, by relabeling a_1, \dots, a_m suitably, we may assume that*

$$[n]_2 = [a_1]_2 < [a_2]_2 < \dots < [a_m]_2.$$

Proof. There is nothing to prove for $m = 1$. We will proceed by induction on $m \geq 2$. Relabeling the a_i 's if necessary, we may assume that

$$(2.1) \quad [a_1]_2 \leq [a_2]_2 \leq \dots \leq [a_m]_2.$$

Assume $m = 2$, and consider the decompositions

$$n = \sum_{i=k}^t 2^i n_i, \quad a_1 = \sum_{i=0}^t 2^i b_i, \quad a_2 = \sum_{i=0}^t 2^i c_i,$$

with $0 \leq b_i, c_i, n_i \leq 1$, $k \geq 0$ and $n_k = 1$. By Lemma 2.2 we have $b_i = c_i = 0$ for $0 \leq i < k$ and, moreover, relabeling a_1 and a_2 if necessary, we may assume that $(b_k, c_k) = (1, 0)$. Thus $[n]_2 = [a_1]_2 < [a_2]_2$ as needed.

For the induction step when $m > 2$, first we apply the case $m = 2$ to $n = a_1 + (n - a_1)$ and (2.1) to get

$$[a_1]_2 \neq [n - a_1]_2 \geq [a_1]_2,$$

whence $[n - a_1]_2 > [a_1]_2 = [n]_2$. Now the statement follows by applying the induction hypothesis to $n - a_1 = \sum_{i=2}^m a_i$. \square

Corollary 2.4. *Suppose that $a, b \in \mathbb{Z}_{>0}$, $n = a + b$, and $\binom{n}{a}$ is odd. Then there is a unique $c \in \{a - 1, a\}$ such that $\binom{n-1}{c}$ is odd. Moreover, if we assume additionally that $[a]_2 \leq [b]_2$, then $\binom{n-1}{a-1}$ is odd.*

Proof. As $\binom{n}{a} = \binom{n-1}{a-1} + \binom{n-1}{a}$, the first claim follows. For the second claim, let $c \in \{a - 1, a\}$ be such that $\binom{n-1}{c}$ is odd. By Corollary 2.3, the assumption $[a]_2 \leq [b]_2$ implies that $[n]_2 = [a]_2 < [b]_2$. If n is odd, then a is odd and b is even, and, by Lemma 2.2, c is even, whence $c = a - 1$. If n is even, we consider the decompositions

$$n = \sum_{i=k}^t 2^i n_i, \quad a = \sum_{i=0}^t 2^i a_i, \quad b = \sum_{i=0}^t 2^i b_i,$$

with $0 \leq a_i, b_i, n_i \leq 1$, $k \geq 1$ and $n_k = 1$. By Lemma 2.2,

$$(a_0, \dots, a_k) = (0, \dots, 0, 1), \quad (b_0, \dots, b_k) = (0, \dots, 0, 0),$$

and so $\binom{n-1}{a}$ is even. Hence again we must have that $c = a - 1$. \square

Recall that complex irreducible characters of S_n are labelled by partitions $\lambda \vdash n$: $\chi = \chi^\lambda$. By Theorem 1.2, there is a canonical map $\lambda \mapsto \lambda^*$ such that, if $\chi^\lambda \in \text{Irr}(S_n)$ is of odd degree then $\chi^* = \chi^{\lambda^*}$ is the unique odd-degree irreducible constituent of $\chi_{S_{n-1}}$.

From now on, we fix an odd prime power q . For any $n \geq 1$, let $G = GL_n(q)$, with a natural module $V = \mathbb{F}_q^n = \langle e_1, \dots, e_n \rangle_{\mathbb{F}_q}$. As in [KT₂], it is convenient for us to use the Dipper-James classification of complex irreducible characters of G , as described in [J₂]. Namely, every $\chi \in \text{Irr}(G)$ can be written uniquely, up to a permutation of the pairs $\{(s_1, \lambda_1), \dots, (s_m, \lambda_m)\}$, in the form

$$(2.2) \quad \chi = S(s_1, \lambda_1) \circ S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m).$$

Here, $s_i \in \bar{\mathbb{F}}_q^\times$ has degree d_i over \mathbb{F}_q , $\lambda_i \vdash k_i$, $\sum_{i=1}^m k_i d_i = n$, and the m elements s_i have pairwise distinct minimal polynomials over \mathbb{F}_q . In particular, $S(s_i, \lambda_i)$ is an irreducible

character of $GL_{k_i d_i}(q)$. Furthermore, there is a parabolic subgroup $P_\chi = U_\chi L_\chi$ of G with Levi subgroup $L_\chi = GL_{k_1 d_1}(q) \times \dots \times GL_{k_m d_m}(q)$ and unipotent radical U_χ . The (outer) tensor product

$$\psi := S(s_1, \lambda_1) \otimes S(s_2, \lambda_2) \otimes \dots \otimes S(s_m, \lambda_m)$$

is an L_χ -character, and χ is obtained from ψ via the Harish-Chandra induction $R_{L_\chi}^G$, i.e. we first inflate ψ to a P_χ -character and then induce it to G . The adjoint operation of Harish-Chandra restriction ${}^*R_{L_\chi}^G$ takes any character ρ of G , afforded by a $\mathbb{C}G$ -module W to the L_χ -character afforded by W^{U_χ} , the fixed point subspace for U_χ on W .

Let $P = UL$ be a maximal parabolic subgroup of G with Levi subgroup $L = GL_1(q) \times GL_{n-1}(q)$ and unipotent radical U . Conjugating suitably in G and applying the transpose-inverse automorphism if necessary, we may assume that $P = \text{Stab}_G(\langle e_1 \rangle_{\mathbb{F}_q})$ and the second factor $GL_{n-1}(q)$ of L fixes both e_1 and $\langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}$.

Given the above notation, we can now prove the following theorem which implies Theorem A:

Theorem 2.5. *Let q be an odd prime power, $n \geq 2$, $G = GL_n(q)$, $P = \text{Stab}_G(\langle e_1 \rangle_{\mathbb{F}_q})$. Suppose that $\chi \in \text{Irr}_{2'}(G)$. Then the following statements hold:*

- (i) *One can choose a label (2.2) for χ such that $s_i \in \mathbb{F}_q^\times$ (so that $d_i = 1$) and $\chi^{\lambda_i} \in \text{Irr}_{2'}(\mathsf{S}_{k_i})$ for all $i = 1, \dots, m$, and*

$$[n]_2 = [k_1]_2 < [k_2]_2 < \dots < [k_m]_2;$$

- (ii) *$\chi_P = \chi^* + \Delta$, where $\chi^* \in \text{Irr}_{2'}(P)$ and either $\Delta = 0$ or Δ is a P -character all irreducible constituents of which are of even degree;*
- (iii) *χ^* is trivial on U , and equal to $S(s_1, (1)) \otimes (S(s_1, \lambda_1^*) \circ S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m))$ when viewed as a character of $GL_1(q) \times GL_{n-1}(q)$;*
- (iv) *If n is odd, then the map $\chi \mapsto \chi^*$ is a bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(P)$.*

Note that in 2.5(iii), the symbol $S(s_1, \lambda_1^*)$ is considered void if $k_1 = 1$. We proceed in a series of lemmas.

Lemma 2.6. *Statement (i) of Theorem 2.5 holds.*

Proof. Since the degree of χ is odd, so are the degrees of each $S(s_i, \lambda_i)$, which implies that $d_i = \deg(s_i) = 1$, for example by [KT₂, Lemma 5.7(ii)], i.e. $s_i \in \mathbb{F}_q^\times$ for all $i = 1, \dots, m$. Next, we also must have that $|G : P_\chi|$ is odd, which implies by a repeated application of [NT₁, Lemma 4.4(i)] that $n! / \prod_{i=1}^m k_i!$ is odd. So we may assume by Corollary 2.3 that $[n]_2 = [k_1]_2 < [k_2]_2 < \dots < [k_m]_2$. Finally, it is well known (and follows from the hook formula for the degree of unipotent characters of G , see [FS, (1.15)]) that

$$(2.3) \quad \chi^\lambda(1) \equiv \deg(S(s, \lambda)) \pmod{2}$$

if $s \in \mathbb{F}_q^\times$, and so we conclude that $\chi^{\lambda_i} \in \text{Irr}_{2'}(\mathsf{S}_{k_i})$. □

Lemma 2.7. *Let $X = X_1 \times X_2$, where $X_1 \cong GL_m(q)$, $X_2 \cong GL_n(q)$, $P_1 = UL_1$ a parabolic subgroup of X_1 with unipotent radical U and Levi subgroup L_1 , and let $L = L_1 \times X_2$.*

- (i) *If α is a character of X_1 and β is a character of X_2 , then ${}^*R_L^X(\alpha \otimes \beta) = {}^*R_{L_1}^{X_1}(\alpha) \otimes \beta$.*
- (ii) *If γ is a character of L_1 and δ is a character of X_2 , then $R_L^X(\gamma \otimes \delta) = R_{L_1}^{X_1}(\gamma) \otimes \delta$.*

Proof. (i) Let α , respectively β , be afforded by a $\mathbb{C}X_1$ -module A , respectively a $\mathbb{C}X_2$ -module B . Then ${}^*R_L^X(\alpha \otimes \beta)$ is afforded by the L -module $(A \otimes B)^U = A^U \otimes B$ and so equal to ${}^*R_{L_1}^{X_1}(\alpha) \otimes \beta$.

(ii) Inflate γ to the character $\tilde{\gamma}$ of P_1 using $P_1/U \cong L_1$, and inflate $\gamma \otimes \delta$ to the character $\rho = \tilde{\gamma} \otimes \delta$ of $P = P_1 \times X_2$ using $P/U \cong L$. Then

$$\begin{aligned} R_L^X(\gamma \otimes \delta) &= \rho^X = ((\tilde{\gamma} \otimes 1_{X_2}) \cdot (1_{X_1} \otimes \delta)_P)^X = \\ &= (\tilde{\gamma} \otimes 1_{X_2})^X \cdot (1_{X_1} \otimes \delta) = (R_{L_1}^{X_1}(\gamma) \otimes 1_{X_2}) \cdot (1_{X_1} \otimes \delta) = R_{L_1}^{X_1}(\gamma) \otimes \delta. \end{aligned}$$

□

Proposition 2.8. *Statements (ii) and (iii) of Theorem 2.5 hold.*

Proof. (a) First we note that, since L acts transitively on the $q^{n-1} - 1$ non-principal irreducible characters of U and q is odd, any irreducible character of P which is nontrivial on U has even degree. Hence all the odd-degree irreducible constituents of χ_P are contained in ${}^*R_L^G(\chi)$. Next, by Lemma 2.6, we already know that $s_i \in \mathbb{F}_q^\times$ for all $i = 1, \dots, m$.

Suppose that $m = 1$. Then χ is a unipotent character of G tensored with a linear character; in particular, χ belongs to the principal series. By the Comparison Theorem [HL, Theorem 5.9] (see also [C, Theorem 5.1] for the case of the principal series), the computation of ${}^*R_L^G(\chi)$ can be replaced by the computation of $(\chi^{\lambda_1})_{S_{n-1}}$, where we identify S_n , respectively S_{n-1} , with the Weyl group of $\mathcal{G} = GL_n(\bar{\mathbb{F}}_q)$, respectively of $\mathcal{L} = GL_1(\bar{\mathbb{F}}_q) \times GL_{n-1}(\bar{\mathbb{F}}_q)$. (See also [FS, Proposition (1C)] for the explicit formula in the case of G .) Applying Theorem 1.2 and (2.3), we are done in this case.

(b) Now we will assume $m \geq 2$ and set $a = k_1$, $b = n - k_1$, where k_1, \dots, k_m satisfy 2.5(i); in particular,

$$(2.4) \quad [a]_2 < [b]_2,$$

and so $a \neq b$. Let

$$M = Stab_G(\langle e_1, \dots, e_a \rangle_{\mathbb{F}_q}) \cap Stab_G(\langle e_{a+1}, \dots, e_n \rangle_{\mathbb{F}_q}) \cong GL_a(q) \times GL_b(q),$$

so that $\chi = R_M^G(\alpha \otimes \beta)$, where

$$\alpha = S(s_1, \lambda_1), \quad \beta = S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m).$$

This follows by the transitivity of the Harish-Chandra induction [DM, Proposition 4.7]. By the Mackey formula for Harish-Chandra induction and restriction (see e.g. [DF, Theorem 1.14]),

$$(2.5) \quad \begin{aligned} {}^*R_L^G(\chi) &= {}^*R_L^G(R_M^G(\alpha \otimes \beta)) = R_{L \cap M}^L \left({}^*R_{L \cap M}^M(\alpha \otimes \beta) \right) \\ &\quad \oplus R_{L \cap w M w^{-1}}^L \left(\text{conj}_w({}^*R_{M \cap w^{-1} L w}^M(\alpha \otimes \beta)) \right), \end{aligned}$$

where w is the permutation matrix corresponding to the cycle $(1, 2, \dots, a+1)$ and conj_w denotes the conjugation by w .

(c) Here we consider the first summand in the right hand side of (2.5). By Lemma 2.7(i),

$${}^*R_{L \cap M}^M(\alpha \otimes \beta) = {}^*R_{GL_1(q) \times GL_{a-1}(q)}^{GL_a(q)}(\alpha) \otimes \beta.$$

It is well known (see e.g. [L, p. 70]) that the Harish-Chandra induction and restriction respect the Lusztig series, which in our case is labeled by the semisimple element in the dual group $G^* \cong G$ that has each $s_i \in \mathbb{F}_q^\times$ as eigenvalue with multiplicity k_i . Hence we can write

$$(2.6) \quad {}^*R_{GL_1(q) \times GL_{a-1}(q)}^{GL_a(q)}(\alpha) = \sum_{j=1}^r S(s_1, (1)) \otimes S(s_1, \mu_j)$$

for some $r \geq 1$ and some partitions $\mu_j \vdash (a-1)$. It then follows by Lemma 2.7(ii) and the transitivity of the Harish-Chandra induction that

$$\begin{aligned} R_{L \cap M}^L({}^*R_{L \cap M}^M(\alpha \otimes \beta)) &= R_{GL_1(q) \times GL_{a-1}(q) \times GL_b(q)}^L \left(\left(\sum_{j=1}^r S(s_1, (1)) \otimes S(s_1, \mu_j) \right) \otimes \beta \right) \\ &= \sum_{j=1}^r S(s_1, (1)) \otimes R_{GL_{a-1}(q) \times GL_b(q)}^{GL_{n-1}(q)}(S(s_1, \mu_j) \otimes \beta) = \sum_{j=1}^r S(s_1, (1)) \otimes \gamma_j \end{aligned}$$

where $\gamma_j := S(s_1, \mu_j) \circ S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m) \in \text{Irr}(GL_{n-1}(q))$. Certainly, the irreducible constituent $S(s_1, (1)) \otimes \gamma_j$ can be of odd degree only when $\deg(S(s_1, \mu_j))$ and

$$|GL_{n-1}(q) : (GL_{a-1}(q) \times GL_b(q))|$$

are both odd. The former implies, by applying (a) to (2.6) that $\mu_j = \lambda_1^*$; furthermore, this happens for exactly one $j \in \{1, 2, \dots, r\}$. The latter implies by [NT₁, Lemma 4.4(i)] that $\binom{n-1}{a-1}$ is odd.

We have shown that the first summand in (2.5) contains at most one irreducible constituent of odd degree, namely

$$S(s_1, (1)) \otimes (S(s_1, \lambda_1^*) \circ S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m));$$

moreover, if such a constituent occurs, then it occurs with multiplicity 1 and $\binom{n-1}{a-1}$ is odd.

(d) Arguing as in (c), we can show that each irreducible constituent in the second summand of (2.5) is of form

$$S(s'_1, (1)) \otimes (S(s'_2, \nu_1) \circ S(s'_3, \nu_2) \circ \dots \circ S(s'_{m-1}, \nu_{m-1}) \circ S(s_1, \lambda_1)),$$

where $s'_1 \in \{s'_2, \dots, s'_m\} = \{s_2, \dots, s_m\}$ and ν_1, \dots, ν_{m-1} are some partitions with the lengths totalling to $b-1$. By [NT₁, Lemma 4.4(i)], such an irreducible constituent can have odd degree only when $\binom{n-1}{b-1} = \binom{n-1}{a}$ is odd.

Given the condition (2.4), Corollary 2.4 implies that $\binom{n-1}{a}$ is even. Thus the second summand in (2.5) contains no odd-degree irreducible constituent. Since $\chi(1)$ is odd, we are done by the result of (c). \square

Completion of the proof of Theorem 2.5. In view of Lemma 2.6, Proposition 2.8, and Lemma 2.1, it remains to prove that $|\text{Irr}_{2'}(G)| = |\text{Irr}_{2'}(P)|$ when $n = 2k+1$. We can choose a Sylow 2-subgroup $S = S_1 \times S_2 < L$ of G , where $S_1 = \mathbf{O}_2(GL_1(q))$, and $S_2 \in \text{Syl}_2(GL_{2k}(q))$ has the form $S_2 = A \wr B$, where $A \in \text{Syl}_2(GL_2(q))$ and $B \in \text{Syl}_2(\mathbf{S}_k)$. Then the S -module V decomposes as $V_1 \oplus V'$, where $V_1 = \langle e_1 \rangle_{\mathbb{F}_q}$ and all irreducible constituents of $V' = \langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}$

are of even dimension. It follows that $\mathbf{N}_G(S)$ is contained in $\text{Stab}_G(V_1) \cap \text{Stab}_G(V') = L$ and so $\mathbf{N}_G(S) = \mathbf{N}_L(S)$. By the main result of [O1],

$$(2.7) \quad |\text{Irr}_{2'}(G)| = |\text{Irr}_{2'}(\mathbf{N}_G(S))| = |\text{Irr}_{2'}(\mathbf{N}_L(S))| = |\text{Irr}_{2'}(L)|.$$

As mentioned in part (a) of the proof of Proposition 2.8, $|\text{Irr}_{2'}(L)| = |\text{Irr}_{2'}(P)|$, and so we are done. \square

Proof of Theorem B. Let $G := GL_n(q)$, and consider normal subgroups $S := SL_n(q)$ and $H := \mathbf{Z}(G)S$ of G .

(i) Consider any $\theta \in \text{Irr}_{2'}(S)$. Since $H = \mathbf{Z}(G) * S$ is a central product, θ extends to H which has odd index $\text{gcd}(n, q-1)$ in G . Hence θ lies under some $\chi \in \text{Irr}_{2'}(G)$. Now we can find a label (2.2) for χ that satisfies 2.5(i). Applying [KT₁, Theorem 1.1] to χ , we then see that χ_S is irreducible, i.e. $\chi_S = \theta$. We have shown that every $\theta \in \text{Irr}_{2'}(S)$ extends to G , whence

$$(2.8) \quad |\text{Irr}_{2'}(S)| = |\text{Irr}_{2'}(G)|/(q-1)$$

as $G/S \cong C_{q-1}$.

(ii) We may assume that $Q = P \cap S$, where P is the maximal parabolic subgroup of G considered in Theorem 2.5. Then $Q = UL_1$ with $L_1 := L \cap S \cong GL_{n-1}(q)$. Note that the normal subgroup

$$K := \text{Stab}_S(e_1) \cap \text{Stab}_S(\langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}) \cong SL_{n-1}(q)$$

of L_1 acts transitively on $\text{Irr}(U) \setminus \{1_U\}$ since $n \geq 3$. It follows that every irreducible character of Q that is nontrivial on U has even degree. In particular,

$$(2.9) \quad |\text{Irr}_{2'}(Q)| = |\text{Irr}_{2'}(L_1)|.$$

(iii) Now we keep the notation of (i) and consider any odd-degree irreducible constituent ζ of θ_Q (which exists since θ has odd degree). We have shown that ζ is trivial on U and so is a constituent of ${}^*R_L^G(\chi)$. We can view ζ as an L_1 -character. Any irreducible character of $\mathbf{Z}(G)L_1 = \mathbf{Z}(G) * L_1$ lying above ζ then also has degree equal to $\zeta(1)$. Next, $\mathbf{Z}(G)L_1$ has odd index $\text{gcd}(n, q-1)$ in L . Thus any irreducible character of L lying above ζ must have odd degree. Applying Theorem 2.5, we now see that ζ is a constituent of $(\chi^*)_{L_1}$, and

$$(\chi^*)_{\tilde{K}} = S(s_1, \lambda_1^*) \circ S(s_2, \lambda_2) \circ \dots \circ S(s_m, \lambda_m)$$

for

$$\tilde{K} := \text{Stab}_G(e_1) \cap \text{Stab}_G(\langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}) \cong GL_{n-1}(q).$$

Since $\chi^*(1)$ is odd, by [NT₁, Lemma 4.4(i)] and Corollary 2.3 we have that $[k_1 - 1]_2, [k_2]_2, \dots, [k_m]_2$ are pairwise different; in particular, $k_1 - 1, k_2, \dots, k_m$ are pairwise different. It then follows by [KT₁, Theorem 1.1] that $(\chi^*)_{\tilde{K}}$ is irreducible. Hence, $(\chi^*)_{L_1}$ is irreducible and $\zeta = (\chi^*)_{L_1}$.

(iv) To complete the proof of Theorem B, by Lemma 2.1 it remains to show that $|\text{Irr}_{2'}(S)| = |\text{Irr}_{2'}(Q)|$. Since

$$|\text{Irr}_{2'}(L)| = |\text{Irr}_{2'}(GL_1(q) \times GL_{n-1}(q))| = (q-1)|\text{Irr}_{2'}(GL_{n-1}(q))| = (q-1)|\text{Irr}_{2'}(L_1)|,$$

we are done by combining (2.7), (2.8), and (2.9). \square

3. FINITE GROUPS WITH SELF-NORMALIZING SYLOW SUBGROUPS

In this section we prove Theorem C. In fact, in this case the key hypothesis is that Sylow p -subgroups are self-normalizing and not the nature of the prime p . The next result was proved in [N₁] for $P = H$ (with a more complicated proof).

Theorem 3.1. *Suppose that G is a p -solvable group, $P \in \text{Syl}_p(G)$, and assume that $P = \mathbf{N}_G(P)$. Let $P \leq H \leq G$. If $\chi \in \text{Irr}(G)$ has p' -degree, then the restriction of χ to H contains a unique p' -degree irreducible constituent $\psi \in \text{Irr}(H)$. This establishes a natural bijection between the p' -degree irreducible characters of G and H .*

Proof. We proceed by induction on $|G|$. Let $\chi \in \text{Irr}(G)$ be of p' -degree. Let N be a normal subgroup of p' -order. Since $\chi(1)$ is not divisible by p , using the Clifford correspondence (Theorem (6.11) of [Is₂]), let $\theta \in \text{Irr}(N)$ be P -invariant under χ . Then $\mathbf{C}_N(P) = 1$ by hypothesis, and so $\theta = 1_N$, by the Glauberman correspondence (Theorem (13.1) of [Is₂]). Therefore, $N \leq \ker(\chi)$. Now consider $\bar{\chi} \in \text{Irr}(G/N)$ the corresponding character in the factor group. By induction $\bar{\chi}_{NH}$ contains a unique p' -irreducible constituent ψ . Now, all the constituents of this character restrict irreducibly to H , so we are done in this case. So we may assume that $N = 1$.

Let now N be a normal p -subgroup of G . Then χ_P contains a linear constituent λ and $\lambda_N = \nu \in \text{Irr}(N)$. Let $P \leq T$ be the stabilizer of ν in G , and let $\psi \in \text{Irr}(T)$ be the Clifford correspondent of χ over ν . It is well known by Clifford theory (see [S, Lemma 2.5]) that

$$\chi_H = (\psi_{T \cap H})^H + \Delta$$

where no irreducible constituent of Δ lies over ν . We claim that no irreducible constituent of Δ has p' -degree. If ξ is an irreducible constituent of Δ of p' -degree, then ξ_N has some linear P -invariant constituent ϵ . But then χ_N has P -invariant irreducible constituents ν and ϵ , and by a standard argument they are $\mathbf{N}_G(P)$ -conjugate. However, $\mathbf{N}_G(P) = P$, and since both are P -invariant, they are equal. This contradicts the choice of Δ . Suppose now that $T < G$. Notice that T has a self-normalizing Sylow p -subgroup. By induction,

$$\psi_{T \cap H} = \mu + \rho,$$

where μ has p' -degree and every irreducible constituent of ρ has degree divisible by p . All these constituents lie over ν , so all of them induce irreducibly to H by the Clifford correspondence. We conclude that

$$\chi_H = (\psi_{T \cap H})^H + \Delta = \mu^H + \rho^H + \Delta,$$

and the theorem is again proved in this case.

Hence the last case is that $T = G$. We claim that ν extends to G . This is because ν extends to P , by coprimeness it extends to every $Q/N \in \text{Syl}_q(G/N)$ if $q \neq p$ (by Corollary (6.27) of [Is₂]), and therefore it extends to G by Theorem (6.26) of [Is₂]. Let $\lambda \in \text{Irr}(G)$ be a linear extension of ν . By Gallagher's Corollary (6.17) of [Is₂], we can write $\chi = \beta\lambda$, where $\beta \in \text{Irr}(G/N)$, has p' -degree. By induction,

$$\beta_H = \tau + \Xi,$$

where every irreducible constituent of Ξ has degree divisible by p . Now

$$\chi_H = \lambda_H \beta_H = \lambda_H \tau + \lambda_H \Xi,$$

and this proves half of the theorem. The bijectivity of the map is proved similarly. \square

The fact that $\mathbf{N}_G(P) = P$ is key. For instance, take the semidirect product G of an extraspecial group $F = 3_+^{1+2}$ of exponent 3 acted on by C_4 in such a way that C_4 acts trivially on the center of F . The normalizer of a Sylow 2-subgroup in G is $N = C_4 \times C_3$, is even maximal. Every restriction of the irreducible characters of degree 3 has three irreducible linear constituents.

Proof of Theorem C. In view of Theorem 3.1, we may assume $p > 2$. Decompose $\chi_H = \sum_{i=1}^t \rho_i + \sum_{j=1}^s \psi_j$, where $t \geq 1$, $s \geq 0$, $\rho_i, \psi_j \in \text{Irr}(H)$, $p \nmid \rho_i(1)$, $p \mid \psi_j(1)$. Now each $\rho_i|_P$ contains a linear character, and by [NTV, Corollary B], $\chi_P = \lambda + \delta$ contains a unique linear character λ (so that either $\delta = 0$ or all irreducible constituents of δ have degree divisible by p). Hence $t = 1$, and we have a well-defined map

$$* : \chi \mapsto \chi^* := \rho_1$$

from $X := \text{Irr}_{p'}(G)$ to $Y := \text{Irr}_{p'}(H)$.

Now, for any $\rho \in Y$, ρ^G contains a p' -degree $\chi \in \text{Irr}(G)$ and then χ_H contains ρ , so $\chi^* = \rho$. Thus $*$ is onto. By [NTV, Corollary B], $|X| = |P/P'| = |Y|$. So $*$ is a bijection. \square

4. CANONICAL CHARACTER CORRESPONDENCES IN SYMMETRIC GROUPS

Note that the natural character correspondences described in [APS] (see Theorem 1.2) and in Theorems A, B, and C are given by restriction. In the case of $G = S_n$, one may ask whether the restriction to a Young subgroup H of odd index in G would also give such a correspondence between the odd-degree characters of G and those of H . Unfortunately, the answer is no: if $G = S_7$ and $H = S_5 \times S_2$, then the restriction to H of any irreducible character of degree 35 of G have three odd-degree irreducible constituents.

When $G = S_n$ and $P \in \text{Syl}_2(G)$, a canonical bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(\mathbf{N}_G(P))$ was defined in [G] (although given by restriction to P only when n is a 2-power). Another type of correspondence (given for an arbitrary prime) is described in [E].

We will construct a new explicit canonical bijection, where, in addition, the correspondent $\chi^\sharp \in \text{Irr}_{2'}(P)$ of $\chi \in \text{Irr}_{2'}(G)$ is an irreducible constituent of χ_P (although not necessarily occurring with multiplicity one). Let $\mathcal{H}(n)$ denote the set of n hook partitions $(n-i, 1^i)$, $0 \leq i < n$, of n . If $\lambda \vdash n$, then $Y(\lambda)$ is the Young diagram of λ . Furthermore, we will say that λ is an *odd* partition of n (and write $\lambda \vdash_o n$) exactly when $\chi^\lambda \in \text{Irr}(S_n)$ has odd degree.

Lemma 4.1. *Let $n \in \mathbb{Z}_{\geq 1}$ and let γ be a partition of n with a hook H of length m in $Y(\gamma)$. Let $\beta \in \mathcal{H}(m)$ be the hook partition corresponding to H , and let $\alpha \vdash (n-m)$ be such that $Y(\alpha)$ is obtained from $Y(\gamma)$ by removing the rim m -hook R of $Y(\gamma)$ corresponding to H . Then $(\chi^\gamma)_{S_{n-m} \times S_m}$ contains $\chi^\alpha \otimes \chi^\beta$ as an irreducible constituent with multiplicity one.*

Proof. We apply the Littlewood-Richardson rule as given in [JK, Corollary 2.8.14]. Write $\beta = (k, 1^{m-k})$ for some $1 \leq k \leq m$ and consider the symbols a_{ij} where the (i, j) -node belongs to $Y(\beta)$:

$$a_{11}, a_{12}, \dots, a_{1k}, a_{21}, a_{31}, \dots, a_{m-k+1,1}.$$

We need to count the number of ways of adding symbols a_{ij} to $Y(\alpha)$ to get $Y(\gamma)$, that is, to fill up the rim m -hook R of $Y(\gamma)$ with these symbols a_{ij} in such a way that all conditions (i), (ii), (iii) of [JK, Corollary 2.8.14] are fulfilled. First we have to put the k symbols $a_{11}, a_{12}, \dots, a_{1k}$ in k different columns of R consecutively starting from the rightmost to get a proper diagram. Since R has exactly k columns, we have to put these symbols at the top of these k columns, and there is exactly one way to do it. Then we need to put the $m - k$ symbols $a_{21}, a_{31}, \dots, a_{m-k+1,1}$ in the $m - k$ remaining rows of R consecutively starting from the highest row. Since there remain only $m - k$ rows, each with one node, there is again exactly one way to do it. \square

We refer to [JK, §2.7] for the notion of the m -core of any partition $\lambda \vdash n$ and any $m \in \mathbb{Z}_{\geq 1}$.

Lemma 4.2. *Let $m \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}$ be such that $m \leq n \leq 2m - 1$, and let $\alpha = (a_1, a_2, \dots, a_r) \vdash (n - m)$ with $a_1 \geq a_2 \geq \dots \geq a_r > 0$.*

- (i) *For any $\beta = (k, 1^{m-k}) \in \mathcal{H}(m)$, there is exactly one $\gamma \vdash n$ such that γ has a hook H of length m that corresponds to β , and, furthermore, $Y(\alpha)$ is obtained from $Y(\gamma)$ by removing the rim m -hook corresponding to H .*
- (ii) *Suppose in addition that m is a 2-power and that $\alpha \vdash_o (n - m)$. Then, for each $\beta = (k, 1^{m-k}) \in \mathcal{H}(m)$, there is a unique $\lambda \vdash_o n$ such that γ has a hook H of length m that corresponds to β , and, furthermore, α is the m -core of γ .*

Proof. (i) We determine in how many ways $Y(\alpha)$ can be obtained from $Y(\gamma)$ by removing a rim m -hook R that corresponds to H , a hook of length m with associated partition $\beta \in \mathcal{H}(m)$; in particular, R spans $m - k + 1$ rows and k columns. Since $n \leq 2m - 1$, H must be the hook corresponding to a node that belongs either to the first row or the first column of $Y(\gamma)$. Thus R must touch either the first row or the first column of $Y(\alpha)$. We consider the outer rim S of $Y(\alpha)$, which consists of all the (i, j) -nodes, where at least one of the $(i, j-1)$ -node, $(i-1, j-1)$ -node, $(i, j-1)$ -node belongs to the rim of $Y(\alpha)$. Then S spans $r + 1$ rows and $a_1 + 1$ columns and so has length $L := a_1 + r + 1$. Note that

$$a_1 + r \leq a_1 + a_2 + \dots + a_r + 1 = n - m + 1 \leq m.$$

Suppose first that $m - k + 1 \leq r$. Then R spans $k \geq m - r + 1 \geq a_1 + 1$ columns. If R does not touch the first row of $Y(\alpha)$, then it must touch the first column of $Y(\alpha)$ and so spans at most a_1 columns, a contradiction. Thus R must begin in the first row of $Y(\alpha)$. In this case, since $k \leq r$, R consists precisely of the N_1 nodes that belong to the first $m - k + 1$ rows of S , together with $m - N_1$ more nodes in the first row, to the right of the top node of S . (Note that, since the $(r+1)$ th row of S has at least two nodes, we have $N_1 \leq L - 2 \leq m - 1$.) Thus γ exists and is unique in this case.

Next suppose that $m - k + 1 \geq r + 1$. If R does not touch the first column of $Y(\alpha)$, then R spans at most r rows, a contradiction. Hence R must start from the first column

of $Y(\alpha)$. If in addition $k \leq a_1$, then R consists precisely of the N_2 nodes that belong to the first k columns of S , together with $m - N_2$ more nodes in the first column, below the last row of S . (Note that, since the $(a_1 + 1)$ th column of S has at least two nodes, we have $N_2 \leq L - 2 \leq m - 1$.) Thus γ exists and is unique in this case. Finally, if $k \geq a_1 + 1$, then R consists of S together with $k - (a_1 + 1)$ more nodes in the first row (to the right of the top node of S), and $(m - k + 1) - (r + 1)$ additional nodes in the first column (below the last row of S). Again, γ exists and is unique in this case.

(ii) By [APS, Lemma 1] (see also [O2, §6]), there are exactly m odd partitions $\lambda_i \vdash_o n$, $1 \leq i \leq m$, such that the m -core of λ_i is α , and, furthermore, each λ_i has a unique hook H_i of length m . As $n \leq 2m - 1$, $Y(\alpha)$ is obtained from $Y(\lambda_i)$ by removing the rim m -hook corresponding to H_i . Each H_i corresponds to exactly one of m hook partitions $\beta_i \in \mathcal{H}(m)$, and, as we showed in (i), λ_i is uniquely determined by β_i . It follows that $\beta_1, \beta_2, \dots, \beta_m$ are pairwise distinct and so are precisely the m hook partitions of m . \square

Theorem 4.3. *Let $n \in \mathbb{Z}_{\geq 1}$ and let $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$ be the 2-adic decomposition of n , where $n_1 > n_2 > \dots > n_r \geq 0$. Let $G = S_n$ and $P \in \text{Syl}_2(G)$.*

(i) *There are canonical bijections*

$$\text{Irr}_{2'}(G) \xrightarrow{\alpha} \Theta(n) \xleftarrow{\beta} \text{Irr}_{2'}(P)$$

with $\Theta(n) := \mathcal{H}(2^{n_1}) \times \mathcal{H}(2^{n_2}) \times \dots \times \mathcal{H}(2^{n_r})$.

- (ii) *If $\alpha(\chi) = (\mu_1, \mu_2, \dots, \mu_r)$ for $\chi \in \text{Irr}_{2'}(G)$, then the restriction of χ to the Young subgroup $S_{2^{n_1}} \times S_{2^{n_2}} \times \dots \times S_{2^{n_r}}$ contains $\chi^{\mu_1} \otimes \chi^{\mu_2} \otimes \dots \otimes \chi^{\mu_r}$ as an irreducible constituent.*
- (iii) *The map $\chi \mapsto \chi^\sharp := \beta^{-1}(\alpha(\chi))$ is a canonical bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(P)$. Furthermore, χ^\sharp is an irreducible constituent of χ_P .*

Proof. We can take $P = P_1 \times P_2 \times \dots \times P_r$ with $P_i \in \text{Syl}_2(S_{2^{n_i}})$. Then any $\lambda \in \text{Irr}_{2'}(P)$ is of the form $\lambda = \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_r$ with $\lambda_i \in \text{Irr}_{2'}(P_i)$. By Lemma 3.1 and Theorem 3.2 of [G], there is a unique $\nu_i \in \mathcal{H}(2^{n_i})$ such that λ_i is the unique odd-degree irreducible constituent of the restriction of $\chi^{\nu_i} \in \text{Irr}_{2'}(S_{2^{n_i}})$ to P_i . Now we can define

$$\beta(\lambda) := (\nu_1, \nu_2, \dots, \nu_r).$$

Next, consider any $\chi = \chi^\pi \in \text{Irr}_{2'}(S_n)$, so that $\pi \vdash_o n$. By [APS, Lemma 1], π contains a unique 2^{n_1} -hook corresponding to $\mu_1 \in \mathcal{H}(2^{n_1})$, and $\pi_2 \vdash_o (n - 2^{n_1})$, where π_2 is the 2^{n_1} -core of π . Conversely, any such pair (μ_1, π_2) can be obtained in this way from a unique $\pi \vdash_o n$ by Lemma 4.2. As $Y(\pi_2)$ is obtained from $Y(\pi)$ by removing the rim 2^{n_1} -hook corresponding to μ_1 , by Lemma 4.1 we have that the restriction of χ^π to $S_{2^{n_1}} \times S_{n-2^{n_1}}$ contains $\chi^{\mu_1} \otimes \chi^{\pi_2}$ as an irreducible constituent. Applying this process to π_2 we then get a pair (μ_2, π_3) with $\mu_2 \in \mathcal{H}(2^{n_2})$ and $\pi_3 \vdash_o (n - 2^{n_1} - 2^{n_2})$. Repeating this process successively, we get $\mu_i \in \mathcal{H}(2^{n_i})$ for all i , and can then define

$$\alpha(\chi) := (\mu_1, \mu_2, \dots, \mu_r).$$

One easily checks that both α and β are bijections, and that the claims in (ii) and (iii) concerning restrictions of χ to Y and P are fulfilled.

Note that irreducible characters of $G = S_n$ are rational, and the same is true for odd-degree irreducible characters of $P \in \text{Syl}_2(G)$, see [NT₂, Lemma 3.3]. Furthermore, if $n \neq 6$, then all automorphisms of S_n are inner and $P = N_G(P)$, so every automorphism of G stabilizing P is induced by a conjugation $j_x : g \mapsto xgx^{-1}$ for some $x \in P$. Hence the bijection $\chi \mapsto \chi^\sharp$ in Theorem 4.3 commutes with the actions of Galois automorphisms and group automorphisms stabilizing P . Direct calculations show that the same conclusion holds in the case $n = 6$. \square

Remark 4.4. In the notation of Theorem 4.3, the multiplicity of χ^\sharp in χ_P is not always equal to one. For example, if $n = 12$ and $\chi = \chi^{(6,2^3)}$ (so $\chi(1) = 275$), direct computation shows that the multiplicity of χ^\sharp in χ_P is equal to 4.

Corollary 4.5. *Let $G = S_n$ and let $Y = S_{k_1} \times S_{k_2} \times \dots \times S_{k_m}$ be a Young subgroup of odd index in G . Then there is a canonical bijection $\chi \mapsto \chi^*$ between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(Y)$.*

Proof. For each i , choose $P_i \in \text{Syl}_2(S_{k_i})$. Then $P := P_1 \times P_2 \times \dots \times P_m \in \text{Syl}_2(G)$. Consider any $\chi \in \text{Irr}_{2'}(G)$ and let $\lambda := \chi^\sharp \in \text{Irr}_{2'}(P)$ as in Theorem 4.3. Next, write $\lambda = \lambda_1 \otimes \lambda_2 \otimes \dots \otimes \lambda_m$ with $\lambda_i \in \text{Irr}_{2'}(P_i)$. For each i there is a unique $\psi_i \in \text{Irr}_{2'}(S_{k_i})$ such that $(\psi_i)^\sharp = \lambda_i$ by Theorem 4.3. Now we can define

$$\chi^* = \psi := \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_m \in \text{Irr}_{2'}(Y).$$

It is easy to check that the map $\chi \mapsto \chi^*$ is a bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(Y)$.

Note that all irreducible characters in question are rational. Furthermore, k_1, k_2, \dots, k_m are pairwise distinct by Corollary 2.3, and so $N_G(Y) = Y$. Hence, if $n \neq 6$ then all automorphisms of G stabilizing Y are induced by conjugations $j_x : g \mapsto xgx^{-1}$ with $x \in Y$. If $n = 6$, then $Y = S_4 \times S_2$ and the same conclusion holds. It follows that the constructed bijection commutes with the actions of Galois automorphisms and group automorphisms stabilizing Y . \square

We are now ready to prove Theorem D.

Proof of Theorem D. (i) Let M be a maximal subgroup of $G = S_n$ of odd index. We will construct a canonical bijection $\chi \mapsto \chi^*$ between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(M)$. If $1 \leq n \leq 4$, then $M \in \text{Syl}_2(G)$ and so we can set $\chi^* = \chi^\sharp$ using Theorem 4.3. Thus we may assume that $n \geq 5$. Maximal subgroups of odd index of G are determined by the main result of [LS] and independently by [K, Theorem C], and they are either maximal Young subgroups or stabilizers $S_k \wr S_t$ of set partitions of $\{1, 2, \dots, n = kt\}$ into t subsets of size k . In the former case we can apply Corollary 4.5 (with $m = 2$).

So we will assume that $M = S_k \wr S_t$, and write $M = N \rtimes S_t$ where

$$N := S_k \times S_k \times \dots \times S_k \cong (S_m)^t$$

and S_t naturally permutes the t direct factors of N .

(ii) Consider any $\varphi \in \text{Irr}_{2'}(M)$. Since $\varphi(1)$ is odd, the length of the S_t -orbit of any irreducible constituent of φ_N is odd. Hence, by Corollary 2.3 and using the action of S_t on N , we can find an irreducible constituent

$$\psi = \underbrace{\psi_1 \otimes \dots \otimes \psi_1}_{t_1} \otimes \underbrace{\psi_2 \otimes \dots \otimes \psi_2}_{t_2} \otimes \dots \otimes \underbrace{\psi_m \otimes \dots \otimes \psi_m}_{t_m}$$

of φ_N , where $m \geq 1$, $1 \leq [t_1]_2 < [t_2]_2 < \dots < [t_m]_2$, $\sum_{i=1}^m t_i = t$, and $\psi_1, \dots, \psi_m \in \text{Irr}_{2'}(\mathsf{S}_k)$ are pairwise distinct. Note that each S_t -orbit of odd length on $\text{Irr}_{2'}(N)$ contains a unique representative ψ satisfying these conditions. Then $Y := \text{Stab}_{\mathsf{S}_t}(\psi) = \mathsf{S}_{t_1} \times \mathsf{S}_{t_2} \times \dots \times \mathsf{S}_{t_m}$ is a Young subgroup of odd index in S_t , and the inertia group J of ψ in M is precisely $N \rtimes Y$. Note that ψ has a canonical extension $\tilde{\psi}$ to J : if ψ_i is afforded by a $\mathbb{C}\mathsf{S}_k$ -module V_i , then we can let Y act on

$$V := \underbrace{V_1 \otimes \dots \otimes V_1}_{t_1} \otimes \underbrace{V_2 \otimes \dots \otimes V_2}_{t_2} \otimes \dots \otimes \underbrace{V_m \otimes \dots \otimes V_m}_{t_m}$$

via naturally permuting the tensor factors, and then take $\tilde{\psi}$ to be the J -character afforded by V . By the Clifford correspondence and Gallagher's Corollary (6.17) of [Is₂] (or directly by [JK, Theorem 4.4.3]), we have

$$\varphi = (\tilde{\psi}\alpha)^M,$$

where $\alpha \in \text{Irr}_{2'}(Y)$. Thus each $\varphi \in \text{Irr}_{2'}(M)$ defines a unique pair (ψ, α) in a canonical way.

(iii) Fix $R \in \text{Syl}_2(\mathsf{S}_k)$. Then we can find $Q \in \text{Syl}_2(\mathsf{S}_t)$ such that $P = R \wr Q = R^t \rtimes Q$ is a Sylow 2-subgroup of M .

Consider any $\lambda \in \text{Irr}_{2'}(P)$. Then, as in (ii), after a suitable S_t -conjugation, we can write

$$\mu := \lambda_{R^t} = \underbrace{\mu_1 \otimes \dots \otimes \mu_1}_{s_1} \otimes \underbrace{\mu_2 \otimes \dots \otimes \mu_2}_{s_2} \otimes \dots \otimes \underbrace{\mu_{m'} \otimes \dots \otimes \mu_{m'}}_{s_{m'}},$$

where $m' \geq 1$, $1 \leq [s_1]_2 < [s_2]_2 < \dots < [s_{m'}]_2$, $\sum_{i=1}^{m'} s_i = t$, and $\mu_1, \dots, \mu_{m'} \in \text{Irr}_{2'}(R)$ are pairwise distinct. Note that each S_t -orbit of odd length on $\text{Irr}_{2'}(R^t)$ contains a unique representative μ satisfying these conditions. Then $Y' := \text{Stab}_{\mathsf{S}_t}(\mu) = \mathsf{S}_{s_1} \times \mathsf{S}_{s_2} \times \dots \times \mathsf{S}_{s_{m'}}$ is a Young subgroup of S_t . Since μ is Q -invariant, Y' has odd index in S_t , and the inertia group J' of μ in $R^t \rtimes \mathsf{S}_t$ is precisely $R^t \rtimes Y'$. We can replace Q by $Q' := Q_1 \times Q_2 \times \dots \times Q_{m'}$ where $Q_i \in \text{Syl}_2(\mathsf{S}_{s_i})$. As in (ii), μ has a canonical extension $\tilde{\mu}$ to J' . We now have that

$$\lambda = (\tilde{\mu})_P \cdot \beta,$$

where $\beta \in \text{Irr}_{2'}(Q')$. Thus each $\lambda \in \text{Irr}_{2'}(P)$ defines a unique pair (μ, β) in a canonical way.

(iv) Now we can define χ^* for each $\chi \in \text{Irr}_{2'}(G)$ as follows. First, let $\lambda := \chi^\sharp \in \text{Irr}_{2'}(P)$ as in Theorem 4.3. Then we can apply the analysis in (iii) to λ to get the canonical pair (μ, β) . We will now rename m' by m , s_i by t_i , and Y' by Y . For each i , there is a unique $\psi_i \in \text{Irr}_{2'}(\mathsf{S}_k)$ such that $\psi_i^\sharp = \mu_i$. Next,

$$\beta = \beta_1 \otimes \beta_2 \otimes \dots \otimes \beta_m,$$

where $\beta_i \in \text{Irr}_{2'}(Q_i)$. Again, for each i there is a unique $\alpha_i \in \text{Irr}_{2'}(\mathsf{S}_{t_i})$ such that $\alpha_i^\sharp = \beta_i$, and then we take

$$\alpha := \alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_m \in \text{Irr}_{2'}(Y).$$

Now we define $\chi^* := \varphi$, where φ corresponds to (ψ, α) as in (ii). It is straightforward to check that this map is a bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(M)$.

Note that the irreducible characters of S_n are all rational. Furthermore, all the automorphisms of S_n that stabilize a fixed Young subgroup Y of odd index are induced by

conjugations $j_x : g \mapsto xgx^{-1}$ with $x \in Y$. This is obvious for $n \neq 6$ as all automorphisms of S_n are inner and Y is maximal. If $n = 6$, then one can check that $Y \cong S_4 \times S_2$ and again all automorphisms of S_n stabilizing Y are inner. Hence the defined bijection commutes with the action of Galois automorphisms and group automorphisms stabilizing Y . \square

5. CANONICAL CHARACTER CORRESPONDENCES IN FINITE GENERAL LINEAR AND UNITARY GROUPS

For $\kappa = \pm$, we let $GL_n^\kappa(q)$ denote $GL_n(q)$ when $\kappa = +$ and $GU_n(q)$ when $\kappa = -$. Let C_κ denote the unique subgroup of order $q - \kappa 1$ of $\mathbb{F}_{q^2}^\times$, and let \tilde{C}_κ denote the character group of C_κ (under the pointwise product). We will fix a generator ε of C_κ , a primitive $(q - \kappa 1)$ th root of unity $\tilde{\varepsilon} \in \mathbb{C}$. Then, for $s = \varepsilon^j \in C_\kappa \leq \mathbb{F}_{q^2}^\times$, let \hat{s} denote the character that sends ε to $\tilde{\varepsilon}^j$. We can identify

$$(5.1) \quad \text{Irr}(\mathbf{O}_2(C_\kappa)) \times \text{Irr}(\mathbf{O}_{2'}(C_\kappa)) \longleftrightarrow \tilde{C}_\kappa.$$

Let Γ denote the Galois group $\text{Gal}(\mathbb{Q}(\tilde{\varepsilon})/\mathbb{Q})$, and let Γ act on C_κ via $\sigma(\varepsilon) = \varepsilon^i$ whenever $\sigma(\tilde{\varepsilon}) = \tilde{\varepsilon}^i$. We will fix a suitable basis of the natural module $V = \mathbb{F}_q^n$, resp. $\mathbb{F}_{q^2}^n$, for G , and use it to define the Frobenius automorphism F_p sending any matrix $X := (a_{ij})$ to (a_{ij}^p) in that basis (where p is the unique prime divisor of q) and, additionally, the inverse-transpose automorphism $\tau : X \mapsto {}^t X^{-1}$ when $\kappa = +$. If $\kappa = -$ we set $\tau := (F_p)^f$ for $q = p^f$. Let

$$(5.2) \quad D := \langle \tau, F_p \rangle,$$

and let D act on C_κ and correspondingly on \tilde{C}_κ via

$$F_p(s) = s^p, \quad F_p(\hat{s}) = \hat{s}^p, \quad \tau(s) = s^{-1}, \quad \tau(\hat{s}) = \bar{\hat{s}} = \hat{s}^{-1}.$$

We also let D act trivially on $\mathcal{H}(n)$, the set of hook partitions of n .

For $\lambda \vdash n$, let φ^λ denote the unipotent character of $GL_n^\kappa(q)$ labeled by λ . By [DM, Proposition 13.30], $S(s, \lambda) = \varphi^\lambda(\hat{s} \circ \det)$ if $\kappa = +$; note that $\hat{s} \in \tilde{C}_\kappa$ is uniquely determined by $S(s, \lambda)$, so in what follows we also use $S(\hat{s}, \lambda)$ to denote $S(s, \lambda)$. If $\kappa = -$, we also define

$$S(\hat{s}, \lambda) := \varphi^\lambda(\hat{s} \circ \det).$$

Lemma 5.1. *Let $n = 2^m$ for some $m \in \mathbb{Z}_{\geq 0}$, q be an odd prime power, $\kappa = \pm$, $G = GL_n^\kappa(q)$, and $P \in \text{Syl}_2(G)$.*

- (i) *The field of values of any character in $\text{Irr}_{2'}(\mathbf{N}_G(P))$ is contained in $\mathbb{Q}(\tilde{\varepsilon})$.*
- (ii) *There is a canonical bijection between $\text{Irr}_{2'}(\mathbf{N}_G(P))$ and $\tilde{C}_\kappa \times \mathcal{H}(n)$ that commutes with the action of Γ . Furthermore, we can choose P to be D -invariant and the canonical bijection to be D -equivariant.*
- (iii) *$\text{Irr}_{2'}(\mathbf{N}_G(P))$ contains exactly 2^{m+1} real characters, and all of them are also rational.*

Proof. It is well known that

$$(5.3) \quad \mathbf{N}_G(P) = ZP = Z_1 \times P,$$

where $Z := \mathbf{Z}(G)$ and $Z_1 := \mathbf{O}_{2'}(Z)$ (see eg. [Ko, Theorem 1] for $n > 2$). Note that Z can be canonically identified with C_κ . The case $m = 0$ is obvious, so we will assume $m \geq 1$.

(a) Consider the case $n = 2$. Suppose $q \equiv \kappa 1 \pmod{4}$. Then we can take a basis, which is orthonormal if $\kappa = -$, of $V_1 = \mathbb{F}_q^2$ or $\mathbb{F}_{q^2}^2$, define F_p and τ in this basis, and choose $P = T_2 \rtimes \langle z \rangle$, where $T_2 = \mathbf{O}_2(T)$, $T := \{\text{diag}(x, y) \mid x, y \in C_\kappa\}$, and $|z| = 2$. It is easy to see that each $\lambda \in \text{Irr}_{2'}(P)$ is uniquely determined by $\gamma \in \text{Irr}(\mathbf{O}_2(C_\kappa))$ and $j \in \{0, 1\}$, where $\lambda(g) = \gamma(xy)$ for $g = \text{diag}(x, y) \in T_2$ and $\lambda(z) = (-1)^j$. Also, P is D -invariant.

Suppose $q \equiv -\kappa 1 \pmod{4}$. Then we can find $\beta \in \mathbb{F}_{q^2}^\times$ of order $(q^2 - 1)_2$. First we consider the case $\kappa = +$. Then $(q^2 - 1)_2 = (p^2 - 1)_2$ and so $\beta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_q$, $\beta^{p+1} = -1$, and $\beta + \beta^p \in \mathbb{F}_p$. Now we can identify $V_1 = \mathbb{F}_q^2$ with \mathbb{F}_{q^2} and take x to be the multiplication by β , z to be the Frobenius map $v \mapsto v^q$. We then choose $(1, \beta)$ to be a basis (over \mathbb{F}_q) for V_1 , and use this basis to define F_p and τ . This ensures that F_p fixes both x and z (as they have matrices $\begin{pmatrix} 0 & 1 \\ 1 & \beta + \beta^p \end{pmatrix}$ and $\begin{pmatrix} 1 & \beta + \beta^p \\ 0 & -1 \end{pmatrix}$ in the given basis), and furthermore $\tau(x) = x^{-1}$, $\tau(z) = x^{-2}z$. If $\kappa = -$, then we can choose a Witt basis for $V_1 = \mathbb{F}_{q^2}^2$, and take

$$x = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-q} \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in this basis. Using the same basis to define F_p , we get $F_p(x) = x^p$, $F_p(z) = z$. In either case, P is D -invariant, and D acts trivially on $P/P' \cong C_2^2$, and again each $\lambda \in \text{Irr}_{2'}(P)$ is uniquely determined by $\gamma \in \text{Irr}(\mathbf{O}_2(C_\kappa))$ and $j \in \{0, 1\}$.

Now any $\alpha \in \text{Irr}_{2'}(\mathbf{N}_G(P))$ is of the form $\alpha = \delta \otimes \lambda$, where $\delta \in \text{Irr}(Z_1)$ and $\lambda \in \text{Irr}_{2'}(P)$. Then we send α to $(\hat{s}, (2-j, j))$, where $\hat{s} \in \tilde{C}_\kappa$ corresponds to (γ, δ) via (5.1) and certainly $(2-j, j) \in \mathcal{H}(2)$. One easily checks that (i) and (ii) hold in this case. Furthermore, $\alpha = \bar{\alpha}$ exactly when $\delta = 1_{Z_1}$ and $\gamma = \bar{\gamma}$, which then also imply that α is rational. It also follows that there are exactly 4 of such α 's.

(b) In the general case, we can fix a direct sum decomposition

$$V = \underbrace{V_1 \oplus V_1 \oplus \dots \oplus V_1}_{2^{m-1}},$$

where V_1 is considered with the basis chosen in (a), and fix a basis of V compatible with that basis of V_1 and this decomposition. We then use this basis to define F_p , τ , and choose $P = P_1 \rtimes Q$, where $P_1 = R \times R \times \dots \times R \cong R^{2^{m-1}}$, $R \in \text{Syl}_2(GL_2^\kappa(q))$, and $Q \in \text{Syl}_2(\Sigma)$ with $\Sigma \cong S_{2^{m-1}}$ naturally permuting the 2^{m-1} direct summands in V and, correspondingly, the 2^{m-1} direct factors in P_1 . Note that Q also permutes these direct factors transitively; furthermore, D acts trivially on Q and stabilizes both P_1 and P . It follows that any $\theta \in \text{Irr}_{2'}(P)$ is uniquely determined by $\lambda \in \text{Irr}_{2'}(R)$ and $\nu \in \text{Irr}_{2'}(Q)$, where $\theta_{P_1} = \lambda \otimes \lambda \otimes \dots \otimes \lambda$ and $\theta_Q = \nu$. As in (i), λ corresponds to (γ, j) with $\gamma \in \text{Irr}(\mathbf{O}_2(C_\kappa))$ and $j \in \{0, 1\}$. Each $\alpha \in \text{Irr}_{2'}(\mathbf{N}_G(P))$ is of the form $\alpha = \delta \otimes \theta$, where $\delta \in \text{Irr}(Z_1)$ and $\theta \in \text{Irr}_{2'}(P)$. By Lemma 3.1 and Theorem 3.2 of [G], there is a unique hook partition $\pi = (2^{m-1} - k, 1^k) \vdash 2^{m-1}$ such that ν is the unique linear constituent of the restriction to Q of $\chi^\pi \in \text{Irr}(S_{2^{m-1}})$; in particular, ν is rational. Hence $\alpha = \bar{\alpha}$ exactly when $\delta = 1_{Z_1}$ and $\lambda = \bar{\lambda}$, which, as mentioned in (i), then imply that α is rational, and there are exactly 2^{m+1} of such α 's. Now we send α to (\hat{s}, ξ) , where $\hat{s} \in \tilde{C}_\kappa$ corresponds to (γ, δ) via (5.1),

and $\xi \in \mathcal{H}(n)$ is defined as follows

$$\xi = \begin{cases} (2^m - 2k, 1^{2k}), & j = 0 \\ (2^m - 2k - 1, 1^{2k+1}), & j = 1. \end{cases}$$

It is straightforward to see that the defined map is a bijection between $\text{Irr}_{2'}(\mathbf{N}_G(P))$ and $\tilde{C}_\kappa \times \mathcal{H}(n)$, and that both (i) and (ii) hold. \square

Lemma 5.2. *Let q be an odd prime power, $n \geq 2$, $G = GU_n(q)$, and let $\chi \in \text{Irr}_{2'}(G)$. Then there is a unique label of the form*

$$\chi = S(\hat{s}_1, \lambda_1) \circ S(\hat{s}_2, \lambda_2) \circ \dots \circ S(\hat{s}_m, \lambda_m),$$

where for $1 \leq i \leq m$ we have that $\hat{s}_i \in \tilde{C}_-$ are pairwise different, $S(\hat{s}_i, \lambda_i) \in \text{Irr}_{2'}(GU_{k_i}(q))$, $\chi^{\lambda_i} \in \text{Irr}_{2'}(\mathbf{S}_{k_i})$, and

$$[n]_2 = [k_1]_2 < [k_2]_2 < \dots < [k_m]_2.$$

Proof. We can identify the dual group G^* with G and consider the Jordan correspondent (s, φ) of χ , where $s \in G$ is semisimple and φ is a unipotent character of $\mathbf{C}_G(s)$, cf. [DM]. It is easy to see that the condition $2 \nmid \chi(1)$ implies that

$$\mathbf{C}_G(s) \cong GU_{k_1}(q) \times GU_{k_2}(q) \times \dots \times GU_{k_m}(q)$$

with $\sum_{i=1}^m k_i = n$ and furthermore, $n!/\prod_{i=1}^m k_i!$ is odd by [NT₁, Lemma 4.4(i)]. Hence by Corollary 2.3 there is a unique relabeling of the k_i 's such that

$$[n]_2 = [k_1]_2 < [k_2]_2 < \dots < [k_m]_2.$$

Now we can write $s = \text{diag}(s_1, s_2, \dots, s_m)$ with $s_i \in \mathbf{Z}(GU_{k_i}(q))$ and then identify $\mathbf{Z}(GU_{k_i}(q))$ with C_- . Note that the s_i 's are pairwise different because of the structure of $\mathbf{C}_G(s)$. Also, we can write $\varphi = \varphi_1 \otimes \varphi_2 \otimes \dots \otimes \varphi_m$ with $\varphi_i = \varphi^{\lambda_i} \in \text{Irr}(GU_{k_i}(q))$ being the unipotent character labeled by $\lambda_i \vdash k_i$. Since $2 \nmid \varphi(1)$, we see that $\chi^{\lambda_i}(1) \equiv \varphi_i(1) \equiv 1 \pmod{2}$ by [FS, (1.15)]. As before, $S(\hat{s}_i, \lambda_i)$ is the irreducible character of $GU_{k_i}(q)$ corresponding to $(s_i, \varphi^{\lambda_i})$. We also note by [DM, Theorem 13.25] that

$$(5.4) \quad \chi = R_{\mathbf{C}_G(s)}^G(S(\hat{s}_1, \lambda_1) \otimes S(\hat{s}_2, \lambda_2) \otimes \dots \otimes S(\hat{s}_m, \lambda_m)),$$

where $R_{\mathbf{C}_G(s)}^G$ is now the Lusztig induction. \square

We can now prove the following result which implies Theorem E:

Theorem 5.3. *Let $n \in \mathbb{Z}_{\geq 1}$, q be an odd prime power, $G = GL_n(q)$ or $GU_n(q)$, and $P \in \text{Syl}_2(G)$. Then*

- (i) *The field of values of any character in $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(\mathbf{N}_G(P))$ is contained in $\mathbb{Q}(\tilde{\varepsilon}) = \mathbb{Q}(\exp(\frac{2\pi i}{q-\kappa 1}))$, where $\kappa = +$ if $G = GL_n(q)$ and $\kappa = -$ if $G = GU_n(q)$.*
- (ii) *There is a canonical bijection $\chi \mapsto \chi^\sharp$ between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(\mathbf{N}_G(P))$ that commutes with the action of $\Gamma = \text{Gal}(\mathbb{Q}(\tilde{\varepsilon})/\mathbb{Q})$. Furthermore, we can choose P to be D -invariant and $\chi \mapsto \chi^\sharp$ to be D -equivariant, where D is defined in (5.2).*

Proof. Write $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$ with $n_1 > \dots > n_r \geq 0$, and define

$$\Omega(n) := \tilde{C}_\kappa \times \mathcal{H}(2^{n_1}) \times \tilde{C}_\kappa \times \mathcal{H}(2^{n_2}) \times \dots \times \tilde{C}_\kappa \times \mathcal{H}(2^{n_r}).$$

We will construct the desired bijection $\chi \mapsto \chi^\sharp$ by composing two canonical bijections

$$\alpha : \text{Irr}_{2'}(G) \rightarrow \Omega(n), \quad \beta : \Omega(n) \rightarrow \text{Irr}_{2'}(\mathbf{N}_G(P)).$$

(a) First we consider the case $r = 1$. Then β^{-1} is the inverse of the map constructed in Lemma 5.1. Next, if $\chi \in \text{Irr}_{2'}(G)$, then by Theorem 2.5(i) and Lemma 5.2, $\chi = S(\hat{s}, \lambda)$ with $\hat{s} \in \tilde{C}_\kappa$ and $\lambda \vdash n$, and moreover $\chi^\lambda \in \text{Irr}(\mathbf{S}_n)$ has odd degree by (2.3). Hence $\lambda \in \mathcal{H}(n)$ by [G, Lemma 3.1] and we can define $\alpha(\chi) = (\hat{s}, \lambda)$. Note that, since G is uniform, unipotent characters of G are \mathbb{Q} -linear combinations of the Deligne-Lusztig character $R_T^G(1_T)$, $T < G$ any maximal torus, whence they are rational. Furthermore, $\hat{s} \circ \det$ takes values in $\mathbb{Q}(\tilde{\varepsilon})$. Hence, $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\tilde{\varepsilon})$, and for any $\sigma \in \Gamma$,

$$\chi^\sigma = S(\hat{s}, \lambda)^\sigma = S(\hat{s}^\sigma, \lambda).$$

Next, the unipotent character φ^λ is D -invariant (see eg. [M, Theorem 2.5]), and F_p , respectively τ , sends $\hat{s} \circ \det$ to $\hat{s}^p \circ \det$, respectively $\hat{s}^{-1} \circ \det$. Hence, (i) and (ii) hold in this case.

(b) In the general case, we can fix a decomposition (orthogonal if $\kappa = -$)

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

where $V_i = \mathbb{F}_q^{2^{n_i}}$ if $\kappa = +$ and $V_i = \mathbb{F}_{q^2}^{2^{n_i}}$ if $\kappa = -$. We also fix a basis in V compatible with this decomposition and define F_p , τ in this basis. Then we choose $P = P_1 \times P_2 \times \dots \times P_r$ with $P_i \in \text{Syl}_2(G_i)$ being D -invariant for $G_i := GL^\kappa(V_i) \cong GL_{2^{n_i}}^\kappa(q)$, cf. Lemma 5.1. Note that P_i is an irreducible subgroup of G_i and so

$$(5.5) \quad \mathbf{N}_G(P) = \mathbf{N}_{G_1}(P_1) \times \dots \times \mathbf{N}_{G_r}(P_r).$$

So if $\theta = \theta_1 \otimes \dots \otimes \theta_r \in \text{Irr}_{2'}(\mathbf{N}_G(P))$, then we can define

$$\beta(\theta) := (\hat{t}_1, \mu_1, \hat{t}_2, \mu_2, \dots, \hat{t}_r, \mu_r)$$

if the bijection in Lemma 5.1 sends $\theta_i \in \text{Irr}_{2'}(\mathbf{N}_{G_i}(P_i))$ to (\hat{t}_i, μ_i) . Lemma 5.1 also implies that $\mathbb{Q}(\theta) \subseteq \mathbb{Q}(\tilde{\varepsilon})$ and that β commutes with the action of Γ and D .

Now consider any $\chi \in \text{Irr}_{2'}(G)$ and apply Theorem 2.5(i) and Lemma 5.2 to χ . Assume first that $m = 1$, so that $\chi = S(\hat{s}, \lambda)$. Then $2 \nmid \chi^\lambda(1)$ by (2.3). Applying Corollary 4.5 to the Young subgroup $Y = \mathbf{S}_{2^{n_1}} \times \dots \times \mathbf{S}_{2^{n_r}}$, we obtain

$$(\chi^\lambda)^\star = \chi^{\nu_1} \otimes \chi^{\nu_2} \otimes \dots \otimes \chi^{\nu_r},$$

where $\nu_i \in \mathcal{H}(2^{n_i})$ by [G, Lemma 3.1]. In this case, we define

$$(5.6) \quad \alpha(\chi) := (\hat{s}, \nu_1, \hat{s}, \nu_2, \dots, \hat{s}, \nu_r).$$

In the case of general m , note that, by Lemma 2.2, each k_i is the sum of some 2^{n_j} 's. Moreover, when we express all k_i , $1 \leq i \leq m$, this way, each 2^{n_j} occurs in precisely one of these m expressions. Now we can apply (5.6) to each $S(\hat{s}_i, \lambda_i)$ and then define $\alpha(\chi)$ by putting all $\alpha(S(\hat{s}_i, \lambda_i))$ together. It is easy to check that the resulting map is a bijection.

Furthermore, as in (a), $\hat{s}_i \circ \det$ take values in $\mathbb{Q}(\tilde{\varepsilon})$. Hence, (2.2), (5.4), and [DM, Proposition 12.2] (and the paragraph right before it) imply that $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\tilde{\varepsilon})$ and that, if

$$\chi = R_L^G(S(\hat{s}_1, \lambda_1) \otimes S(\hat{s}_2, \lambda_2) \otimes \dots \otimes S(\hat{s}_m, \lambda_m))$$

(for a suitable Levi subgroup L which in our case can be chosen to be D -invariant), then for any $\sigma \in \Gamma$ we have

$$\chi^\sigma = R_L^G(S(\hat{s}_1^\sigma, \lambda_1) \otimes S(\hat{s}_2^\sigma, \lambda_2) \otimes \dots \otimes S(\hat{s}_m^\sigma, \lambda_m)).$$

Thus α commutes with the action of Γ . The fact that α commutes with the action of F_p follows from [NTT, Corollary 2.3] and the arguments in (a). In the case $\kappa = +$, α also commutes with τ as R_L^G is just the Harish-Chandra induction. Hence, (i) and (ii) hold in this case as well. \square

Note that if $P \in \text{Syl}_p(G)$ satisfies $\mathbf{N}_G(P) = P\mathbf{C}_G(P)$ for some odd prime p , then the restriction from G to $\mathbf{N}_G(P)$ yields a natural correspondence between the p' -degree irreducible characters of the principal p -block of G and those of $\mathbf{N}_G(P)$, see [NTV, Theorem A]. Theorem E yields a canonical correspondence but with $p = 2$.

Proof of Corollary F. The number of real odd-degree irreducible characters of $\mathbf{N}_G(P)$ can be easily computed using Lemma 5.1 and (5.5). Since the correspondence in Theorem E preserves fields of values of characters, the statement follows. \square

Corollary 5.4. *Let $n \in \mathbb{Z}_{>1}$, q be an odd prime power, $G = GL_n(q)$, and let P be a parabolic subgroup of odd index in G with Levi subgroup L . Then there is a canonical bijection between $\text{Irr}_{2'}(G)$, $\text{Irr}_{2'}(P)$, and $\text{Irr}_{2'}(L)$.*

Proof. We may assume that P is a standard parabolic subgroup with Levi subgroup $L = GL_{k_1}(q) \times GL_{k_2}(q) \times \dots \times GL_{k_m}(q)$, where $m > 1$. As in the proof of Lemma 2.6, $2 \nmid |G : P|$ implies that we may relabel the k_i 's so that $[k_1]_2 < [k_2]_2 < \dots < [k_m]_2$. We also write $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$ where $n_1 > n_2 > \dots > n_r \geq 0$. As in the proof of Theorem 5.3, note by Lemma 2.2 that each k_i is the sum of some 2^{n_j} 's. Moreover, when we express all k_i , $1 \leq i \leq m$, this way, each 2^{n_j} occurs in precisely one of these m expressions. For each j , choose $Q_j \in \text{Syl}_2(GL_{2^{n_j}}(q))$. Then, if $k_i = 2^{n_{j_1}} + 2^{n_{j_2}} + \dots + 2^{n_{j_a}}$, we can choose $R_i := Q_{j_1} \times Q_{j_2} \times \dots \times Q_{j_a} \in \text{Syl}_2(GL_{k_i}(q))$, and $R := R_1 \times R_2 \times \dots \times R_m \in \text{Syl}_2(G)$. The formula (5.5) shows that

$$\mathbf{N}_G(R) = \mathbf{N}_{GL_{k_1}(q)}(R_1) \times \mathbf{N}_{GL_{k_2}(q)}(R_2) \times \dots \times \mathbf{N}_{GL_{k_m}(q)}(R_m) < L.$$

Consider any $\chi \in \text{Irr}_{2'}(G)$ and let $\theta := \chi^\sharp \in \text{Irr}_{2'}(\mathbf{N}_G(R))$ as given by the character correspondence in Theorem 5.3. Next, write $\theta = \theta_1 \otimes \theta_2 \otimes \dots \otimes \theta_m$ with $\theta_i \in \text{Irr}_{2'}(\mathbf{N}_{GL_{k_i}(q)}(R_i))$. By Theorem 5.3, for each i there is a unique $\psi_i \in \text{Irr}_{2'}(GL_{k_i}(q))$ such that $(\psi_i)^\sharp = \theta_i$. Now we can define

$$\chi^* = \psi := \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_m \in \text{Irr}_{2'}(L)$$

and check that the map $\chi \mapsto \chi^*$ is a bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(L)$. Note that L can be chosen to be D -invariant, and $\chi \mapsto \chi^\sharp$ then commutes with the actions of Γ and D by Theorem 5.3.

Finally, we show that the inflation (from L to P) gives a natural bijection between $\text{Irr}_{2'}(L)$ and $\text{Irr}_{2'}(P)$. For, consider any $\varphi \in \text{Irr}_{2'}(P)$ and suppose that φ is nontrivial at U , the unipotent radical of P . Then the L -orbit on the irreducible constituents of φ_U has odd size, and so $R < L$ fixes some $1_U \neq \lambda \in \text{Irr}(U)$. On the other hand, (5.3) and (5.5) show that $\mathbf{C}_G(R) \cap U = 1$. As R acts coprimely on U , the Glauberman correspondence [Is₂, Theorem 13.1] implies that 1_U is the only R -invariant irreducible character of U . \square

The same proof as above yields an analogue of Corollary 5.4 for $G = GU_n(q)$ with $2 \nmid q$:

Corollary 5.5. *Let $n \in \mathbb{Z}_{>1}$, q be an odd prime power, $G = GU_n(q)$, and let*

$$L \cong GU_{k_1}(q) \times GU_{k_2}(q) \times \dots \times GU_{k_m}(q)$$

be a Levi subgroup of odd index in G . Then there is a canonical bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(L)$. \square

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