CHARACTER RESTRICTIONS AND MULTIPLICITIES IN SYMMETRIC GROUPS

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Abstract. We give natural correspondences of odd-degree characters of the symmetric groups and some of their subgroups, which can be described easily by restriction of characters, degrees and multiplicities.

1. Introduction

In this paper, we present some natural maps from the set of odd-degree complex irreducible characters of a symmetric group S_n into the set of odd-degree irreducible characters of the natural subgroups S_m with $m < n$ (that is, the pointwise stabilizers of $n-m$ points in the natural action of S_n on n points). The image of a character of S_n under one of these maps is easily described in terms of its restriction to S_m.

As an example of another map on characters of S_n that is defined in terms of restrictions and which is relevant here, we mention a result of J. L. Alperin. Alperin discovered, that if χ is an odd-degree irreducible character of the symmetric group $G = \mathsf{S}_{2^k}$ and $P \in \mathrm{Syl}_2(G)$, then the restriction χ_P contains a unique linear character λ and the map $\chi \mapsto \lambda$ is a bijection between $\mathrm{Irr}_{2'}(G)$, the set of odd-degree irreducible characters of G, and the set $\mathrm{Irr}_{2'}(P)$ of linear characters of P. This fact, whose proof was finally given in [G], was essential to construct in [GKNT] a canonical bijection $\mathrm{Irr}_{2'}(\mathsf{S}_n) \to \mathrm{Irr}_{2'}(P)$ for any n.

Recently, an astonishing fact about symmetric groups was proved. It was shown in [APS] that if $\chi \in \mathrm{Irr}_{2'}(\mathsf{S}_n)$, then the restriction $\chi_{\mathsf{S}_{n-1}}$ contains a unique odd-degree irreducible constituent. This simply does not happen in the restriction from S_n to any arbitrary S_m if $m < n$, in general. Our main new idea here is to consider not only odd-degree irreducible constituents but also odd multiplicities. In this way, we

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can construct canonical character correspondences between S_n and S_{n-2^k}. (The case $k=0$ is [APS].)

Theorem A. *Let $n>1$ be an integer, and let $k \geq 0$ with $2^k < n$. If $\chi \in \mathrm{Irr}_{2'}(\mathsf{S}_n)$, then the restriction $\chi_{\mathsf{S}_{n-2^k}}$ to the natural subgroup S_{n-2^k} contains a unique odd-degree irreducible constituent with odd multiplicity.*

We will denote the unique odd-degree irreducible constituent with odd multiplicity in the restriction $\chi_{\mathsf{S}_{n-2^k}}$ in Theorem A as $f_k(\chi)$. If $2^k+2^l<n$, then both $f_k(f_l(\chi))$ and $f_l(f_k(\chi))$ are defined, but unfortunately, these characters may be different. We provide examples below.

Recall that irreducible characters $\chi \in \mathrm{Irr}(\mathsf{S}_n)$ are canonically labeled by partitions of n: $\chi = \chi^\lambda$ for $\lambda \vdash n$. We can precisely describe the partition μ of $n-2^k$ that affords the character $f_k(\chi) \in \mathrm{Irr}(\mathsf{S}_{n-2^k})$. We believe that the following theorem has interest on its own and, unlike Theorem A, it works for arbitrary primes.

Theorem B. *Let p be a prime, and let k,n be integers with $1 \leq p^k < n$. Suppose that λ is a partition of n and that μ is a partition of $n-p^k$. Let $Y(\lambda)$ be the Young diagram of λ. Then χ^μ has p'-multiplicity in the restriction of χ^λ to a natural subgroup S_{n-p^k} of S_n if and only if $Y(\mu)$ is obtained from $Y(\lambda)$ by removing a rim hook of length p^k. In this case, the multiplicity is congruent to ± 1 modulo p. Moreover, if $\beta \vdash p^k$ is the hook corresponding to the rim hook $Y(\lambda) \setminus Y(\mu)$, then $\chi^\mu \otimes \chi^\beta$ is the unique irreducible constituent θ of $\chi^\lambda|_{\mathsf{S}_{n-p^k} \times \mathsf{S}_{p^k}}$ that lies above χ^μ and satisfies the condition $p \nmid (\theta(1)/\chi^\mu(1))$.*

With Theorems A and B, we can provide an easy description without using combinatorics of the canonical map $\mathrm{Irr}_{2'}(\mathsf{S}_n) \to \mathrm{Irr}_{2'}(P)$ given in [GKNT, Theorem 4.3]. See Theorem 5.1.

2. Some Preliminaries

For any partition $\lambda \vdash n$, χ^λ denotes the complex irreducible character of S_n labeled by λ, and $Y(\lambda)$ denotes the Young diagram of λ. For brevity, we call a hook of length m an *m-hook*, and similarly for *rim m-hooks*.

A character $\chi = \chi^\lambda \in \mathrm{Irr}(\mathsf{S}_n)$ afforded by the partition λ of n has a degree given by the so-called hook length formula. For details of the following facts, see [JK], [O1], [O2]. Let p be a positive integer, *not necessarily a prime number*. To a partition λ of n one may associate its p-core $C_p(\lambda)$ and p-quotient $Q_p(\lambda) = (\lambda_0^{(1)}, \cdots, \lambda_{p-1}^{(1)})$. The p-core is a partition without p-hooks obtained by removing a sequence of p-hooks from λ. The p-quotient is a p-tuple of partitions. One may recover λ from $C_p(\lambda)$ and $Q_p(\lambda)$. The following are basic facts.

Lemma 2.1. *Let $H_p(\lambda)$ be the (multi-)set of hooks of λ having length divisible by p. There is a canonical bijection between $H_p(\lambda)$ and $\cup_{i=0}^{p-1} H_p(\lambda_i^{(1)})$ such that an lp-hook is* *mapped onto an l-hook. This bijection is compatible with hook removals. In particular we have*

$$|\lambda| = |C_p(\lambda)| + pw_p(\lambda)$$

where $w_p(\lambda) = \sum_{i=0}^{p-1} |\lambda_i^{(1)}|$. *The number of hooks in λ of length divisible by p equals* $w_p(\lambda)$.

We may repeat the process of taking cores and quotients to obtain the *p-quotient tower* $\mathcal{Q}_p(\lambda)$ and the *p-core tower* $\mathcal{C}_p(\lambda)$ of λ. They have rows numbered by $i \geq 0$. The ith row of $\mathcal{Q}_p(\lambda)$ contains p^i partitions and the ith row of $\mathcal{C}_p(\lambda)$ contains the p-cores of these partitions in the same order. The 0th row of $\mathcal{Q}_p(\lambda)$ contains λ itself, the 1st row contains the partitions $\lambda_0^{(1)}, \dots, \lambda_{p-1}^{(1)}$ occurring in the p-quotient $Q_p(\lambda)$. The 2nd row contains the partitions occurring in the p-quotients of partitions occurring in the 1st row, and so on.

Remark 2.2. A partition λ may be recovered from its p-core tower. For $k > 0$ it may also be recovered from the knowledge of rows 0 to $(k-1)$ of $\mathcal{C}_p(\lambda)$ and row k of $\mathcal{Q}_p(\lambda)$, because the rows l with $l \geq k$ of $\mathcal{C}_p(\lambda)$ consist of the p-core towers of the partitions in row k of $\mathcal{Q}_p(\lambda)$.

If $\alpha_i(\lambda)$ is the sum of the sizes of the partitions in the ith row in the p-core tower $\mathcal{C}_p(\lambda)$, then we have

$$n = \sum_i \alpha_i(\lambda) p^i.$$

Now let p be a prime integer and let

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k, \ 0 \leq a_i \leq p-1, a_k \neq 0$$

be the p-adic decomposition of n.

A partition affording a character in $\mathrm{Irr}_{p'}(\mathsf{S}_n)$ will be called *p'-partition.* A $2'$-partition will be called *odd*. We write $\lambda \vdash_o n$ if λ is an odd partition of n.

The following result is essentially due to I.G. Macdonald [M].

Proposition 2.3. *A partition λ of n is p' if and only if $a_i = \alpha_i(\lambda)$ for all $i \geq 0$. In particular for $p = 2$ we have that λ is odd if and only if all partitions in the 2-core tower $\mathcal{C}_2(\lambda)$ are (0) or (1) and there is at most one entry (1) in each row.*

Lemma 2.4. *Suppose that $p^k \leq n < p^{k+1}$ and the partition λ of n is p'. We have*

(i) *λ contains a p^k-hook and any partition obtained by removing a p^k-hook from λ is still p'.*
(ii) *If $k < l$ then any partition obtained from λ by adding a p^l-hook is still p'.*

Proof. This is a consequence of Proposition 2.3 and Lemma 2.1. □

3. Proof of Theorem B

Let p be a prime and let $n = \sum_{i=0}^{r} a_i p^i$ be the p-adic decomposition of the nonnegative integer n (however, we do not require the last coefficient a_r to be > 0). Following [O1], we say that $m \in \mathbb{Z}_{\geq 0}$ with p-adic decomposition $m = \sum_{i=0}^{r} b_i p^i$ is a *p-adic subsum of n* if $0 \leq b_i \leq a_i$ for all i.

Lemma 3.1. *For any nonnegative integer n with p-adic decomposition $n = \sum_{i=0}^{r} a_i p^i$ and any prime p, the following statements hold.*

(i) *If $0 \leq m \leq n$ then $p \nmid \binom{n}{m}$ if and only if m is a p-adic subsum of n.*

(ii) *Suppose $n_j \geq 0$ has p-adic decomposition $n_j = \sum_{i=0}^{r} b_{ij} p^i$ for $1 \leq j \leq k$ and $n = \sum_{j=1}^{k} n_j$. Then $p \nmid n!/\prod_{j=1}^{k} n_j!$ if and only if $\sum_{j=1}^{k} b_{ij} \leq a_i$ for all $0 \leq i \leq r$.*

Proof. (i) is well known, see [J, Lemma 22.4] or [O1, Lemma (1.1)].

(ii) follows from (i) by an induction on k. (Note, however, that the condition *each n_i is a p-adic subsum of n* is not enough to guarantee that $p \nmid (n!/\prod_{j=1}^{k} n_j!)$. Take, for instance, $k = p+1$ and $n_1 = n_2 = \dots n_k = 1$.) □

Lemma 3.2. *Let $n = p^k \in \mathbb{Z}_{>0}$ and let $\lambda \vdash n$. Then $p \nmid \chi^\lambda(1)$ if and only if λ is a hook partition, in which case $\chi^\lambda(1) \equiv \pm 1 (\bmod\, p)$.*

Proof. First suppose that $\lambda = (r, 1^{n-r})$ is a hook partition, with $1 \leq r \leq n$. Then the hook length formula implies that

$$\chi^\lambda(1) = \binom{n-1}{r-1}. \tag{3.1}$$

By Lemma 3.1(i), $p \nmid \chi^\lambda(1)$. Moreover,

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r},$$

and $p | \binom{n}{r}$ by Lemma 3.1(i), provided that $1 \leq r \leq n-1$. Now an easy induction on $n-r$ shows that

$$\binom{n-1}{r-1} \equiv (-1)^{r-1} (\bmod\, p).$$

Next, by Macdonald's formula [M], the number of irreducible characters of p'-degree of S_n is exactly n if $n = p^k$. Hence the statements follow. □

Proof of Theorem B. (a) Write $m = n - p^k$ and decompose

$$\chi^\lambda|_{\mathsf{S}_m \times \mathsf{S}_{p^k}} = \sum_{\alpha \vdash m,\ \beta \vdash p^k} c_{\alpha\beta} \chi^\alpha \otimes \chi^\beta,$$

where $c_{\alpha\beta} = c^\lambda_{\alpha\beta} \in \mathbb{Z}_{\geq 0}$ are the Littlewood-Richardson coefficients. It follows that

$$[\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu] = \sum_{\beta \vdash p^k} c_{\mu\beta} \chi^\beta(1).$$

By Lemma 3.2, $p|\chi^\beta(1)$ unless $\beta = (b, 1^{p^k-b})$ is a hook partition. Hence

$$[\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu] \equiv \sum_{\beta=(b,1^{p^k-b}),\ 1\le b\le p^k} c_{\mu\beta}\chi^\beta(1) (\mathrm{mod}\, p). \tag{3.2}$$

(b) Suppose $\beta = (b, 1^{p^k-b})$ is a hook partition. According to [J, Lemma 21.5], $c_{\mu\beta} = 0$ unless $Z := Y(\lambda) \smallsetminus Y(\mu)$ is a disjoint union of, say, s skew hooks, in which case

$$c_{\mu\beta} = \binom{s-1}{c-b}, \tag{3.3}$$

where c is the total number of columns that Z spans.

Hence, $p \nmid [\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu]$ only when Z is a disjoint union of s skew hooks. Assuming $p \nmid [\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu]$, we will show that in fact $s = 1$. Altogether, the formulae (3.1), (3.2), and (3.3) imply that

$$[\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu] \equiv \sum_{b=1}^{c} \binom{s-1}{c-b}\binom{p^k-1}{b-1} = \binom{p^k+s-2}{c-1} (\mathrm{mod}\, p), \tag{3.4}$$

(where the second equality follows by comparing the coeffient of t^{c-1} in $(t+1)^{p^k+s-2}$ and $(t+1)^{s-1}(t+1)^{p^k-1}$).

Assume by contradiction that $s \ge 2$. Then

$$p^k \le p^k + s - 2 \le p^k + p^k - 2$$

and so the p-adic decomposition of p^k+s-2 is $\sum_{i=0}^{k-1} x_i p^i + p^k$ for some $0 \le x_i < p$. In particular, the p-adic decomposition of $s-2$ is $s-2 = \sum_{i=0}^{k-1} x_i p^i$. As $0 \le c-1 \le p^k-1$, the assumption $p \nmid \binom{p^k+s-2}{c-1}$ implies by Lemma 3.1(i) that the p-adic decomposition of $c-1$ is $c-1 = \sum_{i=0}^{k-1} y_i p^i$ with $0 \le y_i \le x_i$. It follows that

$$c \le s-1. \tag{3.5}$$

Recall that $Z = Y(\lambda) \smallsetminus Y(\mu)$ is the disjoint union of s skew hooks $S_1, \dots, S_s$, and it spans c columns. The inequality (3.5) now implies that some skew hooks S_i, S_j with $i \ne j$ intersect the same column of $Y(\lambda)$. Thus S_i contains a node A and S_j contains a node B, where A and B belong to the same column of $Y(\lambda)$.

We may assume that S_i lies higher in $Y(\lambda)$ than S_j. Suppose that the nodes between A and B in the rim of $Y(\lambda)$ all belong to the same column. Since μ is a proper partition, it follows that Z contains all the nodes between A and B in the rim of $Y(\lambda)$, and so S_i and S_j (and all skew hooks $S_{j'}$ above B) are connected in the rim of $Y(\lambda)$, a contradiction. Hence there exists a node C in the rim of $Y(\lambda)$ between A and B that lies to the left of B. But then the row R_C of C in $Y(\lambda)$ lies above, but is shorter than, the row R_B of B in $Y(\lambda)$, and so λ is not a proper partition, again a contradiction.

(c) We have shown that $s = 1$, i.e. $Y(\lambda) \setminus Y(\mu)$ is a rim p^k-hook. In this case, (3.4) shows that $[\chi^\lambda|_{\mathsf{S}_m}, \chi^\mu] \equiv \binom{p^k-1}{c-1} (\bmod\, p)$, and so it is congruent to $\pm 1 (\bmod\, p)$ by Lemma 3.2.

Suppose now that $\beta \vdash p^k$ is such that $c_{\mu\beta} \neq 0$ and $p \nmid \chi^\beta(1)$. By Lemma 3.2, $\beta = (b, 1^{p^k-b})$ is a hook partition. Now (3.3) implies that $c_{\mu\beta} = 1$ and $\beta = (c, 1^{p^k-c})$, that is, β is the hook corresponding to the rim hook $Y(\lambda) \setminus Y(\mu)$. Thus the last statement of the theorem is proved. □

Theorem B and Lemma 2.1 yield the following immediate consequence:

Corollary 3.3. *Let p be a prime, let k, n be integers with $1 \leq p^k < n$, and let $m = n - p^k$. Suppose that λ is a partition of n. Then the number of $\mu \vdash m$ such that χ^μ has p'-multiplicity in the restriction of χ^λ to S_m is equal to the number of hooks of length p^k in the Young diagram of λ, and so it is at most $\lfloor n/p^k \rfloor$. Furthermore, each of these p'-multiplicities is congruent to ± 1 modulo p.*

Theorem 3.4. *Let n be any positive integer prime, $1 \leq m < n$, and let α be any partition of $n - m$. Then there are at least two partitions $\lambda = \lambda^\pm$ of n such that $\chi^\lambda|_{\mathsf{S}_{n-m}}$ contains χ^α with multiplicity one. Here, λ^+ is obtained from α by adding m nodes to the end of the first row, and λ^- is obtained from α by adding m nodes to the end of the first column.*

Suppose in addition that p is a prime and $m = p^k > n/2$. Then the following statements hold.

(i) *$\lambda^\pm$ are the only partitions of n such that $\chi^\lambda|_{\mathsf{S}_{n-m}}$ contains χ^α with multiplicity one.*

(ii) *There is a bijection $\pi_\alpha : \beta \mapsto \lambda(\beta)$ between the set $\mathcal{H}(m)$ of hook partitions β of m and the set $\mathcal{O}(\alpha)$ of partitions λ of n such that χ^α has p'-multiplicity in the restriction of χ^λ to S_{n-m}. Here, $\lambda(\beta)$ has an m-hook of shape β and $Y(\alpha)$ is obtained from $Y(\lambda(\beta))$ by removing the corresponding m-hook. In particular, $\lambda^+ = \lambda((m))$ and $\lambda^- = \lambda((1^m))$. Furthermore,*

$$[\chi^{\lambda(\beta)}|_{\mathsf{S}_{n-m} \times \mathsf{S}_m}, \chi^\alpha \otimes \chi^\beta] = 1.$$

(iii) *For any $\chi \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$, $\chi|_{\mathsf{S}_{n-m}}$ contains a unique irreducible constituent $\chi^\sharp$ that has both p'-degree and p'-multiplicity. The map $\chi \mapsto \chi^\sharp$ is an m-to-1 surjective map between $\mathrm{Irr}_{p'}(\mathsf{S}_n)$ and $\mathrm{Irr}_{p'}(\mathsf{S}_{n-m})$. Furthermore, if $\alpha \in \mathrm{Irr}_{p'}(\mathsf{S}_{n-m})$, then $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$ for all $\lambda \in \mathcal{O}(\alpha)$, and*

$$(\chi^{\pi_\alpha(\beta)})^\sharp = \chi^\alpha$$

for all $\beta \in \mathcal{H}(m)$.

Proof. (a) As in the proof of Theorem B, we have

$$N_{\lambda\alpha} := [\chi^\lambda|_{\mathsf{S}_{n-m}}, \chi^\alpha] = \sum_{\beta \vdash m} c^\lambda_{\alpha\beta} \chi^\beta(1). \tag{3.6}$$

First we consider the case $\lambda = \lambda^+$ and consider any $\gamma \vdash m$. We will apply the Littlewood-Richardson formula [JK, Cor. 2.8.14] to find $c^\lambda_{\alpha\gamma}$. In particular, if γ has at least two rows, then the $(1,1)$-node and the $(2,1)$-node of $Y(\gamma)$ cannot be put in different rows of $Y(\lambda) \smallsetminus Y(\alpha)$, which consists of only one row. Thus $c^\lambda_{\alpha\gamma} = 0$ unless $\gamma = (m)$, in which case the Littlewood-Richardson rule shows that $c^\lambda_{\alpha\gamma} = 1$. It follows from (3.6) that $N_{\lambda^+,\alpha} = 1$.

Next let $\lambda = \lambda^-$ and consider any $\gamma \vdash m$. If γ has at least two columns, then the $(1,1)$-node and the $(1,2)$-node of $Y(\gamma)$ cannot be put in different columns of $Y(\lambda) \smallsetminus Y(\alpha)$, which consists of only one column. Thus $c^\lambda_{\alpha\gamma} = 0$ unless $\gamma = (1^m)$, in which case the Littlewood-Richardson rule shows that $c^\lambda_{\alpha\gamma} = 1$. It follows from (3.6) that $N_{\lambda^-,\alpha} = 1$.

(b) From now on we assume that $m = p^k$.

Suppose $\lambda \vdash n$ and $N_{\lambda\alpha} = 1$. In particular, $p \nmid N_{\lambda\alpha}$. By Theorem B, $Y(\alpha)$ is obtained from $Y(\lambda)$ by removing a rim m-hook, say of shape $\beta \in \mathcal{H}(m)$. In this case, $c^\lambda_{\alpha\beta} = 1$ by [GKNT, Lemma 4.1], whence

$$1 = N_{\lambda\alpha} \geq \chi^\beta(1)$$

by (3.6). It follows that $\beta = (m)$ or (1^m).

From now on we also assume that $n < 2m$. Then the only proper $Y(\lambda)$ that we can get by adding a rim m-hook of shape β to $Y(\alpha)$ is (the Young diagram of) λ^+, respectively λ^-. Hence (i) follows.

The example of $\alpha = (m)$ shows that the condition $n < 2m$ is necessary: $N_{\lambda\alpha} = 1$ for $\lambda = \lambda^+ = (2m)$, $\lambda^- = (m, 1^m)$, and also (m^2).

(c) Again by Theorem B, $p \nmid N_{\lambda\alpha}$ if and only if $Y(\alpha)$ is obtained from $Y(\lambda)$ by removing a rim m-hook, say of shape $\beta \in \mathcal{H}(m)$. Since $n < 2m$, in this case α is the m-core of λ, cf. [JK, Theorem 2.7.16]. Now [GKNT, Lemma 4.2(i)] shows that the map $\pi_\alpha : \mathcal{H}(m) \to \mathcal{O}(\alpha)$ sending β to $\lambda(\beta)$ is well-defined and surjective. This map is injective since α is the m-core of $\lambda(\beta)$ and then β is the shape of the rim m-hook $Y(\lambda(\beta)) \smallsetminus Y(\alpha)$. The equality

$$[\chi^{\lambda(\beta)}|_{\mathsf{S}_{n-m}\times\mathsf{S}_m}, \chi^\alpha \otimes \chi^\beta] = 1$$

follows from [GKNT, Lemma 4.1]. Thus we have proved (ii).

(d) First we note by Macdonald's formula [M] that

$$|\operatorname{Irr}_{p'}(\mathsf{S}_n)| = |\operatorname{Irr}_{p'}(\mathsf{S}_{n-m}) \times \mathcal{H}(m)|. \tag{3.7}$$

Consider any $\chi = \chi^\lambda \in \operatorname{Irr}_{p'}(\mathsf{S}_n)$. Then $\chi|_{\mathsf{S}_{n-m}}$ must contain some irreducible constituent $\chi^\mu \in \operatorname{Irr}_{p'}(\mathsf{S}_{n-m})$ of p'-multiplicity. In other words, $\lambda \in \mathcal{O}(\mu)$. We already mentioned in (c) that in this case μ is the m-core of any $\lambda' \in \mathcal{O}(\mu)$. In particular, the sets $\mathcal{O}(\nu)$ with $\nu \vdash n-m$ are disjoint, and μ is uniquely determined by λ. Thus

χ^μ is uniquely determined by χ, and we can set

$$\chi^\sharp = \chi^\mu.$$

Let us write

$$\mathcal{O}^*(n) = \{\tau \vdash n \mid p \nmid \chi^\tau\}, \quad \text{and} \quad \mathcal{O}^*(\nu) = \mathcal{O}^*(n) \cap \mathcal{O}(\nu)\,.$$

We have shown that

$$\mathcal{O}^*(n) \subseteq \bigcup_{\mu \in \mathcal{O}^*(n-m)} \mathcal{O}^*(\mu) \subseteq \bigcup_{\mu \in \mathcal{O}^*(n-m)} \mathcal{O}(\mu). \tag{3.8}$$

Recall by (ii) that $|\mathcal{O}(\mu)| = |\mathcal{H}(m)|$. This, together with (3.7) and (3.8), implies that $\mathcal{O}(\mu) = \mathcal{O}^*(\mu)$ for all $\mu \in \mathcal{O}^*(n-m)$, and the map $\chi \mapsto \chi^\sharp$ is onto between $\mathrm{Irr}_{p'}(\mathsf{S}_n)$ and $\mathrm{Irr}_{p'}(\mathsf{S}_{n-m})$. The fibers of this map are exactly the sets $\{\chi^\tau \mid \tau \in \mathcal{O}(\mu)\}$ and so have size m.

By the definitions of our maps, if $\beta \in \mathcal{H}(m)$ and $\lambda = \pi_\alpha(\beta)$, then $Y(\alpha)$ is obtained from $Y(\lambda)$ by removing a rim m-hook (corresponding to an m-hook of shape β), whence $(\chi^{\pi_\alpha(\beta)})^\sharp = \chi^\alpha$. □

Note that Theorem 3.4(iii) implies Theorem A in the case $2^k \leq n < 2^{k+1}$.

4. Proof of Theorem A - Odd degree characters in S_n

We call two nonnegative integers 2-*disjoint* if there is no common summand in their 2-adic decompositions. The following result may also be found as [APS, Lemma 6].

Lemma 4.1. *Let λ be a partition with $Q_2(\lambda) = (\lambda_0^{(1)}, \lambda_1^{(1)})$. We have that λ is odd, if and only if*

(i) *$|C_2(\lambda)| \leq 1$,*
(ii) *$\lambda_0^{(1)}$ and $\lambda_1^{(1)}$ are both odd, and*
(iii) *$|\lambda_0^{(1)}|$ and $|\lambda_1^{(1)}|$ are 2-disjoint.*

Proof. For the 2-core towers we have from the definition that $\mathcal{C}_2(\lambda_0^{(1)})$ and $\mathcal{C}_2(\lambda_1^{(1)})$ are embedded as adjacent towers, starting in row 1 of $\mathcal{C}_2(\lambda)$. Thus row $i+1$ of $\mathcal{C}_2(\lambda)$ is the union of the ith rows of $\mathcal{C}_2(\lambda_0^{(1)})$ and $\mathcal{C}_2(\lambda_1^{(1)})$. The lemma therefore follows easily from Lemma 2.3 for $p = 2$. □

Let us call a hook in an odd partition *odd* if the partition obtained be removing the hook is still odd.

Proposition 4.2. *Suppose that $\lambda \vdash_{\mathrm{o}} n$. If $2^k \leq n$ then λ contains a unique* odd *2^k-hook.*

Proof. Suppose that $2^{b_1}, 2^{b_2}, \ldots, 2^{b_t}$, where $b_1 < b_2 < \ldots < b_t$, are the 2-adic summands of n, so that $2^{b_t} \leq n < 2^{b_t+1}$. Thus $k \leq b_t$. Let $Q_2(\lambda) = (\lambda_0^{(1)}, \lambda_1^{(1)})$. We use induction on $k \geq 0$. The case $k = 0$ is [APS, Theorem 1]. Assume that $k \geq 1$ and that the result is proved for $k-1$. Let i be minimal with the property that $k \leq b_i$. Now 2^{b_i-1} is a 2-adic summand of $|\lambda_j|$, for $j = 0$ or $j = 1$, say for $j = 0$. The induction hypothesis shows that the odd partition $\lambda_0^{(1)}$ contains a unique odd 2^{k-1}-hook. Removing this we get an odd partition $\lambda_0^{(1)*}$. The partition λ^* with $C_2(\lambda^*) = C_2(\lambda)$ and $Q_2(\lambda^*) = (\lambda_0^{(1)*}, \lambda_1^{(1)})$ is then obtained from λ by removing a 2^k-hook, by Lemma 2.1. Now $|\lambda_0^{(1)*}| = |\lambda_0^{(1)}| - 2^{k-1}$. The difference between the 2-adic decompositions of $|\lambda_0^{(1)}|$ and $|\lambda_0^{(1)*}|$ is that 2^{b_i-1} is replaced by $2^{k-1}, \ldots, 2^{b_i-2}$. Therefore, since $|\lambda_0^{(1)}|$ and $|\lambda_1^{(1)}|$ are 2-disjoint we get that $|\lambda_0^{(1)*}|$ and $|\lambda_1^{(1)}|$ are 2-disjoint, due to the choice of i. Thus λ^* is odd, by Lemma 2.4. This shows the existence of an odd 2^k-hook in λ.

We now show the uniqueness of such an odd 2^k-hook. A 2^k-hook in λ corresponds to a 2^{k-1}-hook in $\lambda_0^{(1)}$ or $\lambda_1^{(1)}$. If the 2^k-hook is odd, the corresponding 2^{k-1}-hook should be odd in the 2-quotient partition containing it, by Lemma 2.4. There is at most one possibility for this in each of the two 2-quotient partitions, by the induction hypothesis. If again i is minimal with the property that $k \leq b_i$ and 2^{b_i-1} is a 2-adic summand of $|\lambda_0|$ then we have seen that removing an odd 2^{k-1}-hook in $\lambda_0^{(1)}$ results in an odd partition λ^*, obtained from λ by removing an odd 2^k-hook. If $\lambda_1^{(1)}$ also contains an odd 2^{k-1}-hook and $\lambda_1^{(1)*}$ is obtained by removing it, then the partition λ^{o} with $C_2(\lambda^{\mathrm{o}}) = C_2(\lambda)$ and $Q_2(\lambda^{\mathrm{o}}) = (\lambda_0^{(1)}, \lambda_1^{(1)*})$ is *not* odd. The reason is that $|\lambda_0^{(1)}|$ and $|\lambda_1^{(1)*}|$ are *not* 2-disjoint. If j is minimal such that $k \leq b_j$ and 2^{b_j-1} is a 2-adic summand of $|\lambda_1^{(1)}|$ then the difference between the 2-adic decompositions of $|\lambda_1^{(1)}|$ and $|\lambda_1^{(1)*}|$ is that 2^{b_j-1} is replaced by $2^{k-1}, \ldots, 2^{b_j-2}$. Now 2^{b_i-1} occurs in this sequence, because $k \leq b_i < b_j$, so it is a 2-adic summand of $|\lambda_1^{(1)*}|$. But 2^{b_i-1} is also a 2-adic summand of $|\lambda_0^{(1)}|$. ◻

Proof of Theorem A. Consider any $\chi \in \mathrm{Irr}_{2'}(\mathsf{S}_n)$, so that $\chi = \chi^\lambda$ for some $\lambda \vdash_{\mathrm{o}} n$. Let $2^k < n$ and $\mu \vdash (n-2^k)$. By Theorem B, χ^μ has odd multiplicity in $\chi|_{\mathsf{S}_{n-2^k}}$ precisely when $Y(\mu)$ is obtained from $Y(\lambda)$ by removing a 2^k-hook. By Proposition 4.2, λ has a unique such *odd* 2^k-hook. Hence the statement follows. ◻

Proposition 4.2 gives rise to maps from the set of odd partitions of $n, n \geq 2^k$, to the set of odd partitions of $n-2^k$, which, in light of the proof of Theorem A and abusing the notation, we can also denote by f_k. If $n \geq 2^k + 2^l$ one may ask whether $f_k f_l = f_l f_k$ as maps. This is not the case. A very small example is when $n = 5, k = 1, l = 0$. We have

$$f_1((3,2)) = (3), f_0((3)) = (2)$$
$$f_0((3,2)) = (3,1), f_1((3,1)) = (1,1).$$

However we have the following result:

Proposition 4.3. *Suppose that $2^k \leq n < 2^{k+1}$ and that $0 \leq l < k$ satisfies that $2^k + 2^l \leq n$. Then $f_k f_l = f_l f_k$.*

Proof. We prove the claim by induction on $l \geq 0$. For $l = 0$ the statement follows from [APS, Lemma 2(1)]. Indeed, if λ is an odd partition of n and $\mu = f_0(\lambda)$, then the odd partition $f_k f_0(\lambda) = f_k(\mu) = C_{2^k}(\mu)$ is obtained from $f_k(\lambda) = C_{2^k}(\lambda)$ by removing a 1-hook, i.e. it equals $f_0 f_k(\lambda)$.

Assume that $l \geq 1$ and that the claim has result is proved for $l - 1$. Odd hooks of length 2^k and 2^l correspond to odd hooks of length 2^{k-1} and 2^{l-1} in the 2-quotient $Q_2(\lambda) = (\lambda_0^{(1)}, \lambda_1^{(1)})$.

If the odd hooks of length 2^{k-1} and 2^{l-1} are *not* in the same partition of $Q_2(\lambda)$ then their removals obviously commute so that $f_k f_l(\lambda) = f_l f_k(\lambda)$. If the odd hooks of length 2^{k-1} and 2^{l-1} are in the same partition of $Q_2(\lambda)$ then their removals commute by the induction hypothesis and again $f_k f_l(\lambda) = f_l f_k(\lambda)$. □

Remark 4.4. If $0 \leq l < k$ satisfies that $2^k + 2^l \leq n$, then we may have $f_k f_l = f_l f_k$ without 2^k being the highest power of 2 less that n. An example of this is $n = 18$, $k = 3$, $l = 2$. Also, when n is a power of 2 then $f_k f_l = f_l f_k$ for all k, l with $0 \leq l < k$ and $2^k + 2^l \leq n$.

Proposition 4.5. *Suppose that 2^k is a 2-adic summand of n. Then f_k induces a* 2^k-to-1 *surjective map between the sets of odd partitions of n and of odd partitions of $n - 2^k$. Equivalently, the map $\chi \mapsto f_k(\chi)$, where $f_k(\chi)$ is the unique odd-degree irreducible constituent of odd multiplicity in $\chi_{\mathsf{S}_{n-2^k}}$ (see Theorem A), induces a* 2^k-to-1 *surjective map between* $\mathrm{Irr}_{2'}(\mathsf{S}_n)$ *and* $\mathrm{Irr}_{2'}(\mathsf{S}_{n-2^k})$.

Proof. If $k = 0$, then n is odd and the result follows from [APS, Theorem 2]. Indeed the theorem applied to $n - 1$ shows that f_0 is injective as a map from the set of odd partitions of n to the set of odd partitions of $n - 1$. But then it is bijective, since these sets have the same cardinality, by [M].

Suppose that $k > 0$ and that $\lambda \vdash_{\mathrm{o}} n$ and $f_k(\lambda) = \mu \vdash_{\mathrm{o}} n - 2^k$. Then μ is obtained from λ by removing an (odd) 2^k-hook H. By Lemma 2.1 this hook corresponds to an odd 2^{k-1}-hook in a partition $\lambda_i^{(1)}$ in row 1 of the p-quotient tower $\mathcal{Q}_p(\lambda)$. Continuing we see that H corresponds to an odd 1-hook in a partition $\lambda_j^{(k)}$ in row k of $\mathcal{Q}_p(\lambda)$. The position j of this partition is also the position of the unique non-zero entry in row k of $\mathcal{C}_p(\lambda)$. (See Proposition 2.3.) The kth row of $\mathcal{Q}_p(\mu)$ coincides with that of $\mathcal{Q}_p(\lambda)$ except that $\lambda_j^{(k)}$ is replaced by $f_0(\lambda_j^{(k)})$. Also note that rows 0 to $(k-1)$ in $\mathcal{C}_p(\lambda)$ and $\mathcal{C}_p(\mu)$ coincide because removing hooks of length divisible by 2 does not change the 2-core of a partition.

This analysis makes it possible for a given $\mu \vdash_{\mathrm{o}} n - 2^k$ to describe all $\lambda \vdash_{\mathrm{o}} n$ with $f_k(\lambda) = \mu$. Since 2^k is not in the 2-adic decomposition of $n - 2^k$, row k of $\mathcal{C}_p(\mu)$

contains only zero partitions (0). Thus the partitions in row k of $\mathcal{Q}_p(\mu)$ are all of even cardinality, since they have 2-core (0). Choose a position j and replace $\mu_j^{(k)}$ by $f_0^{-1}(\mu_j^{(k)})$. Then we get a $\lambda \vdash_{\text{o}} n$ with $f_k(\lambda) = \mu$ as follows: Rows 0 to $k-1$ of $\mathcal{C}_p(\lambda)$ and $\mathcal{C}_p(\mu)$ coincide. Row k of $\mathcal{Q}_p(\lambda)$ and $\mathcal{Q}_p(\mu)$ coincide except that $f_0^{-1}(\mu_j^{(k)})$ replaces $\mu_j^{(k)}$. The fact that $f_k(\lambda) = \mu$ follows from the above analysis and Remark 2.2. Since there are 2^k choices for j, the result is proved. □

Remark 4.6. The map f_k may induce a surjective map between the sets of odd partitions of n and odd partitions of $n - 2^k$, without 2^k being a 2-adic summand of n. An example is $n = 11$, $k = 2$. For $n = 17$ both f_1 and f_2 are neither surjective nor injective.

5. A canonical McKay correspondence

As mentioned above, a canonical bijection between $\mathrm{Irr}_{2'}(\mathsf{S}_n)$ and $\mathrm{Irr}_{2'}(P)$ for $P \in \mathrm{Syl}_2(\mathsf{S}_n)$ was constructed in [GKNT, Theorem 4.3]. Using our results, we can now give a representation-theoretic description of this McKay bijection.

Let $n = 2^{k_1} + \ldots + 2^{k_t}$, with $k_1 > k_2 > \ldots > k_t \geq 0$ be the 2-adic decomposition of n, $P_i \in \mathrm{Syl}_2(\mathsf{S}_{2^{k_i}})$, so that

$$P := P_1 \times P_2 \times \ldots \times P_t \in \mathrm{Syl}_2(\mathsf{S}_n).$$

Given any $\chi \in \mathrm{Irr}_{2'}(\mathsf{S}_n)$, we form the sequences $(n_1, \ldots, n_t)$, $(\chi_1, \ldots, \chi_t)$, $(\lambda_1, \ldots, \lambda_t)$, and a linear character $\chi^\sharp \in \mathrm{Irr}(P)$ as follows. Set $\chi_0 := \chi$, $n_0 := n$. For any $1 \leq i \leq t$, by Theorem A there is a unique odd-degree character χ_i of S_{n_i} with $n_i := n_{i-1} - 2^{k_i} = \sum_{j=i+1}^{t} 2^{k_j}$, such that χ_i occurs with odd multiplicity in the restriction of χ_{i-1} to S_{n_i}. By Theorem B, the restriction of χ_{i-1} to $\mathsf{S}_{n_i} \times \mathsf{S}_{2^{k_i}}$ contains a unique odd-degree irreducible constituent $\chi_i \otimes \varphi_i$ that lies above χ_i.

By [G, Theorem 3.2], the restriction of φ_i to P_i contains a unique linear character λ_i. Set

$$\chi^\sharp := \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_t \in \mathrm{Irr}(P).$$

Theorem 5.1. *The map* $\chi \mapsto \chi^\sharp$ *is the canonical McKay bijection* $\mathrm{Irr}_{2'}(\mathsf{S}_n) \to \mathrm{Irr}(P)$ *constructed in* [GKNT, Theorem 4.3].

Proof. Let χ_i be labeled by the odd partition $\pi_i \vdash_{\text{o}} n_i$. By Theorem B, $Y(\pi_i)$ is obtained from $Y(\pi_{i-1})$ by removing a rim 2^{k_i}-hook, which is unique by [APS, Lemma 1]. Let $\mu_i \vdash 2^{k_i}$ denote the corresponding hook partition of 2^{k_i}, and note that $\varphi_i = \chi^{\mu_i}$. Now the map α in [GKNT, Theorem 4.3(i)] is given by

$$\alpha(\chi) = (\mu_1, \ldots, \mu_t),$$

and, furthermore, the map β in [GKNT, Theorem 4.3(i)] satisfies

$$\beta^{-1}(\mu_1, \ldots, \mu_t) = \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_t.$$

Thus $\lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_t$ is exactly $\beta^{-1}(\alpha(\chi)) = \chi^\sharp$ as stated in [GKNT, Theorem 4.3(iii)]. □

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