IRREDUCIBLE CHARACTERS OF EVEN DEGREE AND NORMAL SYLOW 2-SUBGROUP

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ABSTRACT. The classical Itô-Michler theorem on character degrees of finite groups asserts that if the degree of every complex irreducible character of a finite group G is coprime to a given prime p, then G has a normal Sylow p-subgroup. We propose a new direction to generalize this theorem by introducing an invariant concerning character degrees. We show that if the average degree of linear and even-degree irreducible characters of G is less than $4/3$ then G has a normal Sylow 2-subgroup, as well as corresponding analogues for real-valued characters and strongly real characters. These results improve on several earlier results concerning the Itô-Michler theorem.

1. INTRODUCTION

The celebrated Itô-Michler theorem [Itô, Mic1] is one of the deep and fundamental results on the relation between character degrees and local structure of finite groups. It asserts that if a prime p does not divide the degree of every complex irreducible character of a finite group G, then G has a normal abelian Sylow p-subgroup.

Over the past decades, there have been several variations and refinements of this result by considering Brauer characters [Man, MW, Mic2], nonvanishing elements [DPSS, NT2], fields of character values [DNT, NT2, IN], and Frobenius-Schur indicator [MT, T]. One primary direction is to weaken the condition that all irreducible characters of G have degree coprime to p, and assume instead only that a subset of characters with a specified field of values has this property, see [DNT, MT] for real-valued characters and [NT2] for p-rational characters.

In this paper, we introduce an invariant concerning character degrees and propose to generalize the Itô-Michler theorem in a completely new direction.

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Given a finite group G, let $\mathrm{Irr}(G)$ denote the set of all complex irreducible characters of G, then write

$$\mathrm{Irr}_p(G) := \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1 \text{ or } p \mid \chi(1)\}$$

and

$$\mathrm{acd}_p(G) := \frac{\sum_{\chi \in \mathrm{Irr}_p(G)} \chi(1)}{|\mathrm{Irr}_p(G)|}$$

so that $\mathrm{acd}_p(G)$ is the average degree of linear characters and irreducible characters of G with degree divisible by p. Then the Itô-Michler theorem can be reformulated in the following way:

If $\mathrm{acd}_p(G) = 1$ *then* G *a has normal abelian Sylow* p*-subgroup.*

Our first result significantly improves this for the prime $p = 2$.

Theorem 1.1. *Let* G *be a finite group. If* $\mathrm{acd}_2(G) < 4/3$ *then* G *has a normal Sylow* 2*-subgroup.*

Theorem 1.1 basically says that even when a group has some irreducible characters of even degree, it still has a normal Sylow 2-subgroup as long as the number of linear characters of the group is large enough. This of course implies the Itô-Michler theorem for $p = 2$ where it is required that the group has no irreducible characters of even degree at all.

One of the key steps in the proof of Theorem 1.1 is to establish the solvability of the groups in consideration. In fact, we can do more.

Theorem 1.2. *Let* G *be a finite group. If* $\mathrm{acd}_2(G) < 5/2$ *then* G *is solvable.*

We remark that both bounds in Theorems 1.1 and 1.2 are optimal, as shown by the groups S_3 and A_5. Furthermore, $\mathrm{acd}_2(G)$ of non-abelian 2-groups G can get as close to 1 as we wish – just consider the extraspecial 2-groups, and therefore one can not get the commutativity of the Sylow 2-subgroup in Theorem 1.1 as in the Itô-Michler theorem.

In fact, we can also improve on some main results of [DNT] and [MT], by restricting our attention to only *real-valued* characters or even *strongly real* characters. A character $\chi \in \mathrm{Irr}(G)$ is called *real-valued* if $\chi(g) \in \mathbb{R}$ for all $g \in G$, and *strongly real* if it has Frobenius-Schur indicator 1, equivalently, if it is afforded by a real representation. Let

$$\mathrm{Irr}_{p,\mathbb{R}}(G) := \{\chi \in \mathrm{Irr}_p(G) \mid \chi \text{ is real-valued}\},\ \mathrm{acd}_{p,\mathbb{R}}(G) := \frac{\sum_{\chi \in \mathrm{Irr}_{p,\mathbb{R}}(G)} \chi(1)}{|\mathrm{Irr}_{p,\mathbb{R}}(G)|},$$

and

$$\mathrm{Irr}_{p,+}(G) := \{\chi \in \mathrm{Irr}_p(G) \mid \chi \text{ is strongly real}\},\ \mathrm{acd}_{p,+}(G) := \frac{\sum_{\chi \in \mathrm{Irr}_{p,+}(G)} \chi(1)}{|\mathrm{Irr}_{p,+}(G)|}$$

Theorem 1.3. *Let G be a finite group. We have:*
 (i) *If $\mathrm{acd}_{2,+}(G) \leq 2$ then G is solvable.*
 (ii) *If $\mathrm{acd}_{2,+}(G) < 4/3$ then G has a normal Sylow 2-subgroup.*

Theorem 1.3(ii) immediately implies [DNT, Theorem A] and [MT, Theorem B]. Furthermore, since any real-valued character of degree 1 is automatically strongly real, it has the following consequence.

Corollary 1.4. *Let G be a finite group. We have:*
 (i) *If $\mathrm{acd}_{2,\mathbb{R}}(G) \leq 2$ then G is solvable.*
 (ii) *If $\mathrm{acd}_{2,\mathbb{R}}(G) < 4/3$ then G has a normal Sylow 2-subgroup.*

Again, the example of S_3 shows that the bounds in Theorem 1.3(ii) and Corollary 1.4(ii) are optimal.

To prove Theorems 1.2 and 1.3(i), we have to use the classification of finite simple groups to show that every nonabelian finite simple group S possesses an irreducible character of even and large enough degree which is extendible to its stabilizer in $\mathrm{Aut}(S)$, cf. Theorem 2.1. Together with Proposition 2.3, this result allows us to bound the number of (strongly real) irreducible characters of small degrees of a finite group with a nonabelian minimal normal subgroup, and then to control the invariant $\mathrm{acd}_2(G)$ of such a group, see Section 2 and Proposition 3.1. We hope that the techniques developed here will be useful in the future study of other problems involving the average degree of a certain set of characters.

One obvious question that one may ask is: is there an analogue of Theorem 1.1 for odd primes? Although our ideas in the proof for the prime 2 do not carry out smoothly to odd primes, we believe that the following is true.

Conjecture 1.5. *Let p be a prime and G a finite group. If $\mathrm{acd}_p(G) < 2p/(p+1)$ then G has a normal Sylow p-subgroup.*

The bound in Conjecture 1.5 perhaps is not optimal for all primes. If C_p (the cyclic group of order p) can act nontrivially on an abelian group of order $p+1$, then the bound clearly can not be lower (for instance $p = 3, 7$, or any Mersenne prime). But when $p = 5$ for example, C_5 can only act trivially on an abelian group of order 6, and we think that the best possible bound is not 10/6, but instead is 15/7, which is attained at $C_{11} \rtimes C_5$.

Theorems 1.2, 1.1, and 1.3 are respectively proved in Sections 3, 4, and 5.

2. Extendibility of some characters of even degree

Throughout the paper, for a finite group G and a positive integer k, we write $n_k(G)$ to denote the number of irreducible complex characters of G of degree k, and $n_{k,+}(G)$ to denote the number of strongly real, irreducible complex characters of G of degree k. Furthermore, if N is a normal subgroup of G, then $n_k(G|N)$ denotes the number of irreducible characters of G of degree k whose kernels do not contain N, and similarly for $n_{k,+}(G|N)$. If θ is a character of a normal subgroup of G, we write $I_G(\theta)$ to denote the stabilizer or the inertia subgroup of θ in G. Other notation is standard (and follows [Isa1]) or will be defined when needed.

We need the following result, whose proof relies on the classification of finite simple groups.

Theorem 2.1. *Every nonabelian finite simple group S has an irreducible character θ of even degree such that $\theta(1) \geq 4$ and θ is extendible to a strongly real character of $I_{\mathrm{Aut}(S)}(\theta)$. Furthermore, if $S \not\cong \mathsf{A}_5$ then θ can be chosen so that $\theta(1) \geq 8$.*

Proof. (i) The cases where $S \cong \mathsf{A}_n$ with $5 \leq n \leq 8$, or $S \cong \mathrm{PSL}_2(q)$ with $q \leq 19$, or $S \cong \mathrm{PSU}_3(3)$, $\mathrm{PSp}_4(3)$, $\mathrm{Sp}_6(2)$, ${}^2F_4(2)'$, or S is one of the 26 sporadic finite simple groups, can be checked directly using [Atl]. (We note that in all these cases but $S = \mathsf{A}_6$, we can always find θ so that it has a rational-valued extension to $I_{\mathrm{Aut}(S)}(\theta)$.) In what follows, we may therefore assume that S is not isomorphic to any of the listed groups. In particular, it follows from the main results of [R, SZ] that the degree of any nontrivial complex irreducible character of S is at least 8.

(ii) Certainly, one can find many different choices for the desired character θ. In what follows, having in mind some other applications, we will try to construct θ in such a way that its extension to $I_{\mathrm{Aut}(S)}(\theta)$ is rational-valued if possible.

Assume now that $S \cong \mathsf{A}_n$ with $n \geq 9$. Consider the irreducible characters $\alpha, \beta \in \mathrm{Irr}(\mathsf{S}_n)$ labeled by the partitions $(n-2, 2)$ and $(n-2, 1^2)$, of degree $n(n-3)/2$ and $(n-1)(n-2)/2$ respectively. Since the given partitions are not self-conjugate, α and β both restrict irreducibly to S. Furthermore, $\beta(1) = \alpha(1) + 1 \geq 28$, and so exactly one of α, β has even degree. As $\mathrm{Aut}(S) \cong \mathsf{S}_n$, we are done in this case by choosing $\theta \in \{\alpha_S, \beta_S\}$ of even degree.

Next we consider the case S is a finite simple group of Lie type in characteristic 2. As shown in [F], the Steinberg character of S, of degree $|S|_2$, extends to a character of a rational representation of $\mathrm{Aut}(S)$, whence we are done again.

(iii) From now on we may assume that S is a finite simple group of Lie type, defined over a finite field $\mathbb{F}_q$ of odd characteristic p.

Consider the cases where $S = \mathrm{PSU}_n(q)$ with $n \geq 4$, $\mathrm{PSp}_{2n}(q)$ with $n \geq 2$, $\Omega_{2n+1}(q)$ with $n \geq 3$, $\mathrm{P}\Omega^+_{2n}(q)$ with $n \geq 5$, or $\mathrm{P}\Omega^-_{2n}(q)$ with $n \geq 4$. As shown in pp. 1–4 of the proof of [DNT, Theorem 2.1], $\mathrm{Aut}(S)$ has a rank 3 permutation character $\rho = 1 + \alpha + \beta$, where the characters α and β both restrict nontrivially and irreducibly to S. Furthermore, exactly one of α and β is of even degree (and both are afforded by rational representations). Hence we are done by choosing $\theta \in \{\alpha_S, \beta_S\}$ of even degree.

The same argument, but applied to a rank 5 permutation character of $\mathrm{Aut}(S)$, see p. 5 of the proof of [DNT, Theorem 2.1], handles the cases $S = E_6(q)$ or ${}^2E_6(q)$.

Suppose that $S = \mathrm{PSL}_n(q)$ with $n \geq 4$. As shown in the proof of [NT1, Proposition 5.5], $\mathrm{Aut}(S)$ has a permutation representation, whose character contains a rational-valued irreducible character γ of (even) degree

$$\begin{cases} (q^n-1)(q^{n-1}-1)/(q-1)^2, & q \equiv 1 (\bmod 4), \\ (q^n-1)(q^{n-1}-1)/(q^2-1), & q \equiv 3 (\bmod 4), \end{cases}$$

with multiplicity one that restricts irreducibly to S. It follows that γ is also afforded by a rational representation, and we can choose $\theta = \gamma_S$.

(iv) In the remaining cases, our choice of θ yields a not necessarily rational, but still admits a strongly real extension to $J := I_{\mathrm{Aut}(S)}(\theta)$. First, if $S = \mathrm{PSL}_3(q)$, then the (unique) character $\theta \in \mathrm{Irr}(S)$ of degree q^2+q extends to $\mathrm{Aut}(S)$ by [T, Lemma 6.2]. Next, if $S = G_2(q)$, ${}^2G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, or $E_8(q)$, then the proof of [MT, Proposition 4.4] yields a strongly real character $\theta \in \mathrm{Irr}(S)$ of even degree, with $J = S$.

If $S = \mathrm{P}\Omega_8^+(3)$ then by p. 1 of the proof of [MT, Proposition 4.9] we can choose θ of degree 300.

Suppose now that $S = \mathrm{PSL}_2(q)$, $\mathrm{PSU}_3(q)$, or (a simple group of type) $D_4(q)$ with $q \geq 5$, or $E_7(q)$. Then we can view S as the derived subgroup of a finite Lie-type group G of adjoint type: $G = \mathrm{PGL}_2(q)$, $\mathrm{PGU}_3(q)$, $D_4(q)_{\mathrm{ad}}$, or $E_7(q)_{\mathrm{ad}}$, respectively. As shown in the proof of [MT, Proposition 4.5] (for types A_1 and E_7), Case IIb of the proof of [MT, Proposition 4.7] (for PSU_3), and the proof of [MT, Proposition 4.9] (for type D_4), G contains a strongly real character of even degree that restricts to an irreducible character $\theta \in \mathrm{Irr}(S)$ with $J = G$. So we are done in these cases as well. $\square$

When a finite group G has a nonabelian minimal normal subgroup N, by using Theorem 2.1, we can produce an irreducible character of N of even degree that is extendible to (a strongly real character of) its stabilizer in G.

Theorem 2.2. *Let G be a finite group with a nonabelian minimal normal subgroup $N \not\cong \mathsf{A}_5$. Then there exists $\varphi \in \mathrm{Irr}(N)$ of even degree such that $\varphi(1) \geq 8$ and φ is extendible to a strongly real character of $I_G(\varphi)$.*

Proof. Since N is a nonabelian minimal normal subgroup of G, we have $N \cong S \times S \times \cdots \times S$, a direct product of k copies of a nonabelian simple group S. Replacing G by $G/\mathbf{C}_G(N)$ if necessary, we may assume that $\mathbf{C}_G(N) = 1$. Then we have

$$N \trianglelefteq G \leq \mathrm{Aut}(N) = \mathrm{Aut}(S) \wr \mathsf{S}_r.$$

Let θ be an irreducible character of S found in Theorem 2.1 and let $\mathcal{O}_S$ be the orbit of θ in the action of $\mathrm{Aut}(S)$ on $\mathrm{Irr}(S)$. Consider the character $\varphi := \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$. Then the orbit of φ under the action of $\mathrm{Aut}(N)$ is

$$\mathcal{O}_N := \{\theta_1 \times \theta_2 \times \cdots \times \theta_k \in \mathrm{Irr}(N) \mid \theta_i \in \mathcal{O}_S\}.$$

Clearly φ is invariant under $I_{\mathrm{Aut}(S)}(\theta) \wr \mathsf{S}_k$. On the other hand,

$$|\mathrm{Aut}(N) : I_{\mathrm{Aut}(S)}(\theta) \wr \mathsf{S}_k| = |\mathrm{Aut}(S) : I_{\mathrm{Aut}(S)}(\theta)|^k = |\mathcal{O}_S|^k = |\mathcal{O}_N|.$$

Therefore we deduce that $I_{\mathrm{Aut}(N)}(\varphi) = I_{\mathrm{Aut}(S)}(\theta) \wr \mathsf{S}_r$.

Assume that θ extends to a strongly real character α of $J := I_{\mathrm{Aut}(S)}(\theta)$, say afforded by an $\mathbb{R}J$-module V. Then $I_{\mathrm{Aut}(N)}(\varphi)$ acts naturally on the $\mathbb{R}$-space $V^{\otimes k}$, on which N acts with character φ. It follows that φ is extendible to the strongly real character of $I_G(\varphi) = G \cap I_{\mathrm{Aut}(N)}(\varphi)$ afforded by $V^{\otimes k}$.

Since $\theta(1) \geq 4$ in general and $\theta(1) \geq 8$ when $S \not\cong \mathsf{A}_5$, we observe that $\varphi(1) = \theta(1)^k \geq 8$ as long as $(S, k) \neq (\mathsf{A}_5, 1)$, and we are done. □

Theorem 2.2 can be used to bound the number of irreducible characters of degrees 1 and 2 of finite groups with a nonabelian minimal normal subgroup, as shown in the next proposition.

Proposition 2.3. *Let G be a finite group with a nonabelian minimal normal subgroup N. Assume that there is some $\varphi \in \mathrm{Irr}(N)$ such that φ is extendible to $I_G(\varphi)$. Let $d := \varphi(1)|G : I_G(\varphi)|$. Then the following hold.*

(i) $n_1(G) \leq n_d(G)|G : I_G(\varphi)|$.

(ii) *If φ extends to a strongly real character of $I_G(\varphi)$, then*

$$n_{1,+}(G) \leq n_{d,+}(G)|G : I_G(\varphi)|.$$

(iii) $n_2(G) \leq n_{2d}(G)|G : I_G(\varphi)| + \frac{1}{2}n_d(G)|G : I_G(\varphi)|$.

Moreover, if φ is invariant in G then $n_2(G) \leq n_{2d}(G)$.

Proof. For simplicity we write $I := I_G(\varphi)$.

(i) First, since $n_1(G) = |G : G'|$ and $n_1(I) = |I : I'|$, we have $n_1(G) \leq |G : I|n_1(I)$. Therefore, we wish to show that $n_1(I) \leq n_d(G)$ where $d := \varphi(1)|G : I|$.

As $N = N' \subseteq I'$, the normal subgroup N is contained in the kernel of every linear character of I so that

$$n_1(I) = n_1(I/N).$$

Recall that $\varphi \in \mathrm{Irr}(N)$ is extendible to I and so we let $\chi \in \mathrm{Irr}(I)$ be an extension of φ. Using Gallagher's theorem and Clifford's theorem (see [Isa1, Corollary 6.17 and Theorem 6.11]), we see that each linear character λ of I/N produces the irreducible character $\lambda\chi$ of T of degree $\varphi(1)$, and this character in turns produces the irreducible character $(\lambda\chi)^G$ of G of degree $(\lambda\chi)^G(1) = \varphi(1)|G : I| = d$. It follows that

$$n_1(I/N) \leq n_d(G)$$

and we therefore have $n_1(I) \leq n_d(G)$, as desired.

(ii) For any group X, let X^* denote the subgroup generated by all x^2, $x \in X$. Then note that $n_{1,+}(G) = |G : G^*|$, $n_{1,+}(I) = |I : I^*|$, and $I^* \leq G^*$, whence $n_{1,+}(G) \leq |G : I|n_{1,+}(I)$. Furthermore, for any $\chi \in \mathrm{Irr}(G)$ and any strongly real linear character λ of G, $\lambda^2 = 1_G$ and so χ and $\chi\lambda$ have the same Frobenius-Schur indicator. In particular, $\chi\lambda$ is strongly real if and only if χ is. Furthermore, if ρ is a strongly real character of a subgroup $T \leq G$, then so is the induced character ρ^G. Now we can argue as in (i) to complete the proof.

(iii) We first claim that

$$n_2(G) \leq n_2(I)|G : I| + \frac{1}{2}n_1(I)|G : I|.$$

Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$. Take ϕ to be an irreducible constituent of $\chi \downarrow_I$. Frobenius reciprocity then implies that χ in turn is an irreducible constituent of ϕ^G. If $\phi(1) = 2$ then as $\phi^G(1) = 2|G : I|$, there are at most $|G : I|$ irreducible constituents of degree 2 of ϕ^G. We deduce that there are at most $n_2(I)|G : I|$ irreducible characters of degree 2 of G that arise as constituents of ϕ^G with $\phi(1) = 2$. On the other hand, if $\phi(1) = 1$ then, as $\phi^G(1) = |G : I|$, there are at most $|G : I|/2$ irreducible constituents of degree 2 of ϕ^G. As above, we deduce that there are at most $n_1(I)|G : I|/2$ irreducible characters of degree 2 of G that arise as constituents of ϕ^G with $\phi(1) = 1$. So the claim is proved.

Since we have already proved in (i) that $n_1(I) = n_1(I/N) \leq n_d(G)$, to prove Proposition 2.3(iii) it suffices to show that $n_2(I) \leq n_{2d}(G)$.

We claim that

$$n_2(I) = n_2(I/N)$$

or in other words, N is contained in the kernel of every irreducible character of degree 2 of I. Let $\phi \in \mathrm{Irr}(I)$ with $\phi(1) = 2$. Since N has no irreducible character of degree 2 and has only one linear character, which is the trivial one, it follows that $\phi_N = 2 \cdot 1_N$. We then have $N \subseteq \mathrm{Ker}(\phi)$, as claimed.

Recall that $\chi \in \mathrm{Irr}(I)$ is an extension of ψ. Using Gallagher's theorem and Clifford's theorem again, we obtain that each irreducible character $\mu \in \mathrm{Irr}(I/N)$ of degree 2 produces the character $(\mu\chi)^G \in \mathrm{Irr}(G)$ of degree $(\mu\chi)^G(1) = 2\psi(1)|G : I| = 2d$. It follows that

$$n_2(I/N) \leq n_{2d}(G),$$

and thus $n_2(I) \leq n_{2d}(G)$, as desired.

If φ is invariant in G then $G = I$, yielding immediately that $n_2(G) \leq n_{2d}(G)$. □

3. Solvability - Theorem 1.2

In this section, we use the results in Section 2 to prove Theorem 1.2. The next proposition handles an important case of this theorem.

Proposition 3.1. *Let G be a finite group with a nonabelian minimal normal subgroup. Then* $\mathrm{acd}_2(G) \geq 5/2$.

Proof. Let N be a nonabelian minimal normal subgroup of G. First we assume that $N \not\cong \mathsf{A}_5$. Then Theorem 2.2 guarantees that there is $\varphi \in \mathrm{Irr}(N)$ of even degree such that $\varphi(1) \geq 8$ and φ is extendible to the inertia subgroup $I_G(\varphi)$. Using Proposition 2.3, we then have

$$n_1(G) \leq n_d(G)|G : I_G(\varphi)|$$

and

$$n_2(G) \leq n_{2d}(G)|G : I_G(\varphi)| + \frac{1}{2}n_d(G)|G : I_G(\varphi)|,$$

where $d := \varphi(1)|G : I_G(\varphi)|$. It follows that

$$3n_1(G) + n_2(G) \leq \frac{7}{2}n_d(G)|G : I_G(\varphi)| + n_{2d}(G)|G : I_G(\varphi)|.$$

Since $\varphi(1) \geq 8$, we have $d \geq 8|G : I_G(\varphi)|$ and hence we can check that $(7/2)|G : I_G(\varphi)| < 2d - 5$ and $|G : I_G(\varphi)| < 4d - 5$. It follows that

$$3n_1(G) + n_2(G) < \sum_{2|k\geq 4}(2k - 5)n_k(G),$$

and thus $\mathrm{acd}_2(G) > 5/2$, as desired.

So it remains to consider $N \cong \mathsf{A}_5$. Then N has an irreducible character φ of degree 4 that is extendible to $\mathrm{Aut}(N)$ (see [Atl, p. 2]), and hence extendible to G as well. It follows from Proposition 2.3 that $n_1(G) \leq n_4(G)$ and $n_2(G) \leq n_8(G)$. Thus

$$3n_1(G) + n_2(G) \leq 3n_4(G) + n_8(G) \leq \sum_{2|k\geq 4}(2k - 5)n_k(G).$$

Again this yields $\mathrm{acd}_2(G) \geq 5/2$ and the proof is completed. □

We are now ready to prove Theorem 1.2, which we restate below.

Theorem 3.2. *Let G be a finite group. If $\mathrm{acd}_2(G) < 5/2$ then G is solvable.*

Proof. Assume, to the contrary, that the theorem is false, and let G be a minimal counterexample. In particular $\mathrm{acd}_2(G) < 5/2$ and G is nonsolvable. Let $H \trianglelefteq G$ be minimal such that H is nonsolvable. Then clearly H is perfect and contained in the last term of the derived series of G. Choose a minimal normal subgroup N of G such that $N \subseteq H$, and when $[H, \mathbf{O}_\infty(H)] \neq 1$ we choose $N \subseteq [H, \mathbf{O}_\infty(H)]$, where $\mathbf{O}_\infty(H)$ denotes the largest normal solvable subgroup of H.

In view of Proposition 3.1, we can assume that N is abelian. It follows that the quotient G/N is nonsolvable since G is nonsolvable. By the minimality of G, we must have $\mathrm{acd}_2(G/N) \geq 5/2$. So

$$\mathrm{acd}_2(G) < 5/2 \leq \mathrm{acd}_2(G/N).$$

Since $n_k(G/N) \leq n_k(G)$ for every positive integer k and $n_1(G/N) = n_1(G)$ as $N \subseteq G'$, it follows that $n_2(G/N) < n_2(G)$ and thus there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2$ and $N \nsubseteq \mathrm{Ker}(\chi)$.

It then has been shown in the proof of [ILM, Theorem 2.2] that $G = LC$ is a central product with the amalgamated subgroup $N = \mathbf{Z}(L)$ of order 2, where

$$L = \mathrm{SL}_2(5) \text{ and } C/\mathrm{Ker}(\chi) := \mathbf{Z}(G/\mathrm{Ker}(\chi)).$$

Since $G = LC$ is a central product with the central amalgamated subgroup N, there is a bijection $(\alpha, \beta) \mapsto \tau$ from $\mathrm{Irr}(L|N) \times \mathrm{Irr}(C|N)$ to $\mathrm{Irr}(G|N)$ such that $\tau(1) = \alpha(1)\beta(1)$. If $(\alpha, \beta) \mapsto \chi$ under the above bijection, we must have $\beta(1) = 1$ since $L \cong \mathrm{SL}(2, 5)$ and there are only three possibilities for $\alpha(1)$, namely 2, 4, and 6. So $\beta \in \mathrm{Irr}(C|N)$ is an extension of the unique non-principal linear character of N. Using Gallagher's theorem, we then have a degree-preserving bijection from $\mathrm{Irr}(C/N)$ to $\mathrm{Irr}(C|N)$. In particular, $n_1(C|N) = n_1(C/N)$. Since $G/L \cong C/N$ and $L \subseteq G'$, we have

$$n_1(C|N) = n_1(C/N) = n_1(G).$$

Employing the arguments in the proof of [MN, Theorem B], we can evaluate and estimate $n_2(G)$, $n_4(G)$, and $n_6(G)$ in terms of $n_1(G)$ as follows:

(i) $n_2(G) = 2n_1(G) + n_2(C/N)$,
(ii) $n_4(G) \geq 2n_1(G)$, and
(iii) $n_6(G) \geq n_1(G) + 2n_2(C/N)$.

Now putting all things together, we have

$$\begin{aligned}\sum_{2|k\geq 4} (2k-5)n_k(G) &\geq 3n_4(G) + 7n_6(G) \\ &\geq 6n_1(G) + 7(n_1(G) + 2n_2(C/N) \\ &= 13n_1(G) + 14n_2(C/N) \\ &> 3n_1(G) + n_2(G).\end{aligned}$$

It then follows that $\mathrm{acd}_2(G) > 5/2$ and this is a contradiction. □

4. Normal Sylow 2-subgroup - Theorem 1.1

The next lemma is crucial in the proof of Theorem 1.1.

Lemma 4.1. *Let $G = N \rtimes M$ where N is an abelian group. Assume that no non-principal irreducible character of N is invariant under M. If $\mathrm{acd}_2(G) < 4/3$ then there is no orbit of even size in the action of M on the set of irreducible characters of N.*

Proof. Let $\{1_N = \alpha_0, \alpha_1, \ldots, \alpha_f\}$ be a set of representatives of the action of M on $\mathrm{Irr}(N)$. For each $1 \leq i \leq f$, let I_i be the inertia subgroup of α_i in G. Since no nonprincipal irreducible character of N is invariant under M, we observe that every I_i is a proper subgroup of G.

Assume, to the contrary, that there is some orbit of even size in the action of M on $\mathrm{Irr}(N)$. Then there exists some index $1 \leq i \leq f$ such that $|G : I_i|$ is even. For $0 \leq i \leq f$, set

$$n_{i,1} := n_1(I_i/N),\ n_{i,\mathrm{even}} := \sum_{2|k} n_k(I_i/N),\ s_{i,\mathrm{even}} := \sum_{\lambda \in \mathrm{Irr}(I_i/N),\ 2|\lambda(1)} \lambda(1).$$

Since G splits over N, it is clear that every I_i also splits over N. It follows that α_i extends to a linear character, say β_i, of I_i as α_i is linear. Gallagher's theorem then implies that the mapping $\lambda \mapsto \lambda\beta_i$ is a bijection from $\mathrm{Irr}(I_i/N)$ to the set of irreducible characters of I_i lying above α_i. Using Clifford's theorem, we then obtain a bijection $\lambda \mapsto (\lambda\beta_i)^G$ from $\mathrm{Irr}(I_i/N)$ to the set of irreducible characters of G lying above α_i. We note that $(\lambda\beta_i)^G(1) = |G : I_i|\lambda(1)$ and hence $(\lambda\beta_i)^G(1)$ is even if and only if either $|G : I_i|$ is even or $\lambda(1)$ is even.

We have

$$\begin{aligned}\sum_{\chi \in \mathrm{Irr}(G), \chi(1)=1 \text{ or even}} \chi(1) &= n_1(G/N) + \sum_{|G:I_i| \text{ even}} |G : I_i| n_{i,1} + \sum_{i=0}^{f} |G : I_i| s_{i,\mathrm{even}} \\ &\geq n_1(G/N) + \sum_{|G:I_i| \text{ even}} |G : I_i| n_{i,1} + 2\sum_{i=0}^{f} |G : I_i| n_{i,\mathrm{even}}.\end{aligned}$$

On the other hand,

$$\sum_{\chi \in \mathrm{Irr}(G), \chi(1)=1 \text{ or even}} \chi(1) = \mathrm{acd}_2(G) \left(n_1(G/N) + \sum_{|G:I_i| \text{ even}} n_{i,1} + \sum_{i=0}^{f} n_{i,\mathrm{even}} \right).$$

Therefore, we deduce that

$$\begin{aligned}\sum_{|G:I_i| \text{ even}} (|G : I_i| - \mathrm{acd}_2(G)) n_{i,1} + \sum_{i=0}^{f} (2|G : I_i| - \mathrm{acd}_2(G)) n_{i,\mathrm{even}} \\ \leq (\mathrm{acd}_2(G) - 1) n_1(G/N).\end{aligned}$$

Since $\mathrm{acd}_2(G) < 4/3$ and $|G : I_i| \geq 1$ for every $0 \leq i \leq f$, it follows that

$$\sum_{|G:I_i| \text{ even}} (|G : I_i| - \mathrm{acd}_2(G)) n_{i,1} \leq (\mathrm{acd}_2(G) - 1) n_1(G/N)$$

and hence

$$(|G : I_j| - \mathrm{acd}_2(G)) n_{j,1} \leq (\mathrm{acd}_2(G) - 1) n_1(G/N)$$

for some $1 \leq j \leq f$ such that $|G : I_j|$ is even.

Observe that $n_1(G/N) = |(G/N) : (G/N)'|$ and $n_{j,1} = n_1(I_j/N) = |(I_j/N) : (I_j/N)'|$. Therefore $n_1(G/N) \le |G : I_j|n_{j,1}$. It follows from the above inequality that

$$(|G : I_j| - \mathrm{acd}_2(G))n_{j,1} \le (\mathrm{acd}_2(G) - 1)|G : I_j|n_{j,1},$$

and thus

$$|G : I_j| \le \frac{\mathrm{acd}_2(G)}{2 - \mathrm{acd}_2(G)}.$$

This is impossible since $|G : I_j| \ge 2$ and $\mathrm{acd}_2(G) < 4/3$, and the proof is complete. $\square$

We now prove the main Theorem 1.1, which is restated below.

Theorem 4.2. *Let G be a finite group. If $\mathrm{acd}_2(G) < 4/3$ then G has a normal Sylow 2-subgroup.*

Proof. We will argue by induction on the order of G. We have $\mathrm{acd}_2(G) < 4/3$ and therefore by Theorem 1.2, G is solvable. If G is abelian then there is nothing to prove, so we assume that G is nonabelian. We then choose a minimal normal subgroup N of G such that $N \subseteq G'$. As G is solvable, we have that N is abelian.

Since $N \subseteq G'$, if an irreducible character of G has kernel not containing N, its degree must be at least 2. We therefore deduce that $\mathrm{acd}_2(G/N) \le \mathrm{acd}_2(G) < 4/3$ and it follows from the induction hypothesis that G/N has a normal Sylow 2-subgroup, say Q/N.

If N is a 2-group then Q is a normal Sylow 2-subgroup of G, and we are done. So we assume from now on that N is an elementary abelian group of odd order. The Schur-Zassenhaus theorem then implies that $Q = N \rtimes P$ where P is a Sylow 2-subgroup of Q (and G as well). By Frattini's argument, we have $G = Q\mathbf{N}_G(P) = N\mathbf{N}_G(P)$. Therefore, if N is contained in the Frattini subgroup of G, we would have $G = \mathbf{N}_G(P)$ and we are done. So we assume that N is not contained in the Frattini subgroup of G, and thus there exists a maximal subgroup M of G such that $N \not\subseteq M$. We then have $G = NM$ and $N \cap M < N$. As N is abelian, it follows that $N \cap M$ is a normal subgroup of G, and hence $N \cap M = 1$ by the minimality of N. We conclude that $G = N \rtimes M$.

If $N \subseteq \mathbf{Z}(G)$, then we would have $Q = N \times P$, so that $P \trianglelefteq G$, and we are done again. So we assume that N is noncentral in G. Thus, by the minimality of N, we have $[N, G] = N$. It follows that no nonprincipal irreducible character of N is invariant under M.

We are now in the situation of Lemma 4.1, and therefore we conclude that there is no orbit of even size in the action of M on the set of irreducible characters of N. In particular, there is no orbit of even size in the action of P on the set of irreducible characters of N. This means that P acts trivially on N since P is a 2-group. Now we have $Q = N \times P \trianglelefteq G$ and, as N has odd order, we deduce that $P \trianglelefteq G$ and this completes the proof of the theorem. $\square$

5. Strongly real characters - Theorem 1.3

In this section we prove Theorem 1.3. We first prove Theorem 1.3(i).

Theorem 5.1. *Let G be a finite group. If $\mathrm{acd}_{2,+}(G) \leq 2$ then G is solvable.*

Proof. Since there is nothing to prove if G is abelian, we assume that G' is nontrivial. Let $N \subseteq G'$ be a minimal normal subgroup of G. If a strongly real character $\chi \in \mathrm{Irr}(G)$ has kernel not containing N, its degree must be at least 2, which is above the average $\mathrm{acd}_{2,+}(G)$. We therefore deduce that

$$\mathrm{acd}_{2,+}(G/N) \leq \mathrm{acd}_{2,+}(G) \leq 2.$$

Working by induction, we now can assume that N is nonabelian.

By Theorem 2.2, there exists $\varphi \in \mathrm{Irr}(N)$ of even degree such that $\varphi(1) \geq 4$ and φ extends to a strongly real character of $I_G(\varphi)$. (Note that we assumed $N \not\cong \mathsf{A}_5$ in Theorem 2.2, but when $N \cong \mathsf{A}_5$ one can choose φ to be the irreducible character of degree 4, and this character is extendible to a strongly real character of S_5.) We then apply Proposition 2.3 to have

$$n_{1,+}(G) \leq n_{d,+}(G)|G : I_G(\varphi)|,$$

where $d := \varphi(1)|G : I_G(\varphi)|$. Since $\varphi(1) \geq 4$, it follows that

$$|G : I_G(\varphi)| < 4|G : I_G(\varphi)| - 2 \leq d - 2,$$

and so

$$n_{1,+}(G) < (d-2)n_{d,+}(G) \leq \sum_{2|k}(k-2)n_{k,+}(G).$$

Therefore we have

$$\mathrm{acd}_{2,+}(G) = \frac{n_{1,+}(G) + \sum_{2|k} k n_{k,+}(G)}{n_{1,+}(G) + \sum_{2|k} n_{k,+}(G)} > 2,$$

and this completes the proof. □

To prove Theorem 1.3(ii), we begin with two known observations on strongly real characters.

Lemma 5.2. *Let N be a normal subgroup of a finite group G such that G/N has odd order. Then every strongly real character $\varphi \in \mathrm{Irr}(N)$ lies under a unique strongly real irreducible character of G.*

Proof. This is [MT, Lemma 2.1(ii)]. □

Lemma 5.3. *Let $N \lhd G$ be such that G/N is a 2-group. Assume that $\varphi \in \mathrm{Irr}(N)$ has 2-defect 0, and that $\overline{\varphi}$ is G-conjugate to φ. Then there exists a strongly real character $\chi \in \mathrm{Irr}(G)$ such that $[\chi_N, \varphi]_N = 1$.*

Proof. This is [T, Lemma 2.5]. □

We also need the following observation.

Lemma 5.4. *Let P be a 2-group acting on an abelian group N of odd order such that N is the unique minimal normal subgroup of $N \rtimes P$. Then $|N|-1 \geq |P : \Phi(P)|$, where $\Phi(P)$ is the Frattini subgroup of P.*

Proof. Suppose that N is a product of n copies of the cyclic groups C_q, where q is a prime. Since N is the only minimal normal subgroup of $N \rtimes P$, we have $\mathbf{C}_P(N) = 1$, and the action of P on N induces a faithful irreducible representation $\mathfrak{X}$ of P over the field $\mathbb{F}_q$. We extend this representation to the representation $\mathfrak{X}^{\overline{\mathbb{F}}_q}$ over the algebraically closure $\overline{\mathbb{F}}_q$. Since $|P|$ is even and q is odd, by Maschke's theorem, $\mathfrak{X}^{\overline{\mathbb{F}}_q}$ is completely reducible, and moreover, is faithful since $\mathfrak{X}$ is faithful. Using the Fong-Swan theorem [Nav, Theorem 10.1] on lifts of irreducible Brauer characters in solvable groups, we conclude that P has a complex faithful character, say χ, of degree n.

Now we apply [Isa2, Theorem A] to deduce that the number of generators in a minimal generating set for P, say $d(P)$, is at most $(3/2)(n-s)+s$, where s is the number of linear constituents of χ. In particular, $d(P) \leq [3n/2]$, and it follows that

$$|P : \Phi(P)| \leq 2^{[3n/2]}.$$

As it is easy to check that $2^{[3n/2]} \leq 3^n - 1$ for every positive integer n, we then have

$$|P : \Phi(P)| \leq 3^n - 1 \leq q^n - 1 = |N| - 1,$$

and the lemma follows. □

Lemmas 5.3 and 5.4 allow us to control $\mathrm{acd}_{2,+}(G)$ in the following special situation.

Proposition 5.5. *Let $G = N \rtimes P$ be a split extension of an elementary abelian group of odd order N by a nontrivial 2-group P. Assume that N is the unique minimal normal subgroup of G. Then $\mathrm{acd}_{2,+}(G) \geq 4/3$.*

Proof. It is well known that groups of odd order have no non-principal real irreducible character. Therefore 1_N is the only real irreducible character of N. It follows that every (strongly) real linear character of G has N inside its kernel. In particular,

$$n_{1,+}(G) = n_{1,+}(G/N) = n_{1,+}(P) = |P : \Phi(P)|.$$

Together with Lemma 5.4, this implies that

$$n_{1,+}(G) \leq |N| - 1.$$

Hence, we can find s disjoint P-orbits $\Omega_1, \ldots, \Omega_s$ on $\mathrm{Irr}(N) \setminus \{1_N\}$ of size $k_1, \ldots, k_s$, such that

$$k_1 + \ldots + k_s \geq n_{1,+}(G).$$

From the hypotheses, we know that P acts irreducibly and faithfully on N, and therefore on $\mathrm{Irr}(N)$ as well since the two actions are permutationally isomorphic. Therefore, each nontrivial character in $\mathrm{Irr}(N)$ is inverted by a central involution of P.

In other words, for any nontrivial $\varphi \in \mathrm{Irr}(N)$, φ and $\overline{\varphi}$ are P-conjugate. Lemma 5.3 then guarantees that there is a strongly real character $\chi_{\{\varphi,\overline{\varphi}\}} \in \mathrm{Irr}(G)$ lying above both φ and $\overline{\varphi}$ with $[\chi_N, \varphi]_N = 1$. Since $I_G(\varphi)$ splits over N, φ is extendible to $I_G(\varphi)$, and hence $\chi_{\{\varphi,\overline{\varphi}\}}(1) = |G : I_G(\varphi)|$, which is an even number since φ is not G-invariant.

Applying the above argument to $\lambda_i \in \Omega_i$, we get a strongly real character $\chi_i \in \mathrm{Irr}(G)$ of even degree $\geq k_i \geq 2$; in particular, $s \leq (\sum_{i=1}^s k_i)/2$ and so

$$3\sum_{i=1}^{s} k_i - 4s \geq \sum_{i=1}^{s} k_i \geq n_{1,+}(G).$$

It follows that the average degree of these s characters $\chi_1, \dots, \chi_s$ and the (strongly) real linear characters of G is at least

$$\frac{n_{1,+}(G) + \sum_{i=1}^s k_i}{n_{1,+}(G) + s} \geq \frac{4}{3}.$$

Since any other strongly real irreducible character of G has degree at least 2, we conclude that $\mathrm{acd}_{2,+}(G) \geq 4/3$. □

The proof of Theorem 1.3 is completed by

Theorem 5.6. *Let G be a finite group. If $\mathrm{acd}_{2,+}(G) < 4/3$ then G has a normal Sylow 2-subgroup.*

Proof. We assume that the statement is false, and let G be a minimal counterexample. In particular, G is non-abelian. Let N be a minimal normal subgroup of G such that $N \subseteq G'$. As before, it follows that the degree of every strongly real, irreducible character of G whose kernel does not contain N is at least 2, and hence $\mathrm{acd}_{2,+}(G/N) \leq \mathrm{acd}_{2,+}(G) < 4/3$. By the minimality of G, we then deduce that G/N has a normal Sylow 2-subgroup, and thus $NP \trianglelefteq G$, where P is a Sylow 2-subgroup of G.

Since $\mathrm{acd}_{2,+}(G) < 4/3 < 2$, Theorem 5.1 guarantees that G is solvable, and so N is elementary abelian. If N has even order, then $NP = P$ and we would be done. So we may assume that N is an elementary abelian group of odd order, and moreover, G has no non-trivial normal 2-subgroup.

We observe that each strongly real linear character of G restricts to a strongly real linear character of NP, and moreover, by Lemma 5.2, each strongly real linear character of NP lies under a unique strongly real linear character of G. It follows that $n_{1,+}(NP) = n_{1,+}(G)$. This and Lemma 5.2 imply that $\mathrm{acd}_{2,+}(NP) \leq \mathrm{acd}_{2,+}(G) < 4/3$. Thus, if $NP < G$, then by the minimality of G, we have $P \lhd NP$ so that $NP = N \times P$, and hence $P \lhd G$, a contradiction. So we may assume that $G = N \rtimes P$. Since G has no non-trivial normal 2-subgroup, we see that N is the unique minimal normal subgroup of G.

Now we have all the hypotheses of Proposition 5.5, and therefore we conclude that $\mathrm{acd}_{2,+}(G) \geq 4/3$. This contradiction completes the proof. □

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