

W-ALGEBRAS FROM HEISENBERG CATEGORIES

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Abstract The trace (or zeroth Hochschild homology) of Khovanov’s Heisenberg category is identified with a quotient of the algebra $W_{1+\infty}$. This induces an action of $W_{1+\infty}$ on the center of the categorified Fock space representation, which can be identified with the action of $W_{1+\infty}$ on symmetric functions.

Keywords: Heisenberg algebra; categorification; W-algebras; Hochschild homology; trace; Heisenberg category

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1. Introduction

The algebra Sym of symmetric functions in infinitely many variables is important in classical representation theory in part because it describes the characters of representations of symmetric groups in characteristic 0. More precisely, there are

isomorphisms

$$\mathrm{Sym} \cong \bigoplus_{n=0}^{\infty} Z(\mathbb{C}[S_n]) \cong \bigoplus_{n=0}^{\infty} K_0(\mathbb{C}[S_n]\text{-mod}),$$

where $Z(\mathbb{C}[S_n])$ is the center of the group algebra, and $K_0(\mathbb{C}[S_n]\text{-mod})$ is the Grothendieck group of the category of finite-dimensional representations. Since $Z(\mathbb{C}[S_n])$ is isomorphic to the center of the category $\mathbb{C}[S_n]\text{-mod}$, the rightmost isomorphism says that two basic decategorifications of the categories $\mathbb{C}[S_n]\text{-mod}$, the Grothendieck group and the center, are isomorphic to each other.

It is fruitful to consider a structure of interest for symmetric functions and then try to interpret it in the language of symmetric groups. For example, Gessinger realized the Hopf algebra structure on Sym by considering the maps on characters induced by induction and restriction [14]. Closely related to this Hopf algebra structure is the fact that Sym is the underlying vector space of the canonical Fock space representation of the Heisenberg algebra $\mathfrak{h}$, where generators of $\mathfrak{h}$ act on the Grothendieck group as the maps induced by induction and restriction between different symmetric groups. The algebra of symmetric functions Sym thus becomes the canonical level one Fock space module, which we denote by $\mathcal{V}_{1,0}$, for the Heisenberg algebra.

The module $\mathcal{V}_{1,0}$ can also be equipped with an action of the Virasoro algebra (see [13] where an explicit action of the Virasoro on $\bigoplus_{n=0}^{\infty} Z(\mathbb{C}[S_n])$ is given). More generally, both these actions are unified in the action of the W -algebra $W_{1+\infty}$ [23].

One of the aims of this paper is to explain how this action $W_{1+\infty}$ on $\mathcal{V}_{1,0}$ arises naturally from a categorification of the Heisenberg algebra $\mathfrak{h}$. More precisely, Khovanov introduced in [17] a monoidal category $\mathcal{H}$ whose Grothendieck group contains the Heisenberg algebra $\mathfrak{h}$.¹ Moreover, he defines an action of $\mathcal{H}$ on $\bigoplus_{n=0}^{\infty} \mathbb{C}[S_n]\text{-mod}$.

The main result of this paper is a calculation of the trace $\mathrm{Tr}(\mathcal{H})$ (or zeroth Hochschild homology) of $\mathcal{H}$ and its action on the trace (or center) of $\bigoplus_{n=0}^{\infty} \mathbb{C}[S_n]\text{-mod}$. More precisely, we prove the following.

Theorem 1. *There is an isomorphism $\mathrm{Tr}(\mathcal{H}) \cong W_{1+\infty}/I$ where I is the two-sided ideal generated by $C-1$ and $w_{0,0}$.² Moreover, the action of $\mathrm{Tr}(\mathcal{H})$ on the center of $\bigoplus_{n=0}^{\infty} \mathbb{C}[S_n]\text{-mod}$ is identified with the canonical level one representation $\mathcal{V}_{1,0}$ of $W_{1+\infty}$.*

The theorem gives a graphical construction of both $W_{1+\infty}/I$ itself, and of its canonical representation. As a corollary, any categorical representation of Khovanov's Heisenberg category $\mathcal{H}$ gives rise to a representation of $W_{1+\infty}/I$. For example, a closely related categorical action of $\mathcal{H}$ on the category of polynomial functors [15] allows us to realize a $W_{1+\infty}$ -algebra action in this context.

From the point of view of Theorem 1, it is perhaps equally reasonable to refer to $\mathcal{H}$ as a categorification of $W_{1+\infty}$. The difference between this statement and the statement that $\mathcal{H}$ categorifies the Heisenberg algebra is that we have made a different choice of

¹Conjecturally, there is an algebra isomorphism $K_0(\mathcal{H}) \cong \mathfrak{h}$.

²The quotient of $W_{1+\infty}$ we see here is not exactly the same as the quotients of $W_{1+\infty}$ that have appeared in closely related literature. For example, in [20], the quotient of $W_{1+\infty}$ which appears has $w_{0,0} = \frac{1}{2}$ instead of $w_{0,0} = 0$.

decategorification procedure. Note however that the question of constructing a monoidal category whose Grothendieck group is $W_{1+\infty}$ remains open.

In the process of identifying $\mathrm{Tr}(\mathcal{H})$ with $W_{1+\infty}/I$, we are also able to find natural graphical realizations of the standard Heisenberg algebra generators. By contrast, in the Grothendieck decategorification the standard Heisenberg generators do not admit an obvious categorical lift and therefore an alternative presentation for the Heisenberg algebra is required.

An analog of Theorem 1 (and of this difference between the Grothendieck group and the trace) also arises in the context of quantum groups associated to type ADE Kac–Moody algebras. Associated to an arbitrary symmetrizable Kac–Moody algebra $\mathfrak{g}$, a 2-category $\mathcal{U}(\mathfrak{g})$ was defined in [18] whose split Grothendieck group is isomorphic to the integral form of Lusztig’s idempotent form of the quantum group $\mathbf{U}_q(\mathfrak{g})$. However, the trace of the 2-category $\mathcal{U}(\mathfrak{g})$ is then identified with the idempotent form of quantum current algebra $\mathbf{U}_q(\mathfrak{g}[t])$ [4, 5, 21].

1.1. Further directions

More generally, one can associate a Heisenberg category $\mathcal{H}_F$ to any finite-dimensional Frobenius algebra F . This is the categorical analog of the fact that one can associate a Heisenberg algebra $\mathfrak{h}_L$ to any $\mathbb{Z}$ -lattice L . If one takes the simplest Frobenius algebra, namely the one-dimensional algebra $\mathbb{C}$, then one recovers $\mathcal{H}$, and for this reason we first study the trace of $\mathcal{H}$. In future work we plan to compute the traces of these more general Heisenberg categories, and thus associate a W-algebra W_F to any Frobenius algebra F . Of particular interest are the Heisenberg categories studied in [8] and associated to zig-zag algebras (these categorify the Heisenberg algebras associated to quantum lattices of type A, D, E). Via categorical vertex operators [6, 7] these Heisenberg categories were related to categorified quantum groups. So one might expect their trace to be related to [3, 5] or to W-algebras associated to quantum groups [12]. Finally, one may also consider the trace on the categorification of twisted Heisenberg algebras such as the twisted version of Khovanov’s category from [9]. In this particular case the trace should be related to the twisted $W_{1+\infty}$ algebra from [16].

In the recent work [19], the algebra $W_{1+\infty}$ appears in relation to the skein module of the torus. It could also be interesting to relate directly the appearance of $W_{1+\infty}$ in Heisenberg categorification to this skein module or to other invariants in quantum topology.

1.2. Outline

In order to prove that $\mathrm{Tr}(\mathcal{H})$ is a quotient of $W_{1+\infty}$, we follow the proof in [20] of the isomorphism between a limit of spherical degenerate double affine Hecke algebras and a quotient of $W_{1+\infty}$. A complete set of generators and relations of the limit of spherical degenerate double affine Hecke algebras was given in [1]. Composing the isomorphism in [20] with the isomorphism from Theorem 1, we get a complete generators and relations description of $\mathrm{Tr}(\mathcal{H})$. Crucial in this comparison is the relationship between $\mathcal{H}$ and the degenerate affine Hecke algebra DH_n . Roughly speaking, DH_n appears in the morphism spaces of the upper half of $\mathcal{H}$. As a vector space $\mathrm{Tr}(DH_n)$ was determined by Solleveld [22] and this calculation plays a role in our computation of $\mathrm{Tr}(\mathcal{H})$.

In §2, Khovanov's Heisenberg algebra $\mathfrak{h}$ and its associated category $\mathcal{H}$ are introduced and its basic properties are reviewed. In §3, the algebra $W_{1+\infty}$ is defined, and some of its important features are recalled. In §4, the trace of $\mathcal{H}$ is determined in terms of $W_{1+\infty}$.

2. The Heisenberg category $\mathcal{H}$

2.1. The Heisenberg algebra and Fock space

Let $\mathfrak{h}$ be the associative algebra over $\mathbb{C}$ with generators h_n for $n \in \mathbb{Z} - \{0\}$ with relations $[h_m, h_n] = m\delta_{m,-n}$.

Another way to present $\mathfrak{h}$ is as follows. Set $p^{(n)} = q^{(n)} = 0$ for $n < 0$. For $n \geq 0$ define $p^{(n)}$ and $q^{(n)}$ by

$$\sum_{n \geq 0} p^{(n)} z^n = \exp \left(\sum_{n \geq 1} \frac{h_{-n}}{n} z^n \right) \quad \sum_{n \geq 0} q^{(n)} z^n = \exp \left(\sum_{n \geq 1} \frac{h_n}{n} z^n \right).$$

In these generators the relations are:

$$\begin{aligned} p^{(n)} p^{(m)} &= p^{(m)} p^{(n)} \\ q^{(n)} q^{(m)} &= q^{(m)} q^{(n)} \\ q^{(n)} p^{(m)} &= \sum_{k \geq 0} p^{(m-k)} q^{(n-k)}. \end{aligned}$$

The Heisenberg algebra $\mathfrak{h}$ has a natural representation $\mathcal{F}$, known as the Fock space. Let $\mathfrak{h}^+ \subset \mathfrak{h}$ denote the subalgebra generated by the $q^{(n)}$ for $n \geq 0$. Let triv_0 denote the trivial representation of $\mathfrak{h}^+$, where all $q^{(n)}$ ($n > 0$) act by zero. Then

$$\mathcal{F} := \text{Ind}_{\mathfrak{h}^+}^{\mathfrak{h}}(\text{triv}_0)$$

is called the Fock space representation of $\mathfrak{h}$. It inherits a $\mathbb{Z}$ grading $\mathcal{F} = \bigoplus_{m \in \mathbb{N}} \mathcal{F}(m)$ by declaring triv_0 to have degree zero, $p^{(n)}$ degree n and $q^{(n)}$ degree $-n$.

2.2. Heisenberg category $\mathcal{H}$

In [17], Khovanov introduced a categorical framework for the Heisenberg algebra $\mathfrak{h}$. This framework consists of a monoidal category $\mathcal{H}$ which is the Karoubi envelope of a monoidal category $\mathcal{H}'$ whose definition we now sketch (see [17] for more details).

The monoidal category $\mathcal{H}'$ is generated by objects $\mathbf{P}$ and $\mathbf{Q}$. We denote the monoidal unit by $\mathbf{1}$ and the tensor product of objects by concatenating symbols so that $P^2Q = P \otimes P \otimes Q$. The generating objects $\mathbf{P}$ and $\mathbf{Q}$ can be represented diagrammatically by an upward pointing strand and a downward pointing strand. Monoidal composition of such objects is then given by sideways concatenation of diagrams. The space of morphisms between products of $\mathbf{P}$'s and $\mathbf{Q}$'s is a $\mathbb{C}$ -algebra described by certain string diagrams with relations. By convention, composition of morphisms is done vertically from the bottom and going up.

The morphisms are generated by crossings, caps and cups as shown below



Thus, for instance, the left crossing is a map in $\text{End}(\mathbf{PP})$ while the right cap is a map $\mathbf{PQ} \rightarrow \mathbf{1}$. These morphisms satisfy the following relations

$$\begin{array}{ccc}
 \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} & = & \begin{array}{c} \nwarrow \quad \nearrow \\ \searrow \quad \nearrow \end{array} \\
 \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} & = & \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array}
 \end{array} \quad (2.2)$$

$$\begin{array}{ccc}
 \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} & = & \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} \\
 \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} & = & \begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array}
 \end{array} \quad (2.3)$$

$$\begin{array}{ccc}
 \begin{array}{c} \circlearrowleft \end{array} & = & 1. \\
 \begin{array}{c} \circlearrowright \end{array} & = & 0.
 \end{array} \quad (2.4)$$

Moreover, two morphisms which differ by planar isotopies are equal. Relation (2.2) implies that there is a map $\mathbb{C}[S_n] \rightarrow \text{End}(\mathbf{P}^n)$. Since $\mathcal{H}$ is assumed to be idempotent complete this means that we also get objects $\mathbf{P}^{(\lambda)}$, for any partition $\lambda \vdash n$, associated with the corresponding minimal idempotent $e_\lambda \in \mathbb{C}[S_n]$. Likewise, we also have $\mathbf{Q}^{(\lambda)}$ for any $\lambda \vdash n$. We will denote by (m) and (1^m) the unique one-part and m -part partitions of m .

Let $\mathcal{H}^{\geq}$ and $\mathcal{H}^{\leq}$ be the full subcategories of $\mathcal{H}$ generated by $\mathbf{P}$ and $\mathbf{Q}$, respectively. Let $\mathcal{H}^0$ be the full subcategory of $\mathcal{H}$ generated by the monoidal unit $\mathbf{1}$.

Theorem 2 [17]. *Inside $\mathcal{H}$ we have the following relations*

- (1) $\mathbf{P}^{(\lambda)}$ and $\mathbf{P}^{(\mu)}$ commute for any partitions λ, μ ;
- (2) $\mathbf{Q}^{(\lambda)}$ and $\mathbf{Q}^{(\mu)}$ commute for any partitions λ, μ ;
- (3) $\mathbf{Q}^{(n)}\mathbf{P}^{(m)} \cong \bigoplus_{k \geq 0} \mathbf{P}^{(m-k)}\mathbf{Q}^{(n-k)}$ and $\mathbf{Q}^{(1^n)}\mathbf{P}^{(1^m)} \cong \bigoplus_{k \geq 0} \mathbf{P}^{(1^{m-k})}\mathbf{Q}^{(1^{n-k})}$;
- (4) $\mathbf{Q}^{(n)}\mathbf{P}^{(1^m)} \cong \mathbf{P}^{(1^m)}\mathbf{Q}^{(n)} \oplus \mathbf{P}^{(1^{m-1})}\mathbf{Q}^{(n-1)}$ and $\mathbf{Q}^{(1^n)}\mathbf{P}^{(m)} \cong \mathbf{P}^{(m)}\mathbf{Q}^{(1^n)} \oplus \mathbf{P}^{(m-1)}\mathbf{Q}^{(1^{n-1})}$.

Thus, at the level of Grothendieck groups we have a map $\mathfrak{h} \rightarrow K_0(\mathcal{H})$. This map is known to be injective but it is not known if it is surjective.

2.3. The degenerate affine Hecke algebra

Inside $\mathcal{H}$ consider the element $X_i \in \text{End}(\mathbf{P}^n)$, acting on the i th factor $\mathbf{P}$, as illustrated in the left-hand side of (2.5). In [17], this element was studied and it was encoded

diagrammatically by a solid dot, as shown

$$\begin{array}{c} \uparrow \\ \text{loop} \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \bullet \\ \downarrow \end{array} \quad (2.5)$$

In [17], it was shown that these X_i 's together with the symmetric group $\mathbb{C}[S_n] \subset \text{End}(\mathbf{P}^n)$ generate a copy of the degenerate affine Hecke algebra. In particular, using equation (2.2)–(2.4) the equations

$$\begin{array}{c} \uparrow \downarrow \\ \bullet \end{array} - \begin{array}{c} \uparrow \downarrow \\ \bullet \end{array} = \begin{array}{c} \uparrow \\ \uparrow \end{array} = \begin{array}{c} \uparrow \downarrow \\ \bullet \end{array} - \begin{array}{c} \uparrow \downarrow \\ \bullet \end{array} \quad (2.6)$$

follow. More precisely, denote a crossing of the i th and $(i+1)$ st strands by T_i .

Proposition 3 [17]. *We have the following relations inside $\text{End}_{\mathcal{H}}(\mathbf{P}^n)$:*

$$\begin{aligned} T_i X_i &= X_{i+1} T_i + 1 \\ X_i T_i &= T_i X_{i+1} + 1 \\ X_i X_j &= X_j X_i \\ T_i^2 &= 1 \\ T_i T_j &= T_j T_i \quad \text{if } |i-j| > 1 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}. \end{aligned}$$

The algebra generated by T_i for $i = 1, \dots, n-1$ and X_i for $i = 1, \dots, n$ satisfying the relations in Proposition 3 is the degenerate affine Hecke algebra DH_n .

We now define bubbles which are endomorphisms of $\mathbf{1} = P^0$ which can be tensored with endomorphisms of $\mathbf{P}^n$ to give new endomorphisms.

$$c_n = \begin{array}{c} \bullet \\ \text{loop} \end{array} \quad \tilde{c}_n = \begin{array}{c} \bullet \\ \text{loop} \end{array} \quad (2.7)$$

Proposition 4 [17, Proposition 2]. *For $n > 0$*

$$\tilde{c}_{n+1} = \sum_{i=0}^{n-1} \tilde{c}_i c_{n-1-i}.$$

Proposition 5 [17, Proposition 4]. *There is an isomorphism of algebras*

$$\text{End}(\mathbf{P}^n) \cong DH_n \otimes \mathbb{C}[c_0, c_1, \dots].$$

In particular, when $n = 0$ we have $\text{End}(\mathbf{1}) \cong \mathbb{C}[c_0, c_1, \dots]$.

Let $J_{m,n}$ be the ideal of $\text{End}_{\mathcal{H}'}(\mathbf{P}^m \mathbf{Q}^n)$ generated by diagrams which contain at least one arc connecting a pair of upper points.

Proposition 6 [17, Equation (19)]. *There exists a short exact sequence*

$$0 \rightarrow J_{m,n} \rightarrow \text{End}_{\mathcal{H}'}(\mathbf{P}^m \mathbf{Q}^n) \rightarrow DH_m \otimes DH_n^{op} \otimes \mathbb{C}[c_0, c_1, \dots] \rightarrow 0.$$

Furthermore, this sequence splits.

3. The algebra $W_{1+\infty}$

In this section, we define the W -algebra of interest and list several important properties. This algebra is defined to be the universal enveloping algebra of a central extension of differential operators on $\mathbb{C}^*$. The W -algebra, $W_{1+\infty}$ comes equipped with a $\mathbb{Z}$ -grading and a compatible $\mathbb{Z}_{\geq 0}$ -filtration. We will also define a faithful irreducible representation of a quotient of $W_{1+\infty}$. Both of these features will be crucial in proving that the W -algebra is related to the trace of $\mathcal{H}$. This section follows closely the exposition in [20, Appendix F].

Let $W_{1+\infty}$ be the $\mathbb{C}$ -associative algebra generated by $w_{l,k}$ for $l \in \mathbb{Z}$ and $k \in \mathbb{N}$ and C with relations that $w_{0,0}$ and C are central, and

$$w_{l,k} = t^l D^k \quad (l, k) \neq (0, 0) \quad (3.1)$$

$$\begin{aligned} & [t^l \exp(\alpha D), t^k \exp(\beta D)] \\ &= (\exp(k\alpha) - \exp(l\beta)) t^{l+k} \exp(\alpha D + \beta D) + \delta_{l,-k} \frac{\exp(-l\alpha) - \exp(-k\beta)}{1 - \exp(\alpha + \beta)} C \end{aligned} \quad (3.2)$$

where α and β are formal parameters, t is the parameter on $\mathbb{C}^*$ and $D = t\partial_t$.

Note that for $l \in \mathbb{Z} - \{0\}$ the set $\{w_{l,0}\}$ generates a Heisenberg subalgebra because

$$[w_{l,0}, w_{k,0}] = l\delta_{l,-k}C. \quad (3.3)$$

We will need the following relations which are direct consequences of (3.2).

$$[w_{l,1}, w_{k,1}] = (k-l)w_{l+k,1} + \frac{k^3-k}{6}C\delta_{l,-k} \quad (3.4)$$

$$[w_{-1,a}, w_{1,b}] = \sum_{r=1}^a \binom{a}{r} w_{0,a+b-r} - \sum_{s=1}^b (-1)^s \binom{b}{s} w_{0,a+b-s} + \delta_{a,0}(-1)^{b+1}C \quad (3.5)$$

$$[w_{l,1}, w_{k,0}] = kw_{l+k,0} - \delta_{l,-k} \frac{k(k-1)}{2}C \quad (3.6)$$

$$[w_{l,0}, w_{0,2}] = -2lw_{l,1} - l^2w_{l,0}. \quad (3.7)$$

We define Virasoro elements

$$\bar{L}_l = -w_{l,1} - \frac{1}{2}(l+1)w_{l,0}.$$

Then it is easy to check that (3.2) gives the Virasoro relations

$$[\bar{L}_l, \bar{L}_k] = (l-k)\bar{L}_{l+k} + \delta_{l,-k} \frac{l^3-l}{12}C.$$

Let $W_{1+\infty}^>, W_{1+\infty}^0, W_{1+\infty}^<$ be subalgebras of $W_{1+\infty}$ generated as follows:

$$\begin{aligned} W_{1+\infty}^> &:= \mathbb{C}\langle w_{l,k}; l \geq 1, k \geq 0 \rangle \\ W_{1+\infty}^0 &:= \mathbb{C}\langle C, w_{0,l}; l \geq 0 \rangle \\ W_{1+\infty}^< &:= \mathbb{C}\langle w_{-l,k}; l \geq 1, k \geq 0 \rangle. \end{aligned}$$

This algebra has a $\mathbb{Z}$ -grading called the rank grading where $w_{l,k}$ is in degree l and C is in degree 0. The algebra is also $\mathbb{Z}_{\geq 0}$ -filtered where $w_{l,k}$ is in degree $\leq k$. Let $W_{1+\infty}^\omega[r, \leq k]$ denote the set of elements in $\mathbb{Z}$ -degree r and $\mathbb{Z}_{\geq 0}$ -degree $\leq k$ where $\omega \in \{<, >, 0, \emptyset\}$.

Denote the associated graded algebra of $W_{1+\infty}$ with respect to the $\mathbb{Z}_{\geq 0}$ -filtration by

$$\tilde{W}_{1+\infty} = gr(W_{1+\infty}).$$

Since the $\mathbb{Z}$ -grading on $W_{1+\infty}$ is compatible with the $\mathbb{Z}_{\geq 0}$ -filtering, the associated graded $\tilde{W}_{1+\infty}$ is $(\mathbb{Z} \times \mathbb{Z}_{\geq 0})$ -graded. Let $\tilde{W}_{1+\infty}^\omega[r, k]$ denote the subspace of $\tilde{W}_{1+\infty}^\omega$ in bidegree (r, k) . Define a generating series for the graded dimension of $\tilde{W}_{1+\infty}^\omega$ by

$$P_{\tilde{W}_{1+\infty}^\omega}(t, q) = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\geq 0}} \dim \tilde{W}_{1+\infty}^\omega[r, k] t^r q^k.$$

Proposition 7. *The graded dimensions of $\tilde{W}_{1+\infty}^>$ and $\tilde{W}_{1+\infty}^<$ are given by:*

$$P_{\tilde{W}_{1+\infty}^>} = \prod_{r>0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k}, \quad P_{\tilde{W}_{1+\infty}^<} = \prod_{r<0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k}.$$

Proof. The defining relations of $W_{1+\infty}$ given in 3.2 imply that the associated graded algebras $\tilde{W}_{1+\infty}^>$ and $\tilde{W}_{1+\infty}^<$ are freely generated by the images of $w_{l,k}$. The proposition now follows easily since the image of $w_{l,k}$ has bidegree (l, k) . $\square$

Lemma 8 [20, Lemma F.5].

- (1) $W_{1+\infty}$ is generated by $w_{-1,0}, w_{1,0}, w_{0,2}$.
- (2) $W_{1+\infty}^>$ is generated by $w_{1,l}$ for $l \geq 0$.
- (3) $W_{1+\infty}^<$ is generated by $w_{-1,l}$ for $l \geq 0$.

For $c, d \in \mathbb{C}$ let $\mathbb{C}_{c,d}$ be a one-dimensional representation of $W_{1+\infty}^{\geq}$ where $w_{k,l}$ acts by zero for $(k, l) \neq (0, 0)$, C acts by c and $w_{0,0}$ acts by d .

Let $\mathcal{M}_{c,d} := \text{Ind}_{W_{1+\infty}^{\geq}}^{W_{1+\infty}}(\mathbb{C}_{c,d})$.

Proposition 9 [2, 11]. *The induced module $\mathcal{M}_{c,d}$ has a unique irreducible quotient $\mathcal{V}_{c,d}$. As a vector space $\mathcal{V}_{1,0}$ is isomorphic to $\mathbb{C}[w_{-1,0}, w_{-2,0}, \dots]$.*

Proposition 10 [20, Proposition F.6]. *The action of $W_{1+\infty}/(C-1, w_{0,0})$ is faithful on $\mathcal{V}_{1,0}$.*

Proof. This is [20, Proposition F.6] since their proof does not depend on how the central elements C and $w_{0,0}$ act. The proof uses the faithfulness of the action of the Heisenberg algebra on Fock space together with the filtration on $W_{1+\infty}$. $\square$

Lemma 11 [20, Lemma F.8]. *The action of $w_{0,2}$ on $\mathcal{V}_{1,0}$ is given by the operator*

$$\sum_{k,l>0} (w_{-l,0}w_{-k,0}w_{k+l,0} + w_{-l-k,0}w_{l,0}w_{k,0}) - w_{0,1}.$$

Proof. This is a straightforward computation using the relations of $W_{1+\infty}$ and the definition of $\mathcal{V}_{1,0}$. For more details see the proof of [20, Lemma F.8]. There is a correction term of $-w_{0,1}$ in the lemma because we take $w_{0,0} = 0$ while in [20] it is taken to be $-\frac{1}{2}$. $\square$

4. Trace of $\mathcal{H}$

4.1. Definitions and conventions

The trace, or zeroth Hochschild homology, $\mathrm{Tr}(\mathcal{C})$ of a $\mathbb{k}$ -linear category $\mathcal{C}$ is the $\mathbb{k}$ -vector space given by

$$\mathrm{Tr}(\mathcal{C}) = \left(\bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(X, X) \right) / \mathcal{I},$$

where $\mathcal{C}(X, X) = \mathrm{End}_{\mathcal{C}}(X)$ and $\mathcal{I} = \mathrm{Span}_{\mathbb{k}}\{fg - gf\}$ where f and g run through all pairs of morphisms $f: X \rightarrow Y$, $g: Y \rightarrow X$ with $X, Y \in \mathrm{Ob}(\mathcal{C})$. For a morphism f in $\mathcal{C}$ denote its class in $\mathrm{Tr}(\mathcal{C})$ by $[f]$. The trace is invariant under passage to the Karoubi envelope $\mathrm{Kar}(\mathcal{C})$.

Proposition 12 [5, Proposition 3.2]. *The natural map $\mathrm{Tr}(\mathrm{Kar}(\mathcal{C})) \rightarrow \mathrm{Tr}(\mathcal{C})$ induced by inclusion of categories is an isomorphism.*

Proposition 13 [4, Lemma 2.1]. *Let $\mathcal{C}$ be a $\mathbb{k}$ -linear additive category. Let $S \subset \mathrm{Ob}(\mathcal{C})$ be a subset such that every object in $\mathcal{C}$ is isomorphic to the direct sum of finitely many copies of objects in S . Let $\mathcal{C}|_S$ denote the full subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}|_S) = S$. Then, the inclusion functor $\mathcal{C}|_S \rightarrow \mathcal{C}$ induces an isomorphism*

$$\mathrm{Tr}(\mathcal{C}|_S) \cong \mathrm{Tr}(\mathcal{C}) \tag{4.1}$$

The trace of a general category is only a vector space. The monoidal structure of $\mathcal{H}$ endows $\mathrm{Tr}(\mathcal{H})$ with a product. Given morphisms $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ then $f \otimes g: XY \rightarrow X'Y'$ and we may define $[f][g] := [f \otimes g]$ so that the product in $\mathrm{Tr}(\mathcal{H})$ is denoted by juxtaposition of classes in the trace. The algebra $\mathrm{Tr}(\mathcal{H})$ is unital with the identity Id_1 of the monoidal unit $\mathbf{1}$ acting as the unit element.

Recall that a categorical representation of the category $\mathcal{H}$ is a $\mathbb{k}$ -linear category $\mathcal{V}$ and a $\mathbb{k}$ -linear functor $\mathcal{H} \rightarrow \mathrm{End}(\mathcal{V})$ sending each object X to an endofunctor $F(X): \mathcal{V} \rightarrow \mathcal{V}$.

Each morphism $f: X \rightarrow Y$ is sent to a natural transformation of functors $F(f): F(X) \rightarrow F(Y)$. A categorical representation gives rise to a representation

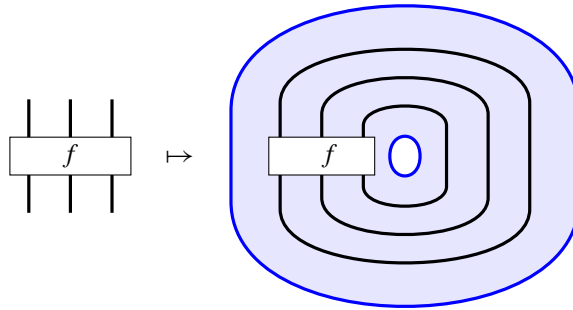
$$\rho_F: \text{Tr}(\mathcal{H}) \rightarrow \text{End}(\mathcal{V}) \quad (4.2)$$

sending the class $[f] \in \text{Tr}(\mathcal{H})$ corresponding to a map $f: X \rightarrow X$ in $\mathcal{H}$ to the endomorphism of the $\mathbb{k}$ -vector space $\text{Tr}(\mathcal{V})$ defined by sending $\psi: U \rightarrow U$ in $\text{Tr}(\mathcal{V})$ to

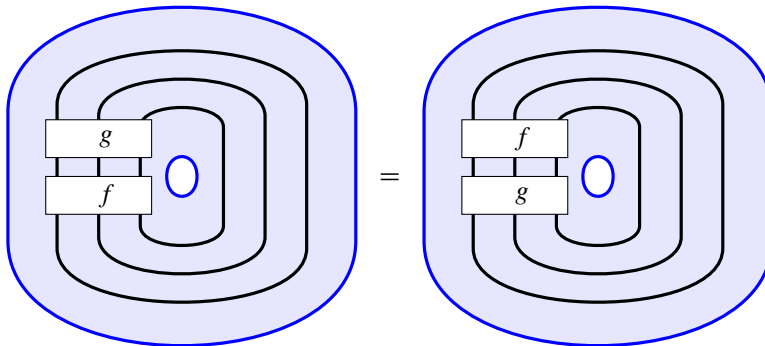
$$\rho_F([f])([\psi]) = [F(X)(\psi) \circ F(f)_U] = [F(f)_U \circ F(X)(\psi)].$$

Here, note that $F(f)_U$ denotes the component of the natural transformation $F(f): F(X) \Rightarrow F(X): \mathcal{V} \rightarrow \mathcal{V}$ corresponding to the object $U \in \text{Ob}(\mathcal{V})$. By naturality this map is well defined.

The morphisms in any cyclic, or strictly pivotal, monoidal category can be described using planar diagrammatics in which any two morphisms that differ by an isotopy are equal. The monoidal category $\mathcal{H}$ is an example such a category. In this context, it is natural to view the element $[f] \in \text{Tr}(\mathcal{H})$ by drawing $f: X \rightarrow X$ in an annulus and then closing up the diagram for f .



The trace relations imposed by the quotient by $\mathcal{I} = \text{Span}_{\mathbb{k}}\{fg - gf\}$ where f and g run through all pairs of morphisms $f: X \rightarrow Y$, $g: Y \rightarrow X$ with $X, Y \in \text{Ob}(\mathcal{H})$ is naturally imposed by the topology:



by sliding diagrams around the annulus.

Let $w \in S_n$. We introduce a formal notation to denote elements in $\text{Tr}(\mathcal{H})$. Define

$$h_w \otimes (x_1^{j_1} \cdots x_n^{j_n}) := [f_{w;j_1, \dots, j_n}]$$

where $f_{w;j_1, \dots, j_n} \in \text{End}(\mathbf{P}^n)$ is given by:

$$f_{w;j_1, \dots, j_n} := \begin{array}{c} \uparrow j_1 \quad \cdots \quad \uparrow j_n \\ \boxed{w} \\ \downarrow \quad \cdots \quad \downarrow \end{array}$$

The tensor product in our notation is strictly formal and is used to emphasize particular elements with permutations at the bottom and some monomial in dots at the top.

Define

$$h_{-w} \otimes (x_1^{j_1} \cdots x_n^{j_n}) := [f_{-w;j_1, \dots, j_n}]$$

where $f_{-w;j_1, \dots, j_n} \in \text{End}(\mathbf{Q}^n)$ is given by:

$$f_{-w;j_1, \dots, j_n} := \begin{array}{c} j_1 \quad \cdots \quad j_n \\ \boxed{w} \\ \downarrow \quad \cdots \quad \downarrow \end{array}$$

Of particular importance is the element $\sigma_n = s_1 \cdots s_{n-1} \in S_n$. We then abbreviate

$$\begin{aligned} h_n \otimes (x_1^{j_1} \cdots x_n^{j_n}) &:= h_{\sigma_n} \otimes (x_1^{j_1} \cdots x_n^{j_n}) = \left[\begin{array}{c} \uparrow j_1 \quad \uparrow j_2 \quad \uparrow j_3 \quad \cdots \quad \uparrow j_n \\ \text{...} \end{array} \right] \\ h_{-n} \otimes (x_1^{j_1} \cdots x_n^{j_n}) &:= h_{-\sigma_n} \otimes (x_1^{j_1} \cdots x_n^{j_n}) = \left[\begin{array}{c} j_1 \quad j_2 \quad j_3 \quad \cdots \quad j_n \\ \downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \end{array} \right] \end{aligned} \quad (4.3)$$

The next lemma allows us to express the elements $h_n \otimes (x_1^{j_1} \cdots x_n^{j_n})$ in terms of the elements $h_r \otimes x_1^k$.

Lemma 14. *For $n \geq 1$ and $1 \leq i \leq n-1$ we have*

$$h_{\pm n} \otimes x_i = h_{\pm n} \otimes x_{i+1} \pm (h_{\pm i} \otimes 1)(h_{\pm(n-i)} \otimes 1).$$

Proof. This is immediate from (2.6) and properties of the trace. $\square$

The next lemma allows us to express generators $[f_{w;j_1, \dots, j_n}]$ in terms of the more elementary generators $h_r \otimes p$ where p is a polynomial in variables $x_1, \dots, x_r$.

Lemma 15. *Let $w \in S_n$ and $(j_1, \dots, j_r)$ a sequence of natural numbers. Then*

$$[f_{\pm w;j_1, \dots, j_n}] = \sum d_{n_1, \dots, n_r} (h_{\pm n_1} \otimes p_{n_1}) \cdots (h_{\pm n_r} \otimes p_{n_r})$$

where the sum is over sequences $(n_1, \dots, n_r)$ of natural numbers that sum to n , for some coefficients $d_{n_1, \dots, n_r} \in \mathbb{C}$ and polynomials p_{n_i} in n_i variables.

Proof. We prove this by induction on $j_1 + \cdots + j_n$. The base case where all of the j_i equal zero is trivial because then $f_{w;j_1,\dots,j_n}$ is just an element in the symmetric group and thus its class $[f_{w;j_1,\dots,j_n}]$ in the trace is determined by its conjugacy class in S_n which of course could be written as a product of disjoint cycles.

Choose an element $g \in S_n$ such that

$$gwg^{-1} = (s_1 \cdots s_{n_1-1}) \cdots (s_{n_1+\cdots n_{r-1}} \cdots s_{n_1+\cdots n_r-1}).$$

Let $p = x_1^{j_1} \cdots x_n^{j_n}$. Thus, $f_{w;j_1,\dots,j_n} = pw$. Now we conjugate this element by g to get $gpwg^{-1} = (g \cdot p)gwg^{-1} + p_Lwg^{-1}$ where p_L is a polynomial of degree less than $j_1 + \cdots + j_n$ and $g \cdot p$ is some other polynomial of degree $j_1 + \cdots + j_n$. The lemma follows by applying induction to the second term and noting that in the first term gwg^{-1} is a product of cycles. $\square$

Corollary 16. Let $w \in S_n$ and $(j_1, \dots, j_r)$ a sequence of natural numbers. Then

$$[f_{\pm w;j_1,\dots,j_n}] = \sum d_{n_1,\dots,n_r} (h_{\pm n_1} \otimes x_1^{l_1}) \cdots (h_{\pm n_r} \otimes x_1^{l_r})$$

where the sum is over sequences $(n_1, \dots, n_r)$ of natural numbers that sum to n , for some constants $d_{n_1,\dots,n_r} \in \mathbb{C}$ and some non-negative integers $l_1, \dots, l_r$.

Proof. This follows from Lemmas 14 and 15. $\square$

4.2. Elements $p^{(n)}$ and $q^{(n)}$

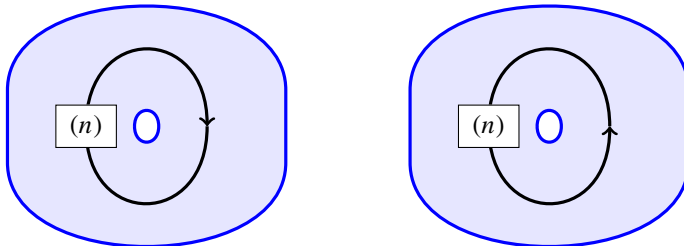
In this section, we will see how the Heisenberg algebra $\mathfrak{h}$ in the presentation given in terms of $p^{(n)}, q^{(n)}$ from § 2 appears in $\text{Tr}(\mathcal{H})$.

Define $p^{(n)} \otimes 1$ and $q^{(n)} \otimes 1$ in $\text{Tr}(\mathcal{H})$ by

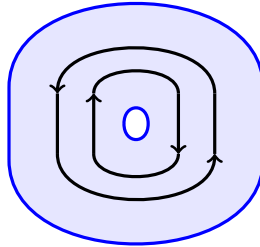
$$p^{(n)} \otimes 1 = \frac{1}{n!} \sum_{w \in S_n} [f_{w;0,\dots,0}] \quad (4.4)$$

$$q^{(n)} \otimes 1 = \frac{1}{n!} \sum_{w \in S_n} [f_{-w;0,\dots,0}] \quad (4.5)$$

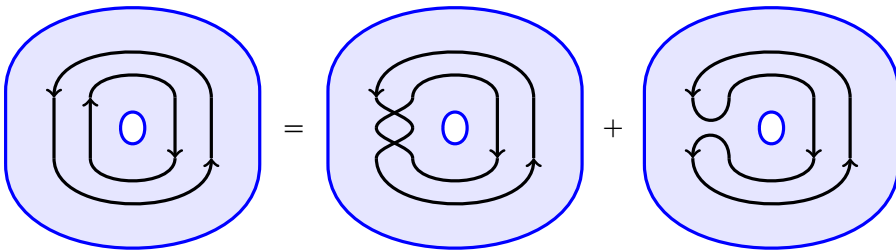
We depict the elements $p^{(n)} \otimes 1$ and $q^{(n)} \otimes 1$ by



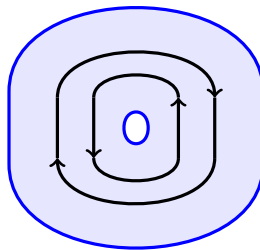
We have for example the relation $(q^{(1)} \otimes 1)(p^{(1)} \otimes 1) = (p^{(1)} \otimes 1)(q^{(1)} \otimes 1) + 1$. The product $(q^{(1)} \otimes 1)(p^{(1)} \otimes 1)$ is



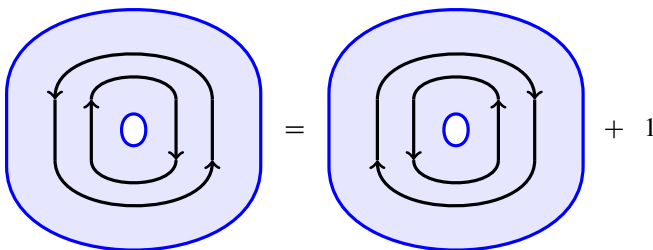
Now, the graphical relations in $\mathcal{H}$ allow us to pull the inner circle outside as follows



The last term above can be simplified using the trace relation to isotope the cup around the annulus creating a circle. Since a counterclockwise circle equals 1, the circle in the last term above can be erased, leaving the identity endomorphism of the monoidal unit in $\mathcal{H}$, or the multiplicative identity $1 \in \text{Tr}(\mathcal{H})$. On the other hand, the first term on the right can be simplified by sliding in the outside circle to obtain



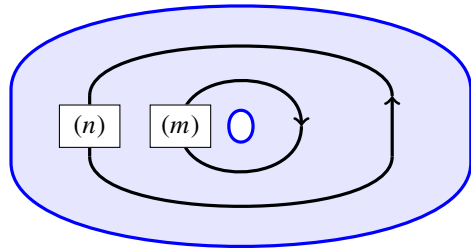
Thus, we end up with



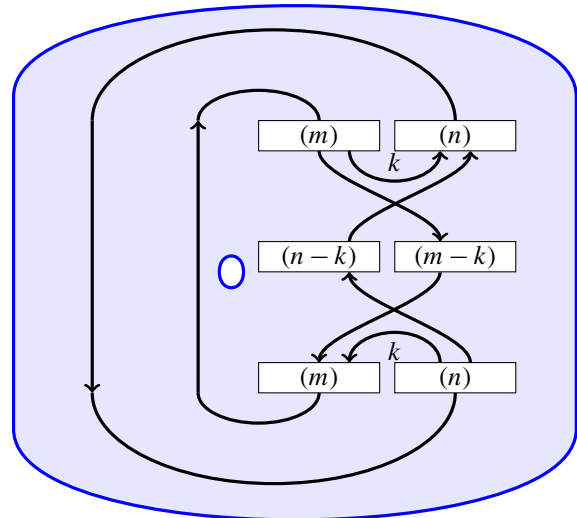
Theorem 17. *The elements $p^{(n)} \otimes 1, q^{(n)} \otimes 1$ for $n \in \mathbb{Z}_{\geq 0}$ satisfy the relations:*

$$\begin{aligned} (p^{(n)} \otimes 1)(p^{(m)} \otimes 1) &= (p^{(m)} \otimes 1)(p^{(n)} \otimes 1) \quad \text{and} \quad (q^{(n)} \otimes 1)(q^{(m)} \otimes 1) = (q^{(m)} \otimes 1)(q^{(n)} \otimes 1) \\ (q^{(n)} \otimes 1)(p^{(m)} \otimes 1) &= \sum_{k \geq 0} (p^{(m-k)} \otimes 1)(q^{(n-k)} \otimes 1). \end{aligned}$$

Proof. This follows fairly directly from the proof of the categorified statement in [17] and its analogous statement in [8]. We sketch the details for the computation of $(q^{(n)} \otimes 1)(p^{(m)} \otimes 1)$. By definition $(q^{(n)} \otimes 1)(p^{(m)} \otimes 1)$ is


(4.6)

Letting $c_{m,n}^k = k! \binom{m}{k} \binom{n}{k}$ we get using a straightforward modification of [8, Equation 31] that (4.6) is equal to:


(4.7)

Slide the top two rectangles and the k cups connecting them counterclockwise to the bottom of the diagram and apply a modification of [8, Equation 32], to get that (4.7) is equal to:

$$\sum_{0 \leq l \leq k} \left(\text{Diagram with nested loops and boxes labeled } (m-k), (n-k), \text{ and } 0 \right).$$

By definition this is equal to $\sum_k (p^{(m-k)} \otimes 1)(q^{(n-k)} \otimes 1)$. $\square$

4.3. Some diagrammatic lemmas

For convenience, we work with classes $[f]$ of endomorphisms in $\text{Tr}(\mathcal{H})$ and omit the annuli from our diagrammatic computations. The reader should keep in mind that all diagrams inside of closed brackets are interpreted on the annulus.

Lemma 18. *For any $a > 1$ we have the following identities*

$$\begin{array}{c} \text{Diagram 1} - \text{Diagram 2} = \text{Diagram 3} - \text{Diagram 4} = \sum_{f+g=a-1} \text{Diagram 5} \end{array} \quad (4.8)$$

$$\begin{array}{c} \text{Diagram 6} = a^{-1} \cdot \text{Diagram 7} + \sum_{f+g=a-3} \tilde{c}_{f+2} \cdot \text{Diagram 8} \end{array} \quad (4.9)$$

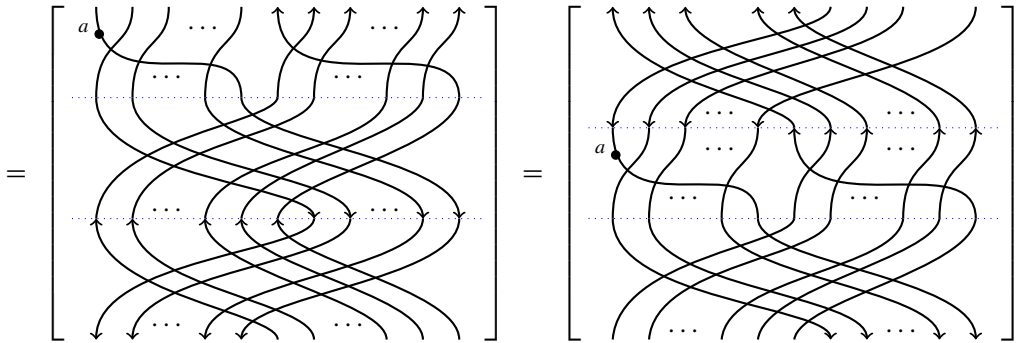
Proof. The first claim follows inductively from (2.6). The second claim follows from the first after observing that a left twist curl is zero and the $\tilde{c}_1 = 0$. $\square$

Lemma 19. *For any $a \geq 0$ we have*

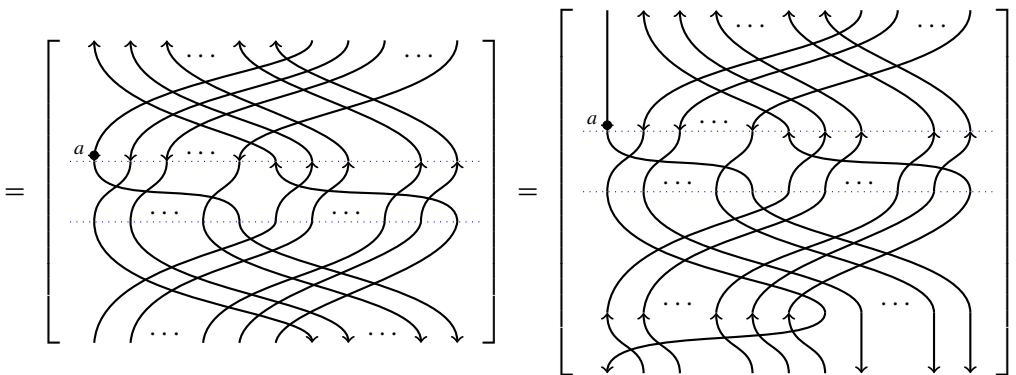
$$\left[\text{Diagram with } m \text{ downward strands and } n \text{ upward strands} \right] = \begin{cases} (h_{-m} \otimes x_1^a)(h_n \otimes 1), & \text{if } m > n \geq 1; \\ (h_{-m} \otimes x_1^a)(h_n \otimes 1) - n\tilde{c}_a, & \text{if } m = n; \\ (h_{-m} \otimes x_1^a)(h_n \otimes 1) - n \sum_{f+g=a-1} \tilde{c}_f(h_{n-m} \otimes x_1^g), & \text{if } n > m, \end{cases}$$

where the left side involves m downward strands and n upward strands.

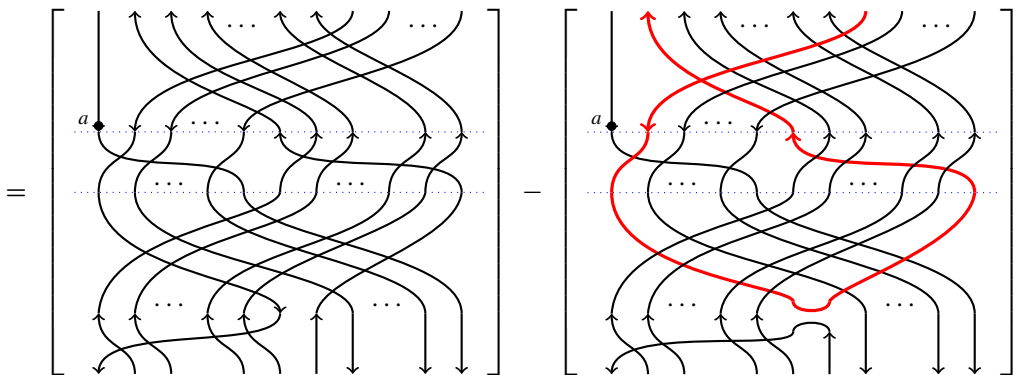
Then using triple point moves to slide the crossings in the middle third of the diagram through the strands at the top third, the diagram becomes



Sliding one layer of crossings from the top third to the bottom third using the trace relation gives



Applying the first equation in (2.3) in the bottom third of the diagram only one term survives



since the second diagram can be simplified to one containing a left twist curl which is zero. It is easy to see that the same thing happens as the left most downward oriented strand at the bottom is pulled through all of the upward oriented lines.

To further simplify the diagram, slide all of the crossings involving the second downward oriented line from the top third of the diagram to the bottom third and simplify using the first equation in (2.3) as above. Each time this equation is applied the resolution term can be simplified so that it contains a left twist curl. Hence, all of these terms vanish and we are left with the following

$$= \left[\begin{array}{c} \text{Diagram with strands labeled } a, \dots, \text{ and crossings} \\ \vdots \\ \text{Diagram with strands labeled } \dots, \text{ and crossings} \end{array} \right].$$

The result follows by Lemma 19. $\square$

Lemma 21. For $1 \leq i < n$ and $a \geq 0$ the identity

$$[(x_1^a T_{i+1} T_{i+2} \dots T_n)(T_1 \dots T_n)(T_{n+1} \dots T_{n+m})(T_n T_{n-1} \dots T_i)] = h_{n+m} \otimes x_1^a$$

holds in the trace.

Proof. The left side of this relation can be expressed by the diagram

$$\left[\begin{array}{c} \text{Diagram with strands labeled } 1, 2, \dots, i, i+1, \dots, n+m \\ \vdots \\ \text{Diagram with strands labeled } 1, i-1, i, i+1, \dots, n, \dots, n+m \end{array} \right]$$

The $T_{i+1} \dots T_n$ at the top can be reordered to $T_n \dots T_{i+1}$ by first sliding the T_{i+1} around the annulus using the trace relation and then back to the top of diagram using equations (2.2) to produce $T_{i+2} T_{i+1} T_{i+3} \dots T_n$. Next use the trace relation and equations (2.2) to bring the crossings $T_{i+2} T_{i+1}$ through the bottom of the diagram back to the top portion of the diagram producing $T_{i+3} T_{i+2} T_{i+1} T_{i+4} \dots T_n$. Continuing in this way all of the crossings at the top third of the diagram can be reordered to produce

$$\left[\begin{array}{c} \text{Diagram with strands labeled } 1, 2, \dots, i, i+1, \dots, n+m \\ \vdots \\ \text{Diagram with strands labeled } 1, i-1, i, i+1, \dots, n, \dots, n+m \end{array} \right] = \left[\begin{array}{c} \text{Diagram with strands labeled } \dots, \dots, \dots, \dots, \dots, \dots \\ \vdots \\ \text{Diagram with strands labeled } \dots, \dots, \dots, \dots, \dots, \dots \end{array} \right]$$

where we have moved the entire top third of the diagram to the bottom using the trace relation. It is now easy to see that using both relations in (2.2) together with the trace relations simplifies the diagram to $h_{n+m} \otimes x_1^a$. $\square$

4.4. A Heisenberg–Virasoro subalgebra

We exhibit generators of $\text{Tr}(\mathcal{H})$ which satisfy the Heisenberg relations for the original generators h_n . There is another set of generators which satisfy Virasoro relations with central charge 1. The Virasoro algebra naturally acts on the Heisenberg algebra and the resulting semi-direct product algebra is known as a Heisenberg–Virasoro algebra. These generators come from images of diagrams which contain at most one right-hand curl.

We recover the original Heisenberg generators by the following result.

Lemma 22. *The elements $h_n \otimes 1$ for $n \in \mathbb{Z} - \{0\}$ satisfy the following Heisenberg relations*

$$[h_m \otimes 1, h_n \otimes 1] = n\delta_{m,-n}.$$

Proof. For $mn > 0$ the underlying morphisms for $(h_m \otimes 1)(h_n \otimes 1)$ and $(h_n \otimes 1)(h_m \otimes 1)$ in the category $\mathcal{H}$ are in the same conjugacy class in $\mathcal{S}_{n+m}$. Thus, their traces are equal.

Suppose then that $m < 0$ and $n > 0$. Then consider the diagram for $(h_n \otimes 1)(h_m \otimes 1)$. Using the trace relation with n applications of the second equation in (2.3) it is easy to see that

$$(h_n \otimes 1)(h_m \otimes 1) = \left[\begin{array}{c} \text{Diagram with } n \text{ strands, } m \text{ strands, and } n \text{ crossings} \end{array} \right]$$

which is equal to $(h_m \otimes 1)(h_n \otimes 1) - n\delta_{n,m}$ by Lemma 20. □

The next Lemma will roughly lead to ‘half’ of a Virasoro algebra.

Lemma 23. *For $m, n \in \mathbb{Z}$ with $mn > 0$,*

$$[h_m \otimes x_1, h_n \otimes x_1] = (n - m)(h_{n+m} \otimes x_1).$$

Proof. Let $\beta_n = h_n \otimes x_1$. We prove by induction on m that $[\beta_m, \beta_n] = (n - m)\beta_{m+n}$. The case $m = 1$ has the following graphical proof.

Without loss of generality assume $n > 0$ so that

$$(h_n \otimes x_1)(h_1 \otimes x_1) = \left[\begin{array}{c} \text{Diagram with } n \text{ strands, } 1 \text{ strand, and } n \text{ crossings} \end{array} \right].$$

Then conjugating by T_n we have

$$\left[\begin{array}{c} \text{Diagram 1} \end{array} \right] \stackrel{(2.6)}{=} \left[\begin{array}{c} \text{Diagram 2} \end{array} \right] - \left[\begin{array}{c} \text{Diagram 3} \end{array} \right]$$

The second term on the right-hand side is just $-(h_{n+1} \otimes x_1)$. The first term can be simplified further by conjugating by T_{n-1} , sliding the dot using (2.6), and observing using Lemma 21 that the crossing resolution term is also equal to $-(h_{n+1} \otimes x_1)$. Continuing this procedure, conjugating by T_{n-2} , sliding the dot, and simplifying the crossing resolution term using Lemma 21, we see that $(h_n \otimes x_1)(h_1 \otimes x_1)$ is equal to

$$\left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] = -n(h_{n+1} \otimes x_1).$$

The first term can be simplified by sliding the lower dot

$$\begin{aligned} \left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] &= \left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] + \left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] \\ &= \left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] + \left[\begin{array}{c} \text{Diagram with } n \text{ crossings and a dot} \end{array} \right] \end{aligned}$$

where the trace relation was used repeatedly on the second term to reorder the crossings at the top of the diagram. The second term simplifies further using the trace relation to reduce the number of crossings so that the second diagram is equal to $(h_{n+1} \otimes x_1)$, completing the induction step.

Now consider the Jacobi identity:

$$[[\beta_1, \beta_{m-1}], \beta_n] + [[\beta_{m-1}, \beta_n], \beta_1] + [[\beta_n, \beta_1], \beta_{m-1}] = 0.$$

Using the base case and the induction hypothesis this becomes

$$(m-2)[\beta_m, \beta_n] + (n-m+1)[\beta_{m+n-1}, \beta_1] + (1-n)[\beta_{n+1}, \beta_{m-1}] = 0.$$

Again by using the base case and the induction hypothesis this becomes

$$(m-2)[\beta_m, \beta_n] + (n-m+1)(2-m-n)\beta_{m+n} + (1-n)(m-n-2)\beta_{m+n} = 0.$$

It now follows easily that $[\beta_m, \beta_n] = (n-m)\beta_{m+n}$. □

We now have some relations leading to an action of a Virasoro algebra on a Heisenberg algebra.

Lemma 24. *Let $m, n \in \mathbb{Z}$ with $mn > 0$. Then*

$$[h_n \otimes x_1, h_m \otimes 1] = m(h_{m+n} \otimes 1).$$

Proof. We only consider the case that m and n are positive. The other case is similar. Using the same method of proof as in Lemma 23 it is not difficult to show

$$\begin{aligned} (h_m \otimes 1)(h_n \otimes x_1) &= \left[\begin{array}{c} \text{Diagram 1: } m \text{ crossings followed by } n \text{ crossings} \end{array} \right] \\ &= \left[\begin{array}{c} \text{Diagram 2: } n \text{ crossings followed by } m \text{ crossings} \end{array} \right] - mh_{n+m} \otimes 1. \end{aligned}$$

Then by sliding the crossings appearing in h_n to the left of those appearing in h_m , the first term can be written as

$$\left[\begin{array}{c} \text{Diagram 3: } n \text{ crossings followed by } m \text{ crossings} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 4: } n \text{ crossings followed by } m \text{ crossings} \end{array} \right]$$

which simplifies to $(h_n \otimes x_1)(h_m \otimes 1)$. □

Lemma 25. *For integers $m, n \geq 1$ we have*

$$[h_{-m} \otimes x_1, h_n \otimes 1] = \begin{cases} n(h_{n-m} \otimes 1) & \text{if } n > m \\ 0 & \text{if } m \geq n \geq 1. \end{cases}$$

Proof. This is proven by direct computation.

$$(h_n \otimes 1)(h_{-m} \otimes x_1) = \left[\begin{array}{c} \text{Diagram 5: } n \text{ crossings followed by } m \text{ crossings} \end{array} \right]$$

The second diagram is zero since the upper cup can be pulled to the bottom of the diagram using the trace, thereby creating a left twist curl. Applying the same technique inductively, the first downward oriented strand and its dot can be brought all the way to the left of the diagram.

and the result claim follows by Lemma 20. $\square$

Lemma 26. *For integers $m, n \geq 1$ we have*

$$[h_n \otimes x_1, h_{-m} \otimes 1] = \begin{cases} -2m(h_{n-m} \otimes 1) & \text{if } n > m \geq 1 \\ 0 & \text{if } n = m \geq 1 \\ -m(h_{n-m} \otimes 1) & \text{if } 1 \leq n < m. \end{cases}$$

Proof. For this proof let $\beta_n = h_n \otimes x_1$ and $\alpha_n = h_n \otimes 1$. We prove this by induction on m . For $m = 1$ we give a graphical proof. Consider $(h_{-1} \otimes 1)(h_n \otimes x_1)$

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$$(2.6) \quad \left[\begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right] + h_{n-1} \otimes 1$$

where we used (4.9) to simplify the dotted left twist curl. Using the trace relation, the second term above is also equal to $h_{n-1} \otimes 1$. For the first term we use the first equation in (2.3) to slide the upward strands to the left.

$$\left[\begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right] = \left[\begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram 6} \end{array} \right]$$

where the first equality holds since all the resolutions terms

$$\left[\begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right]$$

contain left twist curls. Thus, we have proven the base case of our induction.

Graphically it is easy to see that for some constant $\gamma_{n,-m}$ that

$$[\beta_n, \alpha_{-m}] = \gamma_{n,-m} \alpha_{n-m}.$$

In order to compute the constant $\gamma_{n,-m}$ consider the Jacobi identity

$$[\beta_n, [\beta_{-1}, \alpha_{-m+1}]] + [\beta_{-1}, [\alpha_{-m+1}, \beta_n]] + [\alpha_{-m+1}, [\beta_n, \beta_{-1}]] = 0. \quad (4.12)$$

Now applying Lemma 24, (4.12) becomes

$$(1-m)[\beta_n, \alpha_{-m}] - \gamma_{n,-m+1}[\beta_{-1}, \alpha_{n-m+1}] + (n+1)[\beta_{n-1}, \alpha_{-m+1}] = 0.$$

Thus,

$$\begin{aligned} [\beta_n, \alpha_{-m}] &= \frac{\gamma_{n,-m+1}[\beta_{-1}, \alpha_{n-m+1}] - (n+1)\gamma_{n-1,-m+1}\alpha_{n-m}}{1-m} \\ &= \frac{(n-m+1)\gamma_{n,-m+1} - (n+1)\gamma_{n-1,-m+1}}{1-m}\alpha_{n-m}. \end{aligned}$$

The lemma now easily follows by induction. $\square$

Along with Lemma 23, the next lemma will lead to Virasoro relations.

Lemma 27. *Let m and n be positive integers and $T = \min(m, n)$. Then*

$$[h_{-m} \otimes x_1, h_n \otimes x_1] = (n+m)(h_{n-m} \otimes x_1) - \sum_{j=1}^{T-1} j(h_{n-j} \otimes 1)(h_{-m+j} \otimes 1).$$

Proof. For this proof let $\alpha_n = h_n \otimes 1$ and $\beta_n = h_n \otimes x_1$. We proceed by induction on m . The base case is $m = 1$. The proof is similar to Lemma 26. We begin by using the first equation in (2.3) and the trace relation.

$$\left[\begin{array}{c} \downarrow \\ \uparrow \end{array} \right] \stackrel{(2.3)}{=} \left[\begin{array}{c} \uparrow \downarrow \\ \uparrow \end{array} \right] + \left[\begin{array}{c} \uparrow \uparrow \\ \uparrow \end{array} \right]$$

Using (4.9) the second term is equal to $h_{n-1} \otimes x_1$. For the first term we slide the dots upward producing

$$\left[\begin{array}{c} \uparrow \downarrow \\ \uparrow \end{array} \right] + \left[\begin{array}{c} \uparrow \uparrow \\ \uparrow \end{array} \right] + \left[\begin{array}{c} \uparrow \uparrow \\ \uparrow \end{array} \right]$$

The second and third term are both equal to $h_{n-1} \otimes x_1$. For the second diagram pull the cup to the bottom of the diagram producing a right twist curl (which is just another dot), then apply (4.9). Likewise, the third diagram simplifies using the trace relation.

To complete the proof of this claim we must slide the downward oriented line to the right of the upward oriented strands using the first equation in (2.3). Crossing the downward oriented strand past the i th upward oriented strand produces the following sum.

$$\left[\begin{array}{c} \uparrow \downarrow \\ \uparrow \end{array} \right] + \left[\begin{array}{c} \uparrow \uparrow \\ \uparrow \end{array} \right]$$

The second diagram contains a left twist curl with single interior dot. Using (4.9) with $a = 1$ to simplify this dotted curl, the second term above is equal to $h_{n-1} \otimes x_1$. The crossing resolution term resulting from sliding the dot upward in the first term produces a diagram containing a left twist curl, so that these terms all vanish. Continuing this process of moving the downward strand to to the right using the first equation in (2.3) and the dot slide equation we see that $(h_{-1} \otimes x_1)(h_n \otimes x_1)$ is equal to

$$\left[\begin{array}{c} \text{Diagram with strands and dots} \end{array} \right] + (n+1)h_{n-1} \otimes x_1.$$

completing the proof of $m = 1$ case.

Now assume the lemma is true for $[\beta_{-m+1}, \beta_n]$. The Jacobi identity gives

$$0 = [[\beta_{-1}, \beta_{-m+1}], \beta_n] + [[\beta_{-m+1}, \beta_n], \beta_{-1}] + [[\beta_n, \beta_{-1}], \beta_{-m+1}]. \quad (4.13)$$

By the base case and the inductive step (4.13) is:

$$0 = [[\beta_{-1}, \beta_{-m+1}], \beta_n] + \left[(n+m-1)\beta_{-m+1} + \sum_{j=1}^{m-2} j\alpha_{n-j}\alpha_{-m+1+j}, \beta_{-1} \right] + (n+1)[\beta_{-m+1}, \beta_{n-1}].$$

Another application of the base case and the inductive steps gives:

$$\begin{aligned} 0 &= [[\beta_{-1}, \beta_{-m+1}], \beta_n] + (n+m)(m-2)\beta_{n-m} \\ &\quad + \sum_{j=1}^{m-2} j(\alpha_{n-j}\alpha_{-m+1+j}\beta_{-1} - \beta_{-1}\alpha_{n-j}\alpha_{-m+1+j}) \\ &\quad + (n+1) \sum_{j=1}^{m-2} j\alpha_{n-1-j}\alpha_{-m+1+j}. \end{aligned} \quad (4.14)$$

By Lemma 25, we move β_{-1} and (4.14) becomes

$$\begin{aligned} 0 &= [[\beta_{-1}, \beta_{-m+1}], \beta_n] + (m+n)(m-2)\beta_{n-m} \\ &\quad + \sum_{j=1}^{m-2} j\alpha_{n-j}\alpha_{-m+1+j}\beta_{-1} - \sum_{j=1}^{m-2} j\alpha_{n-j}\beta_{-1}\alpha_{-m+1+j} \\ &\quad - \sum_{j=1}^{m-2} j(n-j)\alpha_{n-j-1}\alpha_{-m+1+j} + \sum_{j=1}^{m-2} (n+1)j\alpha_{n-1-j}\alpha_{-m+1+j}. \end{aligned} \quad (4.15)$$

Again moving the element β_{-1} to the right, using Lemma 25 (4.15) becomes

$$\begin{aligned} 0 &= [[\beta_{-1}, \beta_{-m+1}], \beta_n] + (m+n)(m-2)\beta_{n-m} + \sum_{j=1}^{m-2} (n+1)j\alpha_{n-1-j}\alpha_{-m+1+j} \\ &\quad + \sum_{j=1}^{m-2} j\alpha_{n-j}\alpha_{-m+1+j}\beta_{-1} - \sum_{j=1}^{m-2} j\alpha_{n-j}\alpha_{-m+1+j}\beta_{-1} \\ &\quad - \sum_{j=1}^{m-2} j(-m+1+j)\alpha_{n-j}\alpha_{-m+j} - \sum_{j=1}^{m-2} j(n-j)\alpha_{n-j-1}\alpha_{-m+1+j}. \end{aligned}$$

Simplifying the above gives

$$\begin{aligned} 0 &= [[\beta_{-1}, \beta_{-m+1}], \beta_n] + (m+n)(m-2)\beta_{n-m} \\ &\quad + \sum_{j=1}^{m-2} j(j+1)\alpha_{n-1-j}\alpha_{-m+1+j} - \sum_{j=1}^{m-2} j(-m+1+j)\alpha_{n-j}\alpha_{-m+j}. \end{aligned}$$

Combining the last two sums above gives

$$\begin{aligned} 0 &= [[\beta_{-1}, \beta_{-m+1}], \beta_n] + (m+n)(m-2)\beta_{n-m} + \sum_{j=1}^{m-3} (m-2)(j+1)\alpha_{n-1-j}\alpha_{-m+1+j} \\ &\quad + (m-2)\alpha_{n-1}\alpha_{-m+1} + (m-2)(m-1)\alpha_{n-m+1}\alpha_{-1}. \end{aligned}$$

Thus,

$$[[\beta_{-1}, \beta_{-m+1}], \beta_n] = -(m-2) \left((m+n)\beta_{n-m} - \sum_{j=1}^{m-1} j\alpha_{n-j}\alpha_{-m+j} \right).$$

It follows from Lemma 23 that $[\beta_{-1}, \beta_{-m+1}] = (-m+2)\beta_{-m}$. Substituting this into the above equation gives the desired result. $\square$

Remark 28. A slight modification to the $m = 1$ argument of Lemma 27 can be used to prove that for $n, m > 1$ the following relations:

$$\begin{aligned} [h_n \otimes x_1^p, h_1 \otimes x_1^m] &= - \sum_{j=2}^n \sum_{a+b=m-1} h_{n+1} \otimes x_1^p x_j^a x_{j+1}^b \\ &\quad + \sum_{a+b=p-1} h_{n+1} \otimes x_1^a x_2^{b+m} - \sum_{c+d=m-1} h_{n+1} \otimes x_1^c x_2^{p+d}, \\ [h_n \otimes x_1, h_m \otimes x_1^2] &= (m-2n)(h_{n+m} \otimes x_1^2) \\ &\quad + \sum_{j=1}^n (2n-j)(h_j \otimes x_1)(h_{n+m-j} \otimes 1) - \sum_{j=1}^{n-1} j(h_{m+j} \otimes x_1)(h_{n-j} \otimes 1), \end{aligned}$$

hold in $\text{Tr}(\mathcal{H})$. Though we will not need these relations in $\text{Tr}(\mathcal{H})$ in order to identify it with a quotient of $W_{1+\infty}$, they can be helpful in computing explicit formulas for commutators.

We define:

$$\begin{aligned} L_l &= \left| \frac{1}{l} \right| (h_{-l} \otimes (x_1 + \cdots + x_{|l|})) \quad (l \neq 0) \\ L_0 &= c_0 \\ b_l &= h_{-l} \otimes 1. \end{aligned} \tag{4.16}$$

Proposition 29. *The elements L_l for $l \in \mathbb{Z}$ and b_l for $l \neq 0$ generate a Heisenberg–Virasoro algebra with central charge one. That is*

$$\begin{aligned} [b_k, b_l] &= k\delta_{k,-l} \\ [L_k, L_l] &= (k-l)L_{k+l} + \frac{k^3-k}{12}\delta_{k,-l} \\ [L_l, b_k] &= -k(b_{l+k}). \end{aligned}$$

Proof. This follows from Lemmas 22–27. □

4.5. Trace of the degenerate affine Hecke algebra

The trace of DH_n may be computed by a theorem of Solleveld [22] which reduces the problem to computing the trace of a semi-direct product via a spectral sequence argument.

Theorem 30 [22, Theorem 3.4]. *The trace of the degenerate affine Hecke algebra is isomorphic to the trace of the semi-direct product of the symmetric group and a polynomial algebra:*

$$\mathrm{Tr}(DH_n) \cong \mathrm{Tr}(S_n \ltimes \mathbb{C}[x_1, \dots, x_n]).$$

Now let $P(n)$ be the set of partitions of n and, for a partition λ of n , let $p_i(\lambda)$ be the number of times the number i occurs in λ .

Theorem 31 ([10, Theorem 3.1], [22, § 1]). *We have*

$$\mathrm{Tr}(S_n \ltimes \mathbb{C}[x_1, \dots, x_n]) \cong \bigoplus_{\lambda \in P(n)} \bigotimes_{i \geq 1} S^{p_i(\lambda)} \mathbb{C}[x]$$

where $S^k \mathbb{C}[x]$ is the space of S_k invariants of $\mathbb{C}[x_1, \dots, x_k]$.

Theorems 30 and 31 together determine the trace of the degenerate affine Hecke algebra.

Example 32. The degenerate affine Hecke algebra of rank two DH_2 is generated by the group algebra of the symmetric group S_2 and the polynomial algebra $\mathbb{C}[x_1, x_2]$. If s_1 is the generator of S_2 then the only additional relation is $s_1 x_1 = x_2 s_1 + 1$.

There are partitions $\lambda = (2)$ and $\lambda = (1, 1)$ of 2. By the above theorems $\mathrm{Tr}(DH_2)$ is a direct sum of subspaces

$$\bigotimes_{i \geq 1} S^{p_i((2))} \mathbb{C}[x] \bigoplus \bigotimes_{i \geq 1} S^{p_i((1,1))} \mathbb{C}[x] \cong \mathbb{C}[x] \bigoplus S^2 \mathbb{C}[x_1, x_2].$$

This corresponds to a splitting of the trace

$$\mathbb{C}\langle x_1^a s_1 \rangle_{a \in \mathbb{Z}_{\geq 0}} \bigoplus S^2 \mathbb{C}[x_1, x_2].$$

4.6. $\mathrm{Tr}(\mathcal{H})$ as a vector space

Lemma 33. *If $f, g \in DH_n$ with $fg = 1$, then in fact $f, g \in \mathbb{C}[S_n] \subset DH_n$.*

Proof. We consider the non-negative integral filtration on DH_n , with associated graded $\mathrm{gr}(DH_n) \cong \mathbb{C}[S_n] \rtimes \mathbb{C}[x_1, \dots, x_n]$. The degree 0 part of this filtration is precisely $\mathbb{C}[S_n]$. Now $\mathrm{gr}(f)\mathrm{gr}(g) = 1$, which implies that $\mathrm{gr}(f)$ and $\mathrm{gr}(g)$ are in $\mathbb{C}[S_n]$. Thus, f and g are in the degree 0 part of the filtration, as desired. $\square$

Lemma 34. *The indecomposable objects of $\mathcal{H}'$ are of the form $P^m Q^n$ for $m, n \in \mathbb{Z}_{\geq 0}$.*

Proof. An indecomposable object must be of the form $P^m Q^n$ because a subsequence QP produces a decomposition $PQ \oplus 1$.

There is a morphism from $P^m Q^n$ to $P^a Q^b$ if and only if $m - n = a - b$. If $a + b \neq m + n$ then the composition $P^a Q^b \rightarrow P^m Q^n \rightarrow P^a Q^b$ produces cups and caps or circles. By Proposition 6 it follows that this composition cannot be the identity.

Now suppose there are maps $f: P^m Q^n \rightarrow P^m Q^n$ and $g: P^m Q^n \rightarrow P^m Q^n$ such that gf is the identity. We claim that in fact gf is the identity, too. Since a composition containing caps or cups cannot be the identity, Proposition 6 implies f is a monoidal composition of f_1 and f_2 where $f_1: P^m \rightarrow P^m$ and $f_2: Q^n \rightarrow Q^n$. Therefore, f_1 can be identified with an element in DH_m and f_2 may be identified with an element in DH_n^{op} . Similarly, g is a monoidal composition of g_1 and g_2 where $g_1: P^m \rightarrow P^m$ and $g_2: Q^n \rightarrow Q^n$, so that g_1 can be identified with an element in DH_m and g_2 may be identified with an element in DH_n^{op} . But now, by Lemma 33, f_1 and g_1 must belong to $\mathbb{C}[S_m]$. Since $\mathbb{C}[S_m]$ is semi-simple, $g_1 f_1$ is the identity if and only if $f_1 g_1$ is the identity. Thus, f is an isomorphism, and $P^m Q^n$ is indecomposable. $\square$

Lemma 35. *There is an isomorphism of vector spaces*

$$\mathrm{Tr}(\mathcal{H}) \cong \left(\bigoplus_{\substack{m \in \mathbb{Z}_{\geq 0} \\ n \in \mathbb{Z}_{\geq 0}}} \bigoplus_{\substack{\mu \in P(m) \\ \lambda \in P(n)}} \bigotimes_{\substack{i \geq 1 \\ j \geq 1}} S^{p_i(\mu)} \mathbb{C}[x] \otimes S^{p_j(\lambda)} \mathbb{C}[x] \right) \otimes \mathbb{C}[c_0, c_1, \dots].$$

Proof. By Proposition 12, $\mathrm{Tr}(\mathcal{H}) \cong \mathrm{Tr}(\mathcal{H}')$ so by the definition of $\mathrm{Tr}(\mathcal{H}')$ and by Proposition 13 it suffices to consider endomorphisms of all indecomposable objects (m products of P followed by n products of Q) modulo the ideal $\mathcal{I}$ defined in §4.1. That is

$$\mathrm{Tr}(\mathcal{H}') \cong \bigoplus_{m, n \geq 0} \mathrm{End}(P^m Q^n) / \mathcal{I}.$$

Since any map from $P^m Q^n \rightarrow P^{m'} Q^{n'}$ with $m \neq m'$ or $n \neq n'$ must contain a cap or cup, it is clear that the ideal $\mathcal{I}$ is equal to the ideal $J_{m,n}$ from Proposition 6 plus the

ideal generated by $fg - gf$ where $f, g: \mathbf{P}^m \mathbf{Q}^n \rightarrow \mathbf{P}^m \mathbf{Q}^n$. Using the trace relation together with the relations in $\mathcal{H}$, any map contained in the ideal $J_{m,n}$ can be reduced to a sum of endomorphisms of $\mathbf{P}^{m'} \mathbf{Q}^{n'}$ for some m', n' that are not in the ideal $J_{m,n}$ by induction on the number of cap/cups in the diagram. Indeed, each cap/cup can be isotoped around the annulus using the trace relation and relations in $\mathcal{H}$ to eventually create a curl or circle, which can then be further reduced to diagrams containing fewer caps and cups. By applying Proposition 6

$$\mathrm{Tr}(\mathcal{H}) \cong \bigoplus_{m,n \geq 0} \mathrm{Tr}(DH_m \otimes DH_n^{op} \otimes \mathbb{C}[c_0, c_1, \dots]).$$

The Lemma now follows from the results in § 4.5. $\square$

4.7. Towards the trace of $\mathcal{H}$ as an algebra

Denote by $\mathrm{Tr}^>(\mathcal{H})$ (respectively $\mathrm{Tr}^<(\mathcal{H})$) the subalgebra of $\mathrm{Tr}(\mathcal{H})$ generated by $h_l \otimes x_1^k$ (respectively $h_{-l} \otimes x_1^k$) for $l \geq 1, k \geq 0$. Moreover, let $\mathrm{Tr}^0(\mathcal{H}) := \mathrm{Tr}(\mathcal{H}^0)$.

Lemma 36. *The algebra $\mathrm{Tr}(\mathcal{H})$ is $\mathbb{Z}$ -graded where $h_r \otimes x_1^k$ is in degree r .*

Proof. This is clear by an inspection of the defining relations of the category $\mathcal{H}'$. $\square$

We call this $\mathbb{Z}$ -grading the rank grading because it is related to the degenerate affine Hecke algebra associated to a Lie group of a particular rank.

Lemma 37. *The algebra $\mathrm{Tr}(\mathcal{H})$ is $\mathbb{Z}_{\geq 0}$ -filtered where $h_r \otimes x_1^k$ is in degree k .*

Proof. This follows from the fact that a dot may slide through a crossing plus or minus a correction term which is a resolution of the crossing containing no dots. $\square$

For $\omega \in \{<, >, 0\}$, the associated graded of $\mathrm{Tr}^\omega(\mathcal{H})$ with respect to the $\mathbb{Z}_{\geq 0}$ -filtration will be denoted by $gr \mathrm{Tr}^\omega(\mathcal{H})$. This is a $(\mathbb{Z} \times \mathbb{Z}_{\geq 0})$ -graded algebra. We denote the subspace in degree (r, k) by $gr \mathrm{Tr}^\omega(\mathcal{H})[r, k]$ and define the Poincaré series by

$$P_{\mathrm{Tr}^\omega(\mathcal{H})}(t, q) = \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\geq 0}} \dim gr \mathrm{Tr}^\omega(\mathcal{H})[r, k] t^r q^k.$$

Corollary 38. *These Poincaré series are given by*

$$P_{\mathrm{Tr}^>(\mathcal{H})} = \prod_{r>0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k} \quad P_{\mathrm{Tr}^<(\mathcal{H})} = \prod_{r<0} \prod_{k \geq 0} \frac{1}{1 - t^r q^k}.$$

Proof. We prove the first equality (the second follows similarly). Under the isomorphism from Lemma 35 $\mathrm{Tr}^>(\mathcal{H})$ corresponds to

$$\bigoplus_{n \geq 0, \lambda \in P(n)} \bigotimes_{i \geq 1} S^{p_i(\lambda)} \mathbb{C}[x]. \quad (4.17)$$

Now $S^{p_i(\lambda)}\mathbb{C}[x] \cong \mathbb{C}[h_1, \dots, h_{p_i(\lambda)}]$ where the bidegree of the symmetric functions h_i is $(|\lambda|, i)$. It follows that the generating series of (4.17) is given by $\prod_{j \geq 1} \phi(q^j, t)$ where

$$\phi(q, t) := 1 + \frac{q}{(1-t)} + \frac{q^2}{(1-t)(1-t^2)} + \frac{q^3}{(1-t)(1-t^2)(1-t^3)} + \dots$$

But $\phi(q, t)$ is just the basic hypergeometric series ${}_1\phi_0(0; t, q)$ which is known to equal $\prod_{j \geq 0} \frac{1}{1-qt^j}$. The result follows. $\square$

Lemma 39. For $n > 0$ we have

$$\begin{aligned} [h_n \otimes x_1^a, c_0] &= -n(h_n \otimes x_1^a) \\ [h_{-n} \otimes x_1^a, c_0] &= n(h_{-n} \otimes x_1^a). \end{aligned}$$

Proof. By the ‘bubble moves’ [17, page 10] one could move a clockwise circle on the right of an upwards pointing strand to the left of the strand at the cost of subtracting the same diagram with the circle erased. Repeating this move n times to get the circle all the way to the left gives the first part of the lemma.

The second part is proved in a similar fashion. $\square$

Remark 40. Lemma 39 can be generalized to prove the more general identities

$$\begin{aligned} [h_n \otimes x_1^a, c_0^r] &= \sum_{j=1}^r (-1)^j \binom{r}{j} n^j c_0^{r-j} (h_n \otimes x_1^a), \quad \text{and} \\ [h_{-n} \otimes x_1^a, c_0^r] &= \sum_{j=1}^r \binom{r}{j} n^j c_0^{r-j} (h_{-n} \otimes x_1^a). \end{aligned}$$

Lemma 41. For $n > 0$ there are equalities

$$\begin{aligned} [h_n \otimes x_1^a, c_1] &= -2n(h_n \otimes x_1^{a+1}) + \sum_{j=1}^{n-1} 2(n-j)(h_j \otimes x_1^a)(h_{n-j} \otimes 1) \\ [h_{-n} \otimes x_1^a, c_1] &= 2n(h_{-n} \otimes x_1^{a+1}) + \sum_{j=1}^{n-1} 2(n-j)(h_{-j} \otimes x_1^a)(h_{-n+j} \otimes 1). \end{aligned}$$

Proof. For the first part we again use the ‘bubble moves’ [17, page 10] moving a clockwise circle with a dot on the right of n upward pointing strands all the way to the left:

$$(h_n \otimes x_1^a)(c_1) = (c_1)(h_n \otimes x_1^a) - 2h_n \otimes x_1^{a+1} - 2h_n \otimes x_1^a x_2 - \dots - 2h_n \otimes x_1^a x_n.$$

By Lemma 14, this is equal to

$$-2n(h_n \otimes x_1^{a+1}) + \sum_{j=1}^{n-1} 2(n-j)(h_j \otimes x_1^a)(h_{n-j} \otimes 1).$$

The second part is proved in a similar fashion. $\square$

Lemma 42. *For non-negative integers a and b there is an equality*

$$[h_{-1} \otimes x_1^b, h_1 \otimes x_1^a] = \tilde{c}_{a+b} + \sum_{l=0}^{a+b-2} (a+b-1-l) \tilde{c}_l c_{a+b-2-l}.$$

Here we define $c_{-2} = -1$ and $c_{-n} = 0$ for $n \in \mathbb{N}$ where $n \neq 2$.

Proof. The element $(h_1 \otimes x_1^a)(h_{-1} \otimes x_1^b)$ is equal to

$$\begin{aligned} \left[\begin{array}{c} \uparrow \\ a \bullet \\ \downarrow \end{array} \quad \begin{array}{c} \downarrow \\ b \bullet \\ \uparrow \end{array} \right] &= \left[\begin{array}{c} a \bullet \quad b \bullet \\ \nearrow \quad \searrow \\ \searrow \quad \nearrow \end{array} \right] \stackrel{(4.8)}{=} \left[\begin{array}{c} \nearrow \quad \searrow \\ b \bullet \quad a \bullet \\ \searrow \quad \nearrow \end{array} \right] - \sum_{j=0}^{a-1} \left[\begin{array}{c} \nearrow \quad \searrow \\ a+b-1-j \bullet \quad j \bullet \\ \searrow \quad \nearrow \end{array} \right] \\ &= \left[\begin{array}{c} \searrow \quad \nearrow \\ b \bullet \quad a \bullet \\ \nearrow \quad \searrow \end{array} \right] - \sum_{j=0}^{b-1} \left[\begin{array}{c} \searrow \quad \nearrow \\ b-1-j \bullet \quad a+j \bullet \\ \nearrow \quad \searrow \end{array} \right] - \sum_{j=0}^{a-1} \left[\begin{array}{c} \searrow \quad \nearrow \\ j \bullet \quad a+b-1-j \bullet \\ \nearrow \quad \searrow \end{array} \right] \end{aligned}$$

which can be simplified further, again using equation (4.8). Thus,

$$\begin{aligned} (h_1 \otimes x_1^a)(h_{-1} \otimes x_1^b) &= (h_1 \otimes x_1^a)(h_{-1} \otimes x_1^b) - \tilde{c}_{a+b} \\ &\quad - \sum_{j=0}^{b-1} \sum_{k=0}^{a+j-1} \tilde{c}_k c_{a+b-2-k} - \sum_{j=0}^{a-1} \sum_{k=0}^{j-1} \tilde{c}_k c_{a+b-2-k}. \end{aligned}$$

Combining the last two double summations into a single summation gives the lemma. $\square$

Lemma 43. *$\text{Tr}(\mathcal{H})$ is generated by $h_{-1} \otimes 1, h_1 \otimes 1, (c_0 + c_1)$.*

Proof. By Lemmas 39 and 41, we have

$$ad(c_0 + c_1)(h_1 \otimes 1) := [(c_0 + c_1), h_1 \otimes 1] = 2(h_1 \otimes x_1) + h_1 \otimes 1$$

allowing us to generate $h_1 \otimes x_1$.

By Lemma 24, we have $ad^n(h_1 \otimes x_1)(h_1 \otimes 1) = n!(h_n \otimes 1)$ allowing us to also generate $h_n \otimes 1$. Next we could generate $h_n \otimes x_1^k$ for all $n \geq 1, k \geq 0$ inductively by using Lemmas 39 and 41 to calculate $[h_n \otimes x_1^{k-1}, c_0 + c_1]$.

In a similar manner using Lemmas 39 and 41 we can generate $h_{-n} \otimes x_1^k$ for $n \geq 1, k \geq 0$. In particular, we can generate $h_{-1} \otimes x_1^b$ and $h_1 \otimes x_1^a$. Finally we can generate c_n for all $n \geq 0$ using Lemma 42. $\square$

4.8. A representation of the category $\mathcal{H}'$

Let $\mathcal{S}'_n$ denote the category whose objects are all $(\mathbb{C}[S_r], \mathbb{C}[S_n])$ bimodules for some $r \in \mathbb{Z}_{\geq 0}$ corresponding to induction and restriction functors. The morphisms between two bimodules B and B' is the space of bimodule maps between B and B' . Note that if the bimodules B and B' are not bimodules over the same algebra then the space of morphisms is zero.

For each n there is a functor $\mathcal{F}'_n: \mathcal{H}' \rightarrow \mathcal{S}'_n$. The object P is mapped to the $(\mathbb{C}[S_{n+1}], \mathbb{C}[S_n])$ -bimodule $\mathbb{C}[S_{n+1}]$. The object Q is mapped to the $(\mathbb{C}[S_{n-1}], \mathbb{C}[S_n])$ -bimodule $\mathbb{C}[S_n]$ where we view $\mathbb{C}[S_{n-1}]$ as embedded in $\mathbb{C}[S_n]$ by mapping the simple transpositions s_i to s_i for $i = 1, \dots, n-2$. We then extend this action monoidally. For details on how the functor $\mathcal{F}'_n$ acts on morphisms see [17, § 3.3]. Let

$$V = \bigoplus_n \mathrm{Tr}(\mathcal{S}'_n) = \bigoplus_n \mathrm{Tr}(\mathbb{C}[S_n] - \mathrm{mod}).$$

This is naturally a module for $\mathrm{Tr}(\mathcal{H}) \cong \mathrm{Tr}(\mathcal{H}')$ since the category $\mathcal{S}'_n$ is a representation of the category $\mathcal{H}$. For the analogous statement for 2-categories see [3, § 6.0.3].

Proposition 44. *As a vector space V is isomorphic to $\mathbb{C}[h_1 \otimes 1, h_2 \otimes 1, \dots]$.*

Proof. As a vector space $\mathrm{Tr}(\mathbb{C}[S_n])$ is spanned by conjugacy classes of S_n which in turn is isomorphic to the span of irreducible characters of S_n . Thus, V is spanned by the irreducible characters of all symmetric groups. As an algebra under induction it is generated by irreducible representations corresponding to single cycles. These are precisely the elements $h_1, h_2, \dots$ in $\mathrm{Tr}(\mathbb{C}[S_n])$. $\square$

Proposition 45. *The $\mathrm{Tr}(\mathcal{H})$ -module V is irreducible.*

Proof. It was proved in [17] that V categorifies Fock space for the Heisenberg algebra $\mathfrak{h}$. Since $\mathfrak{h}$ acts irreducibly on Fock space and $\mathfrak{h}$ is a subalgebra of $\mathrm{Tr}(\mathcal{H})$, it follows that V must be an irreducible $\mathrm{Tr}(\mathcal{H})$ -module. $\square$

In order to prove the next proposition we recall that the structure of $\mathrm{Tr}(\mathcal{H})$ as a vector space was computed in § 4.6. We follow closely the proof of [20, Proposition F.6] where the faithfulness of $W_{1+\infty}/(C-1, w_{0,0})$ on $\mathcal{V}_{1,0}$ is given.

Proposition 46. *The action of $\mathrm{Tr}(\mathcal{H})$ on V is faithful.*

Proof. In what follows we denote by $\mathrm{Tr}(\mathcal{H})[r, \leq k]$ the set of elements in $\mathbb{Z}$ -degree r and $\mathbb{Z}_{\geq 0}$ -degree $\leq k$. Let I be the annihilator of $\mathrm{Tr}(\mathcal{H})$ in V . There is a filtration on I : $I_0 \subset I_1 \subset \dots$. Since $\bigoplus_r \mathrm{Tr}(\mathcal{H})[r, \leq 0]$ is isomorphic to a Heisenberg algebra and it is known that the Heisenberg algebra acts faithfully on V , it must be the case that

$$I \cap (\bigoplus_r \mathrm{Tr}(\mathcal{H})[r, \leq 0]) = \{0\}.$$

Thus, $I_0 = \{0\}$. If we assume that I is non-zero then the minimal n such that $I_n \neq \{0\}$ is greater than zero.

Notice that for $n \geq 1$ that

$$[h_1 \otimes 1, \mathrm{Tr}(\mathcal{H})[r, \leq n]] \subset \mathrm{Tr}(\mathcal{H})[r+1, \leq n-1].$$

It follows that for $n \geq 1$ that $[I_n, \mathrm{Tr}(\mathcal{H})[r, \leq 0]] = 0$ so that each filtered component I_n is contained in the centralizer the $\bigoplus_r \mathrm{Tr}(\mathcal{H})[r, \leq 0]$ in $\mathrm{Tr}(\mathcal{H})$.

Now we compute the centralizer of $\oplus_r \text{Tr}(\mathcal{H})[r, \leq 0]$ in $\text{Tr}(\mathcal{H})$ and show that it must be trivial. If $g \in \text{Tr}(\mathcal{H})$ then we let its image in the associated graded be denoted by $\bar{g}$. It follows from Lemma 35 that the associated graded is isomorphic to a polynomial algebra $\mathbb{C}[\bar{h}_r \otimes x_1^s]$ for $r \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0}$ and $(r, s) \neq (0, 0)$. This algebra is bigraded with element $\bar{h}_r \otimes x_1^s$ in degree (r, s) .

Consider the map for $n \geq 1$

$$\begin{aligned} \zeta_n &= [h_l \otimes 1, \bullet]: \bigoplus_r \text{Tr}(\mathcal{H})[r, \leq n] / \text{Tr}(\mathcal{H})[r, \leq n-1] \\ &\rightarrow \bigoplus_r \text{Tr}(\mathcal{H})[r, \leq n-1] / \text{Tr}(\mathcal{H})[r, \leq n-2]. \end{aligned}$$

The space $\text{Tr}(\mathcal{H})[r, \leq n] / \text{Tr}(\mathcal{H})[r, \leq n-1]$ is the degree (r, n) piece of the associated graded algebra. It is easy to see that

$$\overline{\zeta_l(h_r \otimes x_1^k)} = -kl \overline{(h_{l+r} \otimes x_1^{k-1})}.$$

Thus, the common kernel of the operators $\cap_l \ker(\zeta_l)$ is zero if $n \geq 1$. This implies that the centralizer of $\oplus_r \text{Tr}(\mathcal{H})[r, \leq 0]$ in $\text{Tr}(\mathcal{H})$ is contained in $\oplus_r \text{Tr}(\mathcal{H})[r, \leq 0]$. But this subalgebra has trivial center. $\square$

4.9. $\text{Tr}(\mathcal{H})$ as an algebra

There is an isomorphism of vector spaces

$$\bar{\psi}: V = \mathbb{C}[h_1 \otimes 1, h_2 \otimes 1, \dots] \rightarrow \mathcal{V}_{1,0} = \mathbb{C}[w_{-1,0}, w_{-2,0}, \dots] \quad (4.18)$$

where

$$(h_{l_1} \otimes 1) \cdots (h_{l_r} \otimes 1) \mapsto (w_{-l_1,0}) \cdots (w_{-l_r,0}).$$

Lemma 47. *The map $\bar{\psi}$ in (4.18) commutes with the action of Heisenberg subalgebras in V and $\mathcal{V}_{1,0}$. That is for any $v \in V$*

$$\bar{\psi}((h_r \otimes 1)v) = w_{-r,0} \bar{\psi}(v).$$

Proof. Propositions 9 and 44 provide vector space realizations of $\mathcal{V}_{1,0}$ and V , respectively. The lemma now follows easily from the definition of $\bar{\psi}$ and the Heisenberg relations calculated in (3.3) and Lemma 22. $\square$

Lemma 48. *For any $v \in V$ we have $\bar{\psi}(c_0 v) = -w_{0,1} \bar{\psi}(v)$.*

Proof. This follows easily using (3.6) to compute $[w_{0,1}, w_{n,0}]$, Lemma 39 to compute $[-c_0, h_n \otimes 1]$ and the definition of the map $\bar{\psi}$. $\square$

Lemma 49. *For any $v \in V$ we have $\bar{\psi}((c_1 + c_0)v) = w_{0,2} \bar{\psi}(v)$.*

Proof. By Lemma 41 and (4.16) $[c_1, h_n \otimes 1] = 2nL_{-n}$. It is well known that the Virasoro operator $2nL_{-n}$ acts on Fock space by $n \sum_{k \in \mathbb{Z}} (h_{n-k} \otimes 1)(h_k \otimes 1)$. Now it is easy to see that the operator c_1 on V acts as

$$\sum_{k,l>0} ((h_l \otimes 1)(h_k \otimes 1)(h_{-l-k} \otimes 1) + (h_{l+k} \otimes 1)(h_{-l} \otimes 1)(h_{-k} \otimes 1)).$$

The result now follows from Lemma 11. □

Now we extend the map $\bar{\psi}$ to a map

$$\psi: \text{Tr}(\mathcal{H}) \rightarrow W_{1+\infty}/(C-1, w_{0,0}) \quad (4.19)$$

by mapping generators

$$h_{-1} \otimes 1 \mapsto w_{1,0} \quad h_1 \otimes 1 \mapsto w_{-1,0} \quad c_1 + c_0 \mapsto w_{0,2}.$$

Lemma 50. *The map ψ is a map of $\mathbb{Z}$ -graded and $\mathbb{Z}_{\geq 0}$ -filtered algebras.*

Proof. This follows since the actions of $\text{Tr}(\mathcal{H})$ and $W_{1+\infty}/(C-1, w_{0,0})$ on V and $\mathcal{V}_{1,0}$ respectively are faithful. □

Remark 51. The map ψ maps the Virasoro subalgebra of $W_{1+\infty}/(C-1, w_{0,0})$ to the Virasoro subalgebra of $\text{Tr}(\mathcal{H})$ from Proposition 29.

Lemma 52. *The map ψ is surjective.*

Proof. This follows directly from the definition of ψ and Lemma 8 which states that $w_{1,0}, w_{-1,0}, w_{0,2}$ generate $W_{1+\infty}$. □

Theorem 53. *The map ψ is an isomorphism of algebras.*

Proof. We first show that ψ restricts to an isomorphism

$$\psi^0: \text{Tr}^0(\mathcal{H}) \rightarrow W_{1+\infty}^0/(C-1, w_{0,0}).$$

For some $a_0, \dots, a_{l-1}, b_0, \dots, b_{l-1} \in \mathbb{C}$ we have

$$\begin{aligned} \psi \left(h_1 \otimes x_1^l + \sum_{k=0}^{l-1} a_k (h_1 \otimes x_1^k) \right) &= \psi \left(ad^l \left(\frac{c_1 + c_0}{2} \right) (h_1 \otimes 1) \right) \\ &= 2^{-l} ad^l(w_{0,2})(w_{-1,0}) \\ &= (-1)^l w_{-1,l} + \sum_{k=0}^{l-1} b_k (w_{-1,k}). \end{aligned} \quad (4.20)$$

The first equality follows from repeated use of Lemmas 39 and 41. The second equality follows from Lemma 49 and the third equality follows from [20, (F.9)]. So

$$\psi(h_1 \otimes x_1^l) = (-1)^l w_{-1,l} + \sum_{k=0}^{l-1} \alpha_k w_{-1,k}$$

for some $\alpha_0, \dots, \alpha_{l-1} \in \mathbb{C}$. By Lemma 42,

$$[h_{-1} \otimes 1, h_1 \otimes x_1^l] = \tilde{c}_l + \sum_{k=0}^{l-2} (l-1-k) \tilde{c}_k c_{l-2-k}. \quad (4.21)$$

Hence we have

$$\psi([h_{-1} \otimes 1, h_1 \otimes x_1^l]) = [w_{1,0}, \psi(h_1 \otimes x_1^l)] = \left[w_{1,0}, w_{-1,l} + \sum_{k=0}^{l-1} \alpha_k w_{-1,k} \right].$$

By Lemma 3.5

$$[w_{1,0}, w_{-1,l}] = -lw_{0,l-1} - \sum_{r=2}^l \binom{l}{r} w_{0,l-r} - 1 \quad (4.22)$$

so that (4.21) and (4.22) give us

$$\psi(c_{l-2}) = -lw_{0,l-1} + \sum_{k=0}^{l-1} d_k w_{0,k-1} + d$$

for some $d_0, \dots, d_{l-1}, d \in \mathbb{C}$. Thus, ψ restricts to an isomorphism ψ_0 .

Next observe that

$$\begin{aligned} \psi(\mathrm{Tr}^>(\mathcal{H})) &\subset W_{1+\infty}^>/(C-1, w_{0,0}) \\ \psi(\mathrm{Tr}^<(\mathcal{H})) &\subset W_{1+\infty}^</(C-1, w_{0,0}). \end{aligned}$$

Since

$$\begin{aligned} \mathrm{Tr}(\mathcal{H}) &= \mathrm{Tr}^>(\mathcal{H}) \otimes \mathrm{Tr}^0(\mathcal{H}) \otimes \mathrm{Tr}^<(\mathcal{H}) \\ W_{1+\infty}/(C-1, w_{0,0}) &= W_{1+\infty}^>/(C-1, w_{0,0}) \otimes W_{1+\infty}^0/(C-1, w_{0,0}) \otimes W_{1+\infty}^</(C-1, w_{0,0}) \end{aligned}$$

it suffices to show that ψ restricted to $\mathrm{Tr}^>(\mathcal{H})$ is an isomorphism (the restriction to $\mathrm{Tr}^<(\mathcal{H})$ is similar). Since we know that ψ is surjective we just need that the graded dimensions of $\mathrm{Tr}^>(\mathcal{H})$ and $W_{1+\infty}^>/(C-1, w_{0,0})$ are the same. This follows from Proposition 7 and Corollary 38. $\square$

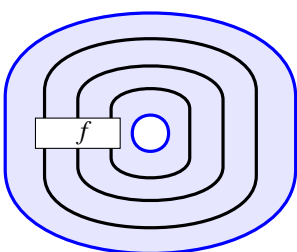
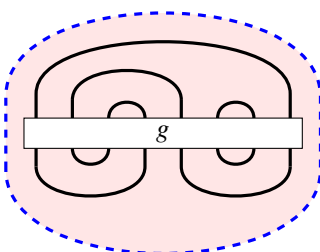
4.10. The action of $\mathrm{Tr}(\mathcal{H})$ on the center

Recall, by Proposition 5, that the center of $\mathcal{H}$ is isomorphic to a polynomial algebra in countably many variables:

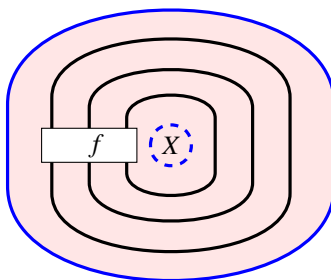
$$Z(\mathcal{H}) = \mathrm{End}_{\mathcal{H}}(\mathbf{1}) \cong \mathbb{C}[c_0, c_1, c_2, \dots].$$

Since all objects of $\mathcal{H}$ have biadjoints and all adjunctions in $\mathcal{H}$ are cyclic, it follows that the trace $\mathrm{Tr}(\mathcal{H})$ acts on $Z(\mathcal{H})$ (see, for example, [3, § 6] or [4, § 9]). Graphically, an element of the center $X \in Z(\mathcal{H})$ can be represented by a closed diagram and an element of the

trace by a diagram on the annulus:

$$f = \text{[diagram of } f \text{ on an annulus]} \in \text{Tr}(\mathcal{H}) \quad X = \text{[diagram of } X \text{ on an annulus]} \in Z(\mathcal{H}).$$



This action of $\text{Tr}(\mathcal{H})$ on $Z(\mathcal{H})$ is given by placing the diagram for X in the interior of the annulus



and interpreting the result as a new central element in $Z(\mathcal{H})$. As a consequence of Theorem 1 and its proof, we have the following.

Corollary 54. *The algebra $\text{Tr}(\mathcal{H})$ acts irreducibly and faithfully on $Z(\mathcal{H})$. Under the isomorphism $\text{Tr}(\mathcal{H}) \cong W_{1+\infty}/I$, this representation is isomorphic to the canonical level one representation $\mathcal{V}_{1,0}$ of $W_{1+\infty}$.*

Proof. The proof that $\text{Tr}(\mathcal{H})$ acts irreducibly and faithfully on $Z(\mathcal{H})$ follows exactly as in Propositions 45 and 46. Those propositions used the action of $\text{Tr}(\mathcal{H})$ on $\text{Tr}(\mathbb{C}[S_n]\text{-mod})$, but we could have equally well worked with the center of $\mathcal{H}$. The identification $\text{End}_{\mathcal{H}}(\mathbf{1}) \cong \mathcal{V}_{1,0}$ follows immediately. $\square$

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