

SUBADDITIVITY OF SYZYGIES OF KOSZUL ALGEBRAS

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ABSTRACT. Estimates are obtained for the degrees of minimal syzygies of quotient algebras of polynomial rings. For a class that includes Koszul algebras in almost all characteristics, these degrees are shown to increase by at most 2 from one syzygy module to the next one. Even slower growth is proved if, in addition, the algebra satisfies Green and Lazarsfeld's condition N_q with $q \geq 2$.

INTRODUCTION

In this paper we study homological properties of commutative graded algebras R , generated over a field k by finitely many elements of degree one. A common approach is to choose a presentation $R \cong S/J$, where S is a standard graded polynomial ring over k and J an ideal containing no linear forms, and use the minimal free resolution F of the graded S -module R . In this context, the numbers

$$t_i^S(R) = \sup\{j \in \mathbb{Z} \mid \operatorname{Tor}_i^S(R, k)_j \neq 0\}$$

are of significant interest, as they bound the degrees of the basis elements of F_i .

Explicit upper bounds exist in terms of the most accessible data: e , the number of variables of S and $t_1^S(R)$, the maximal degrees of the generators of J . They are doubly exponential in i and cannot be strengthened in general; see 5.4 for references.

The situation is dramatically different when R is Koszul; that is, when the minimal free resolution of $k = R/R_+$ over R is linear; equivalently, when $t_i^R(k) \leq i$ for all $i \geq 0$. For Koszul algebras Backelin [3] and Kempf [21] proved inequalities

$$(1) \quad t_i^S(R) \leq 2i \quad \text{for} \quad 1 \leq i \leq \operatorname{pd}_S R.$$

Cases when equality hold are described in [2].

Here we extend and sharpen these results in several directions.

First, we relax the hypothesis, assuming only $t_i^R(k) \leq i$ for $i \leq e + 1$. Such algebras are not necessarily Koszul and—unlike the latter—they are identified by a fixed finite segment of the minimal free resolution of k over R ; see Roos [31]. In Section 4 we prove that they satisfy (1) and determine when equality holds.

Second, for the algebras described above we investigate the existence of upper bounds on $t_i^S(R)$ that depend on $t_a^S(R)$ for $1 \leq a < i$. In Section 5 we prove

$$(2) \quad t_{i+1}^S(R) \leq t_i^S(R) + 2 \quad \text{for} \quad 1 \leq i \leq e - \dim R + 1$$

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when $(i+1)$ is invertible in k ; since then $t_1^S(R) = 2$, this inequality refines (1) under additional hypotheses. It is itself the special case $b = 1$ of the following inequality:

$$(3) \quad t_{a+b}^S(R) \leq t_a^S(R) + t_b^S(R) \quad \text{for } a, b \geq 1 \text{ with } a + b \leq \text{pd}_S R.$$

We conjecture that (3) holds without restrictions on the characteristic of k and review evidence gleaned from a number of sources. In particular, here we show that when R is Cohen-Macaulay and $\binom{a+b}{a}$ is invertible in k one has

$$t_{a+b}^S(R) \leq t_a^S(R) + t_b^S(R) + 1 \quad \text{for } a \geq 1 \text{ and } b \geq 2 \text{ with } a + b \leq \text{pd}_S R.$$

For the algebras described above we study the impact of the Green–Lazarsfeld condition N_q : The differential $F_i \rightarrow F_{i-1}$ is given by a matrix of quadrics for $i = 1$ and by matrices of linear forms for $2 \leq i \leq q$; see [18]. In Section 6 we show that (excluding finitely many specified characteristics) it implies a sharpening of (1):

$$(4) \quad t_i^S(R) \leq 2 \left\lfloor \frac{i}{q+1} \right\rfloor + i + \begin{cases} 0 & \text{if } (q+1) \mid i \\ 1 & \text{otherwise.} \end{cases}$$

This inequality reveals hitherto unknown properties of the resolutions of algebras of geometric interest, even in cases such as Segre products and Veronese subalgebras of polynomial rings. For related recent progress in this context see [32, 34].

Our approach utilizes the homology of the Koszul complex K on a basis of R_1 .

The differential bigraded algebra K provides a bridge between the homology of the R -module k and that of the S -module R ; see [1]. As in [2], the proof of (1) depends on the decomposable classes in $H(K)$ contained in its subalgebra generated by $H_1(K)$. On the other hand, the proofs of (2) and (4) involve an analysis of indecomposable homology classes; namely, of the quotient $H_{a+b}(K)/H_a(K) \cdot H_b(K)$.

This is the main technical innovation in the paper. Its effectiveness depends on the existence (in almost all characteristics) of splitting maps for Koszul cycles, discovered in [8] and described in Section 2 from a more conceptual perspective. These maps are combined with cascades of bounds on regularities of Tor modules, obtained in Sections 1 and 3. The material in the first three sections is presented in its natural generality, which is much greater than the one needed in the sequel.

1. COMPLEXES OF FLAT MODULES

In this section we evaluate how cycles change with the introduction of coefficients, starting from a general situation and moving on to a graded one. Only basic homological considerations are involved.

1.1. Coefficients. Let R be an associative ring and R^c its center. Let

$$(1.1.1) \quad F = \cdots \rightarrow F_{a+1} \xrightarrow{\partial_{a+1}} F_a \xrightarrow{\partial_a} F_{a-1} \rightarrow \cdots$$

be a complex of right R -modules, and for every integer a set

$$Z_a = \text{Ker}(\partial_a), \quad B_a = \text{Im}(\partial_{a+1}), \quad H_a = Z_a/B_a, \quad \text{and} \quad C_a = \text{Coker}(\partial_{a+1}).$$

For each left R -module N and for $Z_a(F \otimes_R N) = \text{Ker}(\partial_a \otimes N)$ the assignment $z \otimes x \mapsto z \otimes x$ defines a natural in N homomorphism of abelian groups

$$(1.1.2) \quad \phi_a: Z_a \otimes_R N \rightarrow Z_a(F \otimes_R N).$$

The next lemma is a sort of “universal coefficients theorem” for cycles.

Lemma 1.2. *When F_a and F_{a-1} are flat there is a natural in N exact sequence*

$$0 \rightarrow \operatorname{Tor}_1^R(B_{a-1}, N) \rightarrow Z_a \otimes_R N \xrightarrow{\phi_a} Z_a(F \otimes_R N) \rightarrow \operatorname{Tor}_1^R(C_{a-1}, N) \rightarrow 0$$

of R^c -modules, and for every $i \geq 1$ there is a natural in N isomorphism

$$(1.2.1) \quad \operatorname{Tor}_{i+1}^R(B_{a-1}, N) \cong \operatorname{Tor}_i^R(Z_a, N).$$

Proof. The maps $F_a \xrightarrow{\pi_a} B_{a-1}$ and $B_{a-1} \xrightarrow{\iota_{a-1}} F_{a-1}$ yield a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ker}(\pi_a \otimes_R N) & \longrightarrow & F_a \otimes_R N & \xrightarrow{\pi_a \otimes_R N} & B_{a-1} \otimes_R N \longrightarrow 0 \\ & & \downarrow & & \downarrow \partial_a \otimes_R N & & \downarrow \iota_{a-1} \otimes_R N \\ 0 & \longrightarrow & 0 & \longrightarrow & F_{a-1} \otimes_R N & \xlongequal{\quad} & F_{a-1} \otimes_R N \end{array}$$

with exact rows. The Snake Lemma then produces an exact sequence

$$0 \rightarrow \operatorname{Ker}(\pi_a \otimes_R N) \rightarrow Z_a(F \otimes_R N) \rightarrow \operatorname{Ker}(\iota_{a-1} \otimes_R N) \rightarrow 0$$

As F_a is flat, the exact sequence $0 \rightarrow Z_a \rightarrow F_a \xrightarrow{\pi_a} B_{a-1} \rightarrow 0$ induces the isomorphisms (1.2.1) and yields an exact sequence

$$0 \rightarrow \operatorname{Tor}_1^R(B_{a-1}, N) \rightarrow Z_a \otimes_R N \xrightarrow{\zeta_a} \operatorname{Ker}(\pi_a \otimes_R N) \rightarrow 0$$

such that the composition of ζ_a with the inclusion $Z_a(F \otimes_R N) \subseteq F_a \otimes_R N$ is equal to ϕ_a . As F_{a-1} is flat we also have exact sequence

$$0 \rightarrow \operatorname{Tor}_1^R(C_{a-1}, N) \rightarrow B_{a-1} \otimes_R N \xrightarrow{\iota_{a-1} \otimes_R N} F_{a-1} \otimes_R N$$

induced by $0 \rightarrow B_{a-1} \xrightarrow{\iota_{a-1}} F_{a-1} \rightarrow C_{a-1} \rightarrow 0$; it computes $\operatorname{Ker}(\iota_{a-1} \otimes_R N)$. \square

Next we describe notation and conventions concerning gradings in this paper.

1.3. Graded objects. Let k be a field and $V = \bigoplus_j V_j$ a graded vector space; set

$$\operatorname{top}(V) = \sup\{j \mid V_j \neq 0\};$$

thus, $\operatorname{top}(V) = -\infty$ if and only if $V = 0$. The notation $\deg(v) = j$ means $v \in V_j$.

A *graded k -algebra* is a graded vector space $R = \bigoplus_{i \in \mathbb{Z}} R_i$ with $R_0 = k$, $R_i = 0$ for $i < 0$, and associative k -bilinear products $R_i \times R_j \rightarrow R_{i+j}$. A *left graded R -module* is a graded vector space $N = \bigoplus_{j \in \mathbb{Z}} N_j$ with $N_j = 0$ for $j \ll 0$ and associative k -bilinear products $R_i \times N_j \rightarrow N_{i+j}$. *Graded right R -modules* are defined similarly.

Homomorphisms of graded R -modules are homogeneous of degree 0. Thus, when L is a right R -module and N a left one, $\operatorname{Tor}_i^R(L, N)$ is a graded R^c -module.

As usual, given an graded R -module N and integer n , we write $N(n)$ for the graded R -module with $N(n)_i = N_{n+i}$ for all $i \in \mathbb{Z}$.

By abuse of notation, k also stands for the module $R/\bigoplus_{i \geq 1} R_i$. The number

$$t_i^R(N) = \operatorname{top}(\operatorname{Tor}_i^R(k, N))$$

then equals the largest degree of a minimal i -syzygy of the R -module N .

Lemma 1.4. *When L is a right graded R -module and N a left one, one has*

$$(1.4.1) \quad \operatorname{top}(\operatorname{Tor}_i^R(L, N)) \leq \operatorname{top}(L) + t_i^R(N) \quad \text{for all } i \in \mathbb{Z}.$$

Proof. Set $t = \text{top}(L)$. We may assume t is finite, and then $s = t - \min\{j \mid L_j \neq 0\}$ is a non-negative integer. If $s = 0$, then $\text{Tor}_i^R(L, N) \cong L \otimes_k \text{Tor}_i^R(k, N)$, so equality holds in (1.4.1). The exact sequence $0 \rightarrow L_t(-t) \rightarrow L \rightarrow L/L_t(-t) \rightarrow 0$ of graded R -modules induces an exact sequence of graded Tor vector spaces, from which the inequality (1.4.1) follows by induction on s . \square

Lemma 1.5. *Let R be a graded algebra and F a complex of flat graded R -modules. For all integers $i \geq 1$ and $a \geq 0$ the following inequalities hold:*

$$(1.5.1) \quad \text{top}(\text{Tor}_i^R(B_a, N)) \leq \max_{0 \leq j \leq a} \{\text{top}(H_j) + t_{a+i+1-j}^R(N)\}$$

$$(1.5.2) \quad \text{top}(\text{Tor}_i^R(Z_a, N)) \leq \max_{0 \leq j < a} \{\text{top}(H_j) + t_{a+i+1-j}^R(N)\}$$

$$(1.5.3) \quad \text{top}(\text{Tor}_i^R(C_{a-1}, N)) \leq \max_{0 \leq j < a} \{\text{top}(H_j) + t_{a-1+i-j}^R(N)\}$$

Proof. Fix some integer $i \geq 1$. In view of (1.2.1), we obtain an exact sequence

$$(1.5.4) \quad \text{Tor}_{i+1}^R(H_a, N) \rightarrow \text{Tor}_i^R(B_a, N) \rightarrow \text{Tor}_{i+1}^R(B_{a-1}, N)$$

from the one induced by $0 \rightarrow B_a \rightarrow Z_a \rightarrow H_a \rightarrow 0$. Now (1.5.4) and (1.4.1) yield

$$\text{top}(\text{Tor}_i^R(B_a, N)) \leq \max\{\text{top}(H_a) + t_{i+1}^R(N), \text{top}(\text{Tor}_{i+1}^R(B_{a-1}, N))\}$$

The proof of (1.5.1) is completed by iterating this procedure.

The isomorphisms (1.2.1) show that formula (1.5.2) follows from (1.5.1).

The exact sequence $0 \rightarrow H_{a-1} \rightarrow C_{a-1} \rightarrow B_{a-2} \rightarrow 0$ induces an exact sequence

$$\text{Tor}_i^R(H_{a-1}, N) \rightarrow \text{Tor}_i^R(C_{a-1}, N) \rightarrow \text{Tor}_i^R(B_{a-2}, N)$$

from which the inequality (1.5.3) follows, due to formulas (1.4.1) and (1.5.1). \square

2. INDECOMPOSABLE KOSZUL HOMOLOGY

In this section R denotes a commutative ring and M an R -module.

2.1. Koszul complexes. Let $E = \bigwedge_R E_1$ be the exterior algebra on a free R -module E_1 sitting in homological degree 1, and $\mu: E \otimes_R E \rightarrow E$ the product map.

The algebra E has following universal property: Each R -linear map $E_1 \rightarrow R$ extends uniquely to an R -linear map $\partial: E \rightarrow E$ of homological degree -1 , satisfying

$$(2.1.1) \quad \partial\mu = \mu(\partial \otimes E + E \otimes \partial).$$

As $\partial^2(E_1) = 0$ holds for degree reasons, this implies $\partial^2 = 0$. The complex K with underlying graded module E and differential ∂ is the *Koszul complex* of $E_1 \rightarrow R$.

Set $K^M = K \otimes_R M$. In view of (2.1.1), K is a DG algebra and K^M is a DG K -module. Thus, $H(K)$ is a graded algebra and $H(K^M)$ is a graded $H(K)$ -module.

If R is a graded k -algebra and $E_1 \rightarrow R$ a homomorphism of graded R -modules, see 1.3, then the differential algebra K and its differential module K^M are *bigraded*: their differentials decrease homological degrees by 1 and preserve internal degrees. Thus, $H(R)$ is naturally a bigraded algebra and $H(M)$ a bigraded module over it

Theorem 2.2. *For each pair $(a, b) \in \mathbb{Z}^2$ there exists a natural R -linear map*

$$(2.2.1) \quad \gamma_{a,b}: \text{Tor}_1^R(C_{a-1}(K), Z_b(K^M)) \rightarrow \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)}$$

described in 2.10, which is surjective when $\binom{a+b}{a}$ is invertible in R .

When R and M are graded $\gamma_{a,b}$ is a homomorphism of graded modules.

The theorem is proved at the end of the section.

Corollary 2.3. *Assume $H_a(K) = 0$ for $a \geq 1$ (for instance, E_1 has a basis e_1, \dots, e_r such that the sequence $\partial(e_1), \dots, \partial(e_r)$ is R -regular) and set $I = \partial(E_1)$.*

If $\binom{a+b}{a}$ is invertible in R , then there is a surjective R -linear map

$$\gamma'_{a,b}: \operatorname{Tor}_a^R(R/I, Z_b(K^M)) \rightarrow \operatorname{Tor}_{a+b}^R(R/I, M)$$

Proof. The hypothesis yields $H_{a+b}(K^M) \cong \operatorname{Tor}_{a+b}^R(R/I, M)$ and an exact sequence

$$0 \rightarrow C_{a-1}(K) \rightarrow K_{a-1} \rightarrow \dots \rightarrow K_0 \rightarrow R/I \rightarrow 0$$

of R -modules. Since K_i is flat for $0 \leq i \leq a-1$, the iterated connecting map $\operatorname{Tor}_a^R(R/I, Z_b(K^M)) \rightarrow \operatorname{Tor}_1^R(C_{a-1}(K), Z_b(K^M))$ is bijective. \square

As a special case, we obtain an (unexpected) inequality between Betti numbers:

Corollary 2.4. *Let R be a regular local ring, K the Koszul complex on a minimal generating set of the maximal ideal of R , and M a finitely generated R -module.*

If $\binom{a+b}{a}$ is invertible in R , then the b -cycles of $K^M = K \otimes_R M$ satisfy

$$\beta_a^R(Z_b(K^M)) \geq \beta_{a+b}^R(M).$$

In the graded setup such inequalities hold for graded Betti numbers. \square

2.5. Diagonal maps. Equipping $E \otimes_R E$ with the product

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (-1)^{|x_2||y_1|} (x_1 y_1 \otimes x_2 y_2).$$

turns it into an R -algebra that is strictly graded-commutative for the homological degree. The universal property of exterior algebras yields a unique homomorphism $\Delta: E \rightarrow E \otimes_R E$ of graded R -algebras, such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in E_1$.

Set $\partial' = \partial \otimes E$ and $\partial'' = E \otimes \partial$. These maps satisfy the equalities

$$(2.5.1) \quad \partial' \Delta = \Delta \partial = \partial'' \Delta$$

$$(2.5.2) \quad \partial' \partial'' + \partial'' \partial' = 0$$

Indeed, the maps in (2.5.1) and (2.5.2) are graded derivations of the R -algebras E and $E \otimes_R E$, respectively. They are generated in homological degree 1, so it suffices to verify agreement on E_1 and $(E \otimes_R E)_1$, respectively. This is straightforward.

2.6. Partial cycles. As E is a graded free R -module, we have

$$E \otimes_R Z(K^M) = E \otimes_R \operatorname{Ker}(\partial \otimes M) = \operatorname{Ker}(\partial'' \otimes M)$$

From (2.5.2) we obtain $(\partial' \otimes M)(\operatorname{Ker}(\partial'' \otimes M)) \subseteq \operatorname{Ker}(\partial'' \otimes M)$, so $\partial' \otimes M$ turns $\operatorname{Ker}(\partial'' \otimes M)$ into a complex. It is equal to $K \otimes_R Z(K^M)$, so we get

$$(2.6.1) \quad Z(K \otimes_R Z(K^M)) = \operatorname{Ker}(\partial' \otimes M) \cap \operatorname{Ker}(\partial'' \otimes M)$$

as graded submodules of $E \otimes_R E \otimes_R M$. For every integer n there is an equality

$$(2.6.2) \quad Z_n(K \otimes_R Z(K^M)) = \bigoplus_{i+j=n} Z_i(K \otimes_R Z_j(K^M))$$

Let $\iota_{i,j}$ and $\pi_{i,j}$ denote the canonical maps from and to $Z_i(K \otimes_R Z_j(K^M))$, respectively, induced by the decomposition (2.6.2).

2.7. The maps α . The equalities (2.1.1) and (2.5.2) imply

$$(\mu \otimes M)(\text{Ker}(\partial' \otimes M + \partial'' \otimes M)) \subseteq \text{Ker}(\partial \otimes M) = Z(K^M).$$

In view of (2.6.1), $\mu \otimes M$ restricts to a homomorphism of graded R -modules

$$\alpha: Z(K \otimes_R Z(K^M)) \rightarrow Z(K^M).$$

Setting $n = a + b$, we define $\alpha_{a,b}$ to be the composed map

$$Z_a(K \otimes_R Z_b(K^M)) \xrightarrow{\iota_{a,b}} Z_n(K \otimes_R Z(K^M)) \xrightarrow{\alpha_n} Z_n(K^M)$$

2.8. The maps β . The equalities (2.5.1) imply

$$(\Delta \otimes_R M)(\text{Ker}(\partial \otimes M)) \subseteq \text{Ker}(\partial' \otimes M) \cap \text{Ker}(\partial'' \otimes M).$$

In view of (2.6.1), $\Delta \otimes_R M$ restricts to a homomorphism of graded R -modules

$$\beta: Z(K \otimes M) \rightarrow Z(K \otimes_R Z(K^M)).$$

Setting $n = a + b$, we define $\beta_{a,b}$ to be the composed map

$$Z_n(K^M) \xrightarrow{\beta_n} Z_n(K \otimes_R Z(K^M)) \xrightarrow{\pi_{a,b}} Z_a(K \otimes_R Z_b(K^M))$$

The next result is proved by Bruns, Conca, and Römer in [8, 2.4] through direct calculations. Our proof mines the graded Hopf algebra structure of E .

Lemma 2.9. *For each pair (a, b) of non-negative integers there is an equality*

$$\alpha_{a,b} \circ \beta_{a,b} = \binom{a+b}{a} \text{id}^{Z_{a+b}(K^M)}.$$

Proof. Set $n = a + b$ and note that $\alpha_{a,b} \circ \beta_{a,b}$ is induced by the composed map

$$\delta_{a,b}: E_n \xrightarrow{\Delta_n} (E \otimes_R E)_n \xrightarrow{\rho_{a,b}} (E \otimes_R E)_n \xrightarrow{\mu_n} E_n$$

where $\rho_{a,b}$ projects $(E \otimes_R E)_n = \bigoplus_{i+j=n} E_i \otimes_R E_j$ onto its summand $E_a \otimes E_b$.

Let n be a positive integer and let \mathbf{n} denote the sequence $(1, \dots, n)$. For each sequence $\mathbf{i} = (u_1, \dots, u_i)$ of integers satisfying $1 \leq u_1 < \dots < u_i \leq n$, write $\mathbf{n} \setminus \mathbf{i} = (v_1, \dots, v_{n-i})$ with $v_1 < \dots < v_{n-i}$ and let $\text{sgn}(\mathbf{i}, \mathbf{n} \setminus \mathbf{i})$ denote the sign of the permutation $(u_1, \dots, u_i, v_1, \dots, v_{n-i})$ of \mathbf{n} . Given a subset $x = \{x_1, \dots, x_n\}$ of E_1 , setting $x_{\mathbf{i}} = x_{u_1} \wedge \dots \wedge x_{u_i} \in E_i$ we obtain the following equalities:

$$\begin{aligned} \delta_{a,b}(x_{\mathbf{n}}) &= \mu_n \circ \rho_{a,b} \left(\sum_{\mathbf{i} \subseteq \mathbf{n}} \text{sgn}(\mathbf{i}, \mathbf{n} \setminus \mathbf{i}) x_{\mathbf{i}} \otimes x_{\mathbf{n} \setminus \mathbf{i}} \right) \\ &= \mu_n \left(\sum_{\text{card } \mathbf{i} = a} \text{sgn}(\mathbf{i}, \mathbf{n} \setminus \mathbf{i}) x_{\mathbf{i}} \otimes x_{\mathbf{n} \setminus \mathbf{i}} \right) \\ &= \binom{n}{a} x_{\mathbf{n}} \end{aligned}$$

It remains to remark that E_n is additively generated by elements of the form $x_{\mathbf{n}}$. \square

2.10. The maps γ . Each pair (a, b) of integers defines a diagram of R -linear maps

$$(2.10.1) \quad \begin{array}{ccccc} & & Z_a(K) \otimes Z_b(K^M) & \xrightarrow{\chi_a \otimes \chi_b} & H_a(K) \otimes H_b(K^M) \\ & \swarrow \phi_a & \downarrow \mu_{a,b}^Z & & \downarrow \mu_{a,b}^H \\ Z_a(K \otimes_R Z_b(K^M)) & \xrightarrow{\alpha_{a,b}} & Z_{a+b}(K^M) & \xrightarrow{\chi_{a+b}} & H_{a+b}(K^M) \end{array}$$

where the maps χ are canonical, $\mu_{a,b}^Z$ and $\mu_{a,b}^H$ are induced by μ in view of (2.1.1), $\alpha_{a,b}$ comes from 2.7, and ϕ_a is given by 1.1 with $F = K$ and $N = Z_b(K^M)$. The definitions show that diagram (2.10.1) commutes, so it yields natural R -linear maps

$$(2.10.2) \quad \text{Coker}(\phi_a) \rightarrow \text{Coker}(\mu_{a,b}^Z) \rightarrow \text{Coker}(\mu_{a,b}^H)$$

The isomorphism $\text{Tor}_1^R(C_{a-1}(K), Z_b(K^M)) \cong \text{Coker}(\phi_a)$ from Lemma 1.2, composed with the maps in (2.10.2), defines the map $\gamma_{a,b}$ in (2.2.1).

Proof of Theorem 2.2. Since χ_{a+b} is surjective, so is the second map in (2.10.2).

If $\binom{a+b}{b}$ is invertible in R , then $\alpha_{a,b}$ is surjective by Lemma 2.9, hence so is the first map in (2.2.1). It follows that $\gamma_{a,b}$ is surjective, as desired.

When R is a graded ring and M is a graded R -module the constructions of $\alpha_{a,b}$, $\beta_{a,b}$, and $\gamma_{a,b}$ show that they are homomorphisms of graded R -modules. \square

3. REGULARITY

In this section k is a field and R a graded commutative k -algebra with $R_0 = k$, $R = k[R_1]$, and $\text{rank}_k R_1$ finite. We write S for the symmetric k -algebra S on R_1 and let $S \rightarrow R$ be the canonical surjective homomorphism of graded k -algebras.

3.1. Regularities. The n th *partial regularity* of an R -module M is the number

$$(3.1.1) \quad \text{reg}_n^R(M) = \max_{i \leq n} \{t_i^R(M) - i\}.$$

Thus, for every $n \in \mathbb{Z}$ there is an inequality

$$(3.1.2) \quad \text{reg}_n(R) \leq \text{reg}_{n+1}(R)$$

and $\sup\{\text{reg}_n^R(M)\}_{n \in \mathbb{Z}}$ is the *Castelnuovo-Mumford regularity* $\text{reg}^R(M)$.

The goal of this section is to prove the following result.

Theorem 3.2. *Let a, b be non-negative integers satisfying $a + b \leq \text{pd}_S R$.*

If $\binom{a+b}{b}$ is invertible in k , then for every R -module M one has

$$t_{a+b}^S(M) \leq \max \left\{ \begin{array}{l} t_a^S(R) + t_b^S(M) \\ \text{reg}_{a-1}^S(R) + \text{reg}_{a+b}^R(M) + a + b \\ \text{reg}_{a-1}^S(R) + \text{reg}_{b-1}^S(M) + \text{reg}_{a+b+1}^R(k) + a + b + 1 \end{array} \right\}$$

We single out a special case of the theorem:

Corollary 3.3. *If $\text{reg}_{a+b+1}^R(k) = 0 = \text{reg}_{a+b}^R(M)$, then there is an inequality*

$$(3.3.1) \quad t_{a+b}^S(M) \leq \max \left\{ \begin{array}{l} t_a^S(R) + t_b^S(M) \\ \text{reg}_{a-1}^S(R) + \text{reg}_{b-1}^S(M) + a + b + 1 \end{array} \right\} \quad \square$$

The proof of Theorem 3.2 utilizes Koszul complexes.

3.4. Koszul homology. The inclusion $R_1(-1) \subseteq R$ defines an R -linear map $R \otimes_k R_1(-1) \rightarrow R$, and hence a Koszul complex K ; see 2.1. Set $K^M = K \otimes_R M$.

For each $i \in \mathbb{N}$ we have an isomorphism

$$(3.4.1) \quad H_i(K^M) \cong \text{Tor}_i^S(k, M)$$

of graded R -modules; it gives an equality

$$(3.4.2) \quad t_i^S(M) = \text{top}(H_i(K^M))$$

As $(R_+)H(K^M) = 0$, for all non-negative integers i, j the preceding formula yields

$$(3.4.3) \quad t_j^R(H_i(K^M)) = t_i^S(M) + t_j^R(k)$$

By applying Theorem 2.2, then formula (1.5.3) with $F = K$ and $N = Z_b(K^M)$, and finally formula (3.4.2) with $M = R$ we obtain:

Lemma 3.5. *If $\binom{a+b}{b}$ is invertible in R , then there is an inequality*

$$(3.5.1) \quad \text{top} \left(\frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \right) \leq \max_{0 \leq j < a} \{t_j^S(R) + t_{a-j}^R(Z_b(K^M))\}. \quad \square$$

Next we estimate the partial regularities of the cycles appearing in (3.5.1).

Lemma 3.6. *For each R -module M the following inequality holds:*

$$(3.6.1) \quad t_a^R(Z_b(K^M)) \leq \max_{1 \leq j \leq b} \left\{ \begin{array}{l} t_a^R(M) + b \\ t_{a+j}^R(M) + b - j \\ t_{b-j}^S(M) + t_{a+j+1}^R(k) \end{array} \right\}$$

Proof. From the exact sequences of R -modules

$$\begin{aligned} 0 &\longrightarrow Z_b(K^M) \longrightarrow K_b^M \longrightarrow B_{b-1}(K^M) \longrightarrow 0 \\ 0 &\longrightarrow B_{b-1}(K^M) \longrightarrow Z_{b-1}(K^M) \longrightarrow H_{b-1}(K^M) \longrightarrow 0 \end{aligned}$$

we obtain the following inequalities:

$$\begin{aligned} t_a^R(Z_b(K^M)) &\leq \max\{t_{a+1}^R(B_{b-1}(K^M)), t_a^R(M) + b\} \\ t_{a+1}^R(B_{b-1}(K^M)) &\leq \max\{t_{a+2}^R(H_{b-1}(K^M)), t_{a+1}^R(Z_{b-1}(K^M))\}. \end{aligned}$$

By concatenating these relations and invoking (3.4.3) we get

$$t_a^R(Z_b(K^M)) \leq \max \left\{ \begin{array}{l} t_a^R(M) + b \\ t_{a+1}^R(Z_{b-1}(K^M)) \\ t_{b-1}^S(M) + t_{a+2}^R(k) \end{array} \right\}$$

The desired result follows from the preceding formula by induction on b . \square

Proof of Theorem 3.2. Products in homology yield an exact sequence

$$H_a(K) \otimes H_b(K^M) \rightarrow H_{a+b}(K^M) \rightarrow \frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \rightarrow 0$$

of graded k -vector spaces. It accounts for the first inequality in the following display, where the equality comes from (3.4.2), the second inequality from (3.5.1), and the

last inequality from (3.6.1):

$$\begin{aligned}
t_{a+b}^S(M) &= \text{top}(H_{a+b}(K^M)) \\
&\leq \max \left\{ \text{top}(H_a(K)) + \text{top}(H_b(K^M)), \text{top} \left(\frac{H_{a+b}(K^M)}{H_a(K)H_b(K^M)} \right) \right\} \\
&\leq \max_{0 \leq i < a} \{ t_a^S(R) + t_b^S(M), t_i^S(R) + t_{a-i}^R(Z_b(K^M)) \} \\
&\leq \max_{\substack{0 \leq i < a \\ 1 \leq j \leq b}} \left\{ \begin{array}{l} t_a^S(R) + t_b^S(M) \\ t_i^S(R) + t_{a-i}^R(M) + b \\ t_i^S(R) + t_{a-i+j}^R(M) + b - j \\ t_i^S(R) + t_{b-j}^S(M) + t_{a-i+j+1}^R(k) \end{array} \right\}
\end{aligned}$$

For each pair (i, j) with $0 \leq i < a$ and $0 \leq j \leq b$, in view of (3.1.1) we have

$$\begin{aligned}
t_i^S(R) + t_{a-i+j}^R(M) + b - j &\leq [\text{reg}_{a-1}^S(R) + i] + [\text{reg}_{a+b}^R(M) + a - i + j] + b - j \\
&= \text{reg}_{a-1}^S(R) + \text{reg}_{a+b}^R(M) + a + b
\end{aligned}$$

Similarly, for each pair (i, j) with $0 \leq i < a$ and $1 \leq j \leq b$, we obtain

$$\begin{aligned}
t_i^S(R) + t_{b-j}^S(M) + t_{a-i+j+1}^R(k) &\leq [\text{reg}_{a-1}^S(R) + i] + [\text{reg}_{b-1}^S(M) + b - j] + [\text{reg}_{a+b+1}^R(k) + a - i + j + 1] \\
&= \text{reg}_{a-1}^S(R) + \text{reg}_{b-1}^S(M) + \text{reg}_{a+b+1}^R(k) + a + b + 1
\end{aligned}$$

It remains to assemble the inequalities in the last three displays. \square

4. DECOMPOSABLE KOSZUL HOMOLOGY

We keep the hypotheses on R and S from Section 3, and let J denote the kernel of the surjection $S \rightarrow R$ of graded k -algebras.

Theorem 4.1. *Let K be the Koszul complex described in 3.4.*

If $\text{reg}_{n+1}^R(k) = 0$ for some $n \geq 0$, then for $0 \leq i \leq n$ there are equalities

$$(4.1.1) \quad H_i(K)_j = 0 \quad \text{when } j > 2i.$$

$$(4.1.2) \quad H_i(K)_{2i} = (H_1(K)_2)^i.$$

Formula (3.4.2) gives a numerical translation of the equality (4.1.1).

Corollary 4.2. *For $0 \leq i \leq n$ the following inequalities hold:*

$$(4.2.1) \quad t_i^S(R) \leq 2i. \quad \square$$

When R is Koszul, that is to say, when $\text{reg}^R(k) = 0$, these results hold for all integers $i \geq 0$. This is known: (4.2.1) is proved by Kempf [21, Lemma 4] and is a special case of a result of Backelin [3, Corollary]; see also [2, 4.1(a) and 1.4]. On the other hand, (4.1.2) is a special case of [2, 4.1(a) and (3.8.3)].

The proof of Theorem 4.1 does not use the material developed in previous sections; rather, it adapts ideas from our proof of [2, 3.1]. We start by presenting the relevant background material.

4.3. Minimal models. Given a set $X = \bigsqcup_{j \geq 1} X_{i,j}$, let $k[X]$ denote the k -algebra $\bigotimes_{i=1}^{\infty} k[X_i]$, where $X_i = \bigsqcup_j X_{i,j}$ and $k[X_i]$ is the exterior algebra on X_i when i is odd, respectively, the symmetric algebra on X_i when i is even.

For $x \in X_{i,j}$ set $|x| = i$ and $\deg(x) = j$. A k -basis of $k[X]$ is given by the element $1 \in k$ and the monomials $x_1^{d_1} \cdots x_u^{d_u}$ with $x_l \in X$ and $d_l = 1$ when $|x_l|$ is odd, respectively, with $d_l \geq 1$ when $|x_l|$ is even. We bigrade $k[X]$ by setting

$$|x_1^{d_1} \cdots x_u^{d_u}| = \sum_{l=1}^u d_l |x_l| \quad \text{and} \quad \deg(x_1^{d_1} \cdots x_u^{d_u}) = \sum_{l=1}^u d_l \deg(x_l).$$

We also bigrade S by $|y| = 0$ and $\deg(y) = i$ for $y \in S_i$.

A *model* of R is a differential bigraded S -algebra $S[X]$ with underlying algebra $S \otimes_k k[X]$ and differential satisfying $|\partial(y)| = |y| - 1$, $\deg(\partial(y)) = \deg(y)$, and

$$\partial(s \otimes yy') = s \otimes \partial(y)y' + (-1)^{|y|} s \otimes y\partial(y'),$$

together with an S -linear map ε that makes the following sequence exact:

$$(4.3.1) \quad \cdots \rightarrow S[X]_{i,*} \xrightarrow{\partial_i} S[X]_{i-1,*} \rightarrow \cdots \rightarrow S[X]_{0,*} \xrightarrow{\varepsilon} R \rightarrow 0$$

A model $S[X]$ is said to be *minimal* if it satisfies $\partial(S[X]) \subseteq (S_+ + X^2)S[X]$. Such a model always exists, and any two are isomorphic as differential bigraded S -algebras; see [1, 7.2.4]. In every minimal model $\partial(X_1)$ is a minimal set of generators of J and $S[X_1]$ is the Koszul complex of the restriction $SX_1 \rightarrow S$ of ∂_1 .

Let $X_{0,0}$ be a k -basis of S_1 , and define a set $X' = \bigsqcup_{j \geq 1} X'_{i,j}$ by the formula

$$(4.3.2) \quad X'_{i,j} = \{x'\}_{x \in X_{i-1,j}}.$$

Set $k\langle X' \rangle = \bigotimes_{i=1}^{\infty} k\langle X'_i \rangle$, where $X'_i = \bigsqcup_j X'_{i,j}$ and $k\langle X'_i \rangle$ denotes the exterior algebra on X'_i when i is odd, respectively, the divided powers algebra on that space when i is even.

A k -basis of $k\langle X' \rangle$ is given by $1 \in k$ and $x'_1{}^{(d_1)} \cdots x'_u{}^{(d_u)}$ with $x'_l \in X'$ and $d_l = 1$ if $|x'_l|$ is odd, respectively, with $d_l \geq 1$ if $|x'_l|$ is even. It is bigraded by setting

$$|x'_1{}^{(d_1)} \cdots x'_u{}^{(d_u)}| = \sum_{l=1}^u d_l |x'_l| \quad \text{and} \quad \deg(x'_1{}^{(d_1)} \cdots x'_u{}^{(d_u)}) = \sum_{l=1}^u d_l \deg(x'_l).$$

By [1, 7.2.6], there exists an isomorphism of bigraded k -vector spaces

$$(4.3.3) \quad \mathrm{Tor}^R(k, k) \cong \bigotimes_{i=1}^{\infty} k\langle X'_i \rangle$$

Proof of Theorem 4.1. Let $S[X]$ be a minimal model of R . Set $k[X] = k \otimes_S S[X]$ and note that (3.4.1) and (4.3.1) yield isomorphisms of bigraded k -algebras

$$(4.4.1) \quad H(k[X]) \cong \mathrm{Tor}^S(k, R) \cong H(K).$$

This induces an isomorphism $kX_{1,2} \cong H_1(K)$, by the minimality of $S[X]$.

Let x' be the element of $X'_{i,j}$ corresponding to the element x of $X_{i-1,j}$; see (4.3.2). For $i+1 \leq n+1$, from $\mathrm{reg}_{n+1}^R(k) = 0$ and (4.3.3) we get $j = i+1$. Thus, $|x| \leq n$ implies $\deg(x) = |x| + 1$. Since $|x| \geq 1$ holds for $i \leq n$, for any monomial $x_1^{d_1} \cdots x_u^{d_u}$

with $|x_1^{d_1} \cdots x_u^{d_u}| \leq n$ we obtain the relations

$$\begin{aligned} \deg(x_1^{d_1} \cdots x_u^{d_u}) &= \sum_{l=1}^u d_l \deg(x_l) = \sum_{l=1}^u d_l(|x_l| + 1) = |x_1^{d_1} \cdots x_u^{d_u}| + \sum_{l=1}^u d_l \\ &\leq 2|x_1^{d_1} \cdots x_u^{d_u}| \end{aligned}$$

For $j > 2i$ these relations yield $(k[X])_{i,j} = 0$. As a consequence, $H_i(k[X])_j = 0$ holds for $j > 2i$ and $H_i(k[X])_{2i}$ is a quotient of $(k[X])_{i,2i}$. In view of (4.4.1), now (4.1.1) is proved, and for (4.1.2) it remains to show $(k[X])_{i,2i} = (kX_{1,2})^i$.

When $\deg(x_1^{d_1} \cdots x_u^{d_u}) = 2|x_1^{d_1} \cdots x_u^{d_u}|$ the inequality in the last display becomes an equality. All d_l and $|x_l|$ being positive integers, for $1 \leq l \leq u$ we get first $|x_l| = 1$, then $|x_l| = \deg(x_l) - |x_l|$; that is, $\deg(x_l) = 2$. As a consequence, we get $k[X]_{i,2i} = k[X_1]_{i,2i} = (kX_{1,2})^i$, as desired. \square

5. SUBADDITIVITY OF SYZYGIES

We keep the hypotheses on R and S from Section 3. In this section we pick up a theme from the introduction—*linearity* at the start of the resolution of k over R implies *subadditivity* of the degree of the syzygies of R over S . We set

$$t_i(R) = t_i^S(R), \quad \text{reg}(R) = \text{reg}^S(R), \quad \text{and} \quad \text{reg}_i(R) = \text{reg}_i^S(R)$$

for $i \in \mathbb{Z}$. Some of our results depend on the number

$$m(R) = \min\{i \in \mathbb{Z} \mid t_i(R) \geq t_{i+1}(R)\}.$$

Lemma 5.1. *The following inequalities hold:*

$$(5.1.1) \quad \text{rank}_k R_1 - \dim R \leq m(R) \leq \text{pd}_S R.$$

Proof. We set $m = m(R)$ and let F be a minimal free resolution of R over S .

For $h = \text{pd}_S R$ we have $t_h(R) > 0$ and $t_{h+1}(R) = -\infty$, hence $m \leq h$.

For $t = t_m(R)$ we have $t_{m-1}(R) < t \leq t_{m+1}(R)$. The functor $?^* = \text{Hom}_S(?, S)$ gives homogeneous R -linear maps of graded R -modules, and hence k -linear maps

$$(F_{m-1})_{-t}^* \xrightarrow{(\partial_m)^*_{-t}} (F_m)_{-t}^* \xrightarrow{(\partial_{m+1})^*_{-t}} (F_{m+1})_{-t}^*$$

The inequalities for t imply $(F_{m-1})_{-t}^* = 0$ and $(F_{m+1})_j^* = 0$ for $j < -t$, hence

$$\text{Im}((\partial_{m+1})_{-t}^*) \subseteq (S_+(F_{m+1})^*)_{-t} = \sum_{i \geq 1} S_i(F_{m+1})_{-t-i}^* = 0$$

Thus, we get $\text{Ext}_S^m(R, S)^t = (F_m)_{-t}^* \neq 0$, and hence $\text{grade}_S R \leq m$. This is the desired lower bound, as $\text{grade}_S R = \text{rank}_k R_1 - \dim R$ because S is regular. \square

Theorem 5.2. *Let a, b be non-negative integers such that $\binom{a+b}{a}$ is invertible in k .*

When $\text{reg}_{a+b+1}^R(k) = 0$ and $\max\{a, b\} \leq m(R)$ hold there is an inequality

$$(5.2.1) \quad t_{a+b}(R) \leq \max \left\{ \begin{array}{l} t_a(R) + t_b(R) \\ t_{a-1}(R) + t_{b-1}(R) + 3 \end{array} \right\}.$$

In particular, there are inequalities

$$(5.2.2) \quad t_{a+1}(R) \leq t_a(R) + 2 \quad \text{when } b = 1.$$

$$(5.2.3) \quad t_{a+b}(R) \leq t_a(R) + t_b(R) + 1 \quad \text{for } b \geq 2.$$

Proof. Formula (3.3.1) and the hypothesis on a, b yield

$$t_{a+b}(R) \leq \max \left\{ \begin{array}{l} t_a(R) + t_b(R) \\ t_{a-1}(R) - (a-1) + t_{b-1}(R) - (b-1) + a + b + 1 \end{array} \right\}$$

This simplifies to (5.2.1). The other inequalities follow, since $t_0(R) = 0$, $t_1(R) = 2$, and $t_{i-1} \leq t_i - 1$ for $i \leq m(R)$. \square

Unless R is a polynomial ring, $\text{reg}_2^R(k) = 0$ implies $t_1(R) = 2$. Thus, (4.2.1) gives an upper bound on $t_a(R)$ by a function depending only on a and $t_1(R)$. Such a property does not hold in general:

Example 5.3. For $R = k[x_1, \dots, x_e]/J$ and J generated by $e + 1$ general quadrics

$$t_1(R) = 2 \quad \text{and} \quad t_2(R) = \left\lfloor \frac{e}{2} \right\rfloor + 2.$$

This is seen from the free resolution given by Migliore and Miró-Roig in [29, 5.4].

On the other hand, we have:

5.4. General upper bounds. For every algebra R the number $t_a(R)$ is bounded above by a function of the numbers a , $t_1(R)$, and $e = \text{rank}_k R_1$; specifically, Bayer and Mumford [5, 3.7] and Caviglia and Sbarra [10, 2.7] give the upper bound

$$t_a(R) \leq (2t_1(R))^{2^{e-2}} + a - 1 \quad \text{for every } a \geq 1.$$

It cannot be tightened substantially, since variations of the Mayr-Meyer ideals [26] yield algebras with the same type of doubly exponential syzygy growth, albeit with different coefficients; see Bayer and Stillman [6, 2.6] and Koh [22, Theorem, p. 233].

The results in this section suggest the following

Conjecture 5.5. If $\text{reg}_{a+b+1}^R(k) = 0$, then the following inequality holds:

$$(5.5.1) \quad t_{a+b}(R) \leq t_a(R) + t_b(R) \quad \text{whenever } a + b \leq \text{pd}_S R.$$

Additional supporting evidence is reviewed below. The algebra $R = S/J$ has *monomial relations* if J can be generated by products of elements in some k -basis of R_1 ; such an algebra is Koszul if and only if J is generated by quadrics; see [15].

Examples 5.6. Set $e = \text{rank}_k R_1$ and $h = \text{pd}_S R$.

The inequality (5.5.1) holds in the following cases:

- (1) $\dim R \leq 1$, $\text{depth } R = 0$, and $a + b = e$: Eisenbud, Huneke, and Ulrich [14, 4.1].
- (2) $b = 1$ when R has monomial relations: Fernández-Ramos and Gimenez [13, 1.9] in the quadratic case, Herzog and Srinivasan [19] in general.
- (3) $e \leq 7$ when R has quadratic monomial relations and $\text{char}(k) = 0$: Verified by McCullough by using MACAULAY2 [17] and NAUTY [28].

In addition, the following weaker inequalities are known to hold:

- (4) $t_{a+b}(R) \leq t_a(R) + t_b(R) + 1$ when R is Cohen-Macaulay, $\text{reg}_{h+1}^R(k) = 0$, and $\binom{a+b}{a}$ is invertible in k : Formulas (5.1.1) and (5.2.3).
- (5) $t_{a+b}(R) \leq \max_{a,b \geq 1} \{t_a(R) + t_b(R)\}$ when $a + b = e$: McCullough [27, 4.4].

When p is a prime number and n a non-negative integer, we say that p is *good* for n if $\binom{n}{i} \not\equiv 0 \pmod{p}$ holds for all $i \in \mathbb{Z}$ with $0 \leq i \leq n$.

Lemma 5.7. *A prime number p is good for an integer $n \geq 0$ if and only if one of the following holds: (a) $p = 0$, or (b) $p > n$, or (c) $n + 1 = p^s u$ for some integers $s \geq 1$ and $2 \leq u \leq p$; furthermore, at most one pair (p, n) satisfies (c).*

Proof. We may assume $0 < p \leq n$. Let $\sum_{j=0}^s n_j p^j$ and $\sum_{j=0}^r i_j p^j$ be the p -adic expansions of n and i , respectively; thus, $s \geq 1$ and $1 \leq n_s < p$. Lucas' congruence

$$\binom{n}{i} \equiv \prod_{j=0}^s \binom{n_j}{i_j} \pmod{p}$$

shows that $\binom{n}{i}$ is invertible mod p if and only if so is $\binom{n_j}{i_j}$ for each j . One has $0 \leq i_j, n_j < p$ by definition, so $\binom{n_j}{i_j}$ is invertible mod p if and only if $n_j \geq i_j$. In conclusion, p is good for n if and only if $n_j \geq i_j$ holds for $j = 0, \dots, s$ and $n \geq i \geq 0$.

An inequality $n \geq i \geq 0$ means $s \geq r$ and there is an integer l with $0 \leq l \leq s$, satisfying $n_j = i_j$ for $j > l$ and $n_l > i_l$. This holds for all i with $n \geq i \geq 0$ if and only if $n_s \geq i_s$ and $n_j = p - 1$ for $s > j \geq 0$; that is, if and only if

$$n = n_s p^s + \sum_{j=0}^{s-1} (p-1)p^j = n_s p^s + (p-1) \frac{p^s - 1}{p-1} = (n_s + 1)p^s - 1.$$

Assume $up^s = u'p'^{s'}$ holds for some prime number p' and integers u', s' satisfying $s' \geq 1$ and $2 \leq u' \leq p$. If $p' > p$, then $p' | u$, and hence $u \geq p' > p$, which is impossible. By symmetry, so is $p > p'$, so we get $p' = p$. Now $s \neq s'$ implies $p | u$ or $p | u'$, which again is impossible. Thus, we must have $s = s'$, and hence $u = u'$. \square

Subadditivity is also reflected in the behavior of partial regularity. Recall from (3.1.2) that one has always an inequality $\text{reg}_a(R) \leq \text{reg}_{a+1}(R)$.

Proposition 5.8. *Let a, b be non-negative integers.*

If $\text{reg}_{a+b+1}^R(k) = 0$, and $\text{char}(k)$ is zero or good for $(a+b)$, then the following inequalities hold:

$$(5.8.1) \quad \text{reg}_{a+1}(R) \leq \text{reg}_a(R) + 1 \quad \text{when } b = 1,$$

$$(5.8.2) \quad \text{reg}_{a+b}(R) \leq \max \left\{ \begin{array}{l} \text{reg}_a(R) + \text{reg}_b(R) \\ \text{reg}_{a-1}(R) + \text{reg}_{b-1}(R) + 1 \end{array} \right\} \quad \text{for } b \geq 2.$$

Proof. Pick any integer i with $0 \leq i \leq a+b$ and write it as $i = a' + b'$ with integer a' and b' satisfying $0 \leq a' \leq a$ and $0 \leq b' \leq b$. The number $\binom{a'+b'}{b'}$ is invertible in k due to the hypothesis on $\text{char}(k)$, so for $b \geq 1$ formulas (3.3.1) and (3.1.2) yield:

$$\begin{aligned} t_i(R) - i &\leq \max \left\{ \begin{array}{l} (t_{a'}(R) - a') + (t_{b'}(R) - b') \\ \text{reg}_{a'-1}(R) + \text{reg}_{b'-1}(R) + 1 \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} \text{reg}_{a'}(R) + \text{reg}_{b'}(R) \\ \text{reg}_{a'-1}(R) + \text{reg}_{b'-1}(R) + 1 \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} \text{reg}_a(R) + \text{reg}_b(R) \\ \text{reg}_{a-1}(R) + \text{reg}_{b-1}(R) + 1 \end{array} \right\} \end{aligned}$$

This implies (5.8.2), and also (5.8.1) because $\text{reg}_j(R) = j$ holds for $j = 0, 1$. \square

6. THE GREEN-LAZARSFELD PROPERTY N_q

We keep the hypotheses on R and S from Section 3, and set

$$p = \text{char}(k), \quad h = \text{pd}_R S, \quad e = \text{rank}_k R_1.$$

Green and Lazarsfeld [18, p. 85] say that R has property N_q for a positive integer q if $t_i(R) = i + 1$ holds for $1 \leq i \leq q$. This is equivalent to $\text{reg}_q^S(R) = 1$, see 3.1.

Theorem 6.1. *Let R be a graded algebra satisfying $\text{reg}_{n+1}^R(k) = 0$ for some $n \geq 2$. When R satisfies N_q with $q \geq 2$ the following inequalities hold:*

$$(6.1.1) \quad t_i(R) \leq 2i - 1 \quad \text{for } 2 \leq i \leq n.$$

If, furthermore, p is good for i , then also the following inequalities hold:

$$(6.1.2) \quad t_i(R) \leq 2 \left\lfloor \frac{i}{q+1} \right\rfloor + i + \begin{cases} 0 & \text{if } (q+1) \mid i \\ 1 & \text{otherwise.} \end{cases}$$

$$(6.1.3) \quad \text{reg}_n(R) \leq 2 \left\lfloor \frac{n}{q+1} \right\rfloor + 1$$

Remark. Both (6.1.1) and (6.1.2) sharpen (4.2.1). The condition $q \geq 2$ is necessary, since a complete intersection of quadrics R has N_1 and $t_a(R) = 2a$ for $a \leq h$.

Proof. The isomorphisms (3.4.2) and (4.1.2) give for each $i \geq 2$ equalities

$$\text{Tor}_i^S(R, k)_{2i} = (\text{Tor}_1^S(R, k)_2)^2 (\text{Tor}_1^S(R, k)_2)^{i-2} = \text{Tor}_2^S(R, k)_4 (\text{Tor}_1^S(R, k)_2)^{i-2}$$

We have $\text{Tor}_2^S(R, k)_4 = 0$ because R satisfies N_2 , and hence $\text{Tor}_i^S(R, k)_{2i} = 0$. Since (4.4.1) gives $t_{2i} \leq 2i$ for $i \geq 1$, we conclude that $t_{2i} \leq 2i - 1$ holds for $i \geq 2$.

If $i \leq q$, then (6.1.2) holds by hypothesis. When $i \geq q + 1$ we may assume by induction on i that for $0 \leq j < i$ the following inequalities hold:

$$(6.1.4) \quad \text{reg}_j(R) \leq 2 \left\lfloor \frac{j}{q+1} \right\rfloor + \begin{cases} 0 & \text{if } (q+1) \mid j \\ 1 & \text{otherwise.} \end{cases}$$

Euclidean division yields integers $0 \leq u \leq q$ and $v \geq 1$, such that

$$i = (q+1)v + u.$$

In case $u = 0$ by applying first formula (3.3.1) with $a = i - 1$ and $b = 1$, then formula (6.1.4) with $j = (q+1)v - 1$, $j = (q+1)v - 2$, and $j = 0$ we obtain

$$\begin{aligned} t_i(R) &\leq \max \left\{ \begin{array}{l} t_{(q+1)v-1}(R) + t_1(R) \\ \text{reg}_{(q+1)v-2}(R) + \text{reg}_0(R) + i + 1 \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} 2(v-1) + (i-1) + 1 + 2 \\ 2(v-1) + 1 + 0 + i + 1 \end{array} \right\} \\ &= 2v + i \end{aligned}$$

In case $1 \leq u \leq q$ by applying formula (3.3.1) with $a = (q+1)v$ and $b = u$, then formula (6.1.4) with $j = (q+1)v$, $j = u$, $j = (q+1)v - 1$, and $j = u - 1$ we get

$$\begin{aligned} t_i(R) &\leq \max \left\{ t_{(q+1)v}(R) + t_u(R) \right. \\ &\quad \left. \text{reg}_{(q+1)v-1}(R) + \text{reg}_{u-1}(R) + i + 1 \right\} \\ &\leq \max \left\{ 2v + (q+1)v + u + 1 \right. \\ &\quad \left. 2(v-1) + 1 + 1 + i + 1 \right\} \\ &= 2v + i + 1. \end{aligned}$$

This concludes the proof of (6.1.2). Formula (6.1.3) is a direct consequence. \square

Conjecture 5.5 predicts that the bounds in Theorem 6.1 can be strengthened:

Lemma 6.2. *Assume R has property N_q for some $q \geq 2$.*

If $t_{i+q}(R) \leq t_i(R) + t_q(R)$ holds for $i \leq h - q$, then there are inequalities

$$(6.2.1) \quad t_i(R) \leq \left\lfloor \frac{i}{q} \right\rfloor + i \quad \text{for } i \geq 1.$$

$$(6.2.2) \quad \text{reg } R \leq \left\lfloor \frac{h}{q} \right\rfloor.$$

Proof. For $1 \leq i \leq q$ the inequality (6.2.1) is just condition N_q . Thus, by induction on i we may assume that the inequality holds for some $i \geq q$. The calculation

$$\begin{aligned} t_{i+1}(R) &\leq t_{i+1-q}(R) + t_q(R) \\ &\leq \left\lfloor \frac{i+1-q}{q} \right\rfloor + i+1-q+q+1 \\ &= \left\lfloor \frac{i+1}{q} \right\rfloor + i+1 \end{aligned}$$

completes the induction step for formula (6.2.1). The latter implies (6.2.2). \square

Next we describe graphical displays that help visualize such inequalities.

6.3. Betti templates. Recall that the *Betti table* of R is the matrix that has $\beta_{i,i+j}(R)$ as the entry in its i th column and j th row. In order to present information on the vanishing of Betti numbers, for each integer $q \geq 1$ we form a *Betti template* by replacing $\beta_{i,i+j}(R)$ with one of the symbols from the following list:

- * non-zero due to condition N_q .
- zero due to condition N_q .
- o zero by (6.1.1).
- zero by (6.1.2) when p is good for i , predicted by (6.2.1) in general.
- ⊗ zero predicted by (6.2.1).
- ▽ no information available.

Thus, columns 0 through i of the template contain the available and the conjectural information on the vanishing of $\beta_{i,i+j}(R)$ when $\text{reg}_{i+1}^R(k) = 0$ and R satisfies N_q .

An additional row at the bottom of the template displays—if it exists—the only good characteristic p for i other than $p = 0$ or $p > i$; see Lemma 5.7.

A natural question is how tightly these templates can be filled in. Precise answers would be interesting both in their own right and for applications to coordinate rings of classical algebraic varieties. We present estimates by drawing on existing

examples, and also record new information obtained from our results. It is worth noting that the algebras in these examples are Koszul and many are Gorenstein.

Recall that the *Hilbert series* $H_R(t) = \sum_{i \geq 0} \text{rank}_k R_i t^i$ is a rational function with numerator in $\mathbb{Z}[t]$ and denominator equal to $(1-t)^{\dim R}$.

6.4. Minimally elliptic singularities. Let $k = \mathbb{C}$ and let R be the associated graded ring of a rationally elliptic surface singularity; see Laufer [24]. Wahl [35, 2.8] proved that R is a Gorenstein ring, $\text{pd}_S(R) = e - 2 \geq 2$, $\beta_{i,j}(R) = 0$ for $j \neq i + 1$ and $1 \leq i \leq e - 3$, $\beta_{e-2,j}(R) = 0$ for $j \neq e$, and $H_R(t) = (1 + et + t^2)/(1 - t)^2$.

When k is infinite Wahl's arguments apply to any Gorenstein k -algebra that has $H_R(t) = (1 + et + t^2)/(1 - t)^{\dim R}$. Furthermore, such an R is Koszul: By factoring out a maximal R -regular sequence of linear forms it suffices to treat the case of dimension zero, where the desired assertion follows from [25, Theorem 3(1)].

In particular, for every $q \geq 2$ there is a Koszul k -algebra R that is a Gorenstein ring, and for which equality holds in (6.2.1) with $i = q + 1$.

Example 6.5. The first 13 columns and 8 rows of the Betti template for N_2 :

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	*	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	*	*	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇
2	-	-	-	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇
3	-	-	-	0	⊙	∇	∇	∇	∇	∇	∇	∇	∇	∇
4	-	-	-	0	0	○	⊙	∇	∇	∇	∇	∇	∇	∇
5	-	-	-	0	0	0	○	⊙	⊙	∇	∇	∇	∇	∇
6	-	-	-	0	0	0	0	○	○	⊙	⊙	∇	∇	∇
7	-	-	-	0	0	0	0	0	○	○	⊙	⊙	⊙	∇
8	-	-	-	0	0	0	0	0	0	○	○	○	⊙	⊙
p						3		2	3	5				7

Let R be a general Gorenstein algebra with Hilbert series $1 + 5z + 5z^2 + z^3$. It is Koszul by [11, 6.3]. When $p \neq 2$, by specializing Schreyer's result [33, p. 107] to the Artinian case one sees that the Betti table of R is given by

	0	1	2	3	4	5	6
0	1	-	-	-	-	-	-
1	-	10	16	-	-	-	-
2	-	-	-	16	10	-	-
3	-	-	-	-	-	1	-
4	-	-	-	-	-	-	-

Thus, when $q = 2$ there is a Koszul k -algebra R that is a Gorenstein ring, and for which equalities hold in (6.2.1) with $i = 3$ and $i = 5$.

Property N_3 appears prominently in recent studies of a basic construction:

6.6. Segre products. Let (e_1, \dots, e_s) be an s -tuple of positive integers with $s \geq 2$.

Let $R^{(e_1, \dots, e_s)}$ be the coordinate ring of the Segre embedding $\mathbb{P}_k^{e_1} \times \dots \times \mathbb{P}_k^{e_s} \rightarrow \mathbb{P}_k^e$, where $e = \prod_{i=1}^s (e_i + 1) - 1$. It is Koszul; see Bărcănescu and Manolache [4, 2.1].

When $k = \mathbb{C}$ Rubei [32, Theorem 10] proved that $R^{(e_1, \dots, e_s)}$ satisfies N_3 , but not N_4 unless $s = 2$ and $e_1 = 1$ or $e_2 = 1$. Snowden developed a functorial method for building its syzygies: "One consequence of this result is that almost any question

about p -syzygies—such as, is the module of 13-syzygies of every Segre embedding generated in degrees at most 20?—can be resolved by a finite computation (in theory);” see [34, p. 226]. Column 13 in the next example gives a positive answer.

Example 6.7. The first 16 columns and 8 rows of the Betti template for N_3 :

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	*	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	-	*	*	*	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇
2	-	-	-	-	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇
3	-	-	-	-	○	⊗	⊗	∇	∇	∇	∇	∇	∇	∇	∇	∇	∇
4	-	-	-	-	0	○	○	○	⊗	⊗	∇	∇	∇	∇	∇	∇	∇
5	-	-	-	-	0	0	○	○	○	⊗	⊗	⊗	⊗	∇	∇	∇	∇
6	-	-	-	-	0	0	0	○	○	○	○	○	○	⊗	⊗	⊗	⊗
7	-	-	-	-	0	0	0	0	○	○	○	○	○	⊗	⊗	⊗	⊗
8	-	-	-	-	0	0	0	0	0	○	○	○	○	○	○	○	⊗
p						3		2	3	5				7	5	2	

The algebra $R^{(1,1,1)}$ has N_3 by 6.6, it is Gorenstein with $H_R = (1+4t+t^2)/(1-t)^4$ by [4, 3.2(7)], and so it is Koszul by 6.4.

Equality can hold in (6.2.2) for any $q = h - 1 \geq 2$ by 6.4 and with $(q, h) = (2, 5)$ by Example 6.5. However, (6.2.2) may be far from optimal when $\text{pd}_S R$ is large. In fact, we know of no family of Koszul algebras satisfying N_2 and with projective dimensions bounded above by some linear function of their regularities. On the other hand, families with quadratic bounds do exist:

6.8. Grassmannians. Let $R^{[n]}$ be the coordinate ring of the Grassmannian of 2-dimensional subspaces of the affine space \mathbb{A}_k^n . It is classically known that the k -algebra $R^{[n]}$ is minimally generated by $\binom{n}{2}$ elements, and Hochster [20, 3.2] proved that it is a Gorenstein ring of Krull dimension $2n - 3$. It is also Koszul, see [21], and when $p = 0$ or $p > \frac{1}{2}(n - 4)$ has property N_2 by Kurano [23, 5.3].

The following hold: $\text{reg}(R^{[n]}) = n - 3$ by [9, 1.4] but $\text{pd}(R^{[n]}) = \binom{n-2}{2}$.

In a different direction, Dao, Huneke, and Schweig [12, 4.8] obtained an upper bound on the regularity of Koszul algebras with monomial relations satisfying N_q for some $q \geq 2$ in terms of the number e of algebra generators: They exhibit an explicit polynomial $f_q(x) \in \mathbb{R}[x]$ and a real number $\alpha_q \in \mathbb{R}$ with $\alpha_q > 1$, such that

$$\text{reg } R \leq \log_{\alpha_q}(f_q(e)).$$

Our last example shows that no bound of similar type can exist for the class of *all* Koszul algebras satisfying N_q , since it contains the following family:

6.9. Veronese subalgebras. Let $R^{q,n}$ be the k -subalgebra generated by all the monomials of degree q in a polynomial ring in qn variables. It is Koszul by [4, 2.1], has N_q by [7, 4.2], and needs $\binom{qn+q-1}{q}$ generators, and its regularity is equal to $(q-1)n$; the last expression is well known: It follows easily from the results of Goto and Watanabe in [16, Ch. 3] and is given explicitly by Nitsche in [30, 4.2].

REFERENCES

- [1] L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math. **166**, Birkhauser, Basel, 1998; 1–118.

- [2] L. L. Avramov, A. Conca, S. B. Iyengar, *Free resolutions over commutative Koszul algebras*, Math. Res. Lett. **17** (2010), 197–210.
- [3] J. Backelin, *Relations between rates of growth of homologies*, Research Reports in Mathematics **25**, Department of Mathematics, Stockholm University, 1988.
- [4] S. Bărcănescu, N. Manolache, *Betti numbers of Segre-Veronese singularities*, Rev. Roumaine Math. Pures Appl. **26** (1981), 549–565.
- [5] D. Bayer, D. Mumford, *What can be computed in algebraic geometry?*, Computational algebraic geometry and commutative algebra (Cortona, 1991), Sympos. Math. XXXIV, Cambridge Univ. Press, Cambridge, 1993; pp. 1–48.
- [6] D. Bayer, M. Stillman, *On the complexity of computing syzygies*, Computational aspects of commutative algebra, J. Symbolic Comput. **6** (1988), 135–147.
- [7] W. Bruns, A. Conca, T. Römer, *Koszul homology and syzygies of Veronese subalgebras*, Math. Ann. **351** (2011), 761–779.
- [8] W. Bruns, A. Conca, T. Römer, *Koszul cycles*, Combinatorial Aspects of Commutative Algebra and Algebraic Geometry, Abel Symposia **6**, Springer, Berlin, 2011; pp. 17–33.
- [9] W. Bruns, J. Herzog, *On the computation of a -invariants*, Manuscripta Math. **77** (1992), 201–213.
- [10] G. Caviglia, E. Sbarra, *Characteristic-free bounds for the Castelnuovo-Mumford regularity*, Compositio Math. **141** (2005), 1365–1373.
- [11] A. Conca, M. E. Rossi, G. Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. **129** (2001), 95–121.
- [12] H. Dao, C. Huneke, J. Schweig, *Bounds on the regularity and projective dimension of ideals associated to graphs*, J. Algebraic Combin. **38** (2013), 37–55.
- [13] O. Fernández-Ramos, P. Gimenez, *Regularity 3 in edge ideals associated to bipartite graphs*, arXiv:1207.5553.
- [14] D. Eisenbud, C. Huneke, B. Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. **128** (2006), 573–605.
- [15] R. Fröberg, *Determination of a class of Poincaré series*, Math. Scand. **37** (1975), 29–39.
- [16] S. Goto, K. Watanabe, *On graded rings, I*, J. Math. Soc. Japan **30** (1978), 179–213.
- [17] D. Grayson, M. Stillman, MACAULAY2, A software system for research in algebraic geometry, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [18] M. L. Green, R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), 73–90.
- [19] J. Herzog, H. Srinivasan, *A note on the subadditivity problem for maximal shifts in free resolutions*, Preprint 2013, arXiv:1303.6214.
- [20] M. Hochster, *Grassmannians and their Schubert varieties are Arithmetically Cohen-Macaulay*, J. Algebra **25** (1973), 40–57.
- [21] G. Kempf, *Some wonderful rings in algebraic geometry*, J. Algebra **134** (1990), 222–224.
- [22] J. Koh, *Ideals generated by quadrics exhibiting double exponential degrees*, J. Algebra **200** (1998), 225–245.
- [23] K. Kurano, *Relations on Pfaffians. I. Plethysm formulas*, J. Math. Kyoto Univ. **31** (1991), 713–731.
- [24] H. B. Laufer, *On minimally elliptic singularities*, Amer. J. Math. **99** (1977), 1257–1295.
- [25] G. L. Levin, L. L. Avramov, *Factoring out the socle of a local Gorenstein ring*, J. Algebra **55** (1978), 74–83.
- [26] E. W. Mayr, A. R. Meyer, *The complexity of the word problems for commutative semigroups and polynomial ideals*, Adv. in Math. **46** (1982), 305–329.
- [27] J. McCullough, *A polynomial bound on the regularity of an ideal in terms of half of the syzygies*, Math. Res. Letters **19** (2012), 555–565.
- [28] B. McKay, NAUTY, User’s Guide (Version 2.4), available at <http://cs.anu.edu.au/~bdm/nauty>
- [29] J. Migliore, R.-M. Miró-Roig, *On the minimal free resolution of $n + 1$ general forms*, Trans. Amer. Math. Soc. **355** (2002), 1–36.
- [30] M. J. Nitsche, *Castelnuovo-Mumford regularity of seminormal simplicial affine semigroup rings*, J. Algebra **368** (2012), 345–357.
- [31] J.-E. Roos, *Commutative non-Koszul algebras having a linear resolution of arbitrarily high order. Applications to torsion in loop space homology*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 1123–1128.

- [32] E. Rubei, *Resolutions of Segre embeddings of projective spaces of any dimension*, J. Pure Appl. Algebra **208** (2007), 29–37.
- [33] F.-O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. **275** (1986), 105–137.
- [34] A. Snowden, *Syzygies of Segre embeddings and Δ -modules*, Duke Math. J. **162** (2013), 225–277.
- [35] J. M. Wahl, *Equations defining rational singularities*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 231–263.

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