

BASS NUMBERS OVER LOCAL RINGS VIA STABLE COHOMOLOGY

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To Jürgen Herzog on his 70th birthday.

ABSTRACT. For any non-zero finite module M of finite projective dimension over a noetherian local ring R with maximal ideal \mathfrak{m} and residue field k , it is proved that the natural map $\mathrm{Ext}_R(k, M) \rightarrow \mathrm{Ext}_R(k, M/\mathfrak{m}M)$ is non-zero when R is regular and is zero otherwise. A noteworthy aspect of the proof is the use of stable cohomology. Applications include computations of Bass series over certain local rings.

INTRODUCTION

Let (R, \mathfrak{m}, k) denote a commutative noetherian local ring with maximal ideal \mathfrak{m} and residue field k ; when R is not regular we say that it is *singular*.

This article revolves around the following result:

Theorem. *If (R, \mathfrak{m}, k) is a singular local ring and M an R -module of finite projective dimension, then $\mathrm{Ext}_R(k, \pi^M) = 0$ for the canonical map $\pi^M: M \rightarrow M/\mathfrak{m}M$.*

Special cases, known for a long time, are surveyed at the end of Section 2. Even in those cases our proof is new. It utilizes a result of Martsinkovsky [11] through properties of Vogel's stable cohomology functors [6, 3] recalled in Section 1. It also suggests extensions to DG modules over certain commutative DG algebras; see [2]. Applications of the theorem include new criteria for regularity of local rings (in Section 2) and explicit computations of Bass numbers of modules (in Section 3).

1. STABLE COHOMOLOGY

In this section we recall the construction of stable cohomology and basic results required in the sequel. The approach we adopt is based on a construction by Vogel, and described in Goichot [6]; see also [3].

Let R be an associative ring and let R^c denote its center. Given left R -modules L and M , choose projective resolutions P and Q of L and M , respectively. Recall that a homomorphism $P \rightarrow Q$ of degree n is a family $\beta = (\beta_i)_{i \in \mathbb{Z}}$ of R -linear maps $\beta_i: P_i \rightarrow Q_{i+n}$; that is, an element of the R^c -module

$$\mathrm{Hom}_R(P, Q)_n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(P_i, Q_{i+n}).$$

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This module is the n -th component of a complex $\text{Hom}_R(P, Q)$, with differential

$$\partial_n(\beta) = \partial_{i+n}^Q \beta_i - (-1)^n \beta_{i-1} \partial_n^P.$$

The maps $\beta: P \rightarrow Q$ with $\beta_i = 0$ for $i \gg 0$ form a subcomplex with component

$$\overline{\text{Hom}}_R(P, Q)_n = \coprod_{i \in \mathbb{Z}} \text{Hom}_R(P_i, Q_{i+n}) \quad \text{for } n \in \mathbb{Z}.$$

We write $\widehat{\text{Hom}}_R(P, Q)$ for the quotient complex. It is independent of the choices of P and Q up to R -linear homotopy, and so is the exact sequence of complexes

$$(1.0.1) \quad 0 \longrightarrow \overline{\text{Hom}}_R(P, Q) \longrightarrow \text{Hom}_R(P, Q) \xrightarrow{\theta} \widehat{\text{Hom}}_R(P, Q) \longrightarrow 0.$$

The *stable cohomology* of the pair (L, M) is the graded R^c -module $\widehat{\text{Ext}}_R(L, M)$ with

$$\widehat{\text{Ext}}_R^n(L, M) = H^n(\widehat{\text{Hom}}_R(P, Q)) \quad \text{for each } n \in \mathbb{Z}.$$

It comes equipped with functorial homomorphisms of graded R^c -modules

$$(1.0.2) \quad \text{Ext}_R^n(L, M) \xrightarrow{\eta^n(L, M)} \widehat{\text{Ext}}_R^n(L, M) \quad \text{for all } n \in \mathbb{Z}.$$

1.1. If $\text{pd}_R L$ or $\text{pd}_R M$ is finite, then $\widehat{\text{Ext}}_R^n(L, M) = 0$ for all $n \in \mathbb{Z}$.

Indeed, in this case we may choose P or Q to be a bounded complex. The definitions then yield $\overline{\text{Hom}}_R(P, Q) = \text{Hom}_R(P, Q)$, and hence $\widehat{\text{Hom}}_R(P, Q) = 0$.

1.2. For a family $\{M_j\}_{j \in J}$ of R -modules and every integer n the canonical inclusions $M_j \rightarrow \coprod_{j \in J} M_j$ induce, by functoriality, a commutative diagram of R^c -modules

$$(1.2.1) \quad \begin{array}{ccc} \text{Ext}_R^n(L, \coprod_{j \in J} M_j) & \xrightarrow{\eta^n(L, \coprod_{j \in J} M_j)} & \widehat{\text{Ext}}_R^n(L, \coprod_{j \in J} M_j) \\ \uparrow & & \uparrow \\ \coprod_{j \in J} \text{Ext}_R^n(L, M_j) & \xrightarrow{\coprod_{j \in J} \eta^n(L, M_j)} & \coprod_{j \in J} \widehat{\text{Ext}}_R^n(L, M_j) \end{array}$$

Proposition 1.3. *Suppose L admits a resolution by finite projective R -modules.*

For every integer n the vertical maps in (1.2.1) are bijective. In particular, the map $\eta^n(L, \coprod_{j \in J} M_j)$ is injective or surjective for some n if and only if $\eta^n(L, M_j)$ has the corresponding property for every $j \in J$.

Proof. Let P be a resolution of L by finite projective R -modules and Q_j a projective resolution of M_j . The complex $\coprod_{j \in J} Q_j$ is a projective resolution of $\coprod_{j \in J} M_j$, and we have a commutative diagram of morphisms of complexes of R^c -modules

$$\begin{array}{ccccccc} 0 \rightarrow \overline{\text{Hom}}_R(P, \coprod_{j \in J} Q_j) & \rightarrow \text{Hom}_R(P, \coprod_{j \in J} Q_j) & \rightarrow \widehat{\text{Hom}}_R(P, \coprod_{j \in J} Q_j) & \rightarrow 0 \\ \uparrow \pi & \uparrow \pi & \uparrow \widehat{\pi} & & & & \\ 0 \rightarrow \coprod_{j \in J} \overline{\text{Hom}}_R(P, Q_j) & \rightarrow \coprod_{j \in J} \text{Hom}_R(P, Q_j) & \rightarrow \coprod_{j \in J} \widehat{\text{Hom}}_R(P, Q_j) & \rightarrow 0 & & & \end{array}$$

with natural vertical maps. The map $H^n(\pi)$ is bijective, as it represents

$$\coprod_{j \in J} \text{Ext}_R^n(L, M_j) \rightarrow \text{Ext}_R^n(L, \coprod_{j \in J} M_j),$$

which is bijective due to the hypothesis on L . As $\widehat{\pi}$ is evidently bijective, $H^n(\widehat{\pi})$ is an isomorphism. The right-hand square of the diagram above induces (1.0.2). \square

2. LOCAL RINGS

The next theorem is the main result of the paper. It concerns the maps

$$\mathrm{Ext}_R^n(k, \beta): \mathrm{Ext}_R^n(k, M) \rightarrow \mathrm{Ext}_R^n(k, V)$$

induced by some homomorphism $\beta: M \rightarrow V$, and is derived from a result of Martsinkovsky [11] by using properties of stable cohomology, recalled above.

Theorem 2.1. *Let (R, \mathfrak{m}, k) be a local ring and V an R -module such that $\mathfrak{m}V = 0$.*

If R is singular and $\beta: M \rightarrow V$ is an R -linear map that factors through some module of finite projective dimension, then

$$\mathrm{Ext}_R^n(k, \beta) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Proof. By hypothesis, there exists an R -module N of finite projective dimension such that β factors as $M \xrightarrow{\gamma} N \xrightarrow{\delta} V$. The following diagram

$$\begin{array}{ccccc} \mathrm{Ext}_R^n(k, M) & \xrightarrow{\mathrm{Ext}_R^n(k, \beta)} & \mathrm{Ext}_R^n(k, V) & \xrightarrow{\eta^n(k, V)} & \widehat{\mathrm{Ext}}_R^n(k, V) \\ & \searrow \mathrm{Ext}_R^n(k, \gamma) & \uparrow \mathrm{Ext}_R^n(k, \delta) & & \uparrow \widehat{\mathrm{Ext}}_R^n(k, \delta) \\ & & \mathrm{Ext}_R^n(k, N) & \xrightarrow{\eta^n(k, N)} & \widehat{\mathrm{Ext}}_R^n(k, N) = 0 \end{array}$$

commutes due to the naturality of the maps involved; the equality comes from 1.1.

The map $\eta^n(k, k)$ is injective by [11, Theorem 6]. Proposition 1.3 shows that $\eta^n(k, V)$ is injective as well, so the diagram yields $\mathrm{Ext}_R^n(k, \beta) = 0$. \square

Note that no finiteness condition on M is imposed in the theorem. This remark is used in the proof of the following corollary, which deals with the maps

$$\mathrm{Tor}_n^R(k, \alpha): \mathrm{Tor}_n^R(k, V) \rightarrow \mathrm{Tor}_n^R(k, M)$$

induced by some homomorphism $\alpha: V \rightarrow M$.

Corollary 2.2. *If R is singular and $\alpha: V \rightarrow M$ is an R -linear map that factors through some module of finite injective dimension, then*

$$\mathrm{Tor}_n^R(k, \alpha) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Proof. Set $(-)^{\vee} = \mathrm{Hom}_R(-, E)$, where E is an injective envelope of the R -module k . Let $V \rightarrow L \rightarrow M$ be a factorization of α with L of finite injective dimension. By Ishikawa [7, 1.5] the module L^{\vee} has finite flat dimension, so it has finite projective dimension by Jensen [9, 5.8]. As $\mathfrak{m}(V^{\vee}) = 0$ and α^{\vee} factors through L^{\vee} , Theorem 2.1 gives $\mathrm{Ext}_R^n(k, \alpha^{\vee}) = 0$. The natural isomorphism $\mathrm{Ext}_R^n(k, -^{\vee}) \cong \mathrm{Tor}_n^R(k, -)^{\vee}$ now yields $\mathrm{Tor}_n^R(k, \alpha)^{\vee} = 0$, whence we get $\mathrm{Tor}_n^R(k, \alpha) = 0$, as desired. \square

Next we record an elementary observation, where $(-)^* = \mathrm{Hom}_R(-, R)$.

Lemma 2.3. *Let (R, \mathfrak{m}, k) be a local ring and $\chi: X \rightarrow Y$ an R -linear map.*

If $\mathrm{Coker}(\chi)$ has a non-zero free direct summand, then $\mathrm{Ker}(\chi^) \not\subseteq \mathfrak{m}Y^*$ holds.*

When Y is free of finite rank the converse assertion holds as well.

Proof. The condition on $\mathrm{Coker}(\chi)$ holds if and only if there is an epimorphism $\mathrm{Coker}(\chi) \rightarrow R$; that is, an R -linear map $v: Y \rightarrow R$ with $v\chi = 0$ and $v(Y) \not\subseteq \mathfrak{m}$.

When such a v exists it is in $\mathrm{Ker}(\chi^*)$, but not in $\mathfrak{m}Y^*$, for otherwise $v(Y) \subseteq \mathfrak{m}$.

When Y is finite free and $\mathrm{Ker}(\chi^*) \not\subseteq \mathfrak{m}Y^*$ holds, pick v in $\mathrm{Ker}(\chi^*) \setminus \mathfrak{m}Y^*$. Since Y^* is finite free, v can be extended to a basis of Y^* , hence $v(Y) = R$. \square

The theorem in the introduction is the crucial implication in the next result:

Theorem 2.4. *Let (R, \mathfrak{m}, k) be a local ring. For each R -module M , let*

$$\varepsilon_M^n = \text{Ext}_R^n(k, \pi^M) : \text{Ext}_R^n(k, M) \rightarrow \text{Ext}_R^n(k, M/\mathfrak{m}M)$$

be the map of R -modules induced by the natural map $\pi^M : M \rightarrow M/\mathfrak{m}M$.

The following conditions are equivalent.

- (i) R is regular.
- (ii) $\varepsilon_R^n \neq 0$ for some integer n .
- (iii) $\varepsilon_M^n \neq 0$ for some R -module M with $\text{pd}_R M < \infty$ and some integer n .
- (iv) $\varepsilon_M^d \neq 0$ for every finite R -module $M \neq 0$ and for $d = \dim R$.
- (v) $\text{Coker}(\partial_n^F)$, where F is a minimal free resolution of k over R , has a non-zero free direct summand for some integer n .

Proof. Set $G = \text{Hom}_R(F, R)$ with F as in (v). From $\text{Hom}_R(F, M) \cong G \otimes_R M$ and $\partial(G \otimes M) \subseteq \mathfrak{m}(G \otimes M)$ (by the minimality of F) we get a commutative diagram

$$\begin{array}{ccc} \text{Ext}_R^n(k, M) & \xrightarrow{\varepsilon_M^n} & \text{Ext}_R^n(k, M/\mathfrak{m}M) \\ \cong \uparrow & & \uparrow \cong \\ \text{H}_{-n}(G \otimes_R M) & \xrightarrow{\text{H}_{-n}(G \otimes_R \pi^M)} & \text{H}_{-n}(G \otimes_R (M/\mathfrak{m}M)) = G_{-n} \otimes (M/\mathfrak{m}M) \end{array}$$

(i) \implies (iv). As R is regular, F is the Koszul complex on a minimal generating set of \mathfrak{m} . This gives $G_d = R$, an isomorphism $\text{H}_{-d}(G \otimes_R \pi^M)$, and an inequality $M/\mathfrak{m}M \neq 0$ by Nakayama's Lemma; now the diagram yields $\varepsilon_M^d \neq 0$.

(iv) \implies (ii) \implies (iii). These implications are tautologies.

(iii) \implies (i). This implication is a special case of Theorem 2.1.

(ii) \iff (v). The preceding diagram shows that the condition $\varepsilon_R^n \neq 0$ is equivalent to $\text{Ker}(\partial_{-n}^G) \not\subseteq \mathfrak{m}G_{-n}$. Thus, the desired assertion follows from Lemma 2.3. \square

Notes 2.5. The equivalence of conditions (i) and (ii) in Theorem 2.4 was proved by Ivanov [8, Theorem 2] when R is Gorenstein and by Lescot [10, 1.4] in general.

The equivalence of (i) and (v) is due to Dutta [4, 1.3]. As shown above, it follows from Lescot's theorem via the elementary Lemma 2.3. Martsinkovsky deduced Dutta's theorem from [11, Theorem 6], and used the latter to prove regularity criteria different from (ii), (iii), and (iv) in Theorem 2.4; see [11, p. 11].

3. BASS NUMBERS OF MODULES

The n th Bass number of a module M over a local ring (R, \mathfrak{m}, k) is the integer

$$\mu_R^n(M) = \text{rank}_k \text{Ext}_R^n(k, M).$$

In what follows, given a homomorphism $\beta : M \rightarrow N$ and an R -submodule $N' \subseteq N$ we let $M \cap N'$ denote the submodule $\beta^{-1}(N')$ of M .

Theorem 3.1. *Let (R, \mathfrak{m}, k) be a local ring, $M \rightarrow N$ an R -linear map, and set*

$$r = \text{rank}_k(M/M \cap \mathfrak{m}N).$$

If R is singular and $\text{pd}_R N$ is finite, then for each $n \in \mathbb{Z}$ there is an equality

$$\mu_R^n(M \cap \mathfrak{m}N) = \mu_R^n(M) + r\mu_R^{n-1}(k).$$

Proof. Set $\overline{M} = M/(M \cap \mathfrak{m}N)$ and $\overline{N} = N/\mathfrak{m}N$, and let $\pi: M \rightarrow \overline{M}$ and $\iota: \overline{M} \rightarrow \overline{N}$ be the induced maps. They appear in a commutative diagram with exact row

$$\begin{array}{ccc} M & \longrightarrow & N \\ \pi \downarrow & & \downarrow \\ 0 & \longrightarrow & \overline{M} \xrightarrow{\iota} \overline{N} \end{array}$$

Since ι is k -linear, it is split, so we get a commutative diagram with exact row

$$\begin{array}{ccc} \mathrm{Ext}_R(k, M) & \longrightarrow & \mathrm{Ext}_R(k, N) \\ \mathrm{Ext}_R(k, \pi) \downarrow & & \downarrow 0 \\ 0 & \longrightarrow & \mathrm{Ext}_R(k, \overline{M}) \xrightarrow{\mathrm{Ext}_R(k, \iota)} \mathrm{Ext}_R(k, \overline{N}) \end{array}$$

and zero map due to Theorem 2.1. It implies $\mathrm{Ext}_R(k, \pi) = 0$.

By definition, there exists an exact sequence of R -modules

$$0 \longrightarrow (M \cap \mathfrak{m}N) \longrightarrow M \xrightarrow{\pi} \overline{M} \longrightarrow 0$$

As $\mathrm{Ext}_R(k, \pi) = 0$, its cohomology sequence yields an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^{n-1}(k, \overline{M}) \longrightarrow \mathrm{Ext}_R^n(k, M \cap \mathfrak{m}N) \longrightarrow \mathrm{Ext}_R^n(k, M) \longrightarrow 0$$

of k -vector spaces for each integer n . Computing ranks over k and using the isomorphism $\mathrm{Ext}_R(k, \overline{M}) \cong \mathrm{Ext}_R(k, k) \otimes_k \overline{M}$, we obtain the desired equality. \square

Recall that the n th *Betti number* of M is the integer $\beta_n^R(M) = \mathrm{rank}_k \mathrm{Ext}_R^n(M, k)$.

Corollary 3.2. *Assume that R is singular and $N \supseteq M \supseteq \mathfrak{m}N$ holds, and set*

$$s = \mathrm{rank}_k(N/M).$$

If N is finite and $\mathrm{pd}_R N = p < \infty$ holds, then for each $n \in \mathbb{Z}$ there is an equality

$$(3.2.1) \quad \mu_R^n(M) = \sum_{i=0}^p \mu_R^{n+i}(R) \beta_i^R(N) + s \beta_{n-1}^R(k).$$

In particular, $\mu_R^n(\mathfrak{m}) = \mu_R^n(R) + \beta_{n-1}^R(k)$ for each $n \geq 0$.

Proof. The hypotheses give $M \cap \mathfrak{m}N = \mathfrak{m}N$ and $r = \mathrm{rank}_k(M/\mathfrak{m}N)$. By applying Theorem 3.1 to the submodules $M \subseteq N$ and $\mathfrak{m}N \subseteq N$ we obtain

$$\begin{aligned} \mu_R^n(M) &= \mu_R^n(\mathfrak{m}N) - r \beta_{n-1}^R(k) \\ &= \mu_R^n(N) + \mathrm{rank}_k(N/\mathfrak{m}N) \beta_{n-1}^R(k) - r \beta_{n-1}^R(k) \\ &= \mu_R^n(N) + s \beta_{n-1}^R(k). \end{aligned}$$

As $\mathrm{pd}_R N$ is finite, Foxby [5, 4.3(2)] yields $\mu_R^n(N) = \sum_{i=0}^p \mu_R^{n+i}(R) \beta_i^R(N)$. \square

Remark 3.3. The hypothesis $p < \infty$ in the corollary is needed, as otherwise the sum in (3.2.1) is not defined. On the other hand, when R is regular—and so p is necessarily finite—formula (3.2.1) may fail. For instance, with $d = \dim R$ one has

$$\mu_R^n(\mathfrak{m}) = \begin{cases} \binom{d}{n-1} & \text{for } n = 1, \dots, d, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$ induces a cohomology long exact sequence where $\mu_R^i(R) = 0$ for $n \neq d$, the map ε_R^d is bijective by Theorem 2.4, and $\mu_R^i(k) = \binom{d}{i}$.

3.4. Bass numbers are often described in terms of the generating formal power series $I_R^M(t) = \sum_{n \in \mathbb{Z}} \mu_R^n(M) t^n$. We also use the series $P_M^R(t) = \sum_{n \in \mathbb{Z}} \beta_n^R(M) t^n$.

In these terms, the formulas (3.2.1) for all $n \in \mathbb{Z}$ can be restated as an equality

$$(3.4.1) \quad I_R^M(t) = I_R^R(t) P_N^R(t^{-1}) + st P_k^R(t).$$

3.5. Let (S, \mathfrak{m}_S, k) and (T, \mathfrak{m}_T, k) be local rings and let $\varepsilon_S: S \rightarrow k \leftarrow T : \varepsilon_T$ denote the canonical maps. The *fiber product* of S and T over k is defined by the formula

$$S \times_k T := \{(s, t) \in S \times T \mid \varepsilon_S(s) = \varepsilon_T(t)\}.$$

It is well known, and easy to see, that this is a subring of $S \times T$, which is local with maximal ideal $\mathfrak{m} = \mathfrak{m}_S \oplus \mathfrak{m}_T$ and residue field k . Set $R = S \times_k T$.

Let N and P be finite modules over S and T , respectively. The canonical maps $S \leftarrow R \rightarrow T$ turn N and P into R -modules, and for them Lescot [10, 2.4] proved

$$(3.5.1) \quad \frac{I_R^N(t)}{P_k^R(t)} = \frac{I_S^N(t)}{P_k^S(t)} \quad \text{and} \quad \frac{I_R^P(t)}{P_k^R(t)} = \frac{I_T^P(t)}{P_k^T(t)}.$$

When $N/\mathfrak{n}_S N = V = P/\mathfrak{m}_T P$ holds for some k -module V the fiber product

$$N \times_V P := \{(n, p) \in N \times P \mid \pi^N(n) = \pi^P(p)\}$$

has a natural structure of finite R -module.

Corollary 3.6. *With notation as in 3.5, set $v = \text{rank}_k V$ and $M = N \times_V P$.*

If S and T are singular and $\text{pd}_S N$ and $\text{pd}_T P$ are finite, then

$$\frac{I_R^{\mathfrak{m}M}(t)}{P_k^R(t)} = \frac{I_S^S(t) P_N^S(t^{-1})}{P_k^S(t)} + \frac{I_T^T(t) P_P^T(t^{-1})}{P_k^T(t)} + 2vt.$$

Proof. We have $\mathfrak{m}M \cong \mathfrak{m}_S N \oplus \mathfrak{m}_T P$ as R -modules, whence the first equality below:

$$\begin{aligned} \frac{I_R^{\mathfrak{m}M}(t)}{P_k^R(t)} &= \frac{I_R^{\mathfrak{m}_S N}(t)}{P_k^R(t)} + \frac{I_R^{\mathfrak{m}_T P}(t)}{P_k^R(t)} \\ &= \frac{I_S^S(t) P_N^S(t^{-1})}{P_k^S(t)} + vt + \frac{I_T^T(t) P_P^T(t^{-1})}{P_k^T(t)} + vt. \end{aligned}$$

The second one comes by applying formulas (3.5.1) and (3.4.1), in this order. \square

Notes 3.7. For all finite R -modules $N \supseteq M \supseteq \mathfrak{m}N \neq 0$, it is proved in [1, Theorem 4] that the Bass numbers of M and $\mathfrak{m}N$ *asymptotically* have the same size, measured on appropriate polynomial or exponential scales. The *closed formula* in Corollary 3.2 is a much more precise statement, but as noted in Remark 3.3 that formula may not hold when $\text{pd}_R N$ is infinite or when R is regular.

The last conclusion in Corollary 3.2 is Lescot's result [10, 1.8(2)]. Combining it with the expression for $I_R^R(t)$ obtained from Corollary 3.6 by setting $N = S$, $P = T$, and $V = k$, one recovers [10, 3.2(1)]. The proof of Corollary 3.6 faithfully transposes Lescot's derivation of [10, 3.2(1)] from [10, 1.8(2)].

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