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Renormalization for a Scalar Field in an External Scalar Potential

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Received: 31 January 2018; Accepted: 22 February 2018; Published: 27 February 2018

Abstract: The Pauli–Villars regularization procedure confirms and sharpens the conclusions reached previously by covariant point splitting. The divergences in the stress tensor of a quantized scalar field interacting with a static scalar potential are isolated into a three-parameter local, covariant functional of the background potential. These divergences can be naturally absorbed into coupling constants of the potential, regarded as a dynamical object in its own right; here, this is demonstrated in detail for two different models of the field-potential coupling. There is a residual dependence on the logarithm of the potential, reminiscent of the renormalization group in fully-interacting quantum field theories; these terms are finite, but numerically dependent on an arbitrary mass or length parameter, which is purely a matter of convention. This work is one step in a program to elucidate boundary divergences by replacing a sharp boundary by a steeply-rising smooth potential.

Keywords: vacuum energy; renormalization; Pauli–Villars; point splitting; boundary; potential; scalar field

1. Introduction

1.1. Regularization and Renormalization in External Fields

Quantum field theories are notoriously afflicted by infinities, also known as divergences or as cutoff dependences. Such mathematical phenomena are understood to arise because of deficiencies, at high frequencies or short length scales, of the models being studied; and the task becomes one of finding well-defined and physically-plausible ways of “sweeping the [offending terms] under the rug” [1] (p. 137). Fortunately, it is often found that the bad terms have the same functional form as other terms that are expected to appear in the equations of motion of the full system, and hence, they can be absorbed by renormalization of coupling constants. Furthermore, any residual ambiguity after a careful surgical removal of the divergences is reflected solely in the numerical values of those coupling constants, which are arbitrary parameters of the theory and should be fixed by experiment.

The most widely-known sort of renormalization is that which takes place in perturbative (usually in momentum space) treatment of interacting fields (hence, nonlinear equations of motion) in flat space-time. The present paper belongs to a different tradition, that where the quantized field satisfies a linear equation of motion against a nontrivial background, which is treated classically. This “external condition” may be a reflecting boundary (as in the Casimir effect), a space-time geometry (as in cosmology), a strong applied electromagnetic field (as in powerful lasers, or in the Klein–Schwinger–Sauter effect) or an equilibrium expectation value of the field itself (as in spontaneous symmetry breaking). The techniques, and some of the conceptual issues, involved here are so different

from those of standard quantum electrodynamics, etc., that the literatures of the two subfields are largely without contact. In particular, most of the observable conclusions are tied to the original configuration space, not a Fourier space.

The identification and removal of divergences usually takes place in two steps, regularization and renormalization. The first of these terms refers to modification of the theory, or of formal expressions arising within the theory, so that they are finite, but somewhat arbitrary and not physically correct. Intuitively, one would expect to recover the true physical situation by taking a limit of some parameter in the regularized quantity, but in fact, that limit will diverge. Renormalization addresses the issue of what to do with the offending terms, after which a finite and physically trustworthy calculated result will remain. Regularization methods fall into several categories as follows.

1.1.1. Analytic Methods

Dimensional regularization and the method of zeta functions are the standard analytic methods. They involve analytic continuation of the regularized quantity back to the physical value of the regularization parameter, where the original expression (usually an integral) was ill-defined. It turns out that often, there is no pole or other singularity at that point, so a finite result is obtained without the necessity of any visible surgery. This is regarded by some people as a great advantage of the method and by others (e.g., [2,3]) as evidence that it begs the question (i.e., reaches a conclusion by inserting it as a hidden, unjustified assumption). In particular, the divergences in total energy that appear at reflecting boundaries arise as integrals of local densities that are pointwise finite after all the usual renormalizations of local field theory have been performed; these integrals presumably have physical meaning, representing energies that would be finite (at least in potential-free regions), but large, if the sharp boundary were replaced by a sufficiently more accurate physical model.

1.1.2. Cutoff Methods

These are less elegant than the analytic methods, but are regarded as more trustworthy physically, because the divergence reemerges in the limit and must be discarded by a conscious act. The divergent terms typically scale as a negative power of the regularization parameter, but in those cases where the analytic methods encounter a pole, the cutoff methods yield a logarithm, which unavoidably introduces a new length scale into the problem. The huge disadvantage of these methods is that they disrupt the Lorentz symmetry of the underlying theory.

The classic method of this type is the ultraviolet cutoff, which forces divergent integrals (or sums) over eigenvalues to converge by damping out the high-frequency contributions. Lorentz covariance is, in a sense, restored by point-splitting [4], wherein the two field operators in a product (a typical term in an energy or charge observable) are evaluated at different (but nearby) space-time points, and the result is covariantly expanded in a power series in the separation. Unfortunately, the singular behavior near the limit depends on the direction of separation, so skeptics may describe point-splitting as a covariant ensemble of non-covariant methods, rather than a covariant method. This situation is sometimes rationalized by the argument that a non-covariant regularization method inevitably gives rise to non-covariant counterterms, which must be chosen to exactly cancel the direction-dependent terms so that the renormalized expression is covariant in form. From that point of view it is covariant terms that are the greater embarrassment, as they are subject to a renormalization ambiguity. In the context of external gravitational fields, in fact, ad hoc adjustments of the leading covariant terms of the expectation value of the stress-energy-momentum tensor, $\langle T_{\mu\nu} \rangle$, were found to be necessary to assure the physically necessary conservation law, $\nabla_\mu T^{\mu\nu} = 0$ [5,6]. We have recently found the same thing for an external scalar potential [7].

In the gravitational case, abstract arguments [8,9] demonstrate that a conserved renormalized stress tensor consistent with various expected formal properties is unique, up to the covariant renormalization terms, and is reproduced by both point splitting and analytic regularizations; in this sense, qualms over any apparent arbitrariness in the point-splitting procedure are unwarranted. On the

other hand, it is pointed out in [10] that various cutoff procedures leading to discrepant results are still in use; therefore, there is value in looking at yet another renormalization approach that again confirms the “axiomatic” consensus.

1.1.3. The Pauli–Villars Method

Pauli–Villars regularization and renormalization have recently been reviewed in [11] (Appendix A and Section 9), to which we refer for references and a detailed explanation. The first main thrust of the present paper is to develop the method for a theory with an external scalar potential, noting its essential consistency with the point splitting conducted in [7]. Then, we use the resulting insight to discuss the renormalization step in a more thorough, cogent manner than was possible in that earlier paper.

In the terminology of [11], we are conducting a formalistic Pauli–Villars regularization by introducing a number of auxiliary masses that will be taken to infinity at the end. That is, we replace the formal expression, in terms of a Green function, for an observable $\langle A \rangle$ (with physical mass $m_1 = 0$) by a sum:

$$\sum_{j=1}^J f_j \langle A(m_j) \rangle \quad (f_1 = 1) \quad (1)$$

and require:

$$\sum_{j=1}^J f_j = 0, \quad \sum_{j=1}^J f_j m_j^2 = 0, \quad \sum_{j=1}^J f_j m_j^4 = 0. \quad (2)$$

Because we do not regard the new terms as corresponding to true physical fields, we, following Anselmi [12], do not require $|f_j| = 1$ for $j \neq 1$ and thus can stop the sum at $J = 4$. The result is cancellation of the quartic, quadratic and logarithmic singularities that would otherwise arise in a field theory in a space-time dimension of four. However, finite renormalization constants (undetermined by the theory) will remain.

More precisely, set $m_4^2 = \alpha m_2^2$ and $m_3^2 = \beta m_2^2$. Then, one finds:

$$f_2 = \frac{-\alpha\beta}{(\alpha-1)(\beta-1)}, \quad f_3 = \frac{\alpha}{(\alpha-\beta)(\beta-1)}, \quad f_4 = \frac{-\beta}{(\alpha-1)(\alpha-\beta)}. \quad (3)$$

The only restrictions are that α and β be positive and distinct from each other and from one. One can now take m_2 to infinity, and m_3 and m_4 will follow it.

The Pauli–Villars method is like the analytic methods in that it preserves local Lorentz covariance (good) and removes the divergences by a trick without clear physical justification (bad). It is like the cutoff methods in that it does not just discard divergent terms, but replaces them with finite terms that can be honestly identified with modifications of the coupling constants in the equations of motion and stress tensors of the background fields in the model. In principle, the method does not require any cutoff, but in practice, it is convenient to start from the short-distance expansion of the Green function that is provided by the point-splitting method. The Pauli–Villars gambit then neatly cancels all the problematical direction-dependent terms in the point-split calculation, while preserving the forms of the covariant terms that allow renormalization.

1.2. Power-Law Potentials

The present work is motivated by the study of boundary effects, specifically the Casimir effect [13–16] or, more precisely, its scalar analog. The divergences in that theory arise because the idealization of a perfectly reflecting boundary is unrealistic at high frequencies; it is natural to search for a model that sufficiently “softens” the boundary, but remains mathematically tractable (see [17] (Appendix and Section 1) for detailed historical remarks and references). This paper is one step in a program [7,17–19] to replace a Dirichlet boundary at $z = 1$ by a scalar potential:

$$V(x, y, z) = \begin{cases} 0 & \text{if } z \leq 0, \\ z^\alpha & \text{if } z > 0, \end{cases} \quad (4)$$

where α is positive (as explained in [18], dimensional parameters have been suppressed to simplify the notation). If α is large, the behavior of a “perfect conductor” at $z = 1$ is expected to be acceptably modeled. A related program was launched independently by Mazzitelli et al. [20,21]. More recently, the analogous problem of an electromagnetic field in a z -dependent dielectric constant has gained attention [22–24].

The divergences associated with a sharp boundary are hereby replaced by new divergences associated with the interaction between the quantized field and the potential. Those divergences, however, are of a familiar type, most similar to those in quantum field theories with a background gravitational field (metric tensor) [4–6]. They were investigated by dimensional regularization in [20] and by point-splitting in [7]. In the latter, some aspects of renormalization were postponed to later work; here, we take them up with the aid of the Pauli–Villars approach. The theory with a background scalar potential is, of course, simpler than that with a background metric. It does present one novel feature, which makes the interpretation of the renormalization process slightly subtle: a potential V that is a constant function of the coordinates is indistinguishable from the square of a Klein–Gordon mass for the to-be-quantized scalar field. In the case of Equation (4), the convention that $V = 0$ on the negative z axis removes any ambiguity in the definition of the field mass. For gravitational and other external potentials, the mass is a separate parameter, not entangled with the potential in any way.

Our problem can be approached on four levels of increasing generality. To complete the study of the concrete problem posed in [7,17], we need only to deal with a potential of the form (4). To understand covariant regularization and renormalization, however, it is necessary to be more general. For certain purposes, it suffices to consider any potential that is a function of z alone, but for others, we consider a general time-independent function of the three spatial coordinates. Finally, to get a full understanding of the stress tensor, we need to contemplate the theory in which a nontrivial space-time metric tensor is included. We have not striven for maximal generality when it was not needed for our ultimate purposes.

In the next section, we set up the basic equations for two interacting scalar fields, one playing the role of the potential, while the other is to be quantized. We treat two different ways of coupling the fields, to demonstrate that the renormalization concept is apparently robust against the details of the dynamics of the potential itself. In Section 3, we record the expansions (in point separation) of the vacuum expectation values of the field squared and of the stress tensor, mostly contained in [7]; these we generalize to fields of positive mass and carry out the Pauli–Villars construction. Then, we note the fate of the stress-tensor conservation law and trace identity under that construction. In Section 4, we interpret the terms with undetermined coefficients as renormalizations of the coupling constants in equations of motion for the potential and for the gravitational field when the potential’s stress tensor is included. There, we also draw some conclusions and project some future work.

2. Scalar Field Models

Let φ be the scalar field to be quantized and V be the potential creating the confining soft wall. Their interaction is implemented through a term $V\varphi$ in the equation of motion of φ and equivalently by a term $-\frac{1}{2}V\varphi^2g_{\mu\nu}$ in the stress tensor [17] ((2.1) and (2.19)). The point of renormalization is that, ultimately, V must itself be a dynamical object determined by this coupling along with other terms that do not involve φ . One possibility, introduced in [18] and discussed in more detail in [25], is that V is just another scalar field, satisfying a standard Klein–Gordon-like equation (but treated semiclassically, with the effects of φ entering through expectation values). Another model, introduced in [20], is that $V = \frac{1}{2}\sigma^2$, where σ is such a scalar field. We further develop both models here.

2.1. Lagrangian and Equations of Motion

The Lagrangian density in flat space for Model 1 is:

$$\mathcal{L}_1 = \frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\nabla \varphi)^2 - m^2 \varphi^2 - \lambda \varphi^2 V + (\partial_t V)^2 - (\nabla V)^2 - M^2 V^2 - 2JV \right\}, \quad (5)$$

where we have given φ a mass to enable Pauli–Villars regularization. We have omitted any terms for V that will not play a role in the renormalization; including such terms would not affect the argument in any way. With a similar understanding, the Lagrangian density for Model 2 is:

$$\mathcal{L}_2 = \frac{1}{2} \left\{ (\partial_t \varphi)^2 - (\nabla \varphi)^2 - m^2 \varphi^2 - \frac{\lambda}{2} \varphi^2 \sigma^2 + (\partial_t \sigma)^2 - (\nabla \sigma)^2 - \frac{\Lambda}{12} \sigma^4 - M^2 \sigma^2 \right\}. \quad (6)$$

In Model 2, the fields φ and σ both have dimension $[\text{length}]^{-1}$, and λ is dimensionless; in Model 1, V and λ have dimension $[\text{length}]^{-1}$.

Each field Φ , whether $\Phi = \varphi, V$ or σ , satisfies an equation of motion that is obtained from the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \sum_{\mu} \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \Phi)} \right) = 0. \quad (7)$$

For Model 1, we obtain:

$$\square \varphi - m^2 \varphi - \lambda V \varphi = 0, \quad (8)$$

for the quantum field φ , while the equation of motion for V is:

$$\square V - \frac{\lambda}{2} \varphi^2 - M^2 V - J = 0, \quad (9)$$

where:

$$\square = -\partial_t^2 + \nabla^2 \quad (10)$$

is the d'Alembertian operator. In Model 2, we have:

$$\square \varphi - m^2 \varphi - \frac{\lambda}{2} \varphi \sigma^2 = 0 \quad (11)$$

for φ and:

$$\square \sigma - \frac{\lambda}{2} \varphi^2 \sigma - \frac{\Lambda}{6} \sigma^3 - M^2 \sigma = 0 \quad (12)$$

for σ .

2.2. Curved-Space Action and Stress-Energy-Momentum Tensor

To define an unambiguous stress tensor, we need the generalization of the Lagrangian density to an action integral in curved space-time. For generality, couplings to the scalar curvature must be included. We write g for the absolute value of the determinant of $g_{\mu\nu}$, and we adopt the sign convention where the minus sign is associated with the time coordinate. In Model 1, we have:

$$S_{1\text{matter}} = \frac{1}{2} \int \sqrt{g} d^4x \left\{ -g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - m^2 \varphi^2 - \zeta R \varphi^2 - \lambda \varphi^2 V - g^{\mu\nu} \partial_{\mu} V \partial_{\nu} V - M^2 V^2 - Z R V - 2JV \right\}, \quad (13)$$

and in Model 2, we have:

$$S_{2\text{matter}} = \frac{1}{2} \int \sqrt{g} d^4x \left\{ -g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2 - \xi R \varphi^2 - \frac{\lambda}{2} \varphi^2 \sigma^2 - g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{\Lambda}{12} \sigma^4 - M^2 \sigma^2 - \Xi R \sigma^2 \right\}. \quad (14)$$

For generality, one could include $\Xi R V^2$ and ΛV^4 terms in S_1 and $ZR\sigma$ and $J\sigma$ terms in S_2 , but we list only terms whose coupling constants may change during renormalization (Section 4). Note that ZRV in (13) is not the standard curvature coupling; it is an unusual counterterm, with a dimensioned coupling constant, that will be forced by the renormalization theory. The stability of Model 1 is questionable, because of the terms with an odd power of V , but it will serve to make our point about renormalizability.

The stress tensor is obtained by varying the action with respect to the metric:

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\partial}{\partial g^{\mu\nu}} S_{\text{matter}}. \quad (15)$$

The full gravitational action contains another term, the Einstein–Hilbert action S_{grav} . Because R , the Ricci curvature scalar, depends on second derivatives of the metric tensor, finding this variation of S requires integration by parts twice. The result of this complicated calculation is well known [26] (231). Here, however, we need only its specialization to flat space, which means that we can redo the calculation to first order in $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. From [27] (8.25),

$$R = \partial_\alpha \partial_\beta h^{\alpha\beta} - \square h^\alpha_\alpha + O(h^2). \quad (16)$$

Note also that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)$ (sign sic!), where indices on h are raised by the flat-space metric η . Thus, for any field quantity Ψ ,

$$-\frac{2}{\sqrt{g}} \frac{\partial}{\partial g^{\mu\nu}} \int \sqrt{g} d^4x \left(-\frac{1}{2} \xi R \Psi \right) = -\frac{\partial}{\partial h^{\mu\nu}} \int \sqrt{1 + O(h)} d^4x \xi R \Psi \quad (17)$$

$$= \xi (g_{\mu\nu} \square - \partial_\mu \partial_\nu) \Psi + O(h), \quad (18)$$

because the only terms that survive in the flat limit are those that come from the double integration by parts of the second derivative of the integrand with respect to $h^{\mu\nu}$. If $\Psi \propto V$, we here encounter a quantity:

$$W_{\mu\nu} \equiv W[V]_{\mu\nu} \equiv (\square V) g_{\mu\nu} - \partial_\mu \partial_\nu V \quad (19)$$

that will be useful in what follows. If $V = \sigma^2$, this works out to:

$$W[\sigma^2]_{\mu\nu} = 2(\sigma \square \sigma g_{\mu\nu} - \sigma \partial_\mu \partial_\nu \sigma + \partial^\alpha \sigma \partial_\alpha \sigma g_{\mu\nu} - \partial_\mu \sigma \partial_\nu \sigma). \quad (20)$$

Similarly, if $\Psi = \varphi^2$, we obtain in Equation (18) the well-known stress-tensor term for nonminimal coupling,

$$T_{\mu\nu}[\xi] = \xi (g_{\mu\nu} \square - \partial_\mu \partial_\nu) \varphi^2 \quad (21)$$

$$= 2\xi (\varphi \square \varphi g_{\mu\nu} - \varphi \partial_\mu \partial_\nu \varphi + \partial^\alpha \varphi \partial_\alpha \varphi g_{\mu\nu} - \partial_\mu \varphi \partial_\nu \varphi). \quad (22)$$

(Although the curvature has gone to zero, the ξ term in the action still makes a contribution to the stress tensor, though not to the field's equation of motion.)

We now have all the ingredients of the stress tensor of φ as given in [17] (2.19). Later, we shall need the full stress tensors (15) of the theories, including the contributions from the potentials (but specialized to flat space). They are:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi + \partial_\mu V \partial_\nu V - g_{\mu\nu} \mathcal{L}_1 + \xi W[\varphi^2]_{\mu\nu} + ZW[V]_{\mu\nu} \quad (23)$$

for Model 1 and:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi + \partial_\mu \sigma \partial_\nu \sigma - g_{\mu\nu} \mathcal{L}_2 + \xi W[\varphi^2]_{\mu\nu} + \Xi W[\sigma^2]_{\mu\nu} \quad (24)$$

for Model 2.

3. Vacuum Expectation Values

3.1. The Square of the Field

According to [7] (2.9), the leading behavior of the expectation value of the product of two field operators in an external potential $v(z)$ is given through the formulas [7] (A8) to be:

$$\langle \varphi^2 \rangle = \frac{1}{4\pi^2 \delta^2} + \frac{v}{16\pi^2} \left(\ln \frac{\mu^2 \delta^2}{4} + \ln \frac{v}{\mu^2} + 2\gamma - 1 \right) - \frac{1}{96\pi^2} \frac{v''}{v} + O(v^{-2}) + O(\delta^2 \ln \delta). \quad (25)$$

Here, the notation $\langle \varphi^2 \rangle$ indicates that one thinks of this quantity as the expectation value of the square of a field operator, temporarily regularized by point-splitting; δ is the distance between the two points, which are in the Euclideanized transverse three-plane $(-it, x, y, z)$ with z fixed; the mass of the quantized field is zero; and an arbitrary mass scale μ has been introduced to split the δ dependence from the v dependence.

In [7], the quantized field was assumed massless, but Equation (25) applies to a field of mass m if v is replaced by $m^2 + \lambda V$, where V is the physical potential; we abbreviate λV as V . We implement the Pauli–Villars method as set forth in Section 1.1.3, with $m_1 = 0$ and three large masses $m_2, \dots, m_4$. The effect of the Pauli–Villars procedure on the various terms in Equation (25) depends on their dependence on m^2 , hence on the undifferentiated v . By virtue of the identities (2) satisfied by the f_j , terms that are polynomial in m^2 are annihilated; these include the quadratic divergence, the $2\gamma - 1$ and the $\ln \delta$ divergence. On the other hand, a term with v in the denominator, such as v''/v or the nonlocal remainder in the expansion, is left unchanged except for replacement of v by V , because the contributions from the regulator masses vanish when the latter are taken to infinity. What remains to be discussed is:

$$\sum_j f_j \frac{m_j^2 + V}{16\pi^2} \ln \frac{m_j^2 + V}{\mu^2}, \quad (26)$$

which is actually independent of μ by virtue of Equations (2) (instead, the results will depend on the arbitrary mass ratios α and β). Henceforth, we can take $\mu = 1$ in most contexts.

For $j > 1$ and V bounded (e.g., z restricted to a bounded interval in the case (4)), the summand can be expanded in a power series:

$$\begin{aligned} \frac{m_j^2 + V}{16\pi^2} \ln(m_j^2 + V) &= \frac{1}{16\pi^2} (m_j^2 + V) \left(\ln m_j^2 + \frac{V}{m_j^2} + \dots \right) \\ &= \frac{1}{16\pi^2} \left(m_j^2 \ln m_j^2 + V \ln m_j^2 + V \right) + O(m_j^{-2}). \end{aligned} \quad (27)$$

Recalling that:

$$\sum_{j>1} f_j = -f_1 = -1$$

and adding on the $j = 1$ terms, we get the regularized quantity:

$$\langle \varphi^2 \rangle_{\text{PV}} = \frac{1}{16\pi^2} \left[V \ln V - \frac{1}{6} \frac{V''}{V} + O(V^{-2}) + C_2 + C_3 V - V \right], \quad (28)$$

where we have defined:

$$C_1 = \sum_{j>1} f_j m_j^4 \ln m_j^2, \quad C_2 = \sum_{j>1} f_j m_j^2 \ln m_j^2, \quad C_3 = \sum_{j>1} f_j \ln m_j^2. \quad (29)$$

The expectation is that after renormalization of coupling constants as $m_j \rightarrow \infty$ for $j > 1$, the C_j will be replaced by arbitrary finite values. The other terms in (28) are finite as $\delta \rightarrow 0$, as we have anticipated by dropping the δ -dependent remainder term.

3.2. The Stress Tensor

The leading behavior of the expectation value of the stress tensor is given by [7] (3.2); it is too complicated to reproduce here, but the complications stem mostly from the need to write its direction-dependent terms in 4×4 matrix notation. Applying the same logic as before, we see that the Pauli–Villars procedure will leave the bottom two lines of [7] (3.2) unchanged while annihilating the divergent and direction-dependent first and third lines. In the second line, we split the logarithmic factor as before and again see that the divergence and the γ terms are annihilated, leaving:

$$\langle T_{\mu\nu} \rangle = \frac{v^2}{64\pi^2} \ln \frac{v}{\mu^2} (-g_{\mu\nu}) - \left(\xi - \frac{1}{6} \right) \frac{v''}{16\pi^2} \ln \frac{v}{\mu^2} \text{diag}(1, -1, -1, 0) + O(v^{-1}).$$

The last explicit term can be written covariantly via:

$$-v'' \text{diag}(1, -1, -1, 0) = (\square v) g_{\mu\nu} - \partial_\mu \partial_\nu v = W[v]_{\mu\nu}. \quad (30)$$

Thus (with the renormalization point $\mu = 1$):

$$\langle T_{\mu\nu} \rangle_{\text{PV}} = \sum_j f_j \left\{ \frac{m_j^4 + 2m_j^2 V + V^2}{64\pi^2} \ln(m_j^2 + V) (-g_{\mu\nu}) + \frac{\xi - \frac{1}{6}}{16\pi^2} [(\square V) g_{\mu\nu} - \partial_\mu \partial_\nu V] \ln(m_j^2 + V) + O((m_j^2 + V)^{-1}) \right\}.$$

Expanding the logarithms as before, we arrive at:

$$\begin{aligned} \langle T_{\mu\nu} \rangle_{\text{PV}} = & -\frac{g_{\mu\nu}}{64\pi^2} V^2 \ln V + \frac{\xi - \frac{1}{6}}{16\pi^2} W_{\mu\nu} \ln V + O(V^{-1}) \\ & -\frac{g_{\mu\nu}}{64\pi^2} (C_1 + 2C_2 V + C_3 V^2) - \frac{g_{\mu\nu}}{64\pi^2} \sum_{j>1} f_j (m_j^2 V + \frac{3}{2} V^2) + \frac{\xi - \frac{1}{6}}{16\pi^2} C_3 W_{\mu\nu}. \end{aligned}$$

(Here, $W = W[V]$.) The $m_j^2 V$ term is polynomial, hence annihilated; but for the $\frac{3}{2} V^2$, the term with $j = 1$ is not in the sum, so like a similar term in $\langle \varphi^2 \rangle$, this one survives with a factor -1 . Thus, we end with:

$$\begin{aligned} \langle T_{\mu\nu} \rangle_{\text{PV}} = & -\frac{g_{\mu\nu}}{64\pi^2} V^2 \ln V + \frac{\xi - \frac{1}{6}}{16\pi^2} W_{\mu\nu} \ln V + O(V^{-1}) \\ & -\frac{g_{\mu\nu}}{64\pi^2} (C_1 + 2C_2 V + C_3 V^2) + \frac{3}{128\pi^2} V^2 g_{\mu\nu} + \frac{\xi - \frac{1}{6}}{16\pi^2} C_3 W_{\mu\nu}. \quad (31) \end{aligned}$$

3.3. The Trace Identity

The classical trace identity [7] (4.1) is:

$$T_\mu^\mu + V \varphi^2 - 3 \left(\xi - \frac{1}{6} \right) \square(\varphi^2) = 0. \quad (32)$$

From (31) and (28), we have (modulo $O(V^{-1})$):

$$\begin{aligned}\langle T_{\mu}^{\mu} \rangle_{PV} &= -\frac{1}{16\pi^2} V^2 \ln V - \frac{C_1}{16\pi^2} - \frac{C_2}{8\pi^2} V - \frac{C_3}{16\pi^2} V^2 + \frac{3}{32\pi^2} V^2 \\ &\quad + \frac{3}{16\pi^2} \left(\xi - \frac{1}{6} \right) [(\square V) \ln V + C_3(\square V)], \\ V \langle \varphi^2 \rangle_{PV} &= \frac{1}{16\pi^2} \left[V^2 \ln V - \frac{1}{6} V'' + C_2 V + C_3 V^2 - V^2 \right], \\ -3 \left(\xi - \frac{1}{6} \right) \square \langle \varphi^2 \rangle_{PV} &= \frac{3}{16\pi^2} \left(\xi - \frac{1}{6} \right) [-(\square V) \ln V - C_3(\square V)].\end{aligned}$$

Adding these three lines, we get:

$$\langle T_{\mu}^{\mu} \rangle_{PV} + V \langle \varphi^2 \rangle_{PV} - 3 \left(\xi - \frac{1}{6} \right) \square \langle \varphi^2 \rangle_{PV} = \frac{1}{16\pi^2} \left(\frac{1}{2} V^2 - \frac{1}{6} (\square V) - C_1 - C_2 V \right). \quad (33)$$

Apart from the terms involving renormalization constants, this is precisely the trace anomaly [7] ((5.10) with (5.6)). The C_1 term represents the trace of the metric tensor and is inevitable in any renormalization prescription that allows an arbitrary finite renormalization of the cosmological constant. Discussion of the C_2 term is delayed to a later section.

The V^2 term in (33) came from the terms in each of $\langle \varphi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ that required the identity $\sum_{j>1} f_j = -1$ for their evaluation.

The $\square V$ term in (33) came about in an interesting way. It arose from the V''/V term in (28), which one might have been inclined to discard from the calculation as “higher order” because of its denominator. In contrast to $V \langle \varphi^2 \rangle_{PV}$, this term would not appear in $\langle v \varphi^2 \rangle_{PV}$. In the context of the stress tensor, V'' is a quantity of the type called “critical order” in [7]. In contrast, the $O(V^{-1})$ term in (31) has numerator $(V')^2$, marking it as clearly higher than critical order. This distinction will become important in Section 4.2.2.

3.4. The Conservation Law

The classical conservation law [7] (4.2) is:

$$\partial_{\mu} T_{\nu}^{\mu} + \frac{1}{2} (\partial_{\nu} V) \varphi^2 = 0. \quad (34)$$

Recall that $g_{\mu\nu}$ and $W_{\mu\nu} = (\square V) g_{\mu\nu} - \partial_{\mu} \partial_{\nu} V$ are conserved. Furthermore, in the soft wall case (V a function only of z), $W_{\mu\nu}$ is orthogonal to $\partial_{\nu} V$ (in the general case, the term in question does not vanish, but it has V in the denominator and hence has higher than critical order; it is absorbed into the nonlocal remainder $O(V^{-1})$ in Equation (31), which must automatically be conserved as a whole).

Therefore, from (31):

$$\partial_{\mu} \langle T_{\nu}^{\mu} \rangle_{PV} = -\frac{1}{64\pi^2} (V \partial_{\nu} V + 2V \partial_{\nu} V \ln V) - \frac{C_2}{32\pi^2} \partial_{\nu} V - \frac{C_3}{32\pi^2} V \partial_{\nu} V + \frac{3}{64\pi^2} V \partial_{\nu} V.$$

On the other hand, modulo $O(V^{-1})$:

$$\frac{1}{2} (\partial_{\nu} V) \langle \varphi^2 \rangle_{PV} = \frac{1}{32\pi^2} (\partial_{\nu} V V \ln V + C_2 \partial_{\nu} V + C_3 V \partial_{\nu} V - V \partial_{\nu} V).$$

These two expressions do add up to zero!

The terms in (31) and (33) that arose from $\sum_{j>1} f_j = -1$, which combined to give the V^2 term in the trace anomaly, are crucial to this result. Without them, the term in the conservation law coming from $V^2 \partial_j \ln V$ could not be canceled.

Note also that there was automatic cancellation of the C_2 terms. Although $C_2 V g_{\mu\nu}$ is not a conserved tensor by itself, it is needed in Equation (34) to cancel the leading arbitrary constant term, proportional to C_2 , in $\langle \varphi^2 \rangle$.

4. Renormalization and Interpretation

4.1. Comparison of Pauli–Villars and Point-Splitting “Renormalization”

As in [7], we try to reserve the technical term renormalization for changing the coupling constants in the Lagrangian (hence the equations of motion and the stress tensor) of the background fields so as to absorb divergences, and we use “renormalization” (including the quotation marks) for the otherwise “unspeakable act” (a variation on [3]) of simply discarding the divergences without a convincing physical explanation of where they went. Renormalization is the subject of the next subsection, but first, we need to make a critical comparison of two candidate “renormalized” stress tensors, the one in [7] that came from point splitting in the spirit of Wald [5,6] and the one that has been provided above by the Pauli–Villars procedure.

Apart from the terms with the constants C_j , which we discuss later, the formulas agree with one exception: the renormalized stress tensor in [7] (5.11) has $V^2/128\pi^2$ instead of the $3V^2/128\pi^2$ of Equation (31) in the “anomalous” term. Yet, both tensors were checked to satisfy the conservation law. How is this possible? The answer is that there is a compensating difference in $\langle \varphi^2 \rangle$: [7] (5.2) (obtained by prescription [7] (5.1)) lacks a term corresponding to the $-V$ in Equation (28).

Let us explore this freedom more systematically. Consider adding to $16\pi^2 \langle \varphi^2 \rangle_{PV}$ the term YV for some constant Y (this amounts to reconstruing the renormalization of the logarithmic term in $\langle \varphi^2 \rangle$ in the spirit of [7] (5.1), but for the Green function alone). To compensate, add to $16\pi^2 \langle T_{\mu\nu} \rangle_{PV}$ the terms:

$$-\frac{Y}{4} V^2 g_{\mu\nu} + Y \left(\xi - \frac{1}{6} \right) W_{\mu\nu}. \quad (35)$$

These quantities are tensorial and of critical order, and they yield zero when inserted (together) into Identities (32) and (34), so the modified renormalized quantities are still consistent with all the formal requirements. (There is a trace anomaly, but it is independent of Y .)

The two most natural choices are:

- $Y = 1$: This removes the term $-V$ from $\langle \varphi^2 \rangle$ and recovers [7] (5.11) for $\langle T_{\mu\nu} \rangle$ modulo a term proportional to the conserved and covariant tensor $W_{\mu\nu}$, so it reproduces the conclusions of [7].
- $Y = \frac{3}{2}$: This completely removes the anomalous $V^2 g_{\mu\nu}$ term from the stress tensor (and again changes the coefficient of the $W_{\mu\nu}$ term). In $\langle \varphi^2 \rangle$, it changes the $-V$ to $+V/2$.

All these pairs $(\langle T_{\mu\nu} \rangle, \langle \varphi^2 \rangle)$ satisfy (34). Which $\langle T_{\mu\nu} \rangle$ to consider the true renormalized stress tensor is a matter of convention, since the definition of the renormalized field-squared is a convention. However, we may think of $T_{\mu\nu}$ as the source of the gravitational field; how can it depend on such a convention? Well, allowing the C coefficients to be nonzero, we observe from Equations (28), (31) and (35) that C_3 absorbs the problem just discussed, as a change in Y can be regarded as effectively the same thing as a change in C_3 (and hence, one can forget Y outside this subsection).

In the Pauli–Villars approach, it is natural to retain the three C_j as arbitrary renormalization constants, rather than to argue them away. C_1 does not appear in (28) and multiplies $g_{\mu\nu}$ in (31), a conserved covariant tensor, which does, however, modify the trace formula (33). C_2 appears in (28) as the leading constant term and appears in (31) multiplying a covariant object that is not conserved by itself, but enters the conservation equation precisely to cancel that constant; it, also, modifies (33). In the gravity context, it is sometimes argued that C_1 and C_2 (which have physical dimensions) ought to be zero because the massless scalar field theory is scale-invariant and that the Pauli–Villars method suggests otherwise only because it manifestly violates the scale invariance. This argument loses some force in the presence of a scalar potential; after all, a constant scalar potential is simply the square of

a Klein–Gordon mass. The case of C_3 is different, since it is dimensionless. C_3 appears in (28) as the coefficient of V (an allowed covariant object in the scalar) and appears twice in (31); the first occurrence again conspires with the V term in (28) to satisfy the conservation law, while the second multiplies the conserved and covariant tensor $W_{\mu\nu}$. Although W has a nonzero trace, C_3 cancels out of (33). Which coefficient of W in (31) corresponds to a vanishing value of C_3 in (28) depends on the choice of Y .

4.2. Renormalization

4.2.1. Model 1

Consider first the equation of motion of V . After taking an expectation value, we may write (9) as the classical equation:

$$\square V = \frac{1}{2}\lambda\langle\varphi^2\rangle + M^2V + J.$$

Henceforth, all expectation values are presumed to be in their Pauli–Villars forms. Thus, we take from Equation (28), with V interpreted as λV and the renormalization mass scale restored,

$$\langle\varphi^2\rangle = \frac{1}{16\pi^2} \left[\lambda V \ln \left(\frac{\lambda V}{\mu^2} \right) + O(V^{-1}) + C_2 + C_3 \lambda V - \lambda V \right].$$

Thus:

$$\square V = M^2V + J + \frac{1}{32\pi^2} \lambda^2 V \ln \left(\frac{\lambda V}{\mu^2} \right) + O(V^{-1}) + \frac{C_2\lambda}{32\pi^2} + \frac{C_3\lambda^2}{32\pi^2} V - \frac{\lambda^2}{32\pi^2} V. \quad (36)$$

This suggests defining renormalized, or effective, coupling constants:

$$J_{\text{ren}} = J + \frac{C_2\lambda}{32\pi^2}, \quad M_{\text{ren}}^2 = M^2 + \frac{C_3\lambda^2}{32\pi^2}. \quad (37)$$

The final term, proportional to $-V$, is left out of the mass renormalization because it is inherently linked to the μ dependence of the logarithmic term; we shall see (Equation (39)) that the corresponding term in the stress tensor has a different coefficient, so including this term in the effective mass would cause inconsistency.

Turn now to the stress tensor. The terms in $T_{\mu\nu}$ that depend on V and not on φ are, from Equation (23),

$$\partial_\mu V \partial_\nu V - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \partial_\alpha V \partial_\beta V + M^2 V^2 + 2JV \right] + ZW_{\mu\nu}.$$

These must be combined with the renormalized stress tensor (31) of φ (which includes the $V\varphi^2$ interaction term):

$$\begin{aligned} \frac{1}{16\pi^2} \left[-\frac{\lambda^2}{4} g_{\mu\nu} V^2 \ln \left(\frac{\lambda V}{\mu^2} \right) + \left(\xi - \frac{1}{6} \right) \lambda W_{\mu\nu} \ln \left(\frac{\lambda V}{\mu^2} \right) + O(V^{-1}) \right. \\ \left. - \frac{1}{4} g_{\mu\nu} (C_1 + 2C_2\lambda V + C_3\lambda^2 V^2) + \frac{3\lambda^2}{8} V^2 g_{\mu\nu} + \left(\xi - \frac{1}{6} \right) C_3 \lambda W_{\mu\nu} \right]. \end{aligned}$$

The combinations (37) again appear, together with:

$$Z_{\text{ren}} = Z + \frac{1}{16\pi^2} \left(\xi - \frac{1}{6} \right) C_3 \lambda, \quad (38)$$

yielding the final formula:

$$\langle T_{\mu\nu} \rangle = \partial_\mu V \partial_\nu V - \frac{1}{2} g_{\mu\nu} \left[g^{\alpha\beta} \partial_\alpha V \partial_\beta V + M_{\text{ren}}^2 V^2 + 2J_{\text{ren}} V \right] + Z_{\text{ren}} W_{\mu\nu} - \frac{C_1}{64\pi^2} g_{\mu\nu} \\ - \frac{\lambda^2}{64\pi^2} g_{\mu\nu} V^2 \ln \left(\frac{\lambda V}{\mu^2} \right) + \left(\xi - \frac{1}{6} \right) \frac{\lambda}{16\pi^2} W_{\mu\nu} \ln \left(\frac{\lambda V}{\mu^2} \right) + O(V^{-1}) + \frac{3\lambda^2}{128\pi^2} V^2 g_{\mu\nu}. \quad (39)$$

The term proportional to $C_1 g_{\mu\nu}$ amounts to a renormalization of the cosmological constant (or dark energy). If we had not specialized to flat space, there would be additional renormalizations of the gravitational constant and the coefficients of terms in the Einstein equation quadratic in curvature (cf. [20]).

4.2.2. Model 2

Similar considerations apply to Model 2. The Equation (12) satisfied by σ is:

$$\square\sigma = \frac{\lambda}{2} \langle \varphi^2 \rangle \sigma + \frac{\Lambda}{6} \sigma^3 + M^2 \sigma, \quad (40)$$

where, from Equation (28), with V interpreted as: $\frac{1}{2} \lambda \sigma^2$,

$$\langle \varphi^2 \rangle = \frac{1}{16\pi^2} \left[\frac{1}{2} \lambda \sigma^2 \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) - \frac{1}{3} \frac{\sigma''}{\sigma} - \frac{1}{3} \frac{(\sigma')^2}{\sigma^2} + O(\sigma^{-4}) + C_2 + \frac{C_3}{2} \lambda \sigma^2 - \frac{1}{2} \lambda \sigma^2 \right]. \quad (41)$$

(The derivatives of σ are with respect to z , because the high order terms in Equation (28) were calculated in [7] under the assumption that V depends only on z .) If we interpret “higher order” as meaning “having a σ in the denominator”, then the second and third terms in (41) can be dismissed as $O(\sigma^{-1})$. That would be consistent with what we did, without incident, for Model 1. However, in (12), we see a “resurgence” phenomenon analogous to the one that created part of the trace anomaly: the expectation value of φ^2 needs to be multiplied by σ , and thereby, the σ'' term might be promoted to a level of significance. The real issue here is which terms are part of the nonlocal remainder whose conservation, etc., is guaranteed by conservation of the integrand before integration over the spectral parameter (see [7] (Section II)) and was verified order-by-order in [7] (Section IV), as opposed to those terms that need to be accounted for in the renormalization theory. To guard against surreptitious error, we shall temporarily carry the σ'' term along.

Thus, we arrive at:

$$\square\sigma = \frac{\Lambda}{6} \sigma^3 + M^2 \sigma + \frac{1}{32\pi^2} \left[\frac{1}{2} \lambda^2 \sigma^3 \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) - \frac{1}{3} \lambda \sigma'' + O(\sigma^{-1}) + C_2 \lambda \sigma + \frac{C_3}{2} \lambda^2 \sigma^3 - \frac{1}{2} \lambda^2 \sigma^3 \right]. \quad (42)$$

Therefore, the renormalized coupling constants are:

$$\frac{\Lambda_{\text{ren}}}{6} = \frac{\Lambda}{6} + \frac{C_3 \lambda^2}{64\pi^2}, \quad M_{\text{ren}}^2 = M^2 + \frac{C_2 \lambda}{32\pi^2}. \quad (43)$$

Because $\sigma'' = \square\sigma$ for the soft wall, it might appear that the coefficient of $\square\sigma$ should be renormalized to $1 + \lambda/96\pi^2$; but we shall decide otherwise.

The stress-tensor terms that depend on σ , but not φ are, from Equation (24),

$$\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} \left[\partial^\alpha \sigma \partial_\alpha \sigma + \frac{\Lambda}{12} \sigma^4 + M^2 \sigma^2 \right] + \Xi W[\sigma^2]_{\mu\nu}.$$

where the last term (henceforth simply $W_{\mu\nu}$) is given by Equation (20). This must be added to Equation (31) with V replaced by $\frac{\lambda}{2} \sigma^2$:

$$\frac{1}{16\pi^2} \left[-\frac{\lambda^2}{16} \sigma^4 \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) + \left(\xi - \frac{1}{6} \right) \frac{\lambda^2}{4} W[\sigma^2]_{\mu\nu} \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) + O(\sigma^{-2}) \right. \\ \left. - \frac{1}{4} g_{\mu\nu} (C_1 + C_2 \lambda \sigma^2 + \frac{1}{4} C_3 \lambda^2 \sigma^4) + \frac{3\lambda^2}{32} g_{\mu\nu} \sigma^4 + \left(\xi - \frac{1}{6} \right) \frac{\lambda^2}{4} C_3 W[\sigma^2]_{\mu\nu} \right].$$

Again the renormalized coupling constants (43) emerge, along with:

$$\Xi_{\text{ren}} = \Xi + \left(\xi - \frac{1}{6} \right) \frac{C_3 \lambda^2}{64\pi^2}. \quad (44)$$

The final formula is:

$$\langle T_{\mu\nu} \rangle = \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} \left[\partial^\alpha \sigma \partial_\alpha \sigma + M_{\text{ren}}^2 \sigma^2 + \frac{\Lambda_{\text{ren}}}{12} \sigma^4 \right] + \Xi_{\text{ren}} W[\sigma^2]_{\mu\nu} - \frac{C_1}{64\pi^2} g_{\mu\nu} \\ - \frac{\lambda^2}{256\pi^2} g_{\mu\nu} \sigma^4 \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) + \left(\xi - \frac{1}{6} \right) \frac{\lambda^2}{4\pi^2} W[\sigma^2]_{\mu\nu} \ln \left(\frac{\lambda \sigma^2}{2\mu^2} \right) + O(\sigma^{-2}) + \frac{3\lambda^2}{512\pi^2} \sigma^4 g_{\mu\nu}. \quad (45)$$

As for Model 1, the last terms in (42) and (45) look like they should have been absorbed into C_3 , but they cannot be because they have different numerical coefficients. The difference is not a mistake; it is necessary to protect the conservation law from an extra term generated from the first logarithmic term in $\langle T_{\mu\nu} \rangle$.

Note that in (45), there is no trace of the problematic σ'' term. The full stress tensor (24) ought to satisfy $\partial_\mu \langle T_{\nu}^{\mu} \rangle = 0$. Direct verification of this from (45) requires substituting from (42) and is successful only if the σ'' term in the latter is ignored. We therefore believe that there is no need to renormalize the kinetic term in the equation of motion.

Our conclusions about Model 2 are qualitatively similar to those of [20], in identifying the renormalized quantities Λ , M , Ξ . The algebraic and numerical details are quite different, because [20] uses dimensional regularization. Furthermore, the object W is visible in their stress tensor [20] (23). The mysterious σ'' is not visible in their equation of motion [20] (21), but probably, it is hidden in $\sigma \langle \varphi^2 \rangle_{\text{ren}}$. We should note that the treatment of [20] is more general than ours, as it is not restricted to flat space, nor (at any stage) to potentials depending on only one coordinate. We, however, in our restricted setting have concrete information about the renormalized remainders, $\langle \varphi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ (see [7,17]).

4.2.3. The Logarithmic Terms

The renormalized stress tensor (39) includes two terms with the structures $V^2 \ln V$ and $W \ln V$ (W reduces to V'' in our intended application, by Equation (30)). These terms are finite, but poorly determined since they depend on μ and any numerical factors included inside the argument of the logarithm. Any change in them is, of course, compensated by the change in M_{ren} and Z_{ren} , which are themselves undetermined by the theory (parallel remarks apply in Model 2 and to the field Equation (36) in Model 1, but we shall not spell them out).

What makes these terms worrisome is that they can become quite large where V is large. In our motivating application, $V(z)$ rises steeply as $z \rightarrow \infty$, and this behavior is intended to reproduce a confining wall, approximating a perfect conductor. One expects the effects of the quantized field φ to be negligible deep inside the wall, yet here, we have a piece of its stress tensor that grows to infinity there. What is its physical meaning?

The choice of μ^2 is at our disposal, and we can always choose it to equal $\lambda V(z_0)$, so that the argument of the logarithm equals unity and the terms disappear at a point z_0 . Because the logarithm is a slowly-varying function, the terms remain small for $z \approx z_0$ (more precisely, for $V(z) - V(z_0) \ll V(z_0)$). We seem to have here something analogous to the “running coupling constant” in the renormalization group of high energy theory, except that the controlling variable is position z , not energy.

For experiments performed in a local region, the ultraviolet problem has been pushed entirely into the renormalized coupling constants.

The situation appears less threatening if the perpetually rising potential is replaced by one that flattens out at large z to a large constant value, which is simply the effective mass squared of φ . Then, choosing $\mu = \lambda^{1/2}m$ removes the logarithmic terms from the entire asymptotic region. Then, Equation (39) (with its undetermined constants M , Z , J and C_1) is the source of the gravitational field, the effects of φ having been integrated out, and the Lagrangian of the field V is of a standard form.

Another potential way of observing $\langle T_{\text{ren}} \rangle$ deep inside the wall is to find the generalized Casimir force on the boundary when the shape of the potential (i.e., some parameter in the profile of V) is varied. This has yet to be investigated. However, the energy density deep inside the potential regions will not have an influence on the attraction between two rigidly moving walls, as calculated in [7] (Section 9).

4.3. Conclusions and Outlook

In the previous paper [7], divergences were regularized by point splitting, and a “renormalized” stress tensor was arrived at, as in gravitational theory [4–6,8,9], by discarding clearly unphysical terms while insisting that the remaining terms of “critical order” satisfy the covariant conservation law (including the contribution from the “renormalized” square of the field). This procedure has some ad hoc features that might leave one uneasy. The Pauli–Villars procedure, although it involves unphysical fields whose entry is hard to motivate, improves this situation by avoiding direction-dependent terms and generating nicely parametrized divergent terms that correspond to counterterms that one would reasonably expect to occur in the background theory, including some that the analytic methods do not produce. It therefore leads automatically to a renormalized theory, which appears more trustworthy than the “black magic” of dimensional or zeta methods. The results appear to be consistent with the dimensional method [20] insofar as they can be compared at present.

The application of the Pauli–Villars method in the context of a scalar field in a (static) scalar potential is the first achievement of this paper. The second, and more important, is the verification that the emergence of “counterterms that one would reasonably expect to occur” is indeed the result of two different models, of rather conventional types, of the dynamics of the scalar potential. We therefore expect this renormalizability to be a robust property.

The work has uncovered two aspects of the renormalized dynamics that may appear unsatisfactory, or at least strange. One is the presence of logarithmic terms that become large in regions where the effects of the quantized field should be suppressed. We suggest that this effect is analogous to the running of coupling constants in standard renormalized quantum field theory and should be no more disturbing, although the effect of these terms on Casimir-like force calculations involving nonrigid walls deserves investigation. The other is the presence of a second derivative of the background field, in the model where it is coupled to the quantized field quadratically, that could combine with, or rival, the kinetic terms in the dynamics of the background field. We think that this is an artifact of carrying the adiabatic expansion of the normal modes a bit too far. (By pushing the asymptotic calculations in [7] to very high order, one could “discover” arbitrarily high order derivatives of the background field in its equation of motion. Clearly, that would be misleading.) However, we mean to remain alert to the possibility that one or the other of these mathematical phenomena indicates a genuine problem in the theory.

With the renormalization theory under control, one can hope to return to the type of calculations carried out in [7] and complete them to the degree of numerical completeness achieved for the exterior region in [17]. The problem is to do numerical calculations in the regimes where the asymptotic (WKB) approximation is inaccurate, while removing the divergences correctly. That work, which will probably be the final installment of the project insofar as the scalar field is concerned, is in progress.

Acknowledgments: We are grateful for valuable discussions with the late Martin Schaden, Alejandro Satz, Prachi Parashar and various participants at the Ginzburg Centennial Conference in Moscow (2017). Furthermore, Robert Wald and F. Diego Mazzitelli commented on the manuscript. T.E.S. received summer support from the Texas A&M Department of Physics and Astronomy's undergraduate research funds. K.A.M. is supported by National Science Foundation Grant No. 1707511.

Author Contributions: S.A.F. conceived of and guided the research and wrote most of the text. T.E.S. performed many of the calculations. K.A.M. was in consultation and provided valuable suggestions and criticisms.

Conflicts of Interest: The authors declare no conflict of interest.

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