

Casimir self-entropy of a spherical electromagnetic δ -function shellKimball A. Milton,^{1,*} Pushpa Kalauni,^{1,†} Prachi Parashar,^{2,1,‡} and Yang Li^{1,§}¹*H. L. Dodge Department of Physics and Astronomy, University of Oklahoma,
Norman, Oklahoma 73019, USA*²*Department of Energy and Process Engineering, Norwegian University of Science and Technology,
7491 Trondheim, Norway*

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In this paper we continue our program of computing Casimir self-entropies of idealized electrical bodies. Here we consider an electromagnetic δ -function sphere (“semitransparent sphere”) whose electric susceptibility has a transverse polarization with arbitrary strength. Dispersion is incorporated by a plasma-like model. In the strong-coupling limit, a perfectly conducting spherical shell is realized. We compute the entropy for both low and high temperatures. The transverse electric self-entropy is negative as expected, but the transverse magnetic self-entropy requires ultraviolet and infrared renormalization (subtraction), and, surprisingly, is only positive for sufficiently strong coupling. Results are robust under different regularization schemes. These rather surprising findings require further investigation.

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The usual expectation, based on the notion that entropy is a measure of disorder, is that entropy should be positive. However, there are circumstances in which entropy can take on negative values. For example, negative entropy is often discussed in connection with biological systems [1]. More interesting physically is the occurrence of negative entropy in black-hole and cosmological physics [2–5].

In Casimir physics, perhaps the first appearance of negative entropy occurred in connection with the description of the quantum vacuum interaction between parallel conducting plates. If dissipation is present, the entropy of the interaction is positive at large distances, $aT \gg 1$, where a is the separation between the plates and T is the temperature, but turns negative for short distances. Considered as a function of temperature, the sign of the entropy changes as the temperature decreases, becoming negative if the dissipation of free electrons is present [6–8], but does tend to zero as the temperature tends to zero, in accordance with the Nernst heat theorem if the residual relaxation is taken into account [9,10]. Although perhaps surprising, this was not thought to be a problem because this phenomenon only referred to the interaction part of the free energy, and the total entropy of the system was expected to be positive. Somewhat later it was discovered that negative Casimir entropies also occurred purely geometrically, for example between a perfectly conducting sphere and a perfectly conducting plane without dissipation [11–13], or between two spheres [14,15]. When the distance times the temperature (in natural units) is of order

unity, typically a negative entropy region was present. Since the effect was dominant in the dipole approximation, this led to a systematic study of the phenomenon of negative entropy arising between polarizable particles, characterized by electric and magnetic polarizabilities, or between such particles and a conducting plate. For appropriate choices of these polarizabilities, these nanoparticles behaved like small conducting spheres. We found that sometimes the entropy started off negative for small aT , before eventually turning positive, and sometimes the entropy was first positive, turned negative for a while, and then turned positive again as aT increased [16,17]. The combined effects of both geometry and dissipation were considered in Refs. [18,19].

The occurrence of negative entropy, geometrically induced, sharpened the puzzle. Again, the suspicion was that the self-entropies of the bodies were much larger, and positive, yielding positive entropies always for the whole system. This was borne out to some extent in the case of perfect conducting spheres. There it turned out that although the self-entropy of a conducting plate vanishes, the self-entropy of a conducting sphere is positive and is such that it precisely cancels the most negative interaction entropy between a sphere and a plate [20,21]. More specifically, the two electromagnetic mode contributions to the entropy, the transverse electric (TE) and transverse magnetic (TM) terms, had opposite signs: as expected [22], the TE was always negative, and the TM positive, the latter dominating the former.

In this paper, we carry the sphere self-entropy problem much further. We consider a simple model of an electromagnetically coupled sphere, represented by a δ -function shell, with arbitrary coupling λ . In the limit as the coupling tends to infinity, this precisely corresponds to a perfectly conducting sphere. This model generalizes the previously

*kmilton@ou.edu†pushpakalauni60@gmail.com‡prachi.parashar@ntnu.no§liyang@ou.edu

described electromagnetic δ -function plate [23], and our electromagnetic δ -function sphere [24], considered at zero temperature. (The closely related plasma spherical shell was considered earlier in Refs. [25,26].) As in our previous works, we model the dispersive property of the shell by a plasma model. We discover that the finite-temperature problem is much more complex than might have been anticipated. Although it is generally supposed that the divergences in the self-free energy are confined to the zero-temperature contribution, this is not the case: the TM contribution to the entropy has both infrared and ultraviolet divergences, which violate the Nernst heat theorem, and hence require some sort of subtraction or “renormalization.” (In this paper, by “renormalization” we mean subtraction of these divergences.) When this is done, the TM self-entropy is no longer always positive. It is positive only for sufficiently strong coupling, while the TE self-entropy is always negative as expected. So we have encountered new phenomena that will require further work to understand.

The outline of this paper is as follows. In Sec. II we set up the formalism and obtain the general expressions for the free energy of the δ -function sphere. The expressions are regulated by point-splitting in time and in the angle on the sphere. We also model the dispersive properties of the shell with a plasma model, characterized by a dimensionless coupling λ_0 . In Sec. III we consider the strong-coupling limit, that of a perfectly conducting spherical shell. Because of the appearance of an infrared singularity, a renormalization of the temperature-dependent part of the free energy is required. After a temperature-dependent infrared-sensitive term is removed, the high- and low-temperature results of Balian and Duplantier [27] are recovered. Finite coupling behaviors are studied first in Sec. IV using the uniform asymptotic expansion for the spherical Bessel functions: at low temperature, this approach only yields the zero-temperature structure with divergences as seen in Ref. [24]. The low-order contributions to the entropy can actually be found exactly in Sec. V, but again the $O(\lambda_0^2)$ TE and the $O(\lambda_0)$ TM parts exhibit divergences which violate the Nernst theorem and must be renormalized away. The low-temperature behavior is studied for both the TE and TM modes in Sec. VI. The TE low-temperature entropy has a simple functional dependence on the coupling, while the TM mode again requires infrared renormalization, and exhibits a dependence on the coupling which is non-monotonic, being positive for strong coupling, but changing to negative as the coupling gets weaker. Finally, the high-temperature limit is discussed in Sec. VII, now using analytic regulation and the Chowla-Selberg formula. It is seen that the order of limits is important; we consider both the limits $aT \gg \lambda_0 \gg 1$, and $\lambda_0 \gg aT \gg 1$, where only the latter corresponds to the perfect conductor. The TM mode again requires a temperature-dependent renormalization. The results of this paper are summarized in Sec. VIII. In the last section of the paper we offer some concluding remarks.

We use natural units $\hbar = c = k_B = 1$, and Heaviside-Lorentz electromagnetic units.

II. ELECTROMAGNETIC δ -FUNCTION PLATE

As in Ref. [20], we can express the Casimir self-free energy of an object with permittivity $\epsilon = \mathbf{1} + \mathbf{V}$ in symbolic form

$$F = \frac{T}{2} \sum_{n=-\infty}^{\infty} \text{Tr} \ln(\mathbf{1} - \Gamma_0 \mathbf{V}), \quad (2.1)$$

expressed as a sum over Matsubara frequencies $\zeta_n = 2\pi nT$, where the trace is over spatial coordinates and internal variables (tensor indices). Here Γ_0 is the free electromagnetic Green’s dyadic, which satisfies

$$\left[-\frac{1}{\zeta_n^2} \nabla \times \nabla \times - \mathbf{1} \right] \Gamma_0(\mathbf{r} - \mathbf{r}') = \mathbf{1} \delta(\mathbf{r} - \mathbf{r}'). \quad (2.2)$$

It is convenient to define a divergence-free Green’s dyadic, which differs from this by a δ -function term [28]:

$$\Gamma_0'(\mathbf{r} - \mathbf{r}') = \Gamma_0(\mathbf{r} - \mathbf{r}') + \mathbf{1} \delta(\mathbf{r} - \mathbf{r}'), \quad \nabla \cdot \Gamma_0' = 0. \quad (2.3)$$

This dyadic can be resolved in terms of vector spherical harmonics

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_{lm}(\Omega), \quad \mathbf{L} = \mathbf{r} \times \frac{1}{i} \nabla, \quad (2.4)$$

as follows:

$$\begin{aligned} \Gamma_0'(\mathbf{r} - \mathbf{r}') &= \sum_{nlm} [-\zeta_n^2 g_l^0(r, r') \mathbf{X}_{lm}(\Omega) \mathbf{X}_{lm}^*(\Omega') \\ &\quad - \nabla \times g_l^0(r, r') \mathbf{X}_{lm}(\Omega) \mathbf{X}_{lm}^*(\Omega') \times \tilde{\nabla}'], \end{aligned} \quad (2.5)$$

where in spherical coordinates $\mathbf{r} = (r, \Omega) = (r, \theta, \phi)$.

For the case of a sphere of radius a described by a semitransparent potential, $\mathbf{V} = \lambda \delta(r - a)(\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})$, the potential is tangent to the surface of the sphere. (This transversality is required by Maxwell’s equations [23].) The trace is then worked out by using the following orthonormality properties of the vector spherical harmonics [29]:

$$\int d\Omega \mathbf{X}_{l'm'}^*(\Omega) \mathbf{X}_{lm}(\Omega) = \delta_{ll'} \delta_{mm'}, \quad (2.6a)$$

$$\int d\Omega f(r') \mathbf{X}_{l'm'}^*(\Omega) \nabla \times g(r) \mathbf{X}_{lm}(\Omega) = 0, \quad (2.6b)$$

$$\begin{aligned} \int d\Omega \nabla' \times f(r') \mathbf{X}_{l'm'}^*(\Omega) \cdot (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}}) \cdot \nabla \times g(r) \mathbf{X}_{lm}(\Omega) \\ = \delta_{ll'} \delta_{mm'} \frac{1}{r r'} \frac{\partial}{\partial r'} (r' f(r')) \frac{\partial}{\partial r} (r g(r)). \end{aligned} \quad (2.6c)$$

The trace in Eq. (2.1) is carried out by expanding the logarithm, doing the trace in each order, and resumming to get

$$F = \frac{T}{2} \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} (2l+1) [\ln(1 + \lambda \zeta_n^2 a^2 g_l^0(a, a)) + \ln(1 - \lambda \partial_r r \partial_{r'} r' g_l^0(r, r'))|_{r=r'=a}]. \quad (2.7)$$

The free reduced spherical Green's function is

$$g_l^0(r, r') = \frac{1}{\kappa r r'} s_l(|\zeta_n| r_{<}) e_l(|\zeta_n| r_{>}), \quad (2.8)$$

where $r_{<(>)}$ represents the lesser or greater of r and r' , and the modified Ricatti-Bessel functions are

$$s_l(x) = \sqrt{\frac{\pi x}{2}} I_{l+1/2}(x), \\ e_l(x) = \sqrt{\frac{2x}{\pi}} K_{l+1/2}(x), \quad (2.9)$$

which have a Wronskian equal to one.

The final form of the expression we must evaluate for the free energy of a semitransparent sphere is defined by 1) inserting point-splitting in (imaginary) time, with parameter τ , and in the spatial directions transverse to the normal of the sphere, with angle parameter δ , and 2) by using the “plasma-model” dispersion relation for the coupling, $\lambda = \lambda_0/(\zeta_n^2 a)$, where λ_0 is a dimensionless constant. These two processes are precisely those followed in Ref. [20]. The regulated free energy is ($x = |\zeta_n|a$)

$$F = \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \\ \times \left[\ln \left(1 + \lambda_0 \frac{e_l(x) s_l(x)}{x} \right) + \ln \left(1 - \lambda_0 \frac{e'_l(x) s'_l(x)}{x} \right) \right]. \quad (2.10)$$

The first term here is the TE free energy and the second is the TM free energy. These two contributions, which reduce to the familiar result for a perfectly conducting shell in the limit $\lambda_0 \rightarrow \infty$, are just those discussed earlier in Ref. [30] [see Eq. (96) there], with the following identifications of the coupling constants there: $\lambda^{\text{TE}} = \lambda_0 a$, $\lambda^{\text{TM}} = -\lambda_0 a/x^2$. In the notation of Ref. [23], $\lambda_0 = \zeta_p a$. It is important to keep this a dependence in computing the self-stress on the sphere from $S = -\frac{\partial}{\partial a} F$; but here we are interested in the entropy, $S = -\frac{\partial}{\partial T} F$, so for notational convenience we keep the coupling as λ_0 .

III. STRONG COUPLING

Since the spherical-shell entropy problem has mostly been considered in the perfectly conducting limit, we begin

with that situation. In strong coupling, $\lambda_0 \rightarrow \infty$, the free energy reduces to

$$F_{\infty} = \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \\ \times \ln \frac{e_l(x) s_l(x) e'_l(x) s'_l(x)}{x^2}, \quad (3.1)$$

where the coupling has disappeared because

$$\sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) = -1 (\delta \neq 0) \quad \text{and} \\ \sum_{n=-\infty}^{\infty} e^{inx} = 0 (x \neq 0 \mod 2\pi). \quad (3.2)$$

To isolate divergences we use the uniform asymptotic expansion (UAE) [31], which gives for the leading behavior of the logarithm¹:

$$\ln e_l(x) s_l(x) e'_l(x) s'_l(x) \sim -\ln 4 - \frac{t^6}{4\nu^2} + \frac{t^6}{32\nu^4} (4 - 54t^2 \\ + 120t^4 - 71t^6) + O(\nu^{-6}), \\ \nu \rightarrow \infty, \quad (3.3)$$

where $x = \nu z$, $t = (1 + z^2)^{-1/2}$, and $\nu = l + 1/2$. Note that there are no odd orders of ν^{-1} in this expansion. This expression is actually valid at $n = 0$ where $t = 1$ [see Eq. (3.14) below].

A. Leading term

The leading term in the expansion of Eq. (3.1),

$$F_{\infty}^{(0)} = \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) (-\ln 4x^2), \quad (3.4)$$

would be thought as *a priori* irrelevant, since it does not refer to the sphere, in view of Eq. (3.2). However, if we take it seriously, we write it as

$$F_{\infty}^{(0)} = \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \ln 4\zeta_n^2 a^2. \quad (3.5)$$

The $n = 0$ term is not defined here, so we regularize this infrared divergence by replacing ζ_n^2 by $\zeta_n^2 + \mu^2$, where dimensionlessly, $p = \mu a$, with μ being a photon “mass.”²

¹This is an astoundingly good approximation. Even for $l = 1$ the discrepancy between the two sides of Eq. (3.3) is within 0.1% for all x .

²In Ref. [32], the case of a massive field was considered. As noted there, when the a_2 heat-kernel coefficient is nonzero, as is the case here, the massless limit cannot be taken.

We will assume p is smaller than any other scale in the problem. Thus, the $n = 0$ term is simply

$$F_{\infty, n=0}^{(0)} = \frac{T}{2} \ln 4p^2, \quad (3.6)$$

but for $n \neq 0$, p will be neglected. Then, the sum can be expressed in terms of polylogarithms, where we now abbreviate $\alpha = 2\pi aT$ ($\tilde{\tau} = \tau/a$):

$$\begin{aligned} F_{\infty, n \neq 0}^{(0)} &= 2T \frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} (2\alpha n)^{\beta} \cos \alpha n \tilde{\tau} |_{\beta=0} \\ &= 2T \Re \frac{\partial}{\partial \beta} [(2\alpha)^{\beta} \text{Li}_{-\beta}(e^{i\alpha \tilde{\tau}})] |_{\beta \rightarrow 0}. \end{aligned} \quad (3.7)$$

At $\beta = 0$, $\text{Li}_0(z) = z/(1-z)$, while it can be shown for small τ that

$$\frac{\partial}{\partial \beta} \text{Li}_{-\beta}(e^{i\alpha \tilde{\tau}}) |_{\beta=0} = \frac{1}{i\alpha \tilde{\tau}} [\ln(-i\alpha \tilde{\tau}) + \gamma] - \frac{1}{2} \ln 2\pi, \quad (3.8)$$

so we find that

$$\begin{aligned} F_{\infty}^{(0)} &= \frac{T}{2} \ln 4p^2 - \frac{1}{2\tau} - T \ln \frac{2\alpha}{2\pi} = -\frac{1}{2\tau} + T \ln \frac{2\pi p}{\alpha} \\ &= -\frac{1}{2\tau} + T \ln \frac{\mu}{T}. \end{aligned} \quad (3.9)$$

The same result is easily obtained by use of the Euler-Maclaurin sum formula,

$$\sum_{n=0}^{\infty} 'g(n) = \int_0^{\infty} dn g(n) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} g^{(2k-1)}(0), \quad (3.10)$$

where the prime means that the $n = 0$ term is counted with half weight. This formula provides a formal asymptotic expansion in terms of the Bernoulli numbers. Even more simply, a corresponding answer is found when we use analytic regulation (we will return to this technique later in Sec. VII), defining

$$\begin{aligned} F_{\infty, n \neq 0}^{(0)} &= -T \sum_{n=1}^{\infty} (2l+1) \ln 4n^2 \alpha^2 = \frac{11}{6} T \frac{\partial}{\partial \beta} \sum_{n=1}^{\infty} (2n\alpha)^{\beta} |_{\beta \rightarrow 0} \\ &= \frac{11}{6} T [\zeta(0) \ln 2\alpha - \zeta'(0)] = -\frac{11}{12} T \ln \frac{2\alpha}{2\pi}, \end{aligned} \quad (3.11)$$

or

$$F_{\infty}^{(0)} = \frac{11}{12} T \ln \frac{2\pi p}{\alpha} \quad (3.12)$$

which uses the replacement $\sum_{l=1}^{\infty} (2l+1) \rightarrow -\frac{11}{12}$ rather than -1 obtained from the point-splitting cutoff. Aside from this fact the ultraviolet finite answer obtained is the

same as that given in Eq. (3.9). Evidently, with the infrared sensitive term included, this contribution is independent of the sphere's radius (which is obvious *a priori*); that fact, together with the sensitivity of the coefficient to the choice of regulation scheme, implies that this contribution is unphysical, and should be disregarded (subtracted).

B. Low-temperature behavior

To see the low-temperature dependence of the free energy, and thus the entropy, we return to the exact strong coupling formula (3.1). As we see from the Euler-Maclaurin formula, to get a temperature correction to the zero-temperature energy we need a contribution odd in n . In fact, the lowest-order odd term in the logarithm occurs for $l = 1$, where $\ln(-e_1 e'_1 s_1 s'_1) \sim \dots -\frac{1}{3}x^3 + \dots$, where the leading omitted terms are even in x . Then the Euler-Maclaurin formula immediately leads to the leading low-temperature correction to the free energy

$$\Delta F_{\infty} = T \frac{B_4}{4!} 3 \cdot 2(2\pi T a)^3 = -\frac{(\pi a)^3}{15} T^4, \quad aT \ll 1, \quad (3.13)$$

as first found in Ref. [27]. [See also Sec. 9.5 of Ref. [33].]

C. High-temperature behavior

To get the high-temperature behavior, we need to consider $n = 0$ separately from $n \neq 0$. The former is

$$\begin{aligned} F_{\infty, n=0} &= \frac{T}{2} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \left(1 - \frac{1}{(2l+1)^2} \right) \\ &= -\frac{T}{2} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{\infty} \frac{P_l(\cos \delta)}{(2l+1)^{2k-1}}. \end{aligned} \quad (3.14)$$

The $k = 1$ term here is divergent as $\delta \rightarrow 0$. The balance of $F_{\infty, n=0}$ sums to

$$\begin{aligned} F_{\infty, n=0} &\rightarrow -T \left(\frac{1}{2} - \frac{1}{4} \gamma + \frac{7}{12} \ln 2 - 3 \ln G \right) \\ &= -\frac{T}{2} (0.027537), \end{aligned} \quad (3.15)$$

where G is Glaisher's constant, $\ln G = \frac{1}{12} - \zeta'(-1)$. For $n \neq 0$, the $O(\nu^{-2})$ term in the UAE expansion of the logarithm seen in Eq. (3.3) is given as

$$F_{\infty}^{(2)} = -T \sum_{n=1}^{\infty} \cos \zeta_n \tau \sum_{l=1}^{\infty} \frac{P_l(\cos \delta)}{2l+1} \frac{\nu^6}{(\nu^2 + \zeta_n^2 a^2)^3}, \quad (3.16)$$

where the sum on n is readily carried out to yield (for simplicity, we have set $\tau = 0$)

$$F_{\infty}^{(2)} = -\frac{T}{2} \sum_{l=1}^{\infty} P_l(\cos \delta) \left\{ -\frac{1}{2\nu} + \frac{1}{32aT} \left[3 \coth \frac{\nu}{2aT} + \frac{3}{2aT} \operatorname{csch}^2 \frac{\nu}{2aT} + \frac{1}{2} \left(\frac{\nu}{aT} \right)^2 \coth \frac{\nu}{2aT} \operatorname{csch}^2 \frac{\nu}{2aT} \right] \right\}. \quad (3.17)$$

The first term in curly brackets in this expression precisely cancels the $k = 1$ term in Eq. (3.14). For large $\nu/2aT$ (this is the low-temperature limit) the hyperbolic cotangent in Eq. (3.17) tends to one, so we should remove that limiting term, which amounts to the zero-energy term,

$$F_{\infty}^{T=0} = -\frac{T}{2} \sum_{l=1}^{\infty} P_l(\cos \delta) \frac{3}{32aT} = \frac{3}{64a} - \frac{3}{64a\delta}. \quad (3.18)$$

In the same limit, the balance of Eq. (3.17) vanishes exponentially fast. The finite part of this is, as noted in Ref. [28], within 2% of the exact Boyer result [27,28,34] for a perfectly conducting spherical shell, $E_B = -0.04617/a$. The divergent term appearing here does not appear in other ways of regulating the zero-point energy. Including higher terms in the UAE, and computing the remainder, indeed gives exactly the Boyer result, at zero temperature.

For high temperature, we evaluate what is left in Eq. (3.17) via the Euler-Maclaurin formula. The integral gives the leading term:

$$\begin{aligned} F_{\infty}^{T \rightarrow \infty} &= -\frac{T}{32} \int_{3/4aT}^{\infty} dx [3(\coth x - 1) + 3x \operatorname{csch}^2 x \\ &\quad + 2x^2 \coth x \operatorname{csch}^2 x] \\ &= -\frac{T}{32} \left[\frac{3}{4aT} \left(3 + 5 \coth \frac{3}{4aT} + \frac{3}{4aT} \operatorname{csch}^2 \frac{3}{4aT} \right) \right. \\ &\quad \left. - 8 \ln \left(2 \sinh \frac{3}{4aT} \right) \right] \\ &\rightarrow -\frac{T}{16} \left(3 - 4 \ln \frac{3}{2aT} \right) - \frac{9}{128a} + O(T^{-5}). \end{aligned} \quad (3.19a) \quad (3.19b)$$

The last limit assumes $aT \gg 1$. The remaining terms in the Euler-Maclaurin series are easily evaluated using Borel summation:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_{n+1}}{n+1} x^n &= \int_0^{\infty} dt e^{-t} \frac{1}{xt} \left(\frac{xt}{e^{xt} - 1} - 1 \right) \\ &= -1 - \frac{1}{x} \ln x - \frac{1}{x} \psi \left(\frac{1}{x} \right), \end{aligned} \quad (3.20)$$

where ψ is the digamma function. This yields the final contribution to the free energy:

$$F_{\infty}^{T \rightarrow \infty} = \frac{T}{4} \left[\ln \frac{2}{3} + \psi \left(\frac{3}{2} \right) \right] + \frac{3}{128a}. \quad (3.21)$$

Adding this result to that in Eqs. (3.15), (3.18), and (3.19b), we see that the finite temperature-independent part cancels, leaving for the high-temperature limit apart from a divergent constant

$$\begin{aligned} F_{\infty}^{T \rightarrow \infty} &\sim T \left(-\frac{1}{4} \ln aT - \frac{3}{16} - \frac{13}{12} \ln 2 + 3 \ln G \right) \\ &= -\frac{T}{4} (\ln aT + 0.768584), \quad aT \gg 1, \end{aligned} \quad (3.22)$$

a result first obtained by Balian and Duplantier [27]. (See also Refs. [33,35].) (To understand this high-temperature limit better, we will break this up into TE and TM parts in Sec. VII C).

To obtain the next term in the high-temperature expansion, we consider the order ν^{-4} term in Eq. (3.3). This term can be written as

$$F_{\infty}^{(4)} = -\frac{T}{16} \sum_{l=1}^{\infty} \nu^3 g(y), \quad (3.23)$$

where, with $y = \nu/2Ta$ and $\alpha = 2\pi Ta$,

$$\begin{aligned} g(y) &= \frac{1}{2(\alpha/\pi)^6} \left[2 \left(\frac{d}{dy^2} \right)^2 + 9y^2 \left(\frac{d}{dy^2} \right)^3 + 5y^4 \left(\frac{d}{dy^2} \right)^4 \right. \\ &\quad \left. + \frac{71}{120} y^6 \left(\frac{d}{dy^2} \right)^5 \right] \left[-\frac{1}{y^2} + \frac{1}{y} \coth y \right]. \end{aligned} \quad (3.24)$$

The leading contribution comes from the first, integral, term in the Euler-Maclaurin formula:

$$F_{\infty}^{(4), T \rightarrow \infty} \sim -\frac{T}{16} \left(\frac{\alpha}{\pi} \right)^4 \int_{3\pi/2\alpha \rightarrow 0}^{\infty} dx x^3 g(x) = -\frac{1}{3840a^2T}, \quad (3.25)$$

or all together,

$$F_{\infty}^{T \rightarrow \infty} \sim -\frac{T}{4} (\ln aT + 0.7686) - \frac{1}{3840a^2T}, \quad (3.26)$$

again as first derived in Ref. [27].

IV. FINITE COUPLING BEHAVIORS

Now let us return to Eq. (2.10) with finite coupling λ_0 , and approximate the logarithm using the UAE:

$$\begin{aligned} \ln \left[\left(1 + \frac{\lambda_0}{x} e_l(x) s_l(x) \right) \left(1 - \frac{\lambda_0}{x} e'_l(x) s'_l(x) \right) \right] &\sim \sum_{k=1}^{\infty} \frac{a^{(k)}(t)}{(2\nu)^k}, \\ \nu &\gg 1, \end{aligned} \quad (4.1)$$

where the first four expansion coefficients are

$$a^{(1)}(t) = 2\lambda_0 t, \quad (4.2a)$$

$$a^{(2)}(t) = -\lambda_0^2 t^2, \quad (4.2b)$$

$$a^{(3)}(t) = \frac{\lambda_0}{3}(-3t^7 + 2\lambda_0^2 t^3), \quad (4.2c)$$

$$a^{(4)}(t) = \frac{\lambda_0^2}{2}(2t^8 - \lambda_0^2 t^4), \quad (4.2d)$$

with again $t = (1 + z^2)^{-1/2}$, $x = \nu z$. Here we have dealt with the infrared divergence in the TM contribution by replacing $1/z^2 t \rightarrow t$, as discussed in Ref. [24], because this substitution does not change the ultraviolet behavior. The idea was that the error introduced is compensated by the remainder, and this substitution is sufficient to capture the divergence structure. However, this would not be expected to be valid for the $n = 0$ term, so we will return to this point later, when discussing the temperature dependence. Note that this expansion is not a power-series expansion in the coupling λ_0 .

The first-order term in this expansion is

$$F^{(1)} = 2\lambda_0 T \sum_{n=0}^{\infty} {}' \cos \zeta_n \tau \sum_{l=1}^{\infty} P_l(\cos \delta) \frac{1}{\sqrt{1 + \zeta_n^2 a^2 / \nu^2}}. \quad (4.3)$$

For low temperature, we again evaluate the sum on n using the Euler-Maclaurin sum formula (3.10). The integral term there is all there is, because the n summand is even about $n = 0$. (This will result in no temperature dependence being revealed, as we saw in the previous section.) That integral is immediately seen to be

$$F^{(1)} = \frac{\lambda_0}{2\pi a} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) K_0(\nu \tilde{\tau}), \quad \tilde{\tau} = \tau/a. \quad (4.4)$$

Since only the τ cutoff is essential here, we set $\delta \rightarrow 0$ and have

$$F^{(1)} = \frac{\lambda_0}{\pi a} \sum_{l=1}^{\infty} \nu K_0(\nu \tilde{\tau}). \quad (4.5)$$

This sum, in turn, may be evaluated using the Euler-Maclaurin formula around $l = 1$. The integral term gives

$$F^{(1a)} = \frac{\lambda_0}{\pi a} \left[\frac{1}{\tilde{\tau}^2} + \frac{9}{16} \left(-1 + 2\gamma + 2 \ln \frac{3\tilde{\tau}}{4} \right) \right]. \quad (4.6)$$

The remainder terms involve, with $g(l) = \nu K_0(\nu \tilde{\tau})$,

$$\frac{1}{2} g(1) = -\frac{3}{4} \left(\ln \frac{3\tilde{\tau}}{4} + \gamma \right), \quad (4.7a)$$

$$g'(1) = -\gamma - 1 - \ln \frac{3\tilde{\tau}}{4}, \quad (4.7b)$$

$$g^{(2k-1)}(1) = \frac{(2k-3)!}{(3/2)^{2k-2}}, \quad k > 1. \quad (4.7c)$$

The resulting series is Borel-summable:

$$\begin{aligned} F(x) &= x \sum_{k=2}^{\infty} B_{2k} \frac{(2k-3)!}{(2k)!} x^{2k-3} \\ &= x \int_0^{\infty} dt \frac{e^{-t}}{(xt)^3} \left[\frac{xt}{e^{xt} - 1} - B_0 - B_1 xt - \frac{B_2}{2} (xt)^2 \right] \\ &= \frac{1}{12x^2} \left[-9 - 6x + (6 + 6x + x^2)(\gamma + \ln x) \right. \\ &\quad \left. - 12x^2(1 - \gamma)\zeta\left(-1, 1 + \frac{1}{x}\right) \right. \\ &\quad \left. - 12x^2\zeta^{(1,0)}\left(-1, 1 + \frac{1}{x}\right) \right], \end{aligned} \quad (4.8)$$

where the integral is evaluated at $x = 2/3$, by analytically continuing in the power of xt in the denominator. The numerical value $F(2/3) = -0.00058434$. Adding together the components, we obtain

$$F^{(1)} = \frac{\lambda_0}{\pi a} \left[\frac{1}{\tilde{\tau}^2} + \frac{11}{24} \ln \tilde{\tau} - 0.345879 \right]. \quad (4.9)$$

The second term in the UAE gives

$$F^{(2)} = -\lambda_0^2 T \sum_{l=1}^{\infty} \frac{P_l(\cos \delta)}{2\nu} \sum_{n=0}^{\infty} {}' \cos \zeta_n \tau \frac{1}{1 + \zeta_n^2 a^2 / \nu^2}. \quad (4.10)$$

Evaluating the n sum again by the Euler-Maclaurin formula, the integral there gives

$$\begin{aligned} F^{(2)} &= -\frac{\lambda_0^2}{8a} \sum_{l=1}^{\infty} P_l(\cos \delta) e^{-\nu \tilde{\tau}} = \frac{\lambda_0^2}{8a} \left(1 - \frac{1}{\Delta} \right), \\ \Delta &= \sqrt{\delta^2 + \tilde{\tau}^2}. \end{aligned} \quad (4.11)$$

There are no remainder terms, because the n summand is even.

The third term

$$F^{(3)} = -\frac{\lambda_0 T}{3} \sum_{n=0}^{\infty} \cos \zeta_n \tau \sum_{l=1}^{\infty} \frac{P_l(\cos \delta)}{(2\nu)^2} \left[-\frac{3}{(1 + \zeta_n^2 a^2 / \nu^2)^{7/2}} + \frac{2\lambda_0^2}{(1 + \zeta_n^2 a^2 / \nu^2)^{3/2}} \right], \quad (4.12)$$

is most easily evaluated when $\tau = 0$. As before the n sum can be replaced by an integral, and the sum on l is easily carried out, with the result

$$F^{(3)} = \frac{\lambda_0}{30\pi a} (4 - 5\lambda_0^2) \left(1 - \frac{3}{2} \ln 2 + \frac{1}{2} \ln \delta \right). \quad (4.13)$$

The fourth term, and those thereafter, are finite:

$$F^{(4)} = \frac{\lambda_0^2}{128a} \left(\frac{\pi^2}{8} - 1 \right) (5 - 4\lambda_0^2). \quad (4.14)$$

The above analysis, based on the uniform asymptotic expansion, and the Euler-Maclaurin summation formula, exhibits no temperature dependence in the free energy. This is because the n summand is even in n , so the Euler-Maclaurin formula (or the Abel-Plana formula) allows the sum to be replaced by an integral, which precisely corresponds to the zero-temperature energy.

V. LOWEST-ORDER COUPLING CONTRIBUTION TO THE FREE ENERGY

As in strong coupling, to get the low-temperature correction, we must return to the exact expression (2.10). In this section we will consider low orders in the coupling, which can be treated exactly. We will first consider the first order in λ_0 contributions.

A. TE $O(\lambda_0)$ behavior

The TE contribution to the free energy in Eq. (2.10) may be expanded to first order in λ_0 as

$$F_{(1)}^{\text{TE}} = \lambda_0 \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \frac{e_l(x) s_l(x)}{x}. \quad (5.1)$$

We can evaluate this exactly. We use the summation theorem [36]

$$\sum_{l=0}^{\infty} (2l+1) P_l(\cos \delta) e_l(x) s_l(y) = \frac{xy}{\rho} e^{-\rho}, \quad \rho = \sqrt{x^2 + y^2 - 2xy \cos \delta} \quad (5.2)$$

to evaluate the l -sum as

$$\sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) e_l(x) s_l(x) = \frac{x}{u} e^{-|x|u} - e_0(x) s_0(x), \quad (5.3)$$

where $u = \sqrt{2(1 - \cos \delta)} \approx \delta$. Then the sum over Matsubara frequencies is ($\alpha = 2\pi T a$, $\tilde{\tau} = \tau/a$)

$$F_{(1)}^{\text{TE}} = \lambda_0 \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{in\tilde{\tau}} \left[\frac{1}{u} e^{-|n|u} + \frac{1}{2|n|\alpha} (e^{-2|n|\alpha} - 1) \right]. \quad (5.4)$$

This is readily evaluated to be

$$F_{(1)}^{\text{TE}} = \frac{\lambda_0}{2\pi a} \left(\frac{1}{u^2 + \tilde{\tau}^2} + \frac{1}{2} \ln \frac{\tilde{\tau}}{2} + \frac{1}{12} \alpha^2 - \frac{1}{2} \ln \frac{\sinh \alpha}{\alpha} \right). \quad (5.5)$$

The free energy diverges as τ and δ tend to zero, but the entropy is finite:

$$S_{(1)}^{\text{TE}} = -\frac{\partial F^{\text{TE}}}{\partial T} = -\lambda_0 \left(\frac{\alpha}{6} + \frac{1}{2\alpha} - \frac{1}{2} \coth \alpha \right). \quad (5.6)$$

For low temperature, the entropy tends to $-\lambda_0 \alpha^3/90$, a result which will be rederived below, while for high temperature,

$$S_{(1)}^{\text{TE}} \sim -\lambda_0 \left(\frac{\alpha}{6} - \frac{1}{2} + \frac{1}{2\alpha} \right). \quad (5.7)$$

B. TE $O(\lambda_0^2)$ behavior

We can also exactly calculate the TE $O(\lambda_0^2)$ behavior. Squaring the identity (5.2) and integrating over angles, we obtain $[w = 2x\sqrt{2(1 - \cos \theta)}]$, where θ is the angle in the sum rule]

$$\sum_{l=0}^{\infty} (2l+1) e_l^2(x) s_l^2(x) = \frac{x^2}{2} \int_0^{4x} \frac{dw}{w} e^{-w}, \quad (5.8)$$

which, unfortunately, loses the angular point-splitting regulator. So now the second-order TE free energy is given by the expression

$$F_{(2)}^{\text{TE}} = -\frac{\lambda_0^2 T}{4} \sum_{n=0}^{\infty} \cos n\tilde{\tau} \left[\int_0^{4n\alpha} \frac{dw}{w} e^{-w} - \frac{2}{x^2} e_0^2(x) s_0^2(x) \right]. \quad (5.9)$$

Because this expression exhibits no ultraviolet divergence, we can set τ to zero. Let us consider the second term (the $l=0$ contribution), first. Since $e_0^2(x) s_0^2(x) = e^{-2x} \sinh^2(x)$,

it is immediately evaluated as

$$F_a^{\text{TE}} = \frac{\lambda_0^2 \alpha}{8\pi a} \left\{ 1 + \frac{1}{2\alpha^2} \left[\frac{\pi^2}{6} - 2\text{Li}_2(e^{-2\alpha}) + \text{Li}_2(e^{-4\alpha}) \right] \right\}. \quad (5.10)$$

Let us similarly consider the low-temperature limit of the first term in Eq. (5.9); we use the Euler-Maclaurin formula to evaluate it, but because the summand is singular at $n = 0$, we do so about $n = 1$:

$$F_b^{\text{TE}} = -\frac{\lambda_0^2 \alpha}{8\pi a} \left[\int_0^\infty dn f(n) - \int_0^1 dn f(n) + \frac{1}{2} f(0) + \frac{1}{2} f(1) - \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} f^{(2k-1)}(1) \right]. \quad (5.11)$$

We can disregard the first term here, the integral from 0 to ∞ , because when the variable is changed from n to $n\alpha$, the integral is seen to be independent of T , and hence does not contribute to the entropy. The second integral from 0 to 1 is

$$\begin{aligned} \frac{\lambda_0^2 \alpha}{8\pi a} \int_0^1 dn f(n) &= \frac{\lambda_0^2 \alpha}{8\pi a} \left[\int_\eta^{4\alpha} \frac{dw}{w} e^{-w} + \frac{1}{4\alpha} (e^{-4\alpha} - 1) \right] \\ &= \frac{\lambda_0^2 \alpha}{8\pi a} \left[\frac{1}{4\alpha} (e^{-4\alpha} - 1) + \text{Ei}(-4\alpha) - \text{Ei}(-\eta) \right], \end{aligned} \quad (5.12)$$

where Ei is the exponential integral function, and we have regulated the divergence at $\cos \theta = 1$ ($w = 0$) by inserting a small positive number η . To evaluate the $f(0)$ term we again need to insert the photon mass parameter p , and then we obtain

$$\frac{\lambda_0^2 \alpha}{8\pi a} \frac{1}{2} f(0) = \frac{\lambda_0^2 \alpha}{16\pi a} \ln \frac{4p}{\eta}. \quad (5.13)$$

The $\frac{1}{2} f(1)$ term is

$$-\frac{\lambda_0^2 \alpha}{16\pi a} \int_\eta^{4\alpha} \frac{dw}{w} e^{-w} = -\frac{\lambda_0^2 \alpha}{16\pi a} [\text{Ei}(-4\alpha) - \text{Ei}(-\eta)]. \quad (5.14)$$

The terms in the Bernoulli series are also readily worked out to all orders:

$$f^{(2k-1)}(1) = \Gamma(2k-1, 4\alpha). \quad (5.15)$$

The leading terms in the Bernoulli expansion are readily evaluated by Borel summation,

$$\begin{aligned} -\sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} (2k-2)! &= -\int_0^\infty \frac{dt}{t^2} e^{-t} \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) \\ &= \frac{1}{2} \ln 2\pi - 1. \end{aligned} \quad (5.16)$$

Then, adding all these components, we find through $O(\alpha^4)$ the second-order TE free energy to be

$$\Delta F_{(2)}^{\text{TE}} = -\frac{\lambda_0^2}{8\pi a} \left\{ \frac{\alpha}{2} \ln \frac{2\pi p}{\alpha} + \frac{\alpha^4}{270} \right\}, \quad (5.17)$$

where the Δ symbol signifies that temperature-independent constants have been dropped. Although the α^2 and α^3 terms canceled, as has the collinear cutoff η , there persists a linear term in T and a $T \ln T$ term dependent on the photon-mass infrared cutoff, which if present would violate the Nernst heat theorem. We have seen precisely such terms appearing in the strong-coupling limit, Eq. (3.9), and will see additional power divergences in the TM contribution to the free energy already in order λ_0 . As we will see below, such divergences are to be omitted. In particular, we will not see this term when we consider the general low-temperature expansion in Sec. VI A.

We can easily extend this calculation to all orders in α . The key observation is that in $O(\alpha^{2k})$, $k \geq 1$, all contributions cancel except that from the B_{2k} term in the Bernoulli sum and the contribution from the $l = 0$ term (5.10). This is a consequence of the identity

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad n > 1. \quad (5.18)$$

Adding up the remainder, leads to the temperature dependence of the free energy in second order:

$$\begin{aligned} \Delta F_{(2)}^{\text{TE}} &= \frac{\lambda_0^2}{8\pi a} \left\{ \alpha + \frac{1}{2\alpha} \left[\frac{\pi^2}{6} - 2\text{Li}_2(e^{-2\alpha}) + \text{Li}_2(e^{-4\alpha}) \right] \right. \\ &\quad \left. - \frac{1}{4} h(4\alpha) - \frac{1}{2} \ln \frac{2\pi p}{\alpha} \right\}, \end{aligned} \quad (5.19)$$

where

$$h(x) = x \int_0^x \frac{dt}{t^2} \left[\frac{t}{e^t - 1} - 1 + \frac{1}{2} t \right], \quad (5.20)$$

which has the limits

$$\begin{aligned} h(x) &\sim \frac{x^2}{12} - \frac{x^4}{2160}, \quad x \ll 1, \\ h(x) &\sim \frac{x}{2} \ln x - Ax, \quad A = 0.63033, \quad x \gg 1. \end{aligned} \quad (5.21)$$

The limiting behaviors of the free energy are

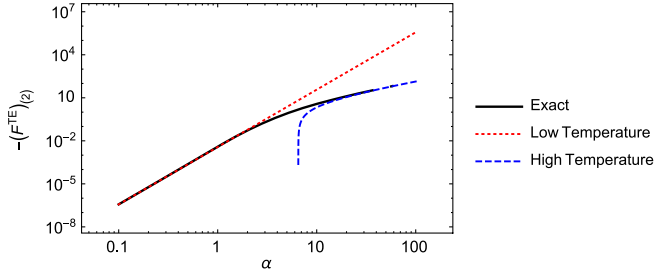


FIG. 1. The negative of the TE free energy in second order in λ_0 , apart from a factor of $\lambda_0^2/(8\pi a)$. The solid line shows the numerically computed free energy (5.19) compared with the low-temperature approximation (dotted curve) and the high-temperature approximation (dashed curve), given in Eq. (5.22).

$$\Delta F_{(2)}^{\text{TE}} \sim \frac{\lambda_0^2}{8\pi a} \begin{cases} -\frac{\alpha^4}{270}, & \alpha \ll 1, \\ 1.63033\alpha - \frac{1}{2}\alpha \ln 4\alpha, & \alpha \gg 1. \end{cases} \quad (5.22)$$

The second-order TE free energy is shown in Fig. 1. Note that because $\Delta F_{(2)}^{\text{TE}}$ has a negative slope, the corresponding entropy contribution is positive, unlike the first-order contribution.

C. TM $O(\lambda_0)$ behavior

Extracting the weak-coupling behavior of the TM contribution is considerably more subtle. The expression we wish to evaluate is

$$F_{(1)}^{\text{TM}} = -\lambda_0 T \sum_{n=0}^{\infty} ' \cos n\alpha \tilde{\tau} \times \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \frac{1}{n\alpha} e'_l(n\alpha) s'_l(n\alpha). \quad (5.23)$$

We first note that the n sum is divergent because of an infrared divergence at $n=0$. To regulate this, we again insert the small photon mass parameter p . Then the $n=0$ contribution is immediately worked out:

$$F_{n=0}^{\text{TM}} = \frac{\lambda_0 \alpha}{4\pi a} \left(-\frac{1+u^2/4}{p^2 u^3} + 1 + \frac{u^2 + \tilde{\tau}^2}{2u^3} \right). \quad (5.24)$$

For the rest, we ignore the photon mass, and use the summation formula (5.2) to write $F_{n \neq 0}^{\text{TM}} = F_a^{\text{TM}} + F_b^{\text{TM}}$, where

$$F_a^{\text{TM}} = \lambda_0 T \sum_{n=1}^{\infty} \cos n\alpha \tilde{\tau} \frac{e'_0(n\alpha) s'_0(n\alpha)}{n\alpha}, \quad (5.25a)$$

$$F_b^{\text{TM}} = -\lambda_0 T \sum_{n=1}^{\infty} \cos n\alpha \tilde{\tau} \frac{e^{-una}}{4u^3(n\alpha)^2} \times [4 + u^2 + u(4 - 3u^2)n\alpha + u^4(n\alpha)^2]. \quad (5.25b)$$

Because $e'_0(x)s'_0(x) = -\frac{1}{2}(1 - e^{-2x})$, we can readily evaluate the subtracted $l=0$ term:

$$F_a^{\text{TM}} = \frac{\lambda_0}{4\pi a} [\ln \tilde{\tau} + \ln \alpha - \alpha + \ln(2 \sinh \alpha)]. \quad (5.26)$$

The n sums in the remaining part $F_{b,n \neq 0}^{\text{TM}}$ are straightforward, leading to, for small u and $\tilde{\tau}$,

$$F_b^{\text{TM}} = -\frac{\lambda_0}{8\pi a} \left\{ \frac{2}{3} \frac{\pi^2}{u^3 \alpha} - \frac{4}{u^2} \left(1 + \frac{\tilde{\tau}}{u} \arctan \frac{\tilde{\tau}}{u} \right) + \frac{1}{u} \left[\left(\frac{\tilde{\tau}^2}{u^2} + 1 \right) \alpha + \frac{\pi^2}{6\alpha} \right] + 2 \ln(u^2 + \tilde{\tau}^2) \alpha^2 - \frac{\tilde{\tau}}{u} \arctan \frac{\tilde{\tau}}{u} - 1 + \frac{1}{1 + \tilde{\tau}^2/u^2} - \frac{\alpha^2}{9} \right\}. \quad (5.27)$$

When Eqs. (5.24), (5.26), and (5.27) are combined, we are left with divergent terms that depend on temperature, as well as a finite remainder:

$$F_{(1)}^{\text{TM}} = -\frac{\lambda_0}{2\pi a} \left\{ \frac{1}{u^3} \left(\frac{\pi^2}{6\alpha} + \frac{\alpha}{2p^2} \right) \left(1 + \frac{u^2}{4} \right) + \left[-\frac{1}{u^2} \left(1 + \frac{u^2}{4} \right) \left(1 + \frac{\tilde{\tau}}{u} \arctan \frac{\tilde{\tau}}{u} \right) - \frac{1}{2} \ln \tilde{\tau} + \frac{1}{2} \ln(u^2 + \tilde{\tau}^2) + \frac{1}{4} \frac{1}{1 + \tilde{\tau}^2/u^2} \right] - \frac{1}{2} \ln \frac{2 \sinh \alpha}{\alpha} - \frac{\alpha^2}{36} \right\}. \quad (5.28)$$

Interestingly, with this way of regulating the infrared divergence, the τ -dependent and finite terms linear in α have canceled between $F_{n=0}^{\text{TM}}$ and $F_{n \neq 0}^{\text{TM}}$.

The entropy is obtained from the free energy by differentiating with respect to α :

$$S = -2\pi a \frac{\partial}{\partial \alpha} F, \quad (5.29)$$

so we see that the terms in $F_{(1)}^{\text{TM}}$ linear in α and inverse linear in α violate the Nernst heat theorem, and therefore seem unphysical. So, there is motivation for simply omitting those terms. We might think that it is the dimensional quantities τ and $a\delta$ which are the fixed regulators, which would suggest a scaling argument for removing these terms as irrelevant, but this is incorrect, since a direct calculation [24] of the stress tensor shows that the principle of virtual work, requiring that the stress on the sphere be the negative derivative of the free energy with respect to a , shows that the quantities τ and δ must be regarded as constant, so that terms in Eq. (5.28) depending on $\tilde{\tau}$ possess a dependence.

However we shall subsequently see two further justifications for “renormalizing” these terms away. The infrared and ultraviolet divergent terms encountered here are quite different from the logarithmic terms seen in Eqs. (3.9) and (5.17). The present divergent terms are power divergences, and so are more convincingly removed. Adopting such a prescription leaves only the finite terms in the free energy and the entropy

$$\begin{aligned}\hat{F}_{(1)}^{\text{TM}} &= \frac{\lambda_0}{4\pi a} \left(\ln \frac{2 \sinh \alpha}{\alpha} + \frac{\alpha^2}{18} \right), \\ \hat{S}_{(1)}^{\text{TM}} &= -\frac{\lambda_0}{2} \left(\coth \alpha - \frac{1}{\alpha} + \frac{\alpha}{9} \right).\end{aligned}\quad (5.30)$$

Unfortunately, this TM contribution to the entropy, like the TE contribution (5.6), is always negative, and the sum of the two is linear in the temperature:

$$\hat{S}_{(1)} = S_{(1)}^{\text{TE}} + \hat{S}_{(1)}^{\text{TM}} = -\frac{2}{9} \lambda_0 \alpha. \quad (5.31)$$

The first-order renormalized entropy is displayed in Fig. 2. Both contributions to the entropy are now negative.

The same result (5.28) is obtained if instead of direct summation, the Euler-Maclaurin formula (around $n = 1$, because the summand is not analytic at the origin) is used. However, remarkably, the Abel-Plana formula gives directly the finite part with all the divergences confined to the temperature-independent part. This is actually not surprising, because analytic regularization techniques omit power divergences. The Abel-Plana formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^{\infty} dt f(t) + i \int_0^{\infty} dt \frac{f(it) - f(-it)}{e^{2\pi t} - 1}, \quad (5.32)$$

concentrates all the divergent terms in the first temperature-independent integral. All the temperature dependence is contained in

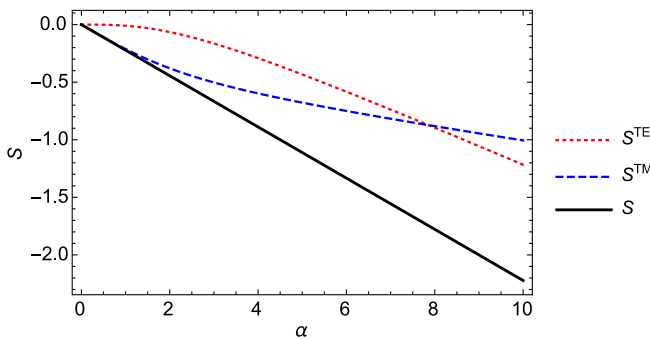


FIG. 2. The TE and “renormalized” TM contributions to the entropy in first order in the coupling. Both the TE and the TM contributions are negative, leading to a sum which is negative and linear in the temperature.

$$\begin{aligned}\Delta F_{(1)}^{\text{TM}} &= -i\lambda_0 T \int_0^{\infty} dn \frac{\cos n\alpha \tilde{\tau}}{e^{2n\pi} - 1} \frac{1}{n\alpha} \\ &\times \left[-e'_0(in\alpha) s'_0(in\alpha) + \frac{\partial}{\partial x} \frac{\partial xy}{\partial y \rho} e^{-i\varphi} + (\text{c.c.}) \right]_{x=y=n\alpha},\end{aligned}\quad (5.33)$$

where ρ is defined in (5.2), which is just the Minkowski version of the Euclidean form in Eq. (5.25). Now, because we have sines and cosines instead of real exponentials, contributions are finite, and we can set $\tau \rightarrow 0$:

$$\Delta F_{(1)}^{\text{TM}} = \frac{\lambda_0}{\pi a} \int_0^{\infty} \frac{dx}{x} \frac{1}{e^{x/aT} - 1} \left(1 - \cos^2 x + \frac{x^2}{3} \right). \quad (5.34)$$

This may be easily shown, for example by Borel summation, to yield precisely the free energy and entropy shown in Eq. (5.30).

VI. LOW-TEMPERATURE BEHAVIOR

In this section we consider arbitrary finite coupling λ_0 , but examine the behavior of the free energy for low temperature. This can be readily extracted by use of the Euler-Maclaurin formula (3.10), which will concentrate all the ultraviolet divergences in the temperature-independent part of the free energy. The integral term in Eq. (3.10), as noted before, contributes only to the temperature-independent part.

A. TE low-temperature behavior

As noted in Sec. III the lowest-order appearance of an odd term in ζ_n occurs for $l = 1$:

$$\frac{e_1(x) s_1(x)}{x} \sim \frac{1}{3} - \frac{2}{15} x^2 + \frac{1}{9} x^3 + \dots, \quad x \ll 1. \quad (6.1)$$

The leading low-temperature correction comes by expanding the logarithm in powers of λ_0 . The k th-order term arising from the Bernoulli series in the Euler-Maclaurin formula is

$$\begin{aligned}\Delta F_{(k)}^{\text{TE}, T \rightarrow 0} &= -T \frac{B_4}{4!} (2\pi T)^3 \left(\frac{\partial}{\partial x} \right)^3 \\ &\times 3(-1)^{k+1} a^3 \lambda_0^k \left(\frac{1}{3} \right)^{k-1} \frac{x^3}{9} \Big|_{x=0} \\ &= -a^3 \left(-\frac{\lambda_0}{3} \right)^k \frac{\pi^3 T^4}{15}.\end{aligned}\quad (6.2)$$

When this is summed over all k we get

$$\Delta F^{\text{TE}, T \rightarrow 0}(\lambda_0) = \frac{(\pi a)^3}{15} T^4 \frac{1}{1 + 3/\lambda_0} \rightarrow \frac{(\pi a)^3}{15} T^4, \quad (6.3)$$

where the last replacement is the strong-coupling limit (which may be directly confirmed). The corresponding entropy is

$$S^{\text{TE}, T \rightarrow 0}(\lambda_0) = -\frac{\alpha^3}{30} \frac{1}{1 + 3/\lambda_0}, \quad (6.4)$$

which for $O(\lambda_0)$ coincides with the result found at the end of Sec. VA, and in $O(\lambda_0^2)$ agrees with the entropy computed from the $O(\alpha^4)$ term in Eq. (5.17), which constitutes further evidence of the irrelevance of the infrared-sensitive logarithmic term there. Comparing with the total low-temperature correction (3.13), we see that the strong-coupling limit of the TM contribution must be

$$\Delta F_{\infty}^{\text{TM}, T \rightarrow 0} = -\frac{2}{15} (\pi a)^3 T^4. \quad (6.5)$$

The corresponding entropies, $S = -\partial F / \partial T$, are negative for the TE contribution, and positive for the TM. As for the plate [20], the latter overwhelm the former.

It might well be objected that since the summand is not analytic at $n = 0$, it would be better to apply the Euler-Maclaurin formula about $n = 1$, as in Eq. (5.11). Doing so in this case yields exactly the same result as found here, but in the next subsection, we will see that an expansion about $n = 1$ is essential to get the result in the TM case, where the singularity at the origin is more severe.

These strong-coupling results were given in Sec. 9.5.1 of Ref. [33] and in Ref. [37], however with an additional T^3 term in the free energy subtracted in the case of the TE contribution, and the same term added to the TM contribution, so that the total contribution remains unchanged. These subtractions, motivated by the heat-kernel analysis of the Weyl expansion, were justified by requiring, perhaps dubiously, that such a term not be present at high temperature. We will see exactly that term when we study the high-temperature strong-coupling limit in Sec. VII C. It seems not possible to make such a subtraction here because we are considering arbitrary coupling, where the geometrical considerations applied in Ref. [33] must be generalized [38].

B. TM low-temperature behavior

For the TM free energy we write

$$F^{\text{TM}} = T \sum_{n=0}^{\infty} g(n),$$

$$g(n) = \cos n \alpha \tilde{\tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \left(1 - \frac{\lambda_0}{x} e'_l(x) s'_l(x) \right). \quad (6.6)$$

For small x , the quantity in the logarithm is singular,

$$\frac{e'_l(x) s'_l(x)}{x} \sim -\frac{l(l+1)}{(2l+1)x^2} - \frac{3+2l(l+1)}{(4l^2-1)(2l+3)} + O(x^2 \text{ or } x^{2l-1}), \quad x \ll 1. \quad (6.7)$$

The special role of $l = 1$ is evident. To define the $n = 0$ term as before we introduce the photon mass parameter p , so up to terms that vanish with p ,

$$\frac{1}{2} g(0) = \frac{1}{2} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \frac{\lambda_0 l(l+1)}{(2l+1)p^2}. \quad (6.8)$$

Because α is also very small, we have the leading term

$$\frac{1}{2} g(1) \approx \frac{1}{2} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \frac{\lambda_0 l(l+1)}{(2l+1)\alpha^2}, \quad (6.9)$$

and the integral term gives

$$-\int_0^1 dn g(n) = -\sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \left[\ln \frac{\lambda_0 l(l+1)}{(2l+1)\alpha^2} + 2 \right]. \quad (6.10)$$

Adding to these the contribution of the first term in the Bernoulli series gives

$$F_{O(T)}^{\text{TM}} \sim T \ln \frac{2\pi p}{\alpha}. \quad (6.11)$$

This term is, of course, identical to the $T \ln T$ term we found in strong coupling in Eq. (3.9), and should be omitted for the same reasons.

It is easy to check that the terms of $O(T^3)$ coming from the integral, the $\frac{1}{2} g(1)$ term, and the first Bernoulli term all cancel. The surviving T^4 behavior again receives canceling contributions from these three places, but arises entirely from the exceptional $l = 1$ term, where

$$\ln \left(1 - \lambda_0 \frac{e'_1(x) s'_1(x)}{x} \right) \sim \ln \frac{2\lambda_0}{3x^2} + x^2 \left(\frac{3}{2\lambda_0} + \frac{7}{10} \right) - \frac{2}{3} x^3 + \dots \quad (6.12)$$

Thus the leading contribution to the TM free energy is

$$F_{O(T^4)}^{\text{TM}} \sim -\frac{2}{3} T \alpha^3 3 \left(-\frac{B_4}{4!} 6 \right) = -\frac{2}{15} (\pi a)^3 T^4, \quad (6.13)$$

just as stated above, in Eq. (6.5).

So we have recovered the strong-coupling limit. Still in the low-temperature context, we can develop an expansion in $\xi = \frac{\alpha}{\sqrt{2\lambda_0/3}}$, which we regard as arbitrary.³ This arises again entirely from the $l = 1$ term in the angular momentum expansion. (Higher terms in l will yield only higher-order terms in T .) This is achieved by expanding

³We recall that in the δ -function plate, we developed the TM free energy in terms of a strong-coupling expansion, as a series in T/λ_0 [20].

$$\ln \left[1 + \left(\frac{3}{2\lambda_0} + \frac{7}{10} \right) x^2 - \frac{2}{3} x^3 \right] = \sum_{p=1}^{\infty} (-1)^{p-1} \frac{1}{p} \left[\left(\frac{3}{2\lambda_0} + \frac{7}{10} \right) x^2 - \frac{2}{3} x^3 \right]^p. \quad (6.14)$$

This leads to the Bernoulli term (about $n = 1$)

$$F_B^{\text{TM}} = -3T \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \alpha^{2k-1} \left(\frac{\partial}{\partial \alpha} \right)^{2k-1} \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} \times \sum_{r=0}^p \binom{p}{r} \left(\frac{3}{2\lambda_0} + \frac{7}{10} \right)^{p-r} \left(-\frac{2}{3} \right)^r \alpha^{2p+r}, \quad (6.15)$$

where the derivative amounts to inserting the factor $(2p+r)!/(2p+r-2k+1)!$. Because only $r = 1$ can result in α^3 dependence (with the remaining α 's absorbed in the definition of ξ), we are led to

$$F_B^{\text{TM}} = -\frac{4\lambda_0^2}{9\pi a} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \sum_{p=1}^{\infty} (-1)^p \frac{(2p+1)!}{(2p-2k+2)!} \xi^{2p+2}. \quad (6.16)$$

The sum on p may be readily carried out:

$$k = 1: \sum_{p=1}^{\infty} (-1)^p \frac{(2p+1)!}{(2p)!} \xi^{2p+2} = -\xi^4 \frac{3 + \xi^2}{(1 + \xi^2)^2}, \quad (6.17a)$$

$$k > 1: \sum_{p=1}^{\infty} (-1)^p \frac{(2p+1)!}{(2p-2k+2)!} \xi^{2p+2} = (-1)^{k+1} (2k-1)! \xi^{2k} (1 + \xi^2)^{-k} \times \cos(2k \arctan \xi). \quad (6.17b)$$

Now we notice that

$$\frac{\xi}{\sqrt{1 + \xi^2}} e^{i \arctan \xi} = \frac{\xi}{1 - i\xi}, \quad (6.18)$$

so then we have

$$F_B^{\text{TM}} = \frac{4\lambda_0^2}{9\pi a} \left\{ \frac{B_2}{2} \frac{\xi^4 (3 + \xi^2)}{(1 + \xi^2)^2} + \sum_{k=2}^{\infty} (-1)^k \frac{B_{2k}}{2k} \frac{1}{2} \times \left[\frac{\xi^{2k}}{(1 + i\xi)^{2k}} + \frac{\xi^{2k}}{(1 - i\xi)^{2k}} \right] \right\}. \quad (6.19)$$

Finally, the sum on k may be recast as an integral,

$$F^{\text{TM}, T \rightarrow 0}(\xi) = \frac{4\lambda_0^2}{9\pi a} \left\{ \frac{B_2}{2} \frac{\xi^4 (3 + \xi^2)}{(1 + \xi^2)^2} + \Re \int_0^{\infty} \frac{dt}{t} e^{-t} \left[-1 + \frac{1}{12} x^2 t^2 + \frac{xt}{2} \cot \frac{xt}{2} \right] - \frac{\xi^4}{2(1 + \xi^2)} + \frac{1}{2} [\xi^2 - \ln(1 + \xi^2)] \right\}, \quad (6.20)$$

where

$$x = \frac{\xi}{1 + i\xi}, \quad \xi = \frac{\alpha}{\sqrt{2\lambda_0/3}}, \quad (6.21)$$

and where we have now added in as the last two terms the contributions from the $\frac{1}{2}g(1)$ term and the integral term in the Euler-Maclaurin formula, respectively. (The latter is worked out similarly to the way we computed the derivatives in the Bernoulli sum.) For strong coupling

$$F^{\text{TM}, T \rightarrow 0}(\xi) \sim -\frac{4\lambda_0^2}{9\pi a} \left(\frac{\xi^4}{120} + \frac{\xi^6}{252} + \frac{\xi^8}{240} + \dots \right) \sim -\frac{\alpha^4}{120\pi a}, \quad \xi \ll 1, \quad (6.22)$$

which reproduces Eq. (6.13), of course. For weak coupling

$$F^{\text{TM}, T \rightarrow 0}(\xi) = \frac{4\lambda_0^2}{9\pi a} \left[\frac{\xi^2}{12} + \gamma - \ln \xi + \sum_{n=1}^{\infty} \frac{(-1)^n}{\xi^{2n}} \zeta(2n+1) \right] \sim \frac{\lambda_0}{18\pi a} \alpha^2 + \frac{4\lambda_0^2}{9\pi a} (\gamma - \ln \xi), \quad \xi \gg 1, \quad (6.23)$$

where the first term exactly agrees with the small-temperature limit of that found in Eq. (5.30), *without the divergent terms seen in Eq. (5.28)*. We can sum the weak-coupling expansion into a closed form:

$$F^{\text{TM}, T \rightarrow 0}(\xi) = \left(\frac{2\lambda_0}{3} \right)^2 \left[\frac{\xi^2}{12} - \ln \xi - \Re \psi \left(1 + \frac{i}{\xi} \right) \right], \quad (6.24)$$

which can be shown numerically to coincide with Eq. (6.20).

Two lessons are thus learned: the subtraction (renormalization) procedure leading to the perturbative free energy and entropy (5.30) apparently is valid, and the free energy develops a positive slope (the entropy becomes negative) for large enough ξ , i.e. for small enough coupling. The low-temperature

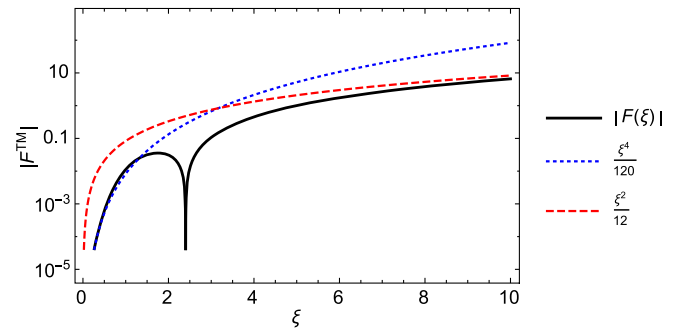


FIG. 3. The absolute value of the TM free energy for low temperature, as a function of the coupling defined through $\xi = \alpha \sqrt{3/2\lambda_0}$. The overall factor $(2\lambda_0/3)^2/\pi a$ has been pulled out. The dotted and dashed lines are the large and small coupling limits, respectively. For strong coupling, the slope is negative, and hence the entropy is positive, but for sufficiently weak coupling, the entropy changes sign. The cusp indicates where the free energy changes sign.

TM free energy is plotted in Fig. 3 as a function of ξ . Now it is seen that the free energy starts with a negative slope as a function of temperature, for large coupling, but at $\xi = 1.75271$ the sign of the slope changes, so the corresponding entropy turns negative. The change occurs for $\lambda_0 = 0.488282\alpha^2$.

VII. HIGH-TEMPERATURE LIMIT

Finally, we return to high temperature, both for finite and large coupling, where unlike in Sec. III we consider the TE and TM contributions separately. These limits are captured completely by the uniform asymptotic expansion for the Bessel functions, in contrast to the low-temperature limit, and because the structures encountered in the sums are quadratic forms, it is particularly convenient to abandon the point-splitting regularization adopted heretofore, and use analytic regularization and the generalized Chowla-Selberg asymptotic formulas. In contrast, point-splitting yields formulas that are rather complicated to evaluate. We start by examining the TE contribution.

A. TE contribution

The high-temperature behavior is captured from the uniform asymptotic expansion, which can be written, as in Eq. (4.1)

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{[c(l+b)^2 + d\alpha^2(n+a)^2]^s} &\equiv E_2(s; c, d\alpha^2; b, a) \\ &= \frac{(d\alpha^2)^{-s}}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma(s+m) \left(\frac{c}{d\alpha^2}\right)^m \zeta(-2m, b) \zeta(2s+2m, a) \\ &\quad + \frac{(d\alpha^2)^{-s}}{2} \left(\frac{\pi d\alpha^2}{c}\right)^{1/2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1, a) \\ &\quad + \frac{2\pi^s}{\Gamma(s)} \cos(2\pi b) c^{-s/2-1/4} (d\alpha^2)^{-s/2+1/4} \\ &\quad \times \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n^{s-1/2} (m+a)^{-s+1/2} K_{s-1/2} \left(2\pi \sqrt{\frac{d\alpha^2}{c}} n(m+a)\right). \end{aligned} \quad (7.3)$$

Here, the various parameters, a , b , c , and d are introduced so that the desired structures can be obtained by appropriate differentiation. Afterwards, we set $a=1$, $b=3/2$, $c=d=1$. We shift the $n > 0$ and $l > 0$ sums so they start at zero. Due to the prefactor of ν in the summand for the free energy, there is always one derivative with respect to b , so the second term above does not contribute. Neither does the last term, because b is an integer plus one-half. Thus, only the first term in Eq. (7.3) survives.

1. High-temperature, fixed coupling

Let us start with the first term in the uniform asymptotic expansion (7.2). Note that the leading term in the uniform asymptotic expansion is exact for $n=0$. By zeta-function regularization, the $n=0$ term yields

$$\ln \left(1 + \frac{\lambda_0}{x} e_l(x) s_l(x)\right) \sim \sum_{k=1}^{\infty} \frac{a_{\text{TE}}^{(k)}(t)}{(2\nu)^k}, \quad \nu \gg 1, \quad (7.1)$$

where the first four expansion coefficients are

$$a_{\text{TE}}^{(1)}(t) = \lambda_0 t, \quad (7.2a)$$

$$a_{\text{TE}}^{(2)}(t) = -\frac{1}{2} \lambda_0^2 t^2, \quad (7.2b)$$

$$a_{\text{TE}}^{(3)}(t) = \lambda_0 \frac{t^3}{2} (1 - 6t^2 + 5t^4) + \lambda_0^3 \frac{t^3}{3}, \quad (7.2c)$$

$$a_{\text{TE}}^{(4)}(t) = -\frac{\lambda_0^2 t^4}{2} (1 - 6t^2 + 5t^4) - \frac{\lambda_0^4 t^4}{4}, \quad \text{etc.} \quad (7.2d)$$

It seems the most effective way to extract the high-temperature dependence is through the use of the Chowla-Selberg formula as generalized by Elizalde [39,40]. That is, we will discard the point-split regularization we have used hitherto, and use the formula

$$F_{n=0}^{(1)\text{TE}} = \frac{\lambda_0 T}{2} \zeta(0, 3/2) = -\frac{\lambda_0 \alpha}{4\pi a}, \quad (7.4)$$

while the remainder of the n sum gives

$$F_{n>0}^{(1)\text{TE}, T \rightarrow \infty} = -\lambda_0 T \frac{1}{2(s-1)} \frac{\partial}{\partial b} E_2(s-1; 1, \alpha^2; b, 1)|_{s \rightarrow 1/2}. \quad (7.5)$$

The $m=0$ term in the Chowla-Selberg formula (7.3) now gives immediately

$$F_{n>0, m=0}^{(1)\text{TE}, T \rightarrow \infty} = \frac{\lambda_0 \alpha^2}{24\pi a}, \quad (7.6)$$

while the $m = 1$ term gives

$$F_{n>0,m=1}^{(1)\text{TE},T\rightarrow\infty} = -\frac{\lambda_0}{4\pi a} \frac{11}{12} \left(\gamma - \ln \alpha + \frac{1}{2s-1} \right), \quad (7.7)$$

where the divergent term is irrelevant, since it is T independent. In this way, we find

$$\Delta F^{(1)\text{TE},T\rightarrow\infty} = \frac{\lambda_0}{4\pi a} \left(\frac{\alpha^2}{6} - \alpha + \frac{11}{12} \ln \alpha \right), \quad (7.8)$$

which is nearly that expected from the exact $O(\lambda_0)$ result (5.5):

$$T \rightarrow \infty: \Delta F^{(1)\text{TE}} \sim \frac{\lambda_0}{4\pi a} \left(\frac{\alpha^2}{6} - \alpha + \ln \alpha \right). \quad (7.9)$$

The missing contribution to the logarithmic term comes from the $O(\lambda_0)$ part of $a_{\text{TE}}^{(3)}$. Using the Chowla-Selberg formula again, but now also differentiating with respect to c and d as well, we find

$$F_{O(\lambda_0)}^{(3)\text{TE},T\rightarrow\infty} = \frac{\lambda_0}{4\pi a} \frac{1}{12} \ln \alpha. \quad (7.10)$$

There are no higher-power temperature corrections in $O(\lambda_0)$ (beyond that displayed, the temperature corrections are exponentially small). Indeed, when the $m = 2$ contribution from the lowest-order UAE is added to the $m = 1$ contribution from the second-order UAE, and the $m = 0$ contribution from the third-order UAE (not displayed in the above formulas), which would potentially give a contribution to the free energy of order α^{-2} , we obtain zero.

Next, we look at the $T \rightarrow \infty$ contribution coming from $a^{(2)}$:

$$F^{(2)\text{TE},T\rightarrow\infty} = -\frac{\lambda_0^2 T}{4} \sum_{n=0}^{\infty} \sum_{l=1}^{\infty} \frac{\nu}{(\nu^2 + n^2 \alpha^2)^s}, \quad s \rightarrow 1, \quad (7.11)$$

where the $n = 0$ contribution is

$$F_{n=0}^{(2)\text{TE},T\rightarrow\infty} = -\frac{\lambda_0^2 \alpha}{16\pi a} [-2^{2s-1} + (2^{2s-1} - 1)\zeta(2s-1)], \quad (7.12)$$

which is divergent as $s \rightarrow 1$. This divergence is canceled by another divergence occurring in the remaining contributions,

$$F_{n>0}^{(2)\text{TE},T\rightarrow\infty} = -\frac{\lambda_0^2 T \alpha^{2-2s}}{8} \zeta(2s-2) + \dots \quad (7.13)$$

When these are added together, we obtain a finite result:

$$F^{(2)\text{TE},T\rightarrow\infty} = -\frac{\lambda_0^2 T}{8} \left(-2 + \gamma + \ln \frac{2\alpha}{\pi} - \frac{11}{12} \zeta(2) \frac{1}{\alpha^2} + O(\alpha^{-4}) \right). \quad (7.14)$$

Here we have kept a subdominant term that will not contribute in the high-temperature, fixed λ_0 regime, because we will see this is important in strong coupling.

All that remains are the higher-order terms in $\lambda_0 t$, which for fixed λ_0 only contribute at $n = 0$. (Again the UAE is exact in this case.) These give a contribution linear in T which is easily summed:

$$\begin{aligned} F_{n=0}^{(>2)\text{TE},T\rightarrow\infty} &= T \sum_{l=1}^{\infty} \nu \left[\ln \left(1 + \frac{\lambda_0}{2\nu} \right) - \frac{\lambda_0}{2\nu} + \frac{\lambda_0^2}{8\nu^2} \right] \\ &= T \left[-\frac{1}{24} - \frac{13}{24} \ln 2 + \frac{1}{2} \ln G + \frac{\lambda_0}{2} - \frac{\lambda_0}{4} \ln 2\pi \right. \\ &\quad \left. - \frac{\lambda_0^2}{8} (1 - \gamma - 2 \ln 2) + \frac{\lambda_0}{2} \ln \Gamma \left(\frac{3 + \lambda_0}{2} \right) \right. \\ &\quad \left. - \zeta^{(1,0)} \left(-1, \frac{3 + \lambda_0}{2} \right) \right]. \quad (7.15) \end{aligned}$$

Adding Eqs. (7.9), (7.14), and (7.15) together, we find

$$\begin{aligned} F^{\text{TE},T\rightarrow\infty}(\lambda_0) &\sim \frac{1}{2\pi a} \left\{ \alpha \left[-\frac{1}{24} - \frac{13}{24} \ln 2 + \frac{1}{2} \ln G \right] \right. \\ &\quad \left. + \lambda_0 \left[\frac{\alpha^2}{12} + \frac{1}{2} \ln \alpha - \frac{\alpha}{4} \ln 2\pi \right] \right. \\ &\quad \left. + \frac{1}{8} \lambda_0^2 \alpha \left(1 - \ln \frac{\alpha}{2\pi} \right) + \frac{1}{2} \lambda_0 \alpha \ln \Gamma \left(\frac{3 + \lambda_0}{2} \right) \right. \\ &\quad \left. - \alpha \zeta^{(1,0)} \left(-1, \frac{3 + \lambda_0}{2} \right) + \frac{11}{96} \frac{\lambda_0^2}{\alpha} \zeta(2) \right\}. \quad (7.16) \end{aligned}$$

The leading term for the high-temperature, fixed coupling regime, is just that given in Eq. (7.6):

$$F^{\text{TE},T\rightarrow\infty}(\lambda_0) \sim \frac{\lambda_0}{24\pi a} \alpha^2, \quad aT \gg 1. \quad (7.17)$$

Particularly interesting here is the strong-coupling limit, $\lambda_0 \rightarrow \infty$:

$$\begin{aligned} F_{\lambda_0 \rightarrow \infty}^{\text{TE},T\rightarrow\infty} &\sim \frac{1}{2\pi a} \left[-\frac{\lambda_0^2 \alpha}{16} \left(1 + 2 \ln \frac{\alpha}{\lambda_0 \pi} \right) + \frac{\lambda_0}{12} (\alpha^2 + 6 \ln \alpha) \right. \\ &\quad \left. - \frac{\alpha}{24} (1 + 2 \ln 2 - 12 \ln G + 11 \ln \lambda_0) + \frac{11}{12} \frac{\lambda_0^2 \pi^2}{\alpha} \right], \\ aT &\gg \lambda_0 \gg 1. \quad (7.18) \end{aligned}$$

Although the $\lambda_0 \alpha$ term canceled, the high-temperature TE free energy does not possess a strong-coupling limit independent of λ_0 . This is because this result is valid for $aT \gg \lambda_0 \gg 1$. That is, we are taking the high-temperature limit before we pass to strong coupling. To reverse the order of limits, we have to consider additional terms.

2. Strong-coupling limit

To get the entropy in the limit $\lambda_0 \gg aT \gg 1$ we have to add the leading correction, because we only included the $n = 0$ term in Eq. (7.15):

$$\begin{aligned} F_{\text{corr1}}^{\text{TE}} &= T \sum_{n,l=1}^{\infty} 2\nu \sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{\lambda_0}{2\nu} \right)^k t^k \\ &= \frac{1}{2\pi a} \left[-\frac{\lambda_0^3}{12} \ln \alpha + 2\alpha^3 f(\lambda_0/2\alpha) - \frac{11}{12} \alpha g(\lambda_0/2\alpha) \right], \end{aligned} \quad (7.19)$$

where the first term arises by taking a limit in the $k = 3$ term, and the sum over the remaining k sum yields

$$\begin{aligned} f(x) &= -\frac{1}{24} \left[x - 3x^2 - 6x^3 + 8\gamma x^3 + 6x^2 \ln 2\pi - \frac{3\zeta(3)}{\pi^2} \right. \\ &\quad \left. - 12\zeta^{(1,0)}(-2, 1+x) + 24x\zeta^{(1,0)}(-1, 1+x) \right] \\ &\rightarrow \frac{1}{9} x^3 (4 - 3\gamma - 3 \ln x) + \frac{1}{8} x^2 (1 - 2 \ln 2\pi x) \\ &\quad - \frac{1}{12} x + \frac{\zeta(3)}{8\pi^2} + \dots, \end{aligned} \quad (7.20)$$

and

$$\begin{aligned} g(x) &= \frac{\pi^2}{12} x^2 - \gamma x - \ln \Gamma(1+x) \\ &\rightarrow \frac{\pi^2}{12} x^2 + (1 - \gamma - \ln x)x - \frac{1}{2} \ln 2\pi x, \end{aligned} \quad (7.21)$$

where the last forms correspond to the strong-coupling limit, $x = \frac{\lambda_0}{2a} \gg 1$. When these are substituted into Eq. (7.19), the $\lambda_0^3 \ln \alpha$ term cancels, and the leading term in $g(x)$ cancels the subleading term in Eq. (7.14),

$$\begin{aligned} \Delta F_{\text{corr1}}^{\text{TE}, T \rightarrow \infty} &= \frac{1}{2\pi a} \left[-\frac{\lambda_0^2}{\alpha} \frac{11\pi^2}{12 \cdot 48} + \alpha \frac{\lambda_0^2}{16} \left(1 - 2 \ln \frac{\pi \lambda_0}{\alpha} \right) \right. \\ &\quad \left. - \alpha^2 \frac{\lambda_0}{12} + \alpha^3 \frac{\zeta(3)}{4\pi^2} + \frac{11}{24} (\lambda_0 + \alpha) \ln \frac{\pi \lambda_0}{\alpha} + \dots \right], \end{aligned} \quad (7.22)$$

where the omitted terms are either independent of T or go to zero as $\lambda_0 \rightarrow \infty$.

We also have to keep the next term in the UAE, which gives

$$\begin{aligned} F_{\text{corr2}}^{\text{TE}, T \rightarrow \infty} &= \frac{T}{2} \sum_{n,l=1}^{\infty} 2\nu \sum_{k=2}^{\infty} (-1)^{k-1} \lambda_0^k \left(\frac{t}{2\nu} \right)^{k+2} (1 - 6t^2 + 5t^4) \\ &\rightarrow \frac{3}{64\pi a} \alpha - \frac{\lambda_0}{48\pi a} \ln \alpha, \end{aligned} \quad (7.23)$$

where again the strong-coupling limit is displayed. Note that the last term here cancels the contribution (7.10).

When these corrections (7.22) and (7.23) are added to the previous result (7.18), we recover the expected perfect-conductor limit for the TE contribution given below:

$$\begin{aligned} F_{\infty}^{\text{TE}, T \rightarrow \infty} &= \frac{1}{2\pi a} \left[\alpha^3 \frac{\zeta(3)}{4\pi^2} - \frac{11}{24} \alpha \ln \frac{\alpha}{2\pi} - \frac{13}{24} \alpha \ln 2 \right. \\ &\quad \left. + \frac{1}{2} \alpha \ln G + \frac{5}{96} \alpha \right]. \end{aligned} \quad (7.24)$$

In Sec. 9.5.2 of Ref. [33] the free energy for a Dirichlet spherical shell was given. If the $l = 0$ term is subtracted from that, $F_{l=0}^{\text{TE}, D} = (T/2) \ln 2aT$, this coincides with our result, except for the leading α^3 term. This is precisely the term that is subtracted off by the procedure advocated in that reference, as mentioned already at the end of Sec. VI A. The objection to this term is that it grows as the area of the sphere; however, such a subtraction does not seem possible in our general analysis, in which the perfect conductor boundary conditions are only obtained as a limit.

B. TM contribution

For $n \neq 0$, the UAE again should capture the leading high-temperature contribution. As in Eq. (7.1),

$$\ln \left(1 - \frac{\lambda_0}{x} e'_l(x) s'_l(x) \right) \sim \sum_{k=1}^{\infty} \frac{a_{\text{TM}}^{(k)}(t)}{(2\nu)^k}, \quad \nu \gg 1, \quad (7.25)$$

where the first four expansion coefficients are

$$a_{\text{TM}}^{(1)}(t) = \frac{\lambda_0}{z^2 t}, \quad (7.26a)$$

$$a_{\text{TM}}^{(2)}(t) = -\frac{\lambda_0^2}{2z^4 t^2}, \quad (7.26b)$$

$$a_{\text{TM}}^{(3)}(t) = -\frac{\lambda_0}{2z^2} t(1 - 6t^2 + 7t^4) + \frac{\lambda_0^3}{3z^6 t^3}, \quad (7.26c)$$

$$a_{\text{TM}}^{(4)}(t) = \frac{\lambda_0^2}{2z^4} (1 - 6t^2 + 7t^4) - \frac{\lambda_0^4}{4z^8 t^4}. \quad (7.26d)$$

We will here content ourselves with the consideration of the $O(\lambda_0)$ contribution. For $n = 0$ we have to use the small-argument expansion (6.7) regularized by putting in a small photon mass p as before. To lowest order in λ_0 , the part sensitive to p vanishes using zeta-function regularization:

$$\begin{aligned} F_{n=0,p}^{\text{TM}} &= -\frac{\lambda_0 T}{2p^2} \sum_{l=1}^{\infty} l(l+1) \\ &= -\frac{\lambda_0 T}{8p^2} [(1-2^2)\zeta(-2) - (1-2^0)\zeta(0)] = 0. \end{aligned} \quad (7.27)$$

This leaves only the finite remainder

$$F_{n=0}^{\text{TM}, T \rightarrow \infty} = \frac{\lambda_0 T}{4} \sum_{l=1}^{\infty} \left[\frac{9}{(2l+3)(2l-1)} + 1 \right] = \frac{\lambda_0 \alpha}{4\pi a}, \quad (7.28)$$

again using zeta-function regularization. This is the linear-in- T behavior expected from the exact result (5.30).

For $n > 0$ we use the UAE (7.26) which gives the leading term

$$F_{(1), n > 0}^{\text{TM}, T \rightarrow \infty} = \lambda_0 T \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\nu}{(n+1)^2 \alpha^2} t^{-2s}, \quad (7.29)$$

$$\nu = l + b, \quad s \rightarrow 1/2,$$

with t^{-2} being the quadratic form seen in Eq. (7.3), where this may be evaluated by differentiating with respect to d , and then using the $m = 0$ term in the Chowla-Selberg formula:

$$F_{m=0}^{(1)\text{TM}, T \rightarrow \infty} = \frac{\lambda_0 \alpha^2}{72\pi a}, \quad (7.30)$$

which is exactly the expected quadratic T dependence seen in Eq. (5.30).

The $m = 1$ term here gives in the same way a contribution to the logarithm term:

$$F_{m=1}^{(1)\text{TM}, T \rightarrow \infty} = \frac{\lambda_0}{4\pi a} \frac{11}{12} \ln \alpha. \quad (7.31)$$

However, there is a similar contribution coming from the order $O(\lambda_0)$ part of the $a_{\text{TM}}^{(3)}$ term

$$F_{n>0}^{(3)\text{TM}, T \rightarrow \infty} = -\frac{\lambda_0 T}{8} \sum_{n,l=0}^{\infty} \frac{\nu}{(n+1)^2 \alpha^2} t^{2s} (1 - 6t^2 + 7t^4), \quad (7.32)$$

$$s \rightarrow 1/2.$$

Again by differentiating with respect to parameters we find, in the large-temperature limit, that only the $m = 0$ term in the Chowla-Selberg formula contributes, yielding

$$F_{(3)\text{TM}, T \rightarrow \infty} = -\frac{\lambda_0}{4\pi a} \frac{23}{12} \ln \alpha. \quad (7.33)$$

When this is added to the previously found logarithmic term (7.31) we obtain exactly the expected result from the exact calculation:

$$F_{O(\lambda_0) \log}^{\text{TM}, T \rightarrow \infty} = -\frac{\lambda_0}{4\pi a} \ln \alpha. \quad (7.34)$$

The net result is just as anticipated from Eq. (5.30),

$$\Delta F_{O(\lambda_0)}^{\text{TM}} \sim \frac{\lambda_0}{4\pi a} \left(\alpha + \frac{\alpha^2}{18} - \ln \alpha \right). \quad (7.35)$$

The fact that we get the same result as in Sec. V C, without the divergent terms seen in Eq. (5.28), is strong evidence that the minimal subtraction scheme there is valid.

C. Strong-coupling TE and TM contributions

In Sec. III we rederived the entropy for a perfectly conducting sphere at high temperatures. Now we want to extract the TE and TM contributions for such a sphere. These seem not to have been presented previously. The former is given by

$$F_{\infty}^{\text{TE}} = \frac{T}{2} \sum_{n=-\infty}^{\infty} e^{i\zeta_n \tau} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \frac{s_l(x) e_l(x)}{x}. \quad (7.36)$$

The zero Matsubara frequency contribution is

$$F_{\infty, n=0}^{\text{TE}} = \frac{T}{2} \sum_{l=1}^{\infty} (2l+1) P_l(\cos \delta) \ln \frac{1}{2l+1}, \quad (7.37)$$

where again we note that the UAE is exact in this case. Proceeding analytically, we drop the point-splitting cutoff, and find

$$F_{\infty, n=0}^{\text{TE}, T \rightarrow \infty} = -\frac{T}{2} \frac{\partial}{\partial \beta} \sum_{l=1}^{\infty} (2l+1)^{1+\beta} \Big|_{\beta=0}$$

$$= -\frac{T}{2} \left[\frac{1}{12} + \frac{1}{6} \ln 2 - \ln G \right]. \quad (7.38)$$

For $n > 0$ we can use the Chowla-Selberg formula (7.3), starting from the leading factor in the UAE,

$$F_{\infty, n > 0}^{(1)\text{TE}, T \rightarrow \infty} = 2T \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \nu \frac{\partial}{\partial \beta} (n^2 \alpha^2 + \nu^2)^{-\beta/2} \Big|_{\beta=0} - \frac{11}{24} T \ln 2$$

$$= T \left[-\alpha^2 \zeta'(-2) - \frac{11}{24} \ln \frac{\alpha}{2\pi} - \frac{11}{24} \ln 2 \right]. \quad (7.39)$$

The second-order term in the UAE yields

$$F_{\infty, n > 0}^{(2)\text{TE}, T \rightarrow \infty} = T \sum_{n,l=1}^{\infty} 2\nu \frac{t^2}{8\nu^2} (1 - 6t^2 + 5t^4)$$

$$= \frac{3}{32} T + O(T^{-1}). \quad (7.40)$$

(This result can also be obtained by the hyperbolic method used in Sec. III.) Adding these components together gives

$$F_{\infty}^{\text{TE}, T \rightarrow \infty} = T \left[\frac{\zeta(3)}{4\pi^2} \alpha^2 - \frac{11}{24} \ln \frac{\alpha}{2\pi} - \frac{13}{24} \ln 2 + \frac{5}{96} + \frac{1}{2} \ln G \right], \quad (7.41)$$

which coincides with the $\lambda_0 \gg aT \gg 1$ limit in Eq. (7.24). Again, recall that the leading $a^2 T^3$ term here is subtracted from the free energy by the method advocated in Sec. 9.5 of Ref. [33]. The same term would be added to the TM

contribution, so there is no effect on the total free energy. If one were only interested in the perfect conductor limit, such a procedure seems sensible, but here, with the context of an arbitrarily coupled semitransparent sphere, it seems unreasonable to implement.

The TM contribution is again more subtle. For $n = 0$ we use the small argument expansion (6.7), and replace x by p ,

$$F_{\infty, n=0}^{\text{TM}, T \rightarrow \infty} = \frac{T}{2} \sum_{l=0}^{\infty} (2l+1) \ln \frac{l(l+1)}{(2l+1)p^2} \\ = T \left[\frac{11}{12} \ln 2p + \frac{1}{12} \ln 2 + \frac{1}{24} - \frac{1}{2} \ln G \right] + F', \quad (7.42)$$

where the finite term displayed is the negative of that seen in Eq. (7.38), and the remainder is evaluated as follows:

$$F' = \frac{T}{2} \sum_{l=1}^{\infty} (2l+1) \ln \left(1 - \frac{1}{(2l+1)^2} \right) \\ = -\frac{T}{2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{2^{2k-1}} \sum_{l=1}^{\infty} \frac{1}{(l+1/2)^{\beta(2k-1)}} \\ = -T \left[\frac{1}{8(\beta-1)} + \frac{13}{12} \ln 2 - 3 \ln G \right], \quad (7.43)$$

where we have regulated the first term in the k sum by introducing an analytic parameter β which tends to 1. [This same sum, regulated by angular point-splitting, was encountered previously in Eq. (3.14).] The singularity here as $\beta \rightarrow 1$ will be canceled by that in the $n > 0$ contributions. The latter are captured by the UAE, as usual, which give, for the leading term

$$F_{\infty, n>0}^{(1)\text{TM}, T \rightarrow \infty} = 2T \sum_{n,l=1}^{\infty} \nu \ln \frac{\sqrt{n^2 \alpha^2 + \nu^2}}{2n^2 \alpha^2} \\ = T \left[\alpha^2 \zeta'(-2) - \frac{11}{24} \ln \frac{\alpha}{2\pi} - \frac{11}{24} \ln 2 \right]. \quad (7.44)$$

The second-order contribution contains a pole when evaluated using the Chowla-Selberg formula (7.3),

$$F_{\infty, n>0}^{(2)\text{TM}, T \rightarrow \infty} = T \sum_{n,l=1}^{\infty} 2\nu \left[-\frac{t^2}{8\nu^2} (1 - 6t^2 + 7t^4) \right] \\ = T \left[\frac{1}{8(\beta-1)} - \frac{9}{32} - \frac{1}{4} \ln \frac{\alpha}{2\pi} \right]; \quad (7.45)$$

the pole here (having been regulated in precisely the same way) cancels that in Eq. (7.43). Adding all these components (7.42), (7.43), (7.44), and (7.45) together gives

$$F_{\infty}^{\text{TM}, T \rightarrow \infty} = T \left[-\alpha^2 \frac{\zeta(3)}{4\pi^2} - \frac{17}{24} \ln \frac{\alpha}{2\pi} + \frac{11}{12} \ln p \right. \\ \left. - \frac{23}{96} + \frac{5}{2} \ln G - \frac{13}{24} \ln 2 \right]. \quad (7.46)$$

The sum of F_{∞}^{TE} and F_{∞}^{TM} , Eqs. (7.41) and (7.46) should yield Eq. (3.22); in fact,

$$F_{\infty}^{\text{TE}, T \rightarrow \infty} + F_{\infty}^{\text{TM}, T \rightarrow \infty} = \frac{11}{12} T \ln \frac{2\pi p}{\alpha} + F_{\infty}^{T \rightarrow \infty}. \quad (7.47)$$

The infrared divergent term is exactly the same as that seen in Eq. (3.12). As argued in Sec. III, that term should be subtracted to obtain the physically meaningful free energy or entropy.

VIII. SUMMARY AND DISCUSSION

In this paper we have investigated the self-entropy of a spherical shell, modeled beyond that of a perfect conductor, as a δ -function (“semitransparent”) shell of radius a . This study was motivated by our desire to better understand the mysterious phenomenon of negative entropy. Earlier, we had considered a similar model of the δ -function plate, where the TE entropy was always negative while the TM entropy was positive, and larger in magnitude, but both approached zero in the strong-coupling limit [20]. Apparently the only limit previously studied for the shell was that of a perfect conductor, without any mode decomposition in Refs. [27,35], while the mode decomposition can be found in Ref. [33]. Here we accomplished a full analysis, for arbitrary coupling, which exhibits many surprising features. We will now give a summary of our findings.

In Sec. II we derived the general expressions (2.10) for the free energy of an electromagnetic δ -function sphere, decomposed into the TE and TM modes. Cancellations occur between the different mode contributions. In Sec. III, which was devoted to strong coupling (perfectly conducting spherical shell), we rederived the total (TE plus TM) free energy correction for low temperature [Eq. (3.13)],

$$\Delta F_{\infty}^{T \rightarrow 0} \sim -\frac{(\pi a)^3}{15} T^4, \quad aT \ll 1 \quad (8.1)$$

(where the Δ refers to the fact this is the correction to the zero-temperature Boyer energy [34]), and for high temperature [Eq. (3.26)],

$$F_{\infty}^{T \rightarrow \infty} \sim -\frac{T}{4} (\ln aT + 0.7686), \quad aT \gg 1. \quad (8.2)$$

However, to obtain these known results [27], we removed a contribution that was infrared sensitive which arose from the leading $\ln \zeta_n^2 a^2$ term ($\zeta_n = 2\pi nT$ being the Matsubara frequency), which seems rather a generalization of a contact term [Eq. (3.9)],

$$F_{\infty}^{(0)} = T \ln \frac{\mu}{T}, \quad (8.3)$$

for point-splitting regularization, while this result is multiplied by 11/12 if analytic regularization is used. Here, μ is a photon mass, introduced to define the TM mode at zero frequency.

We then considered the opposite limit: weak coupling. The TE and TM modes can be exactly evaluated to first order in the coupling λ_0 . The TE self-entropy is [Eq. (5.6)] ($\alpha = 2\pi aT$)

$$S_{(1)}^{\text{TE}} = -\frac{\lambda_0}{2} \left(\frac{1}{3} \alpha - \coth \alpha + \frac{1}{\alpha} \right). \quad (8.4)$$

However, the TM self-entropy exhibits both infrared and ultraviolet divergences, which were regulated by point-splitting in the angular and temporal directions, and by the introduction again of a photon mass. If these divergent terms are simply removed (which is done automatically by analytic regularization, as through the use of the Abel-Plana formula) the result is [Eq. (5.30)]

$$S_{(1)}^{\text{TM}} = -\frac{\lambda_0}{2} \left(\frac{1}{9} \alpha + \coth \alpha - \frac{1}{\alpha} \right), \quad (8.5)$$

which is, like the TE contribution *always negative*. Notice that the total entropy is linear in the temperature in this order, as shown in Fig. 2.

In order λ_0^2 the TE free energy exhibits an infrared divergence, similar to that in Eq. (8.3). This also must be removed in order to avoid a violation of the Nernst theorem, although here, doing so leads to a positive contribution to the entropy.

Next we explored the low-temperature behavior for finite coupling. For the TE contribution, a simple formula was obtained that interpolates between the weak- and strong-coupling regimes [Eq. (6.4)]:

$$S^{\text{TE}}(\lambda_0) \sim -\frac{\alpha^3}{30} \frac{1}{1 + 3/\lambda_0}, \quad aT \ll 1, \quad (8.6)$$

which shows no sign of the infrared divergence in $O(\lambda_0^2)$ mentioned just above. A more complicated formula was found for the TM part [Eq. (6.24)]:

$$S^{\text{TM}}(\lambda_0) = \frac{4}{9} \lambda_0^2 \left[\frac{1}{12} \xi^2 - \ln \xi - \Re \psi(1 + i/\xi) \right],$$

$$\xi = \sqrt{\frac{3}{2\lambda_0}} \alpha, \quad \alpha \ll 1, \quad \xi \text{ arbitrary}. \quad (8.7)$$

This also yields the known strong- and weak-coupling results.

Finally, we considered the high-temperature limit, for arbitrary coupling, for both TE and TM contributions. In this situation, we changed our strategy, and instead of point-splitting, we regulated the double sum over Matsubara frequencies and angular momentum by use of the generalized Chowla-Selberg formula in the form given by Elizalde [39,40], which is particularly convenient for high temperature, since the uniform asymptotic expansions of the Bessel functions capture all the essential physics there. For the TE part, we obtained a rather complicated formula [Eq. (7.16)] for the free energy for fixed λ_0 and

high temperature, which, unsurprisingly, says that the dominant high-temperature entropy is that given by the $O(\lambda_0)$ contribution. If we want to take the strong-coupling limit, additional terms in the UAE must be included. Because of the increased complexity of the TM contributions, we only extracted, and reproduced, the $O(\lambda_0)$ entropy, the process of which, however, vindicated the minimal subtraction procedure used in the exact calculation. The coda consisted of computing, directly in strong coupling, the TE and TM contributions, which should add up to the result (8.2). This they do, except for an extra term [Eq. (7.47)]

$$\Delta F_{\infty}^{\text{IR}} = \frac{11}{12} T \ln \frac{2\pi p}{\alpha}, \quad p = \mu a. \quad (8.8)$$

This term is the same as the “contact term” (8.3), which should be removed *a priori*.

One might think that these surprising findings are a consequence of our use of the plasma model to describe the dispersive character of the coupling. However, if the perhaps more realistic Drude model is used, or even a model of a bound electron so that a characteristic frequency of the binding is introduced, the situation is not more satisfactory. Although these modifications change the infrared behavior, they do not change the ultraviolet behavior, and the appearance of divergent terms in the entropy, and negative TM and total entropies, remains present.

At zero temperature, it is typically argued [41] that the order λ_0 contribution to the energy should be discarded, since it can be canceled by a counterterm. This is probably only possible for the $T = 0$ contribution, and not for the temperature correction. It is true that if the $O(\lambda_0)$ term were removed from the low-temperature result (6.24) the TM entropy would become positive in that regime for all coupling. However, this would wreck the internal consistency of the problem, in particular the passage to the strong-coupling limit described in Sec. VII B. Therefore, this does not appear to be a viable resolution to our difficulty.

IX. CONCLUSIONS

So after somewhat elaborate calculations, we have obtained unsettling results. Contrary to expectations, the TM entropy for a δ -function sphere fails to be finite, both in the infrared and in the ultraviolet. If these divergent terms are merely subtracted (which is somewhat justified by the congruence of the results with analytic calculations based on the Abel-Plana and the Chowla-Selberg formulas) we find that the TM entropy and the total entropy are not necessarily positive. In fact the total entropy is positive only if the coupling is sufficiently strong. The perfectly conducting limit is satisfactory, and overcomes the negative interaction entropy between a perfectly conducting sphere and a

perfectly conducting plate, but apparently for sufficiently imperfect reflectors, this positivity breaks down. The non-monotonicity of the entropy means also that the specific heat need not be positive. The significance of these surprising thermodynamic findings merits further study.

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