

FOURIER COEFFICIENTS ATTACHED TO SMALL AUTOMORPHIC REPRESENTATIONS OF $\mathrm{SL}_n(\mathbb{A})$

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ABSTRACT. We show that Fourier coefficients of automorphic forms attached to minimal or next-to-minimal automorphic representations of $\mathrm{SL}_n(\mathbb{A})$ are completely determined by certain highly degenerate Whittaker coefficients. We give an explicit formula for the Fourier expansion, analogously to the Piatetski-Shapiro–Shalika formula. In addition, we derive expressions for Fourier coefficients associated to all maximal parabolic subgroups. These results have potential applications for scattering amplitudes in string theory.

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2000 *Mathematics Subject Classification*. Primary 11F70, 22E55; Secondary 11F30.

Key words and phrases. Small automorphic forms, Fourier coefficients, Fourier expansion, Nilpotent orbits, String Theory, Scattering amplitudes.

1. INTRODUCTION

1.1. Background and motivation. Let F be a number field and $\mathbb{A}$ be the associated ring of adeles. Let G be a reductive algebraic group defined over F and π an (irreducible) automorphic representation of $G(\mathbb{A})$ as defined in [BJ79, FGKP18].

Fix a Borel subgroup B and let $P \subset G$ be a standard parabolic subgroup with Levi decomposition $P = LU$, and let $\psi : U(F) \backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$ be a global unitary character. Given any automorphic form $\varphi \in \pi$ one can consider the following function on $G(\mathbb{A})$:

$$\mathcal{F}_U(\varphi, \psi; g) = \int_{U(F) \backslash U(\mathbb{A})} \varphi(ug) \psi^{-1}(u) du.$$

This can be viewed as a Fourier coefficient of the automorphic form φ with respect to the unipotent subgroup U . Fourier coefficients of automorphic forms carry a wealth of arithmetic and representation-theoretic information. For example, in the case of classical modular forms on the upper half-plane, Fourier coefficients are well-known to encode information about the count of rational points on elliptic curves. On the other hand, for higher rank Lie groups their arithmetic content is not always transparent, but they always encode important representation-theoretic information. Langlands showed that the constant terms in the Fourier expansion of Eisenstein series provide a source for automorphic L -functions [Lan67], and Shahidi extended this method (now called the Langlands-Shahidi method) to include also the non-constant Fourier coefficients [Sha78, Sha81].

Theta correspondences provide realizations of Langlands functorial transfer between automorphic representations π and π' of two different groups G and G' . In this context automorphic forms attached to *minimal* automorphic representations play a key role [Gin06]. The wave front set of a minimal representation $\pi_{\min}$ of a group G is the closure of the smallest non-trivial nilpotent coadjoint orbit $\mathcal{O}_{\min}$ of G [Jos76, KS90]. The automorphic realizations of minimal representations are characterized by having very few non-vanishing Fourier coefficients [GRS97]. Conversely, the method of descent [GRS11] can be viewed as an inverse to the functorial lifting, in which an automorphic representation of a general linear group GL_n is transferred to a representation of a smaller classical group G . Also in this case do Fourier coefficients of small representations enter in a crucial way.

In general it is a difficult problem to obtain explicit formulas for Fourier coefficients for higher rank groups, let alone settle the question of whether an automorphic form φ can be reconstructed from only a subset of its Fourier coefficients. For cusp forms on GL_n this is possible due to the Piatetski-Shapiro–Shalika formula [Sha74, PS79] that allows to reconstruct φ from its Whittaker coefficients; i.e. the Fourier coefficients with respect to the unipotent radical N of the Borel subgroup $B \subset G$. These coefficients are sums of Eulerian Whittaker coefficients on subgroups of G , and their non-archimedean parts can be obtained from the Casselman–Shalika formula [Shi76, CS80] as described in [FGKP18]. However, even if this gives us complete control of the Fourier expansion with respect to N it does not automatically give us a way of calculating an arbitrary Fourier coefficient

$\mathcal{F}_U(\varphi, \psi; g)$ with respect to some other unipotent subgroup U . Such coefficients play an important role in the construction of L -functions, and also carry information about non-perturbative effects in string theory as described in section 1.3.

Expanding upon the classic results of [GRS97], Miller and Sahi proved in [MS12] that for automorphic forms φ attached to a minimal representation $\pi_{\min}$ of E_6 and E_7 , any Fourier coefficient $\mathcal{F}_U(\varphi, \psi; g)$ is completely determined by maximally degenerate Whittaker coefficients of the form

$$(1.1) \quad \int_{N(F) \backslash N(\mathbb{A})} \varphi(ng) \psi_\alpha(n)^{-1} dn,$$

where ψ_α is non-trivial only on the one-parameter subgroup of N corresponding to the simple root α . This result maybe viewed as a global version of the classic results of Mœglin-Waldspurger in the non-archimedean setting [MW87], and Matumoto in the archimedean setting [Mat87].

For the special cases of SL_3 and SL_4 the Miller–Sahi results were generalized in [GKP16] (following related results in [FKP14]) to automorphic forms attached to a next-to-minimal automorphic representation π_{ntm} . It was shown that any Fourier coefficient is completely determined by (1.1) and coefficients of the following form

$$\int_{N(F) \backslash N(\mathbb{A})} \varphi(ng) \psi_{\alpha, \beta}(n)^{-1} dn,$$

where $\psi_{\alpha, \beta}$ is only supported on strongly orthogonal pairs of simple roots (α, β) which here reduces to that $[E_\alpha, E_\beta] = 0$ [Kna02]. The main goal of the present paper is to use the techniques of [JL13, JLS16, GGS17], in particular the notion of *Whittaker pair*, to extend the above results to all of SL_n .

1.2. Summary of results. We now summarize our main results. In the rest of this paper we will consider SL_n for $n \geq 5$ where we have fixed a Borel subgroup with the unipotent radical N . Let also T be the diagonal elements of $\mathrm{SL}_n(F)$ and, for a character ψ_0 on N , let T_{ψ_0} be the stabilizer of ψ_0 under the action $[h \cdot \psi_0](n) = \psi_0(hnh^{-1})$ for $h \in T$.

Define

$$(1.2) \quad \Gamma_i(\psi_0) := \begin{cases} (\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) & 1 \leq i \leq n-2 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_0} & i = n-1, \end{cases}$$

where $(\mathrm{SL}_{n-i}(F))_{\hat{Y}}$ is the stabilizer of $\hat{Y} = {}^t(1, 0, 0, \dots, 0) \in \mathrm{Mat}_{(n-i) \times 1}(F)$ and consists of elements $\begin{pmatrix} 1 & \xi \\ 0 & h \end{pmatrix}$, with $h \in \mathrm{SL}_{n-i-1}(F)$ and $\xi \in \mathrm{Mat}_{1 \times (n-i-1)}(F)$. When $\psi_0 = 1$ we write $\Gamma_i(1)$ as Γ_i .

Similarly, let $(\mathrm{SL}_j(F))_{\hat{X}}$ be the stabilizer of $\hat{X} = (0, \dots, 0, 1) \in \mathrm{Mat}_{1 \times j}(F)$ with respect to multiplication on the right, ψ_0 a character on N , and define

$$(1.3) \quad \Lambda_j(\psi_0) := \begin{cases} (\mathrm{SL}_j(F))_{\hat{X}} \backslash \mathrm{SL}_j(F) & 2 \leq j \leq n-1 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_0} & j = 1, \end{cases}$$

where, again, we denote $\Lambda_j(1) = \Lambda_j$.

Define also the embeddings $\iota, \hat{\iota} : \mathrm{SL}_{n-i} \rightarrow \mathrm{SL}_n$ for any $0 \leq i \leq n-1$ as

$$(1.4) \quad \iota(\gamma) = \begin{pmatrix} I_i & 0 \\ 0 & \gamma \end{pmatrix} \quad \hat{\iota}(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & I_i \end{pmatrix},$$

where we for brevity suppress their dependence on i . Note that for $i = 0$, they are just the identity maps for SL_n .

The following theorem expands an automorphic form φ attached to a small automorphic representation of SL_n in terms of highly degenerate Whittaker coefficients similar to how cusp forms on GL_n can be expanded in terms of Whittaker coefficients with the Piatetski-Shapiro–Shalika formula [Sha74, PS79]. Expansion of non-cuspidal automorphic forms on GL_n in terms of Whittaker coefficients were discussed in [Yuk93, JL13].

Theorem A. *Let π be a minimal or next-to-minimal irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $n \geq 5$, and let $\varphi \in \pi$.*

(i) *If $\pi = \pi_{\min}$, then φ has the expansion*

$$\varphi(g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(ng) dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n\iota(\gamma)g) \psi_{\alpha_i}^{-1}(n) dn.$$

(ii) *If $\pi = \pi_{ntm}$, then φ has the expansion*

$$\begin{aligned} \varphi(g) = & \int_{N(F) \backslash N(\mathbb{A})} \varphi(ng) dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n\iota(\gamma)g) \psi_{\alpha_i}^{-1}(n) dn + \\ & + \sum_{j=1}^{n-3} \sum_{i=j+2}^{n-1} \sum_{\substack{\gamma_i \in \Gamma_i(\psi_{\alpha_j}) \\ \gamma_j \in \Gamma_j}} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n\iota(\gamma_i)\iota(\gamma_j)g) \psi_{\alpha_j, \alpha_i}^{-1}(n) dn. \end{aligned}$$

Note that the Whittaker coefficients in the last sum of case (ii) have characters supported on two strongly orthogonal (or commuting) simple roots. As mentioned in section 1.1 and further described in [FGKP18], the Whittaker coefficients are sums of Eulerian Whittaker coefficients on smaller subgroups SL_n , whose non-archimedean parts can be computed by the Casselman–Shalika formula [Sha74, CS80]. The more degenerate a Whittaker coefficient is the smaller the subgroup we need to consider (and on which character becomes generic). Thus, maximally degenerate Whittaker coefficients, and the ones with characters supported on two commuting simple roots become particularly simple and are, in principle, one, or a product of two, known SL_2 Whittaker coefficients respectively.

Next, we consider Fourier coefficients on maximal parabolic subgroups. Let P_m the maximal parabolic subgroup of SL_n with respect to the simple root α_m and let $U = U_m$ be the unipotent radical and L_m be the corresponding Levi subgroup which stabilizes U_m under conjugation. For an element $l \in L_m(F)$ and a character ψ_U on U_m we obtain another character ψ_U^l by conjugation as

$$\psi_U^l(u) = \psi_U(lul^{-1}).$$

Fourier coefficients $\mathcal{F}_U$ with conjugated characters are related by l -translates of their arguments

$$\begin{aligned} (1.5) \quad \mathcal{F}_U(\varphi, \psi_U^l; g) &= \int_{U(F) \backslash U(\mathbb{A})} \varphi(ug) \psi_U^{-1}(lul^{-1}) du \\ &= \int_{U(F) \backslash U(\mathbb{A})} \varphi(l^{-1}u'lg) \psi_U^{-1}(u') du' = \mathcal{F}_U(\varphi, \psi_U; lg), \end{aligned}$$

where we have first made the variable substitution $u' = lul^{-1}$ and then used the automorphic invariance since $l \in L_m(F)$. This means that we only need to compute the Fourier coefficients of one character per $L_m(F)$ -orbit.

We show in section 2 that a character can be parametrized by an element $y \in \mathfrak{g}$ by (2.3) denoted by ψ_y , which, under conjugation, satisfies $\psi_y^l = \psi_{l^{-1}yl}$ according to (2.3).

In section 4 and appendix D, we describe these orbits following [Nev11] and construct standard characters $\psi_{y(Y_r(d))}$ on U_m based on anti-diagonal $(n-m) \times m$ rank r matrices $Y_r(d)$, where $d \in F^\times / (F^\times)^2$ for $n = 2r = 2m$ and $d = 1$ otherwise (in which case we suppress the d), and $y(Y_r(d))$ is defined as

$$y(Y_r(d)) = \begin{pmatrix} 0_m & 0 \\ Y_r(d) & 0_{n-m} \end{pmatrix}.$$

Let π be a minimal or next-to-minimal automorphic representations, and

$$r_\pi = \begin{cases} 1 & \text{if } \pi \text{ is a minimal automorphic representation} \\ 2 & \text{if } \pi \text{ is a next-to-minimal automorphic representation.} \end{cases}$$

We will show that only the characters with rank $r \leq r_\pi \leq 2$ give non-vanishing Fourier coefficients. Let us briefly define the characters with rank $r \leq 2$ which will be used in the next theorem, postponing a more general definition to section 4. The rank zero character is the trivial character $\psi_{y(Y_0)} = 1$ and the corresponding Fourier coefficient has been computed in [MW95] as reviewed in [FGKP18]. The rank one character is $\psi_{y(Y_1)} = \psi_{\alpha_m}$ and the rank two character can be defined as follows

$$(1.6) \quad \psi_{y(Y_2)}(u) = \psi(u_{m,m+1} + u_{m-1,m+2}) \quad u \in U_m(\mathbb{A}).$$

The following theorem, together with the known constant term, then allows us to compute any Fourier coefficient with respect to the unipotent radical of a maximal parabolic subgroups for automorphic forms attached to minimal and next-to-minimal automorphic representations in terms of Whittaker coefficients.

Theorem B. *Let π be a minimal or next-to-minimal irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $n \geq 5$, and let r_π be 1 or 2 respectively (which denotes the maximal rank of the character matrix Y_r). Let also, $\varphi \in \pi$, P_m be the maximal parabolic subgroup described above with its associated subgroups $U \equiv U_m$ and L_m , and let ψ_U be a non-trivial character on U with Fourier coefficient*

$$\mathcal{F}_U(\varphi, \psi_U; g) = \int_{U(F) \backslash U(\mathbb{A})} \varphi(ug) \psi_U^{-1}(u) du.$$

Then, there exists an element $l \in L_m(F)$ such that

$$\mathcal{F}_U(\varphi, \psi_U; g) = \mathcal{F}_U(\varphi, \psi_{y(Y_r(d))}; lg)$$

for some standard character $\psi_{y(Y_r(d))}$ described above and in the proof.

Additionally, all $\mathcal{F}_U(\varphi, \psi_{y(Y_r(d))}; lg)$ for $r > r_\pi$ vanish identically. The remaining (non-constant) coefficients can be expressed in terms of Whittaker coefficients on N as follows.

(i) If $\pi = \pi_{min}$:

$$\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi(n g) \psi_{\alpha_m}^{-1}(n) dn.$$

(ii) If $\pi = \pi_{ntm}$:

$$\begin{aligned} \mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = & \int_{N(F) \backslash N(\mathbb{A})} \varphi(n g) \psi_{\alpha_m}^{-1}(n) dn + \\ & + \sum_{j=1}^{m-2} \sum_{\gamma \in \Lambda_j(\psi_{\alpha_m})} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n \hat{\iota}(\gamma) g) \psi_{\alpha_j, \alpha_m}^{-1}(n) dn + \\ & + \sum_{i=m+2}^{n-1} \sum_{\gamma \in \Gamma_i(\psi_{\alpha_m})} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n \iota(\gamma) g) \psi_{\alpha_m, \alpha_i}^{-1}(n) dn. \end{aligned}$$

(iii) If $\pi = \pi_{ntm}$:

$$\mathcal{F}_U(\varphi, \psi_{y(Y_2)}; g) = \int_{C(\mathbb{A})} \int_{N(F) \backslash N(\mathbb{A})} \varphi(n \omega c g) \psi_{\alpha_1, \alpha_3}^{-1}(n) dn dc,$$

where ω is the Weyl element mapping the torus elements

$$(t_1, t_2, \dots, t_n) \mapsto (t_{m-1}, t_{m+2}, t_m, t_{m+1}, t_1, t_2, \dots, t_{m-2}, t_{m+3}, t_{m+4}, \dots, t_n),$$

and the subgroup C of U_m will be detailed in the proof in section 4.

As described in detail in section 2, F -rational nilpotent orbits of SL_n are characterized by $(\underline{p}, d)$ where $\underline{p}$ is a partition of n and $d \in F^\times / (F^\times)^k$ with $k = \mathrm{gcd}(\underline{p})$. If $k = 1$ we will often suppress the extra $d = 1$ and only write out the partition.

There we will also see that, for each orbit, there are natural choices of unipotent subgroups and characters related by conjugations with elements $\gamma \in \mathrm{SL}_n(F)$ and the corresponding Fourier coefficients (2.2) are related by γ -translates of their arguments.

The orbits may be partially ordered and the minimal and next-to-minimal orbits are described by the partitions $[21^{n-2}]$ and $[2^2 1^{n-4}]$, respectively. Besides the trivial partition, these are the only partitions whose associated Fourier coefficients are non-vanishing for φ in a minimal or next-to-minimal irreducible automorphic representation. In section 5 we choose standard representatives for these orbits and specify the associated standard Fourier coefficients which we denote by $\mathcal{F}^{[211 \dots]}$ and $\mathcal{F}^{[221 \dots]}$. For $n \geq 5$, we have that the trivial, minimal and next-to-minimal orbit all have $k = 1$.

The following theorems express these standard Fourier coefficients associated with the two partitions above in terms of Fourier coefficients

on maximal parabolic subgroups that, in turn, were written in terms of Whittaker coefficients in theorem B.

Theorem C. *Let π be an irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $\varphi \in \pi$ and $Y = \prod_{i=3}^n X_{e_i - e_2}$. Then,*

$$\mathcal{F}^{[211\dots]}(\varphi; g) = \sum_{y \in Y(F) \backslash U_1(F) \backslash U_1(\mathbb{A})} \int \varphi(uy^{-1}g) \psi_{\alpha_1}^{-1}(u) du,$$

where U_1 is the unipotent radical of P_1 consisting of the first row of N . The Fourier coefficient $\mathcal{F}^{[211\dots]}$ is for a particular standard choice of orbit representative detailed in the proof; all other choices are related simply by $\mathrm{SL}_n(F)$ translation.

Theorem D. *Let π be an irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $\varphi \in \pi$, $Y' = \prod_{i=5}^n X_{e_i - e_4} \prod_{i=5}^n X_{e_i - e_3}$ and ω be the Weyl element mapping the torus elements*

$$(t_1, t_2, \dots, t_n) \mapsto (t_1, t_3, t_4, t_2, t_5, t_6, \dots, t_n).$$

Then,

$$\mathcal{F}^{[221\dots]}(\varphi; g) = \sum_{y \in Y'(F) \backslash U_2(F) \backslash U_2(\mathbb{A})} \int \varphi(uy^{-1}\omega g) \psi_{y(Y_2)}^{-1}(u) du,$$

where U_2 is the unipotent radical of P_2 consisting of the first two rows of N and $\psi_{y(Y_2)}$ is defined in (1.6) with $m = 2$. The Fourier coefficient $\mathcal{F}^{[221\dots]}$ is for a particular standard choice of orbit representative detailed in the proof; all other choices are related simply by $\mathrm{SL}_n(F)$ translation.

1.3. Applications in string theory. String theory is a quantum theory of gravity describing maps $X : \Sigma \rightarrow M$, where Σ is a Riemann surface (the string worldsheet) and M is a ten-dimensional pseudo-Riemannian manifold (spacetime). Its low-energy limit is a supersymmetric extension of Einstein's theory of gravity in 10 dimensions coupled to additional matter in the form of scalar fields $\Phi : M \rightarrow \mathbb{C}$ and differential forms on spacetime M . Our main focus here will be the scalar fields. The scalar fields parametrize the space of string theory vacua, i.e. the moduli space $\mathcal{M}$.

To make contact with a lower-dimensional world, one choice is to decompose spacetime into

$$M = \mathbb{R}^{1,9-n} \times T^n,$$

where $\mathbb{R}^{1,9-n}$ is the flat Minkowski space in $10 - n$ dimensions and T^n is an n -dimensional torus. In the limit when the size of the torus is small, the physics looks effectively $(10 - n)$ -dimensional and one says that the theory has been *compactified*. As the size of the torus is increased the moduli space $\mathcal{M}$ gets larger and larger due to an increased number of scalar fields Φ . The moduli space for this toroidal compactification is always of the form

$$\mathcal{M} = \mathrm{G}(\mathbb{Z}) \backslash \mathrm{G}(\mathbb{R}) / K,$$

where $\mathrm{G}(\mathbb{R})$ is a semi-simple Lie group in its split real form, K its maximal compact subgroup and $\mathrm{G}(\mathbb{Z})$ an arithmetic subgroup. The group $\mathrm{G}(\mathbb{Z})$ is known as the *U-duality group* and is a symmetry of the full quantum string

theory. The extreme case when $n = 0$, i.e. for no compactification, the moduli space is given by

$$\mathcal{M} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2.$$

Another extreme case is $n = 6$, corresponding to four spacetime dimensions, for which the moduli space is given by [HT95]

$$\mathcal{M} = \mathrm{E}_7(\mathbb{Z}) \backslash \mathrm{E}_7(\mathbb{R}) / (\mathrm{SU}_8 / \mathbb{Z}_2).$$

Here $\mathrm{E}_7(\mathbb{R})$ is the split real form and $\mathrm{E}_7(\mathbb{Z})$ its Chevalley group of integer points. The sequence of groups in between are obtained by successively removing nodes from the E_7 Dynkin diagram; see table 1 for the complete list.

Constraints from U -duality and supersymmetry ensure that certain quantum corrections to Einstein's gravitational theory involve functions $f : \mathcal{M} \rightarrow \mathbb{C}$ that must be eigenfunctions of the ring of $\mathrm{G}(\mathbb{R})$ -invariant differential operators. In particular they are eigenfunctions of the Laplacian on $\mathrm{G}(\mathbb{R})/K$ with specific eigenvalues. In addition, they must have well-behaved growth properties in certain limits corresponding to ‘cusps’ of $\mathcal{M}$. Such quantum corrections are therefore controlled by automorphic forms on $\mathcal{M}$.

It turns out that the relevant automorphic forms are very special and are precisely those attached to a minimal and next-to-minimal automorphic representation of the groups G [GMRV10, Pio10, GMV15]. The Fourier coefficients of such automorphic forms therefore have a direct physical interpretation: the constant terms encode perturbative quantum corrections, while the non-constant terms correspond to non-perturbative, instanton, effects [FK12, FKP14, BV14, BV15a, BV15b, BChP17b, BP17, BChP17a]. For a recent book on automorphic representations and the connection with string theory, see [FGKP18].

Fourier coefficients with respect to different choices of parabolic subgroups $P \subset \mathrm{G}$ correspond to different limits in string theory, and reveal different types of effects. The ones of main interest are certain maximal parabolic subgroups. Let $P_\alpha = L_\alpha U_\alpha$ denote the maximal parabolic whose Levi subgroup is $L_\alpha = M_\alpha \times \mathrm{GL}_1$, where M_α is obtained by removing the node in the Dynkin diagram of G corresponding to the simple root α . There are three types of maximal parabolics of main interest in string theory (the numbering of nodes are according to the Bourbaki convention of the exceptional Lie algebras):

- P_{α_1} : this is the *perturbative, or string theory, limit* where the Levi is of orthogonal type $M_{\alpha_1} = \mathrm{D}_n$;
- P_{α_2} : this is the *M-theory limit* where the Levi is of type $M_{\alpha_2} = \mathrm{A}_n$;
- $P_{\alpha_{n+1}}$: this is the *decompactification limit* where the Levi is of exceptional type $M_{\alpha_{n+1}} = \mathrm{E}_n$ (for $n < 6$ these are strictly speaking not exceptional, but given by table 1).

Theorem B, together with its counterpart in [GKP16], then provides explicit results for the Fourier coefficients of automorphic forms in all these parabolics for the cases $n = 2$ or $n = 3$ when the symmetry groups are $\mathrm{SL}_2 \times \mathrm{SL}_3$ or SL_5 , respectively. The case of SL_5 will be treated in detail in section 6.

TABLE 1. List of U-duality groups in compactifications of (type IIB) string theory on T^n .

n	$G(\mathbb{R})$	$K(\mathbb{R})$	$G(\mathbb{Z})$
0	$SL_2(\mathbb{R})$	SO_2	$SL_2(\mathbb{Z})$
1	$GL_2(\mathbb{R})$	SO_2	$SL_2(\mathbb{Z})$
2	$SL_2(\mathbb{R}) \times SL_3(\mathbb{R})$	$SO_2 \times SO_2$	$SL_3(\mathbb{Z}) \times SL_2(\mathbb{Z})$
3	$SL_5(\mathbb{R})$	SO_5	$SL_5(\mathbb{Z})$
4	$Spin_{5,5}(\mathbb{R})$	$(Spin_5 \times Spin_5)/\mathbb{Z}_2$	$Spin_{5,5}(\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp_8/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
6	$E_7(\mathbb{R})$	$SU_8/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
7	$E_8(\mathbb{R})$	$Spin_{16}/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

Acknowledgements. We have greatly benefitted from many discussions with Guillaume Bossard, Dmitry Gourevitch, Joseph Hundley, Stephen D. Miller and Siddhartha Sahi. We gratefully acknowledge support from the Simons Center for Geometry and Physics, Stony Brook University during the program on “Automorphic forms, mock modular forms and string theory” in the fall of 2016 during which part of the research for this paper was performed. The fourth named author is partially supported by NSF grant DMS-1620329, DMS-1702218 and by a start-up fund from the Department of Mathematics at Purdue University.

2. NILPOTENT ORBITS AND FOURIER COEFFICIENTS

In this section, first, we introduce Whittaker pairs and nilpotent orbits with their associated Fourier coefficients following [GGS17], which is slightly more general and easier to use than the one given in [Gin06]. Then we recall the parametrization of F -rational nilpotent orbits of SL_n in terms of partitions of n from [Nev11] and a lemma for exchanging roots in Fourier integrals from [GRS11].

As before, let F be a number field, $\mathbb{A}$ be the adèle ring of F and fix a non-trivial additive character ψ on $F \backslash \mathbb{A}$. Let also G be a reductive group defined over F , or a central extension of finite degree, and let $\mathfrak{g}$ be the Lie algebra of $G(F)$. For a semi-simple element $s \in \mathfrak{g}$, let $\mathfrak{g}_i^s$ be defined as the eigenspace of s in $\mathfrak{g}$ with eigenvalue i under the adjoint action decomposing $\mathfrak{g}$ to a direct sum of eigenspaces over different eigenvalues. For any $r \in \mathbb{Q}$, we further define $\mathfrak{g}_{\geq r}^s = \bigoplus_{r' \geq r} \mathfrak{g}_{r'}^s$, and similarly for other inequality relations. For an element $X \in \mathfrak{g}$, we will also denote the centralizer of X in $\mathfrak{g}$ as

$$\mathfrak{g}_X = \{x \in \mathfrak{g} \mid [x, X] = 0\}.$$

Furthermore, a semi-simple element s is called *rational semi-simple* if all of its eigenvalues under the adjoint action on $\mathfrak{g}$ are in $\mathbb{Q}$. For such a rational semi-simple element s and a non-trivial nilpotent element $u \in \mathfrak{g}_{-2}^s$ we call the ordered pair (s, u) a *Whittaker pair*. If, for such a pair, s is also called a *neutral element* for u or (s, u) a *neutral pair* if there exists a nilpotent element $v \in \mathfrak{g}$ such that (v, s, u) is an $\mathfrak{sl}_2$ -triple. Recall that an $\mathfrak{sl}_2$ -triple is an ordered triple (v, s, u) of elements in $\mathfrak{g}$ that satisfy the standard commutation

relations for $\mathfrak{sl}_2$,

$$[s, v] = 2v \quad [s, u] = -2u \quad [v, u] = s,$$

where we say that v is a nil-positive element for s , u is a nil-negative element for s .

By the Jacobson–Morozov theorem, there exists an $\mathfrak{sl}_2$ triple for any nilpotent element $u \in \mathfrak{g}$. Moreover, the G -conjugacy classes of $\mathfrak{sl}_2$ -triples are one-to-one with the nilpotent orbits $\mathcal{O}_X = \{gXg^{-1} \mid g \in G(F)\}$ in $\mathfrak{g}$ [CM93].

We will now construct the Fourier coefficient that is associated to a Whittaker pair (s, u) . The pair defines a unipotent subgroup N_s and a character ψ_u on N_s as follows. Following [GGS17, Lemma 3.2.6], let

$$\mathfrak{n}_s = \mathfrak{g}_{>1}^s \oplus \mathfrak{g}_1^s \cap \mathfrak{g}_u,$$

which is a nilpotent subalgebra of $\mathfrak{g}$, and define $N_s = \exp(\mathfrak{n}_s)$ as the corresponding unipotent subgroup of G . Then ψ_u , defined by

$$\psi_u(n) = \psi(\langle u, \log(n) \rangle) \quad n \in N_s(\mathbb{A}),$$

is a character on $N_s(\mathbb{A})$ where $\langle \cdot, \cdot \rangle$ is the Killing form.

Note that if the Whittaker pair (s, u) comes from an $\mathfrak{sl}_2$ -triple (v, s, u) , then, by $\mathfrak{sl}_2$ representation theory, $\text{ad}(s)$ has integer eigenvalues with a graded decomposition of the Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i^s$ and $\mathfrak{g}_u \subset \bigoplus_{i \leq 0} \mathfrak{g}_i^s$ [CM93], and thus,

$$(2.1) \quad \mathfrak{n}_s = \mathfrak{g}_{\geq 2}^s \quad (\text{for neutral } s).$$

Let π be an automorphic representation of $G(\mathbb{A})$ and φ an automorphic form attached to π . The Fourier coefficient associated with a Whittaker pair (s, u) is

$$\mathcal{F}_{s,u}(\varphi)(g) = \int_{N_s(F) \backslash N_s(\mathbb{A})} \varphi(n g) \psi_u^{-1}(n) dn, \quad g \in G(\mathbb{A}),$$

and let $\mathcal{F}_{s,u}(\pi) = \{\mathcal{F}_{s,u}(\varphi) \mid \varphi \in \pi\}$. For convenience, we introduce the following notation for a unipotent subgroup U

$$[U] = U(F) \backslash U(\mathbb{A}).$$

Consider the Fourier coefficient associated with a neutral Whittaker pair (s, u) , and let $(s', u') = (\gamma s \gamma^{-1}, \gamma u \gamma^{-1})$ which is also neutral for any $\gamma \in G(F)$. Because of the invariance of the Killing form we have that $\psi_{u'}(n') = \psi_u(\gamma^{-1} n' \gamma)$ where $n' \in [N_{s'}]$, and because of (2.1) we have that $N_{s'} = \gamma N_s \gamma^{-1}$. Thus, with a variable substitution $n' = \gamma n \gamma^{-1}$,

$$(2.2) \quad \begin{aligned} \mathcal{F}_{s',u'}(\varphi)(g) &= \int_{[\gamma N_{s'} \gamma^{-1}]} \varphi(n' g) \psi_{u'}^{-1}(\gamma^{-1} n' \gamma) dn \\ &= \int_{[N_s]} \varphi(\gamma n \gamma^{-1} g) \psi_u^{-1}(n) dn = \mathcal{F}_{s,u}(\varphi)(\gamma^{-1} g), \end{aligned}$$

using the automorphic invariance of φ . Note the resemblance with (1.5) where we made a conjugation keeping N_s invariant. In particular, (2.2) means that if $\mathcal{F}_{s,u}$ vanishes identically then so do all Fourier coefficients associated to neutral Whittaker pairs (s', u') where $u' \in \mathcal{O}_u$. For an F -rational nilpotent

orbit $\mathcal{O}$, we say that the coefficients $\mathcal{F}_{s,u}$ with neutral s and $u \in \mathcal{O}$ are Fourier coefficients attached to the nilpotent orbit $\mathcal{O}$.

We define the (global) wave-front set $\mathcal{WF}(\pi)$ of an automorphic representation π of $G(\mathbb{A})$ as the set of nilpotent orbits $\mathcal{O}$ such that $\mathcal{F}_{s,u}(\pi)$ is non-zero, for some (and therefore all) neutral Whittaker pairs (s, u) with $u \in \mathcal{O}$. Note that nilpotent orbits can be partially ordered with respect to the inclusion of Zariski closures $\mathcal{O}' \leq \mathcal{O}$ if $\overline{\mathcal{O}'} \subseteq \overline{\mathcal{O}}$.

We recall [GGS17, Theorem C] as follows.

Theorem 2.1 (Theorem C, [GGS17]). *Let π be an automorphic representation of $G(\mathbb{A})$, let (s, u) be a Whittaker pair, and (h, u) a neutral Whittaker pair such that $\mathcal{F}_{h,u}(\pi)$ is zero. Then, $\mathcal{F}_{s,u}(\pi)$ is zero.*

This means that if $u \in \mathcal{O}$ where $\mathcal{O} \notin \mathcal{WF}(\pi)$ then, for any Whittaker pair (s, u) , not necessarily neutral, the associated Fourier coefficient $\mathcal{F}_{s,u}(\varphi)$ vanishes identically for $\varphi \in \pi$.

In this paper, we focus on the group SL_n where we parametrize a character on N_s by $u \in \mathfrak{g}_{-2}^s$ as

$$(2.3) \quad \psi_u(n) = \psi(\mathrm{tr}(u \log(n))) \quad n \in N_s(\mathbb{A}).$$

Then, for any l in the normalizer of $N_s(\mathbb{A})$ in $G(\mathbb{A})$

$$(2.4) \quad \begin{aligned} \psi_y^l(x) &= \psi_y(lxl^{-1}) = \psi(\mathrm{tr}(y \log(lxl^{-1}))) = \psi(\mathrm{tr}(yl \log(x)l^{-1})) \\ &= \psi(\mathrm{tr}(l^{-1}yl \log(x))) = \psi_{l^{-1}yl}(x). \end{aligned}$$

The nilpotent orbits of SL_n can be described by partitions $\underline{p}$ of n . Let us characterize the F -rational orbits of SL_n following [Nev11].

Proposition 2.2 (Proposition 4, [Nev11]). *Let $\underline{p} = [p_1 p_2 \dots p_r]$ be an ordered partition of n , with $p_1 \geq p_2 \geq \dots \geq p_r$ and let $m = \gcd(\underline{p}) = \gcd(p_1, p_2, \dots, p_r)$. For $d \in F^\times$, define $D(d) = \mathrm{diag}(1, 1, \dots, 1, d)$ and let also $J_{\underline{p}}$ be the standard (lower triangular) Jordan matrix corresponding to $\underline{p}$: $J_{\underline{p}} = \mathrm{diag}(J_{[p_1]}, J_{[p_2]}, \dots, J_{[p_r]})$, where $J_{[p]}$ is a $p \times p$ matrix with non-zero elements only on the subdiagonal which are one.*

- (1) *For each $d \in F^\times$, the matrix $D(d)J_{\underline{p}}$ is a representative of an F -rational nilpotent orbit of SL_n parametrized by $\underline{p}$, and conversely, every orbit parametrized by $\underline{p}$ has a representative of this form. We say that the F -rational orbit represented by $D(d)J_{\underline{p}}$ is parametrized by $(\underline{p}, d)$.*
- (2) *The $\mathrm{SL}_n(F)$ -orbits represented by $D(d)J_{\underline{p}}$ and $D(d')J_{\underline{p}'}$ coincide if and only if $\underline{p} = \underline{p}'$ and $d \equiv d'$ in $F^\times / (F^\times)^m$.*

Example 2.3. The F -rational orbit $([322], 1)$ of SL_7 is represented by

$$J_{[322]} = \mathrm{diag}(J_{[3]}, J_{[2]}, J_{[2]}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 2.4. Over $\overline{F}$ the F -rational orbits for different d become the same, meaning that they are completely characterized by partitions of n . There is partial ordering for partitions that agrees with the partial ordering of the $\overline{F}$ -orbits, where $[p_1 p_2 \dots p_r] \leq [q_1 q_2 \dots q_r]$ (possibly padded by zeroes) if [CM93]

$$\sum_{1 \leq j \leq n} p_j \leq \sum_{1 \leq j \leq n} q_j \quad \text{for } 1 \leq n \leq r.$$

The Zarisky topology over F is induced from that of $\overline{F}$ which means that we can use this partial ordering of partitions for the F -rational orbits as well. Thus, when discussing the partial ordering of orbits or the closure of orbits we will sometimes not specify the F -rational orbit, but only the partition, that is, the $\mathrm{SL}_n(\overline{F})$ -orbit.

An automorphic representation π of $\mathrm{SL}_n(\mathbb{A})$ is called *minimal* if $\mathcal{WF}(\pi)$ is the set of orbits in the closure of the minimal (non-trivial) orbit which is represented by the partition $[21^{n-2}]$, and it is called *next-to-minimal* if it is instead the set of orbits in the closure of the next-to-minimal orbit $[2^2 1^{n-4}]$.

We will now recall a general lemma for exchanging roots in Fourier coefficients from [GRS11]. In [GRS11], the groups considered are quasi-split classical groups, but the lemma holds for any connected reductive group with exactly the same proof.

Let G be a connected reductive group defined over F and let C be an F -subgroup of a maximal unipotent subgroup of G . Let also ψ_C be a non-trivial character on $[C] = C(F) \backslash C(\mathbb{A})$, and X, Y two unipotent F -subgroups satisfying the following conditions:

- (1) X and Y normalize C ;
- (2) $X \cap C$ and $Y \cap C$ are normal in X and Y , respectively, $(X \cap C) \backslash X$ and $(Y \cap C) \backslash Y$ are abelian;
- (3) $X(\mathbb{A})$ and $Y(\mathbb{A})$ preserve ψ_C under conjugation;
- (4) ψ_C is trivial on $(X \cap C)(\mathbb{A})$ and $(Y \cap C)(\mathbb{A})$;
- (5) $[X, Y] \subset C$;
- (6) there is a non-degenerate pairing

$$\begin{aligned} (X \cap C)(\mathbb{A}) \times (Y \cap C)(\mathbb{A}) &\rightarrow \mathbb{C}^\times \\ (x, y) &\mapsto \psi_C([x, y]) \end{aligned}$$

which is multiplicative in each coordinate, and identifies $(Y \cap C)(F) \backslash Y(F)$ with the dual of $X(F)(X \cap C)(\mathbb{A}) \backslash X(\mathbb{A})$, and $(X \cap C)(F) \backslash X(F)$ with the dual of $Y(F)(Y \cap C)(\mathbb{A}) \backslash Y(\mathbb{A})$.

Let $B = CY$ and $D = CX$, and extend ψ_C trivially to characters of $[B] = B(F) \backslash B(\mathbb{A})$ and $[D] = D(F) \backslash D(\mathbb{A})$, which will be denoted by ψ_B and ψ_D respectively.

Lemma 2.5 (Lemma 7.1 of [GRS11]). *Assume that (C, ψ_C, X, Y) satisfies all the above conditions. Let f be an automorphic form on $G(\mathbb{A})$. Then for any $g \in G(\mathbb{A})$,*

$$\int_{[B]} f(vg) \psi_B^{-1}(v) dv = \int_{(Y \cap C)(\mathbb{A}) \backslash Y(\mathbb{A})} \int_{[D]} f(vyg) \psi_D^{-1}(v) dv dy.$$

For simplicity, we will use ψ_C to denote its extensions ψ_B and ψ_D when using the lemma.

3. PROOF OF THEOREM A

Before we prove Theorem A in this section, let us first introduce a few definitions and useful lemmas.

Let V_i be the unipotent radical of the parabolic subgroup of type $(1^i, n-i)$, that is, the parabolic subgroup with Levi subgroup $(\mathrm{GL}_1)^i \times \mathrm{GL}_{n-i}$ together with a determinant one condition. Then, $N = V_n = V_{n-1}$ is the unipotent radical of the Borel subgroup and V_i can be seen as the first i rows of N . For $1 \leq i \leq n-1$, let $\alpha_i = e_i - e_{i+1}$ be the i -th simple root of SL_n , and let ψ_{α_i} be the character of N defined by

$$\psi_{\alpha_i}(n) = \psi(n_{i,i+1}), \forall n \in N(\mathbb{A}).$$

For a list of simple roots, we let $\psi_{\alpha_{i_1}, \dots, \alpha_{i_m}} = \psi_{\alpha_{i_1}} \cdots \psi_{\alpha_{i_m}}$ and we also regard ψ_{α_j} for $j \leq i$ as a character of V_i via restriction.

Also, let R_{i+1} be the subgroup of V_{i+1} , consisting of the elements v with conditions that $v_{p,q} = 0$, for all $1 \leq p \leq i$ and $p < q \leq n$, that is R_{i+1} consists of the row $i+1$ in V_{i+1} . It is clear that $R_{i+1} \cong V_i \backslash V_{i+1}$ is an abelian subgroup of V_{i+1} .

For a character ψ_N on N , we say that ψ_N is trivial along a simple root α_i if the restriction of ψ_N to R_i is identically zero.

Example 3.1. For SL_5 we have that

$$V_3 = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\} \quad R_3 = \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & * & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}.$$

Thus, we have that $[R_i] \cong (F \backslash \mathbb{A})^{n-i}$ and the dual of $[R_i]$ is F^{n-i} , which can be identified with the nilpotent subalgebra ${}^t \mathfrak{r}_i(F) = \log({}^t R_i(F))$, where ${}^t R_i(F)$ is the transpose of $R_i(F)$. Given $y \in {}^t \mathfrak{r}_i(F)$, the corresponding character ψ_y on $[R_i]$ is given by (2.3) as

$$\psi_y(x) = \psi(\mathrm{tr}(y \log x)), \quad \forall x \in [R_i].$$

Example 3.2. For SL_5 with R_3 above, let

$$y = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & y_1 & 0 & & \\ & y_2 & & 0 & \\ & & & & 0 \end{pmatrix} \in {}^t \mathfrak{r}_3(F) \quad x = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & x_1 & x_2 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \in [R_3].$$

Then, $\psi_y(x) = \psi(\mathrm{tr}(y \log x)) = \psi(y_1 x_1 + y_2 x_2)$.

Define

$$\mathrm{trdiag}(\cdot) = \mathrm{diag}(\cdot) - \frac{1}{n} \mathrm{tr}(\mathrm{diag}(\cdot))$$

and let $s = s_{V_i}$

$$(3.1) \quad s_{V_i} = \mathrm{trdiag}(2(i-1), 2(i-2), \dots, 0, -2, \dots, -2)$$

for which $\mathfrak{g}_1^s = \emptyset$ and $\mathfrak{n}_s = \mathfrak{g}_{\geq 2}^s$ with the corresponding $N_s = V_i$. In particular, we have $s_N = s_{V_{n-1}} = \mathrm{trdiag}(2(n-2), \dots, 0, -2)$

Lemma 3.3. *Let φ be an automorphic form on $\mathrm{SL}_n(\mathbb{A})$. Then, for $1 \leq i \leq n-2$,*

$$\sum_{\substack{y \in {}^t\mathfrak{r}_i(F) \\ y \neq 0}} \int_{[R_i]} \varphi(xg) \psi_y^{-1}(x) dx = \sum_{\gamma \in \Gamma_i} \int_{[R_i]} \varphi(x\iota(\gamma)g) \psi_{\alpha_i}^{-1}(x) dx,$$

where Γ_i is defined in (1.2) and $\iota(\gamma)$ in (1.4).

We note that the left-hand side of the equation in this lemma equals $\varphi(g)$ up to constant terms corresponding to $y = 0$.

Proof. With $Y \in \mathrm{Mat}_{(n-i) \times 1}(F)$, we parametrize $y \in {}^t\mathfrak{r}_i(F)$ as

$$y(Y) = \begin{pmatrix} 0_{i-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & Y & 0_{n-i} \end{pmatrix}.$$

Let $\hat{Y} = {}^t(1, 0, \dots, 0) \in \mathrm{Mat}_{(n-i) \times 1}(F)$. Then the surjective map $\mathrm{SL}_{n-i}(F) \rightarrow \mathrm{Mat}_{(n-i) \times 1}(F)^\times$ defined by $\gamma \mapsto \gamma^{-1}\hat{Y}$ gives that

$$\mathrm{Mat}_{(n-i) \times 1}(F)^\times \cong (\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) = \Gamma_i$$

from (1.2). (With $\mathrm{Mat}_{(n-i) \times 1}(F)^\times$ we mean the set of all non-zero matrices.) We then have that,

$$(3.2) \quad \sum_{y \neq 0} \int_{[R_i]} \varphi(xg) \psi_y^{-1}(x) dx = \sum_{\gamma \in \Gamma_i} \int_{[R_i]} \varphi(xg) \psi_{y(\gamma^{-1}\hat{Y})}^{-1}(x) dx.$$

We now rewrite the character using that for any $Y \in \mathrm{Mat}_{(n-i) \times 1}(F)$

$$y(\gamma^{-1}Y) = \begin{pmatrix} 0_{i-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \gamma^{-1}Y & 0_{n-i} \end{pmatrix} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} \begin{pmatrix} 0_{i-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & Y & 0_{n-i} \end{pmatrix} \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = l^{-1}yl,$$

where we have introduced $l = \iota(\gamma)$ and denoted $y(Y)$ simply as y , which according to (2.4) gives, for any $x \in [R_i]$, that

$$\psi_{y(\gamma^{-1}Y)}(x) = \psi_{l^{-1}yl}(x) = \psi_y(lxl^{-1}).$$

The element l is in the Levi subgroup of the parabolic subgroup corresponding to V_i , meaning that it preserves V_i under conjugation. In particular, it also normalizes R_i since for $x \in R_i$ parametrized by $X \in \mathrm{Mat}_{1 \times n-1}$

$$lx(X)l^{-1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & X \\ 0 & 0 & I_{n-i} \end{pmatrix} \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma^{-1} \end{pmatrix} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & X\gamma^{-1} \\ 0 & 0 & I_{n-i} \end{pmatrix} = x(X\gamma^{-1}).$$

We can thus make the variable substitution $lxl^{-1} \rightarrow x$ in (3.2) to obtain

$$\sum_{\gamma \in \Gamma_i} \int_{[R_i]} \varphi(xlg) \psi_{y(\hat{Y})}^{-1}(x) dx,$$

where we have used the fact that φ is left-invariant under l^{-1} . Noting that $\psi_{y(\hat{Y})} = \psi_{\alpha_i}$ this proves the lemma. $\square$

We will now state a similar lemma for the last row R_{n-1} , that needs to be treated separately. The freedom in choosing a character ψ_0 in this lemma will be of importance later.

Lemma 3.4. *Let φ be an automorphic form on $\mathrm{SL}_n(\mathbb{A})$. Then, for any character ψ_0 on N trivial on R_{n-1} and either on α_1 or along (at least) two adjacent simple roots not including α_{n-1} ,*

$$\sum_{\substack{y \in {}^t\mathfrak{r}_{n-1}(F) \\ y \neq 0}} \int_{[R_{n-1}]} \varphi(xg) \psi_y^{-1}(x) dx = \sum_{\gamma \in \Gamma_{n-1}(\psi_0)[R_{n-1}]} \int_{[R_{n-1}]} \varphi(x\iota(\gamma)g) \psi_{\alpha_{n-1}}^{-1}(x) dx,$$

where $\Gamma_{n-1}(\psi_0)$ is defined in (1.2).

Proof. With $Y \in F$, we parametrize $y \in {}^t\mathfrak{r}_{n-1}(F)$ as

$$y(Y) = \begin{pmatrix} 0_{n-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}.$$

We recall from page 3 that T_{ψ_0} is the subgroup of diagonal elements in $\mathrm{SL}_n(F)$ stabilizing ψ_0 under conjugation of its argument and that $y \in {}^t\mathfrak{r}_{n-1}(F) \cong F$. The map $T_{\psi_0} \rightarrow {}^t\mathfrak{r}_{n-1}(F)^\times : h \mapsto h^{-1}y(1)h$ is surjective, which can be shown as follows. If the character ψ_0 is trivial along at least two adjacent simple roots not including α_{n-1} . Pick such a pair α_{j-1} and α_j where $2 \leq j \leq n-2$ and for an arbitrary $m \in F^\times$ let $h = \mathrm{diag}(1, \dots, 1, m, 1, \dots, 1, 1/m)$ where the first non-trivial element is at the j th position. Then $h \in T_{\psi_0}$ since $y_0 \in {}^t\mathfrak{n}$ corresponding to ψ_0 is zero at both rows and columns j and n and $h \mapsto y(m)$. If the character ψ_0 is trivial along α_1 , we can choose $h = \mathrm{diag}(m, 1, \dots, 1, \dots, 1, 1/m)$ to arrive at surjectivity.

Because of (2.4) we have that the centralizer of $y(1)$ in T is $T_{\psi_{\alpha_{n-1}}}$, and thus,

$${}^t\mathfrak{r}_{n-1}(F) \cong (T_{\psi_0} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_0} = \Gamma_{n-1}(\psi_0).$$

We then have that

$$\begin{aligned} \sum_{\substack{y \in {}^t\mathfrak{r}_{n-1}(F) \\ y \neq 0}} \int_{[R_{n-1}]} \varphi(xg) \psi_y^{-1}(x) dx &= \sum_{\gamma \in \Gamma_{n-1}(\psi_0)[R_{n-1}]} \int_{[R_{n-1}]} \varphi(xg) \psi_{\gamma^{-1}y(1)\gamma}^{-1}(x) dx \\ &= \sum_{\gamma \in \Gamma_{n-1}(\psi_0)[R_{n-1}]} \int_{[R_{n-1}]} \varphi(xg) \psi_{y(1)}^{-1}(\gamma x \gamma^{-1}) = \sum_{\gamma \in \Gamma_{n-1}(\psi_0)[R_{n-1}]} \int_{[R_{n-1}]} \varphi(x\gamma g) \psi_{\alpha_{n-1}}^{-1}(x), \end{aligned}$$

after making the variable change $\gamma x \gamma^{-1} \rightarrow x$, which concludes the proof. $\square$

Remark 3.5. For $n \geq 5$ any character ψ_0 on N that is non-trivial along at most a single simple root which is not α_{n-1} satisfies the character condition in lemma 3.4.

The following lemma will be used to iteratively expand in rows. The lemma, which is valid for any automorphic representation, will be followed by two corollaries that specialize to a minimal and a next-to-minimal representation respectively.

Lemma 3.6. *Let φ be an automorphic form on $\mathrm{SL}_n(\mathbb{A})$, $1 \leq i \leq n-2$, and ψ_0 be a character on N trivial on the complement of V_i in N . For $i = n-2$ we*

also require that ψ_0 is trivial either along α_1 or along (at least) two adjacent simple roots not including α_{n-1} . Then,

$$(3.3) \quad \int_{[V_i]} \varphi(vg)\psi_0^{-1}(v)dv = \int_{[V_{i+1}]} \varphi(vg)\psi_0^{-1}(v)dv + \sum_{\gamma \in \Gamma_{i+1}(\psi_0)} \int_{[V_{i+1}]} \varphi(v\iota(\gamma)g)\psi_0^{-1}(v)\psi_{\alpha_{i+1}}^{-1}(v)dv.$$

Proof. For $x \in R_{i+1}(F)$ and $v \in V_i(\mathbb{A})$ we have that $\varphi(xvg) = \varphi(vg)$ and can thus Fourier expand along the abelian unipotent R_{i+1} as

$$\varphi(vg) = \sum_{y \in {}^t\mathfrak{r}_{i+1}(F)} \int_{[R_{i+1}]} \varphi(xvg)\psi_y^{-1}(x)dx.$$

Then, using lemma 3.3 (for $i+1 \leq n-2$) or lemma 3.4 (for $i+1 = n-1$)

$$\varphi(vg) = \int_{[R_{i+1}]} \varphi(xvg)dx + \sum_{\gamma \in \Gamma_{i+1}(\psi_0)} \int_{[R_{i+1}]} \varphi(x\iota(\gamma)vg)\psi_{\alpha_{i+1}}^{-1}(x)dx.$$

Let $v \in V_i$ be parametrized as

$$v = \begin{pmatrix} A & B \\ 0 & I_{n-i-1} \end{pmatrix},$$

where $A \in \text{Mat}_{(i+1) \times (i+1)}$ is upper unitriangular and $B \in \text{Mat}_{(i+1) \times (n-i-1)}$ with the elements in the last row being zero. Since B does not intersect the abelianization $[N, N] \setminus N$ (that is, the Lie algebra of B does not contain any generator of a simple root), we have, by assumption, that ψ_0 only depends on A . Similarly, we parametrize $x \in R_{i+1}$ as

$$x = \begin{pmatrix} I_{i+1} & B' \\ 0 & I_{n-i-1} \end{pmatrix},$$

where $B' \in \text{Mat}(i+1) \times (n-i-1)$ with non-zero elements only in the last row. Then,

$$xv = \begin{pmatrix} A & B+B' \\ 0 & I_{n-i-1} \end{pmatrix},$$

which means that $\psi_0(v) = \psi_0(xv)$, and since $\psi_{\alpha_{i+1}}$ only depends on the last row in B' which is the same as for $B+B'$, we also have that $\psi_{\alpha_{i+1}}(x) = \psi_{\alpha_{i+1}}(xv)$.

• For $1 \leq i \leq n-3$ with $\gamma \in \Gamma_{i+1}$, $l = \iota(\gamma)$ is in the Levi subgroup corresponding to V_i and we will now show that $\psi_0(l^{-1}vl) = \psi_0(v)$ for $v \in [V_i]$. We have that

$$l^{-1}vl = \begin{pmatrix} I_{i+1} & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ 0 & I_{n-i-1} \end{pmatrix} \begin{pmatrix} I_{i+1} & 0 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} A & B\gamma \\ 0 & I_{n-i-1} \end{pmatrix}$$

and $\psi_0(v)$ only depends on A .

• For $i = n-2$ with $\gamma \in \Gamma_{n-1}(\psi_0)$, $l = \iota(\gamma) = \gamma$ is in the stabilizer T_{ψ_0} which normalizes V_i and, by definition, means that $\psi_0(v) = \psi_0(lvl^{-1})$.

Thus, for $1 \leq i \leq n-2$,

$$\begin{aligned} \int_{[V_i]} \varphi(vg) \psi_0^{-1}(v) dv &= \int_{[V_i]} \int_{[R_{i+1}]} \varphi(xvg) \psi_0^{-1}(v) dx dv + \\ &+ \sum_{\gamma \in \Gamma_{i+1}(\psi_0)} \int_{[V_i]} \int_{[R_{i+1}]} \varphi(xvlg) \psi_{\alpha_{i+1}}^{-1}(x) \psi_0^{-1}(v) dx dv, \end{aligned}$$

where we have made the variable change $lv l^{-1} \rightarrow v$.

Using that $R_{i+1}V_i = V_{i+1}$ the above expressions simplifies to

$$\int_{[V_{i+1}]} \varphi(vg) \psi_0^{-1}(v) dv + \sum_{\gamma \in \Gamma_{i+1}(\psi_0)} \int_{[V_{i+1}]} \varphi(v\iota(\gamma)g) \psi_0^{-1}(v) \psi_{\alpha_{i+1}}^{-1}(v) dv.$$

□

Corollary 3.7. *Let π be an irreducible minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $\varphi \in \pi$, and ψ_0 be a character on N trivial on the complement of V_i in N , $1 \leq i \leq n-2$. Then, $\mathcal{F}_{\psi_0} := \int_{[V_i]} \varphi(vg) \psi_0^{-1} dv$ can be further expanded as follows.*

(i) *If $\psi_0 = 1$, then*

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) dv + \sum_{\gamma \in \Gamma_{i+1}} \int_{[V_{i+1}]} \varphi(v\iota(\gamma)g) \psi_{\alpha_{i+1}}^{-1}(v) dv,$$

where $\Gamma_{i+1}(\psi_0)$ with $\Gamma_{i+1} = \Gamma_{i+1}(1)$ is defined in (1.2).

(ii) *If $\psi_0 = \psi_{\alpha_j}$ ($1 \leq j \leq i$), then*

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) \psi_0^{-1}(v) dv.$$

Proof. We will use lemma 3.6 where all the considered ψ_0 satisfy the character condition for the last row according to remark 3.5.

For $\psi_0 = 1$, the expression is already in the form of lemma 3.6. This proves case (i).

For $\psi_0 = \psi_{\alpha_j}$ with $1 \leq j \leq i$ we have that $\psi_0(v) \psi_{\alpha_{i+1}}(v) = \psi_{\alpha_j, \alpha_{i+1}}(v) = \psi_u(v)$ for some $u \in \mathfrak{g}$ which is in the next-to-minimal orbit. Theorem 2.1 with the Whittaker pair $(s_{V_{i+1}}, u)$ gives that $\mathcal{F}_{s_{V_{i+1}}, u}(\varphi)$ vanishes for φ in a minimal representation which leaves only the constant (or trivial) mode in lemma 3.6. This proves case (ii). □

Corollary 3.8. *Let π be an irreducible next-to-minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $\varphi \in \pi$, and ψ_0 be a character on N trivial on the complement of V_i in N , $1 \leq i \leq n-2$. Then, $\mathcal{F}_{\psi_0} := \int_{[V_i]} \varphi(vg) \psi_0^{-1} dv$ can be further expanded as follows.*

(i) *If $\psi_0 = 1$, then*

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) dv + \sum_{\gamma \in \Gamma_{i+1}} \int_{[V_{i+1}]} \varphi(v\iota(\gamma)g) \psi_{\alpha_{i+1}}^{-1}(v) dv.$$

(ii) *If $\psi_0 = \psi_{\alpha_j}$ ($1 \leq j < i$), then*

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) \psi_{\alpha_j}^{-1}(v) dv + \sum_{\gamma \in \Gamma_{i+1}(\psi_{\alpha_j})} \int_{[V_{i+1}]} \varphi(v\iota(\gamma)g) \psi_{\alpha_j, \alpha_{i+1}}^{-1}(v) dv.$$

(iii) If $\psi_0 = \psi_{\alpha_i}$, then

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) \psi_{\alpha_i}^{-1}(v) dv.$$

(iv) If $\psi_0 = \psi_{\alpha_j, \alpha_k}$ ($1 < j+1 < k \leq i$), then

$$\mathcal{F}_{\psi_0} = \int_{[V_{i+1}]} \varphi(vg) \psi_{\alpha_j, \alpha_k}^{-1}(v) dv.$$

Where $\Gamma_{i+1}(\psi_0)$ with $\Gamma_{i+1} = \Gamma_{i+1}(1)$ is defined in (1.2).

Proof. We will use lemma 3.6 where the considered ψ_0 in cases (i)–(iii) satisfy the character condition for the last row according to remark 3.5.

- For $\psi_0 = 1$, the expression is already in the form of lemma 3.6. This proves case (i).
- For $\psi_0 = \psi_{\alpha_j}$ with $1 \leq j < i$ we get that $\psi_0(v) \psi_{\alpha_{i+1}}(v) = \psi_{\alpha_j, \alpha_{i+1}}(v)$. This proves case (ii).
- For $\psi_0 = \psi_{\alpha_i}$ we get that $\psi_0(v) \psi_{\alpha_{i+1}}(v) = \psi_{\alpha_i, \alpha_{i+1}}(v) = \psi_u(v)$ for some $u \in \mathfrak{g}$ belonging to an orbit higher than the next-to-minimal. Theorem 2.1 with the Whittaker pair $(s_{V_{i+1}}, u)$ gives that $\mathcal{F}_{s_{V_{i+1}}, u}(\varphi)$ vanishes both for φ in a minimal and next-to-minimal representation which leaves only the constant mode in lemma 3.6. This proves case (iii).
- Lastly, for $\psi_0 = \psi_{\alpha_j, \alpha_k}$ with $2 \leq j+1 < k \leq i$ we first consider $i \leq n-3$ with lemma 3.6. We get that $\psi_0(v) \psi_{\alpha_{i+1}}(v) = \psi_{\alpha_j, \alpha_k, \alpha_{i+1}}(v) = \psi_u(v)$ for some $u \in \mathfrak{g}$ belonging to an orbit higher than the next-to-minimal. Theorem 2.1 with the Whittaker pair $(s_{V_{i+1}}, u)$ gives that $\mathcal{F}_{s_{V_{i+1}}, u}(\varphi)$ vanishes for φ in next-to-minimal representation which leaves only the first term in (3.3).

For $i = n-2$, we expand along the last row and obtain a sum over characters $\psi_u = \psi_0 \psi_y$ on N for all $y \in {}^t \mathfrak{r}_{n-1}(F)$ where only $y = 0$ gives a $u \in \mathfrak{g}$ belonging to an orbit in the closure of the next-to-minimal orbit. Again, using theorem 2.1 only the constant mode remains. This proves case (iv) and completes the proof. $\square$

Proof of theorem A. Since $\varphi(x_1 g) = \varphi(g)$ for $x_1 \in V_1(F)$ we can make a Fourier expansion on V_1 and then use lemma 3.3 to obtain

$$(3.4) \quad \varphi(g) = \int_{[V_1]} \varphi(vg) dv + \sum_{\gamma_1 \in \Gamma_1[V_1]} \int \varphi(v\iota(\gamma_1)g) \psi_{\alpha_1}^{-1}(v) dv.$$

We will now make an iteration in the rows of the nilpotent, starting with the row $i = 1$ and continue until we reach the last row $i = n-1$.

- For case (i), that is, with φ in a minimal representation, the first step, using corollary 3.7, is

$$\begin{aligned} \varphi(g) = \int_{[V_2]} \varphi(vg) dv + \sum_{\gamma_2 \in \Gamma_2[V_2]} \int \varphi(v\iota(\gamma_2)g) \psi_{\alpha_2}^{-1}(v) dv + \\ + \sum_{\gamma_1 \in \Gamma_1[V_2]} \int \varphi(v\iota(\gamma_1)g) \psi_{\alpha_1}^{-1}(v) dv, \end{aligned}$$

where we note that the extra second term comes from the constant term on V_1 . We will, after the iteration end up with

$$\varphi(g) = \int_{[N]} \varphi(ng) dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i[N]} \int \varphi(n\iota(\gamma)g) \psi_{\alpha_i}^{-1}(n) dn.$$

This completes the proof for a minimal representation.

• For case (ii), where φ is in a next-to-minimal-representation, we start again from (3.4) and expand using corollary 3.8. We get, for the first step, that

$$\begin{aligned} \varphi(g) = & \left(\int_{[V_2]} \varphi(vg) dv + \sum_{\gamma_2 \in \Gamma_2[V_2]} \int \varphi(v\iota(\gamma_2)g) \psi_{\alpha_2}^{-1}(vg) dv \right) + \\ & + \sum_{\gamma_1 \in \Gamma_1[V_2]} \int \varphi(v\iota(\gamma_1)g) \psi_{\alpha_1}^{-1}(v) dv, \end{aligned}$$

where the parenthesis comes from the expansion of the constant term in (3.4). Expanding in the next row as well, this becomes

$$\begin{aligned} & \left(\int_{[V_3]} \varphi(vg) dv + \sum_{\gamma_3 \in \Gamma_3[V_3]} \int \varphi(v\iota(\gamma_3)g) \psi_{\alpha_3}^{-1}(v) dv + \sum_{\gamma_2 \in \Gamma_2[V_3]} \int \varphi(v\iota(\gamma_2)g) \psi_{\alpha_2}^{-1}(v) dv \right) + \\ & + \sum_{\gamma_1 \in \Gamma_1[V_3]} \left(\int \varphi(v\iota(\gamma_1)g) \psi_{\alpha_1}^{-1}(v) dv + \sum_{\gamma_3 \in \Gamma_3(\psi_{\alpha_1})[V_3]} \int \varphi(v\iota(\gamma_3)\iota(\gamma_1)g) \psi_{\alpha_1, \alpha_3}^{-1}(v) dv \right). \end{aligned}$$

For each expansion adding a row i , the constant term gives an extra sum over Γ_i of a Fourier integral with character ψ_{α_i} , and from all terms with characters ψ_{α_j} with $j < i - 1$ we get an extra sum over $\Gamma_i(\psi_{\alpha_j})$ together with a character $\psi_{\alpha_j, \alpha_i}$. Corollary 3.8 (iv) implies that these terms with characters non-trivial along two simple roots do not receive any further contributions. Thus, after repeatedly using corollary 3.8 to the last row, we get that

$$\begin{aligned} \varphi(g) = & \int_{[N]} \varphi(ng) dn + \sum_{i=1}^{n-1} \sum_{\gamma \in \Gamma_i[N]} \int \varphi(n\iota(\gamma)g) \psi_{\alpha_i}^{-1}(n) dn + \\ & + \sum_{j=1}^{n-3} \sum_{i=j+2}^{n-1} \sum_{\substack{\gamma_i \in \Gamma_i(\psi_{\alpha_j}) \\ \gamma_j \in \Gamma_j}} \int \varphi(n\iota(\gamma_i)\iota(\gamma_j)g) \psi_{\alpha_j, \alpha_i}^{-1}(n) dn, \end{aligned}$$

which completes the proof of Theorem A. □

4. PROOF OF THEOREM B

In this section, we prove Theorem B which relates Fourier coefficients on a maximal parabolic subgroup with Whittaker coefficients on the Borel subgroup. Recalling that the constant terms are known from [MW95], we only focus on non-trivial characters, but first we need to introduce some notation and lemmas.

For $1 \leq m \leq n - 1$, let U_m be the unipotent radical of the maximal parabolic subgroup P_m with Levi subgroup L_m isomorphic to the subgroup

of $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$ defined by $\{(g, g') \in \mathrm{GL}_m \times \mathrm{GL}_{n-m} : \det(g) \det(g') = 1\}$. U_m is abelian and is isomorphic to the set of all $m \times (n-m)$ matrices. Write U_m as

$$U_m = \left\{ \begin{pmatrix} I_m & X \\ 0 & I_{n-m} \end{pmatrix} : X \in \mathrm{Mat}_{m \times (n-m)} \right\}.$$

Let $\bar{U}_m = {}^t U_m$ be the unipotent radical of the opposite parabolic $\bar{P}_m$. Then the Lie algebra of $\bar{U}_m$ can be written as

$$(4.1) \quad \bar{\mathfrak{u}}_m = {}^t \mathfrak{u}_m = \left\{ y(Y) = \begin{pmatrix} 0_m & 0 \\ Y & 0_{n-m} \end{pmatrix} : Y \in \mathrm{Mat}_{(n-m) \times m} \right\}.$$

It is clear that the character group of U_m can be identified with ${}^t \mathfrak{u}_m$. L_m acts on ${}^t \mathfrak{u}_m$ via conjugation and with (2.4) this becomes a conjugation of the corresponding character's argument. Because of (1.5), the Fourier coefficients for characters in the same $L_m(F)$ -orbit are related by translates of their arguments, which means that we only need to compute one Fourier coefficient for each orbit. We will therefore now describe the $L_m(F)$ -orbits of elements $y(Y) \in {}^t \mathfrak{u}_m$ but leave the details to be proven in appendix D.

Starting first with $\bar{F}$ the number of $L_m(\bar{F})$ -orbits is $\min(m, n-m) + 1$ and the orbits are classified by the rank of the $(n-m) \times m$ matrix Y . A representative of an $L_m(\bar{F})$ -orbit corresponding to rank r can be chosen as $y(Y_r)$ where Y_r is an $(n-m) \times m$ matrix, zero everywhere except for the upper right $r \times r$ submatrix which is anti-diagonal with all anti-diagonal elements equal to one. For each rank r , $0 \leq r \leq \min(m, n-m)$, the corresponding $G(\bar{F})$ -orbit is parametrized by the partition $[2^r 1^{n-2r}]$.

As shown in appendix D, the $L_m(F)$ -orbits are characterized by the same data as the $G(F)$ -orbits with $([2^r 1^{n-2r}], d)$, $0 \leq r \leq \min(m, n-m)$, $d \in F^\times / (F^\times)^k$ and $k \in \gcd([2^r 1^{n-2r}])$ with representatives $y(Y_r(d))$ where $Y_r(d)$ is of the same form as Y_r above, but with the lower left element in the $r \times r$ matrix equal to d

$$Y_r(d) = \begin{pmatrix} 0 & \begin{bmatrix} & & 1 \\ & \ddots & \\ d & 1 & \end{bmatrix} \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{(n-m) \times m}(F).$$

We will continue to write $Y_r(1) = Y_r$. Note that for $0 \leq r \leq 2$ and $n \geq 5$, k is equal to 1. Each such $L_m(F)$ -orbit is also part of the $G(F)$ -orbit of the same data.

From (2.3) the corresponding character on U_m is

$$\psi_{y(Y_r)}(u) = \psi(\mathrm{tr}(y(Y_r) \log(u))), \quad u \in U_m(\mathbb{A}).$$

Let s_m be the semisimple element $\mathrm{trdiag}(1, 1, \dots, 1, -1, -1, \dots, -1)$ with m copies of 1's and $(n-m)$ copies of -1's. Then, for any automorphic form φ on $\mathrm{SL}_n(\mathbb{A})$, the following Fourier coefficient

$$\int_{[U_m]} \varphi(ug) \psi_{y(Y_r(d))}^{-1}(u) du$$

is exactly the degenerate Fourier coefficient $\mathcal{F}_{s_m, y(Y_r(d))}(\varphi)$.

Note that in this paper, we focus on minimal and next-to-minimal representations, hence we only need to consider the cases of $0 \leq r \leq 2$. Indeed, for $3 \leq r \leq \min(m, n-m)$, by definition, the generalized Fourier coefficient attached to the partition $[2^r 1^{n-2r}]$ is identically zero for minimal and next-to-minimal representations. By Theorem 2.1 and since $y(Y_r(d))$ is in the $G(\overline{F})$ -orbit $[2^r 1^{n-2r}]$, all the Fourier coefficients $\mathcal{F}_{s_m, y(Y_r)}(\varphi)$ are also identically zero.

This leaves $r \in \{1, 2\}$ and with our assumption that $n \geq 5$, we thus only need to consider the representatives $y(Y_1)$ and $y(Y_2)$ with $d = 1$ since $\gcd([2^r 1^{n-2r}]) = 1$.

The above arguments proves the first part of theorem B, that there exists an element $l \in L_m(F)$ such that $\mathcal{F}_U(\varphi, \psi_U; g) = \mathcal{F}_U(\varphi, \psi_{y(Y_r)}; lg)$ (note the slight difference in notation $\psi_{y(Y_r)}$ instead of ψ_{Y_r}), and that all $\mathcal{F}_U(\varphi, \psi_{y(Y_r)}; lg)$ for $r > r_\pi$ vanish identically where $r_{\pi_{\min}} = 1$ and $r_{\pi_{\text{ntm}}} = 2$.

We will now determine the remaining Fourier coefficients $\mathcal{F}_U(\varphi, \psi_{y(Y_r)}; g)$ in terms of Whittaker coefficients. For $1 \leq m \leq n-1$, $0 \leq i \leq m-1$, let U_m^i be the unipotent radical of the parabolic of type $(m-i, 1^i, n-m)$. Note that $U_m^0 = U_m$. Note that the character $\psi_{y(Y_1)}$ can be extended to a character of any subgroup of N containing U_m , still denoted by $\psi_{y(Y_1)}$. Let C_{m-i} be the subgroup of U_m^{i+1} consisting of elements with $u_{p,q} = 0$ except when $q = m-i$ and the diagonal elements. Note that C_{m-i} is an abelian subgroup and its character group can be identified with ${}^t\mathfrak{c}_{m-i}$, the Lie algebra of ${}^tC_{m-i}$. Write C_{m-i} as

$$C_{m-i} = \left\{ c(X) = \begin{pmatrix} I_{m-i-1} & X & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-m+i} \end{pmatrix} \right\}$$

and ${}^t\mathfrak{c}_{m-i}$ as

$${}^t\mathfrak{c}_{m-i} = \left\{ y(Y) = \begin{pmatrix} 0_{m-i-1} & 0 & 0 \\ Y & 0 & 0 \\ 0 & 0 & 0_{n-m+i} \end{pmatrix} \right\}.$$

For each $y \in {}^t\mathfrak{c}_{m-i}$, the corresponding character ψ_y of C_{m-i} is defined by $\psi_y(c) = \psi(\text{tr}(y \log(c)))$.

Example 4.1. For SL_5 we have that

$$U_3 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \quad U_3^1 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \quad C_3 = \left\{ \begin{pmatrix} 1 & * \\ & 1 \\ & & 1 \end{pmatrix} \right\}.$$

Note that $U_m^{m-1} = V_m$ and $U_m^{i+1} = C_{m-i}U_m^i$. We will sometimes use $j = m-i$ instead to denote column as follows $U_m^{m-j+1} = C_jU_m^{m-j}$.

We will now construct a semi-simple element $s = s_{U_m^i}$ for which $\mathfrak{g}_1^s = \emptyset$ and such that $\mathfrak{n}_s = \mathfrak{g}_{\geq 2}^s$ corresponds to $N_s = U_m^i$. These conditions are satisfied by

$$(4.2) \quad \text{trdiag}(2i, \dots, 2i, 2(i-1), \dots, 2, 0, -2, \dots, -2)$$

with $m-i$ copies of $2i$ and $n-m$ copies of -2 .

Note that any character ψ on N trivial on the complement of U_m^i in N is also a character on U_m^i by restriction and can be expressed as ψ_y with $y \in \mathfrak{g}_{-2}^s$ where $s = s_{U_m^i}$ such that (s, y) forms a Whittaker pair. Indeed, we have that $y \in \mathfrak{g}_{-2}^{s_N}$ where $s_N = \text{trdiag}(2(n-1), 2(n-2), \dots, 0, -2)$ from (3.1)

and the complement of U_m^i is described by $s - s_N$ meaning that $[y, s - s_N] = 0$ for ψ to be trivial on the complement and thus $[y, s] = [y, s_N] = -2y$.

Lemma 4.2. *Let φ be an automorphic form on $\mathrm{SL}_n(\mathbb{A})$ and $2 \leq j \leq n$. Let also ψ_0 be a character on N which, if $j = 2$, should be trivial along α_1 and either along α_{n-1} or along (at least) two adjacent other simple roots. Then,*

$$\sum_{\substack{y \in {}^t\mathfrak{c}_j(F) \\ y \neq 0}} \int_{[C_j]} \varphi(xg) \psi_y^{-1}(x) dx = \sum_{\gamma \in \Lambda_{j-1}(\psi_0)} \int_{[C_j]} \varphi(x\hat{i}(\gamma)g) \psi_{\alpha_{j-1}}^{-1}(x) dx.$$

where $\Lambda_j(\psi_0)$ is defined in (1.3) and only depends on ψ_0 for $j = 2$. The embedding $\hat{i}$ is defined in (1.4).

Proof. The proof is similar to those of lemmas 3.3 and 3.4.

• For $2 < j \leq n$, we parametrize $y \in {}^t\mathfrak{c}_j(F)$ by row vectors $Y \in \mathrm{Mat}_{1 \times (j-1)}(F)$ with representative $\hat{X} = (0, \dots, 0, 1)$ such that $\psi_{y(\hat{X})} = \psi_{\alpha_{j-1}}$.

The surjective map $\mathrm{SL}_{j-1}(F) \rightarrow {}^t\mathfrak{c}_j(F)^\times : \gamma \mapsto \hat{X}\gamma$ gives that ${}^t\mathfrak{c}_j(F)^\times \cong (\mathrm{SL}_{j-1}(F))_{\hat{X}} \backslash \mathrm{SL}_{j-1}(F) = \Lambda_{j-1}$.

As in lemma 3.3, we can write the action as a conjugation $y(\hat{X}\gamma) = \hat{i}(\gamma)^{-1}y(\hat{X})\hat{i}(\gamma)$ and, using (2.4), $\psi_{y(\hat{X}\gamma)}(x) = \psi_{y(\hat{X})}(\hat{i}(\gamma)x\hat{i}(\gamma)^{-1})$. Since $\hat{i}(\gamma)$ normalizes C_j a variable change gives the wanted expression.

• For $j = 2$, with $y \in {}^t\mathfrak{c}_2(F) \cong F$ we instead consider the map $T_{\psi_0} \rightarrow {}^t\mathfrak{c}_2(F)^\times : h \mapsto h^{-1}y(1)h$ which is surjective by similar arguments as in lemma 3.4 and thus gives ${}^t\mathfrak{c}_2(F)^\times \cong (T_{\psi_0} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_0} = \Lambda_1(\psi_0)$. Writing the conjugation of y as a conjugation of the character's argument and then substituting variables in the Fourier integral thus proves the lemma. $\square$

Lemma 4.3. *Let φ be an automorphic form on $\mathrm{SL}_n(\mathbb{A})$, $1 \leq m \leq n-1$, $2 \leq j \leq m$ and ψ_0 a character on N trivial on the complement of U_m^{m-j} in N . For $j = 2$, ψ_0 should also be trivial either along α_{n-1} or along (at least) two adjacent simple roots other than α_1 .*

$$\begin{aligned} \int_{[U_m^{m-j}]} \varphi(ug) \psi_0^{-1}(u) du &= \\ &= \int_{[U_m^{m-j+1}]} \varphi(ug) \psi_0^{-1}(u) du + \sum_{\gamma \in \Lambda_{j-1}(\psi_0)} \int_{[U_m^{m-j+1}]} \varphi(u\hat{i}(\gamma)g) \psi_0^{-1}(u) \psi_{\alpha_{j-1}}^{-1}(u) du. \end{aligned}$$

Proof. For $2 \leq j \leq m$ we have that $\varphi(xug) = \varphi(ug)$ for $x \in C_i(F)$ and $u \in U_m^i(\mathbb{A})$ and since C_j is abelian

$$(4.3) \quad \varphi(ug) = \sum_{y \in {}^t\mathfrak{c}_j(F)} \int_{[C_j]} \varphi(xug) \psi_y^{-1}(x) dx.$$

Using lemma 4.2, we get that

$$\varphi(ug) = \int_{[C_j]} \varphi(xug) dx + \sum_{\gamma \in \Lambda_{j-1}(\psi_0)} \int_{[C_j]} \varphi(x\hat{i}(\gamma)ug) \psi_{\alpha_{j-1}}^{-1}(x) dx.$$

Let $u \in U_m^{m-j}$ be parametrized as

$$u = \begin{pmatrix} I_{j-1} & B \\ 0 & A \end{pmatrix}$$

where $A \in \text{Mat}_{(n-j+1) \times (n-j+1)}$ is upper unitriangular (with several upper triangular elements being zero) and $B \in \text{Mat}_{(j-1) \times (n-j+1)}$ with elements in the first column being zero. Since B does not intersect the abelianization $[N, N] \backslash N$ (that is, the Lie algebra of B does not contain any generator of a simple root), we have, by assumption, that ψ_0 only depends on A . We also have that $x \in C_j$ can be parametrized as

$$x = \begin{pmatrix} I_{j-1} & B' \\ 0 & I_{n-j+1} \end{pmatrix}$$

where $B' \in \text{Mat}_{(j-1) \times (n-j+1)}$ with only the first column non-zero. Thus,

$$xu = \begin{pmatrix} I_{j-1} & B + B'A \\ 0 & A \end{pmatrix}$$

which means that $\psi_0(u) = \psi_0(xu)$. The first column of B is zero and A is upper unitriangular which means that the first column of $B + B'A$ is the same as the first column of B' and since $\psi_{\alpha_{j-1}}$ only depends on the first column of B' this implies that $\psi_{\alpha_{j-1}}(x) = \psi_{\alpha_{j-1}}(xu)$.

- For $3 \leq j \leq m$ with $\gamma \in \Lambda_{j-1}$ and $l = \hat{l}(\gamma)$,

$$lul^{-1} = \begin{pmatrix} \gamma & 0 \\ 0 & I_{n-j+1} \end{pmatrix} \begin{pmatrix} I_{j-1} & B \\ 0 & A \end{pmatrix} \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & I_{n-j+1} \end{pmatrix} = \begin{pmatrix} I_{j-1} & \gamma B \\ 0 & A \end{pmatrix}$$

and since ψ_0 , by assumption, only depends on A we have that $\psi_0(u) = \psi_0(lul^{-1})$.

- For $j = 2$ with $\gamma \in \Lambda_1$ and $l = \hat{l}(\gamma) = \gamma$ is in the stabilizer T_{ψ_0} which, by definition, means that $\psi_0(u) = \psi_0(lul^{-1})$.

Hence, for $2 \leq j \leq m$, and after making a variable change $lul^{-1} \rightarrow u$, we get that

$$\begin{aligned} & \int_{[U_m^{m-j}]} \int_{[C_j]} \varphi(xlug) \psi_0^{-1}(u) \psi_{\alpha_{j-1}}^{-1}(x) dx du = \\ &= \int_{[U_m^{m-j}]} \int_{[C_j]} \varphi(xulg) \psi_0^{-1}(u) \psi_{\alpha_{j-1}}^{-1}(x) dx du \\ &= \int_{[U_m^{m-j}]} \int_{[C_j]} \varphi(xulg) \psi_0^{-1}(xu) \psi_{\alpha_{j-1}}^{-1}(xu) dx du \\ &= \int_{[U_m^{m-j+1}]} \varphi(ulg) \psi_0^{-1}(u) \psi_{\alpha_{j-1}}^{-1}(u) du. \end{aligned}$$

After similar manipulations for the constant term we obtain

$$\begin{aligned} \int_{[U_m^{m-j}]} \varphi(ug) \psi_0^{-1}(u) du &= \\ &= \int_{[U_m^{m-j+1}]} \varphi(ug) \psi_0^{-1}(u) du + \sum_{\gamma \in \Lambda_{j-1}(\psi_0)} \int_{[U_m^{m-j+1}]} \varphi(ulg) \psi_0^{-1}(u) \psi_{\alpha_{j-1}}^{-1}(u) du. \end{aligned}$$

□

What the above lemma describes is the inclusion of column C_j in the integration domain which is why an additional character along α_{j-1} appears.

Remark 4.4. We note that if ψ_0 is trivial along α_1 but not along at least two adjacent other simple roots we cannot use lemma 4.2, but we could still make an expansion over C_2 and keep the sum over $y \in {}^t\mathfrak{c}_2(F) \cong F$ in the proof above. Since the character ψ_y has the same support as ψ_{α_1} on N we still have that $\psi_y(x) = \psi_y(xu)$ for $x \in C_j(\mathbb{A})$ and $u \in U_m^{m-j}(\mathbb{A})$ and since ψ_0 is still a character on N trivial on the complement of U_m^{m-j} it is still true that $\psi_0(u) = \psi_0(xu)$. Thus, using (4.3)

$$\begin{aligned} \int_{[U_m^{m-2}]} \varphi(ug) \psi_0^{-1}(u) du &= \sum_{y \in {}^t\mathfrak{c}_2(F)} \int_{[U_m^{m-2}]} \int_{[C_2]} \varphi(xug) \psi_0^{-1}(u) \psi_y^{-1}(x) dx du \\ &= \sum_{y \in {}^t\mathfrak{c}_2(F)} \int_{[U_m^{m-2}]} \int_{[C_2]} \varphi(xug) \psi_0^{-1}(xu) \psi_y^{-1}(xu) dx du \\ &= \sum_{y \in {}^t\mathfrak{c}_2(F)} \int_{[V_m]} \varphi(vg) \psi_0^{-1}(v) \psi_y^{-1}(v) dv. \end{aligned}$$

Lemma 4.5. Assume that π is an irreducible minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, $\varphi \in \pi$. For $1 \leq m \leq n-1$, $0 \leq i \leq m-2$, and $g \in \mathrm{SL}_n(\mathbb{A})$,

$$\int_{[U_m^i]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du = \int_{[U_m^{i+1}]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du.$$

Proof. Using lemma 4.3 with $\psi_0 = \psi_{y(Y_1)} = \psi_{\alpha_m}$ (cf. also remark 3.5) we get that

$$(4.4) \quad \int_{[U_m^i]} \varphi(ug) \psi_0^{-1}(u) du = \int_{[U_m^{i+1}]} \varphi(ug) \psi_0^{-1}(u) du + \sum_{\gamma \in \Lambda_{m-i-1}(\psi_0)} \mathcal{F}(\varphi; m, i, \gamma, g),$$

where we have introduced

$$\mathcal{F}(\varphi; m, i, \gamma, g) = \int_{[U_m^{i+1}]} \varphi(u\hat{\iota}(\gamma)g) \psi_0^{-1}(u) \psi_{\alpha_{m-i-1}}(u) du.$$

Let $s = s_{U_m^{i+1}}$ from (4.2), and let $u \in \mathfrak{sl}_n$ with two non-zero entries, both being 1, at positions $(m-i, m-i-1)$ and $(m+1, m)$. Then, $\mathcal{F}(\varphi; m, i, \gamma, g) = \mathcal{F}_{s,u}(\varphi)(\hat{\iota}(\gamma)g)$ and since u is not in the closure of the minimal orbit, theorem 2.1 gives that $\mathcal{F}_{s,u}(\varphi)$ is identically zero leaving only the constant mode in (4.4). □

Proof of Theorem B.

• **Minimal representation.** Assume that π be an irreducible minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, and $\varphi \in \pi$. Applying Lemma 4.5 repeatedly, we get that for each $1 \leq m \leq n-1$,

$$\int_{[U_m]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du = \int_{[U_m^{m-1}]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du.$$

Note that $U_m^{m-1} = V_m$ and $\psi_{y(Y_1)} = \psi_{\alpha_m}$. Applying corollary 3.7 repeatedly, we get that for each $1 \leq m \leq n-1$,

$$\int_{[U_m^{m-1}]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du = \int_{[N]} \varphi(ng) \psi_{y(Y_1)}^{-1}(n) dn,$$

which is exactly

$$\int_{[N]} \varphi(ng) \psi_{\alpha_m}^{-1}(n) dn.$$

• **Next-to-minimal representation - rank 1.** Let π be an irreducible next-to-minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$ and let $\varphi \in \pi$. Recalling that $U_m = U_m^0$ and applying lemma 4.5 with $\psi_0 = \psi_{y(Y_1)} = \psi_{\alpha_m}$ we get

$$\int_{[U_m]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du = \int_{[U_m^1]} \varphi(ug) \psi_0^{-1}(u) du$$

since $\psi_0 \psi_{\alpha_{m-1}} = \psi_{\alpha_m, \alpha_{m-1}} = \psi_u$ for some u that is not in the closure of the next-to-minimal orbit and thus the non-constant modes in lemma 4.5 can be expressed as Fourier coefficients $\mathcal{F}_{s,u}$ with $s = s_{U_m^1}$ from (4.2) which vanish according to theorem 2.1.

Let us make an iteration in $1 \leq i \leq m-2$. Using lemma 4.5 we have that

$$\begin{aligned} (4.5) \quad & \int_{[U_m^i]} \varphi(ug) \psi_0^{-1}(u) du = \\ & = \int_{[U_m^{i+1}]} \varphi(ug) \psi_0^{-1}(u) du + \sum_{\gamma \in \Lambda_{m-i-1}(\psi_{\alpha_m})} \int_{[U_m^{i+1}]} \varphi(u\hat{\iota}(\gamma)g) \psi_{\alpha_{m-i-1}, \alpha_m}^{-1}(u) du. \end{aligned}$$

Since $\psi_{\alpha_m, \alpha_{m-i-1}}$ is a character on N trivial on the complement of U_m^{i+1} we can expand the second term further with lemma 4.3 (or remark 4.4 if $m-i-1 = 2$ and $\psi_{\alpha_m, \alpha_{m-i-1}}$ is not trivial along at least two adjacent roots other than α_1). This would lead to characters $\psi_u = \psi_{\alpha_m, \alpha_{m-i-1}, \alpha_{m-i-2}}$ (or $\psi_u = \psi_{\alpha_m, \alpha_{m-i-1}} \psi_y$ with $y \in {}^t\mathfrak{c}_2(F)$ respectively) where u is not in the closure of the next-to-minimal orbit. Then, $\mathcal{F}_{s,u}$ with $s = s_{U_m^{i+2}}$ from (4.2) vanishes according to theorem 2.1 and the second term only receives the constant mode contribution. Repeating these arguments for the second term in (4.5), it becomes

$$\sum_{\gamma \in \Lambda_{m-i-1}(\psi_{\alpha_m})} \int_{[V_m]} \varphi(u\hat{\iota}(\gamma)g) \psi_{\alpha_{m-i-1}, \alpha_m}^{-1}(u) du.$$

Iterating over i , starting from $i = 1$ above, we get that

$$(4.6) \quad \int_{[U_m]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du = \int_{[V_m]} \varphi(ug) \psi_{\alpha_m}^{-1}(u) du + \sum_{j=1}^{m-2} \sum_{\gamma \in \Lambda_j(\psi_{\alpha_m})} \int_{[V_m]} \varphi(u\hat{\iota}(\gamma)g) \psi_{\alpha_j, \alpha_m}^{-1}(u) du.$$

For $m = 1$, $U_1 = V_1$ and for $m = 2$ we only get the first term in (4.6).

We will now use the methods of section 3 to expand along rows. Using corollary 3.8 case (iv), we see that the second term in (4.6) does not get any further contributions when expanding to N . Starting with the first term in (4.6) and using corollary 3.8 first with case (iii) to V_{m+1} and then repeatedly with cases (ii) and (iv) it becomes

$$\begin{aligned} \int_{[V_{m+1}]} \varphi(ug) \psi_{\alpha_m}^{-1}(u) du &= \int_{[N]} \varphi(ng) \psi_{\alpha_m}^{-1}(n) dn + \sum_{i=m+2}^{n-1} \sum_{\gamma \in \Gamma_i(\psi_{\alpha_m})} \int_{[N]} \varphi(n\iota(\gamma)g) \psi_{\alpha_m, \alpha_i}^{-1}(n) dn. \end{aligned}$$

Lastly,

$$\begin{aligned} \int_{[U_m]} \varphi(ug) \psi_{y(Y_1)}^{-1}(u) du &= \int_{[N]} \varphi(ng) \psi_{\alpha_m}^{-1}(n) dn + \\ &+ \sum_{j=1}^{m-2} \sum_{\gamma \in \Lambda_j(\psi_{\alpha_m})} \int_{[N]} \varphi(n\hat{\iota}(\gamma)g) \psi_{\alpha_j, \alpha_m}^{-1}(n) dn + \\ &+ \sum_{i=m+2}^{n-1} \sum_{\gamma \in \Gamma_i(\psi_{\alpha_m})} \int_{[N]} \varphi(n\iota(\gamma)g) \psi_{\alpha_m, \alpha_i}^{-1}(n) dn. \end{aligned}$$

• **Next-to-minimal representation - rank 2.** Let π be an irreducible next-to-minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$ and let $\varphi \in \pi$. We start from the integral

$$\int_{[U_m]} \varphi(ug) \psi_{y(Y_2)}^{-1}(u) du.$$

For each root α , let X_α be the corresponding one-dimensional root subgroup in SL_n . Let

$$C_1 = X_{e_m - e_{m+2}} \prod_{i=1}^{m-2} X_{e_i - e_{m+2}},$$

and

$$R_1 = X_{e_{m-1} - e_m} \prod_{i=1}^{m-2} X_{e_{m-1} - e_i}.$$

Then C_1 is a subgroup of U_m . Let U'_m be the subgroup of U_m with C_1 -part identically zero. Then one can see that the quadruple

$$(U'_m, \psi_{y(Y_2)}, C_1, R_1)$$

satisfies all the conditions of Lemma 2.5. By this lemma,

$$\begin{aligned} & \int_{[U_m]} \varphi(ug) \psi_{y(Y_2)}^{-1}(u) du \\ &= \int_{C_1(\mathbb{A})} \int_{[R_1 U'_m]} \varphi(ucg) \psi_{y(Y_2)}^{-1}(u) du dc. \end{aligned}$$

Let

$$C_2 = \prod_{i=1}^{m-2} X_{e_i - e_{m+1}},$$

and

$$R_2 = \prod_{i=1}^{m-2} X_{e_m - e_i}.$$

Then C_2 is a subgroup of $R_1 U'_m$. Let U''_m be the subgroup of $R_1 U'_m$ with C_2 -part identically zero. Then one can see that the quadruple

$$(U''_m, C_2, R_2, \psi_{y(Y_2)})$$

satisfies all the conditions of Lemma 2.5. Applying this lemma and by changing of variables,

$$\begin{aligned} & \int_{C_1(\mathbb{A})} \int_{[R_1 U'_m]} \varphi(ucg) \psi_{y(Y_2)}^{-1}(u) du dc \\ (4.7) \quad &= \int_{C_1(\mathbb{A})} \int_{C_2(\mathbb{A})} \int_{[R_2 U''_m]} \varphi(uc_2 c_1 g) \psi_{y(Y_2)}^{-1}(u) du dc_2 dc_1 \\ &= \int_{(C_1 C_2)(\mathbb{A})} \int_{[R_2 U''_m]} \varphi(ucg) \psi_{y(Y_2)}^{-1}(u) du dc. \end{aligned}$$

Let ω be the Weyl element sending torus elements

$$(t_1, t_2, \dots, t_n)$$

to torus elements

$$(t_{m-1}, t_{m+2}, t_m, t_{m+1}, t_1, t_2, \dots, t_{m-2}, t_{m+3}, t_{m+4}, \dots, t_n).$$

Conjugating ω cross from left, the integral in (4.7) becomes

$$(4.8) \quad \int_{C(\mathbb{A})} \int_{[U_m^\omega]} \varphi(u\omega cg) \psi_{y(Y_2)}^{\omega, -1}(u) du dc,$$

where $U_m^\omega = \omega R_2 U''_m \omega^{-1}$, $C = C_1 C_2$, for $u \in U_m^\omega$, $\psi_{y(Y_2)}^\omega(u) = \psi_{y(Y_2)}(\omega^{-1} u \omega)$.

$$U_m^\omega = U_m^{\omega, 1} V_1,$$

where elements $u \in U_m^{\omega, 1}$ have the following form

$$\begin{pmatrix} I_2 & 0 \\ 0 & u' \end{pmatrix},$$

and $U_m^{\omega, 1}$ normalizes V_1 . Recall that V_i be unipotent radical of parabolic subgroup of type $(1^i, n-i)$. Note that $\psi_{y(Y_2)}^\omega|_{V_1} = \psi_{\alpha_1}$, $\psi_{y(Y_2)}^\omega|_{U_m^{\omega, 1}} = \psi_{\alpha_3}$. Recall that $\alpha_1 = e_1 - e_2$, $\alpha_3 = e_3 - e_4$. Hence, the integral in (4.8) becomes

$$(4.9) \quad \int_{C(\mathbb{A})} \int_{[U_m^{\omega, 1}]} \int_{[V_1]} \varphi(vu\omega cg) \psi_{\alpha_1}^{-1}(v) \psi_{\alpha_3}^{-1}(u) dv du dc.$$

Since π is an irreducible next-to-minimal automorphic representation of $\mathrm{SL}_n(\mathbb{A})$, by corollary 3.8, case (iii), the integral in (4.9) becomes

$$(4.10) \quad \int_{C(\mathbb{A})} \int_{[U_m^{\omega,1}]} \int_{[V_2]} \varphi(vu\omega cg) \psi_{\alpha_1}^{-1}(v) \psi_{\alpha_3}^{-1}(u) dv du dc.$$

$U_m^{\omega,1}$ still normalizes V_2 , and

$$U_m^{\omega,1} V_2 = U_m^{\omega,2} V_3,$$

where elements $u \in U_m^{\omega,2}$ have the following form

$$\begin{pmatrix} I_4 & 0 \\ 0 & u'' \end{pmatrix},$$

u'' is in the radical of the parabolic subgroup of type $(m-2, n-m-2)$ in SL_{n-4} , and $U_m^{\omega,2}$ normalizes V_3 . Note that $\psi_{y(Y_2)}^\omega|_{V_3} = \psi_{\alpha_1, \alpha_3}$ and $\psi_{y(Y_2)}^\omega|_{U_m^{\omega,2}}$ is the trivial character. By corollary 3.8, case (iv), the integral in (4.10) becomes

$$(4.11) \quad \int_{C(\mathbb{A})} \int_{[U_m^{\omega,2}]} \int_{[V_4]} \varphi(vu\omega cg) \psi_{\alpha_1, \alpha_3}^{-1}(v) dv du dc.$$

Applying corollary 3.8, case (iv), repeatedly, the integral in (4.11) becomes

$$\int_{C(\mathbb{A})} \int_{[U_m^{\omega,2}]} \int_{[N]} \varphi(nu\omega cg) \psi_{\alpha_1, \alpha_3}^{-1}(n) dn du dc,$$

which becomes

$$\int_{C(\mathbb{A})} \int_{[U_m^{\omega,2}]} \int_{[N]} \varphi(n\omega cg) \psi_{\alpha_1, \alpha_3}^{-1}(n) dn du dc,$$

by changing of variables. Since $\int_{[U_m^{\omega,2}]} du = 1$, we have obtained that

$$\int_{[U_m]} \varphi(ug) \psi_{y(Y_2)}^{-1}(u) du = \int_{C(\mathbb{A})} \int_{[N]} \varphi(n\omega cg) \psi_{\alpha_1, \alpha_3}^{-1}(n) dn dc.$$

This completes the proof of Theorem B. $\square$

5. PROOF OF THEOREMS C AND D

Proof of Theorem C. Let π be any irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$ and let $\varphi \in \pi$. The generalized Fourier coefficient of φ attached to the partition $[21^{n-2}]$ has been defined in Section 2. We recall it as follows.

Let $s = (1, -1, 0, \dots, 0)$, and let $u = J_{[21^{n-2}]}$ which is a matrix zero everywhere except the $(2,1)$ entry being 1. Then the generalized Fourier coefficient of φ attached to the partition $[21^{n-2}]$ is as follows:

$$\mathcal{F}^{[21^{n-2}]}(\varphi; g) = \mathcal{F}_{s,u}(\varphi; g) = \int_{[N_s]} \varphi(ng) \psi_u^{-1}(n) dn,$$

where elements in the one-dimensional unipotent N_s have the form

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

Let $X = \prod_{i=3}^n X_{e_1 - e_i}$ and $Y = \prod_{i=3}^n X_{e_i - e_2}$. Then one can see that $Y(F)$ can be identified with the character space of $[X]$ as follows: given $y \in Y(F)$,

$\psi_y(x) = \psi_u([x, y])$, for any $x \in [X]$. Note that both X and Y normalize N_s . Taking the Fourier expansion of $\mathcal{F}_{s,u}(\varphi)(g)$ along $[X]$, we obtain that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xng) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx.$$

Since $y^{-1} \in Y(F)$ and φ is automorphic, the above integral becomes

$$\begin{aligned} & \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xng) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(y^{-1}xng) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(y^{-1}xn y y^{-1}g) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xn' y^{-1}g) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx, \end{aligned}$$

where $n' = n + [x, y]$. By changing variables, we obtain that

$$\begin{aligned} & \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xn' y^{-1}g) \psi_u^{-1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xn y^{-1}g) \psi_u^{-1}(n) \psi_u^{-1}(-[x, y]) \psi_y^{-1}(x) dn dx. \end{aligned}$$

Note that

$$\begin{aligned} & \psi_u^{-1}(-[x, y]) \psi_y^{-1}(x) \\ &= \psi_u([x, y]) \psi_u(-[x, y]) \\ &= 1. \end{aligned}$$

Hence, we have that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y(F)} \int_{[X]} \int_{[N_s]} \varphi(xn y^{-1}g) \psi_u^{-1}(n) dn dx.$$

Note that $XN_s = U_1$ and $\psi_u = \psi_{\alpha_1}$. Therefore, we have that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y(F)} \int_{[U_1]} \varphi(uy^{-1}g) \psi_{\alpha_1}^{-1}(u) du.$$

This completes the proof of Theorem C. $\square$

Proof of Theorem D. Let π be any irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$ and let $\varphi \in \pi$. The generalized Fourier coefficient of φ attached to the partition $[2^2 1^{n-4}]$ has also been defined in Section 2. We recall it as follows.

Let $s = (1, -1, 1, -1, 0, \dots, 0)$, and let $u = J_{[2^2 1^{n-4}]}$ which is a matrix zero everywhere except the (2,1) and (4,3) entries being 1. Then the generalized Fourier coefficient of φ attached to the partition $[2^2 1^{n-4}]$ is as follows:

$$\mathcal{F}^{[2^2 1^{n-4}]}(\varphi; g) = \mathcal{F}_{s,u}(\varphi; g) = \int_{[N_s]} \varphi(ng) \psi_u^{-1}(n) dn,$$

where elements in N_s have the form

$$\begin{pmatrix} 1 & * & 0 & * & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & * & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-4} \end{pmatrix}.$$

Let ω be the Weyl element sending the torus element

$$(t_1, t_2, \dots, t_n)$$

to the torus element

$$(t_1, t_3, t_4, t_2, t_5, t_6, \dots, t_n).$$

Conjugating ω across from left, we obtain that

$$\mathcal{F}_{s,u}(\varphi; g) = \int_{[N_s^\omega]} \varphi(n\omega g) \psi_u^{\omega, -1}(n) dn,$$

where $N_s^\omega = \omega N_s \omega^{-1}$, and for $n \in N_s^\omega$, $\psi_u^\omega(n) = \psi_u(\omega^{-1}n\omega)$. Elements in $n \in N_s^\omega$ have the following form

$$n = n(z) = \begin{pmatrix} I_2 & z & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{n-4} \end{pmatrix},$$

and $\psi_u^\omega(n) = \psi(z_{1,2} + z_{2,1})$.

Let

$$X' = \prod_{i=5}^n X_{e_1 - e_i} \prod_{i=5}^n X_{e_2 - e_i}$$

and

$$Y' = \prod_{i=5}^n X_{e_i - e_4} \prod_{i=5}^n X_{e_i - e_3}.$$

Then one can see that $Y'(F)$ can be identified with the character space of $[X']$ as follows: given $y \in Y'(F)$, $\psi_y(x) = \psi_u^\omega([x, y])$, for any $x \in [X']$. Note that both X' and Y' normalize N_s . Taking the Fourier expansion of $\mathcal{F}_{s,u}(\varphi)(g)$ along $[X']$, we obtain that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xn\omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) dn dx.$$

Since $y^{-1} \in Y'(F)$ and φ is automorphic, the above integral becomes

$$\begin{aligned} & \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xn\omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(y^{-1}xn\omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(y^{-1}xn y y^{-1} \omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) dn dx \\ &= \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xn' y^{-1} \omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) dn dx, \end{aligned}$$

where $n' = n + [x, y]$. By changing variables, we obtain that

$$\begin{aligned} & \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xn'y^{-1}\omega g) \psi_u^{\omega, -1}(n) \psi_y^{-1}(x) \, dn \, dx \\ &= \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xny^{-1}\omega g) \psi_u^{\omega, -1}(n) \psi_u^{\omega, -1}(-[x, y]) \psi_y^{-1}(x) \, dn \, dx. \end{aligned}$$

Note that

$$\psi_u^{\omega, -1}(-[x, y]) \psi_y^{-1}(x) = \psi_u^\omega([x, y]) \psi_u^\omega(-[x, y]) = 1.$$

Hence, we have that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y'(F)} \int_{[X']} \int_{[N_s^\omega]} \varphi(xny^{-1}\omega g) \psi_u^{-1}(n) \, dn \, dx.$$

Note that $X'N_s^\omega = U_2$ and $\psi_u = \psi_{y(Y_2)}$, using the notation from section 4. Therefore, we have that

$$\mathcal{F}_{s,u}(\varphi; g) = \sum_{y \in Y'(F)} \int_{[U_2]} \varphi(uy^{-1}\omega g) \psi_{y(Y_2)}^{-1}(u) \, du \, dx.$$

This completes the proof of Theorem D. $\square$

6. APPLICATIONS

As is evident from table 1, the case SL_5 appears in the list of symmetry and duality groups in string theory. It is related to compactification of type II string theory on a three-torus T^3 from ten to seven spacetime dimensions. Fourier coefficients of automorphic forms on SL_5 are related to non-perturbative effects as discussed in the introduction. Therefore we analyse here in some detail the structure of Fourier coefficients of automorphic forms, in particular Eisenstein series, attached to a minimal or next-to-minimal automorphic representation of SL_5 that are relevant to the first two higher-derivative corrections in four-graviton scattering amplitudes.

We will first give a detailed description of how the formalism developed above can be used to calculate explicit expressions for Fourier coefficients on maximal parabolic subgroups for automorphic forms attached to a minimal or next-to-minimal automorphic representation of SL_n for $n \geq 5$. Following a general discussion, we will treat two explicit examples for $n = 5$.

6.1. Generalities. With applications to string theory in mind, throughout this section we are restricting to $F = \mathbb{Q}$ and let $\mathbb{A} \equiv \mathbb{A}_{\mathbb{Q}}$. Let $G = \mathrm{SL}_n$ for $n \geq 5$ and P a parabolic subgroup of G with Levi decomposition $P = LU$. The types of expressions that are of interest are of the form:

$$(6.1) \quad \mathcal{F}^{\mathbb{R}}(\varphi, \psi; g) = \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} \varphi(ug) \psi^{-1}(u) \, du,$$

where ψ is some rank-1 or rank-2 character on $U(\mathbb{R})$ and φ is an automorphic form attached to a minimal or next-to-minimal automorphic representations of $G(\mathbb{R})$. Recall that Fourier coefficients with higher rank characters vanish for the representations of interest.

Similar to the adelic case, the real rank-1 and rank-2 Fourier coefficients can be brought to standard forms using the action of the arithmetic Levi

subgroup $L(\mathbb{Z})$. For rank-1 this form is $\psi = \psi_{y(kY_1)}$ for some integer $k \neq 0$ and for rank-2 one has $\psi(k_1 u_{m,m+1} + k_2 u_{m-1,m+2})$ for non-zero integers k_1 and k_2 , cf. (1.6). For simplicity, we will mainly discuss the case $\psi_{y(Y_1)}$ and demonstrate how to apply theorem B, but the techniques demonstrated here can be generalized for the remaining Fourier coefficients of automorphic forms attached to a minimal and next-to-minimal representations on SL_n .

In order to apply theorem B, we first perform an adelic lift as described in [FGKP18]

$$\mathcal{F}^{\mathbb{R}}(\varphi, \psi; g_{\infty}) = \mathcal{F}^{\mathbb{A}}(\varphi, \psi; (g_{\infty}, I_n, I_n, \dots)) = \int_{[U]} \varphi(u(g_{\infty}, I_n, I_n, \dots)) \psi^{-1}(u) du,$$

where we abuse notation and denote the adelic lifts of the automorphic form and character also as φ and ψ . The theorem now gives $\mathcal{F}^{\mathbb{A}}$ in terms of adelic Whittaker functions. For Eisenstein series of weight λ , these Whittaker functions will then be evaluated using the adelic reduction formula

$$(6.2) \quad W_{\psi}(\lambda, a) = \sum_{w_c w'_0 \in \mathcal{C}_{\psi}} a^{(w_c w'_0)^{-1} \lambda + \rho} M(w_c^{-1}, \lambda) W'_{\psi^a}(w_c^{-1} \lambda, 1)$$

of [FKP14]. The power of this formula lies in that it expresses a degenerate Whittaker function evaluated on the Cartan torus of a group $G(\mathbb{A})$ as a sum of generic Whittaker functions on a subgroup $G'(\mathbb{A})$ with $w_c^{-1} \lambda$ orthogonally projected onto its weight space. This subgroup $G'(\mathbb{A})$ is determined by deleting all nodes in the Dynkin diagram of $G(\mathbb{A})$ on which ψ is not supported. Here, w'_0 denotes the longest Weyl word on G' , $\mathcal{C}_{\psi}$ denotes the set

$$\mathcal{C}_{\psi} = \{w \in \mathcal{W} \mid w\Pi' < 0\}$$

where Π' is the set of simple roots of G' and w_c is hence the summation variable and corresponds to a specific representative of the quotient Weyl group $\mathcal{W}/\mathcal{W}'$ described in [FKP14]. ρ denotes the Weyl vector, M denotes the intertwiner

$$M(w, \lambda) = \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(\langle \lambda | \alpha \rangle + 1)}$$

as featured in the Langlands constant term formula, where ξ is the completed Riemann zeta function and ψ^a denotes the “twisted character”, both defined in appendix A.

The evaluation of a real rank-1 Fourier coefficient schematically looks like

$$\begin{aligned} \mathcal{F}^{\mathbb{R}}(\varphi, \psi; g_{\infty}) &= \mathcal{F}^{\mathbb{A}}(\varphi, \psi; (g_{\infty}, I_n, I_n, \dots)) && \text{Adelic lift} \\ &= \sum_{\psi} \sum_{l \in \Lambda \text{ or } l \in \Gamma} W_{\psi}(l(g_{\infty}, I_n, I_n, \dots)) && \text{Theorem B} \\ &= \sum_{\psi} \sum_{l \in \Lambda \text{ or } l \in \Gamma} W_{\psi}((n_{\infty} a_{\infty} k_{\infty}, n_2 a_2 k_2, n_3 a_3 k_3, \dots)) && \text{Iwasawa-decomposition} \\ &= \sum_{\psi} \left(\prod_{p \leq \infty} \psi_p(n_p) \right) \sum_{l \in \Lambda \text{ or } l \in \Gamma} W_{\psi}((a_{\infty}, a_2, a_3, \dots)) && W_{\psi}(nak) = \psi(n)W_{\psi}(a) \end{aligned}$$

$$= \sum_{\psi} \psi_{\infty}(n_{\infty}) \sum_{l \in \Lambda \text{ or } l \in \Gamma} \sum_w a^{\cdots} M(\cdots) W'_{\psi a}(\cdots, 1) \quad \text{Reduction formula (6.2).}$$

The fourth line extracts the unipotent n_p -dependence at each of the local places $p \leq \infty$. In the fifth line we have used that only the archimedean unipotent n_{∞} contributes. The reason that the p -adic unipotent matrices n_p of the p -adic Iwasawa-decomposition of $l \in G(F) \subset G(\mathbb{Q}_p)$ above drop out is as follows. In using theorem B, we will be faced with evaluating Whittaker functions such as

$$\begin{aligned} W_{\alpha_j, \alpha_m}(\hat{l}(\lambda_j)g) & \text{ for } j \leq m-2 \text{ where } \lambda_j \in \Lambda_j \text{ and} \\ W_{\alpha_m, \alpha_i}(\iota(\gamma_i)g) & \text{ for } i \geq m+2 \text{ where } \gamma_i \in \Gamma_i. \end{aligned}$$

We have that γ_i and λ_j are embedded in SL_n as (cf. (1.4))

$$\hat{l}(\lambda_j) = \begin{pmatrix} \lambda_j & \\ & I_{n-j} \end{pmatrix} \quad \text{and} \quad \iota(\gamma_i) = \begin{pmatrix} I_i & \\ & \gamma_i \end{pmatrix}.$$

It is clear from their block-diagonal form that the unipotent n_p in the p -adic Iwasawa-decomposition of $\hat{l}(\lambda_j)$ (and $\iota(\gamma_i)$) will feature the same block-diagonal form. Since W_{α_j, α_m} (and W_{α_m, α_i}) is only sensitive to the unipotent on rows j and $m \geq j+2 > j$ (on rows i and $m \leq i-2 \leq i$), the block diagonal structure of n_p implies $\psi_{\alpha_j, \alpha_m; p}(n_p) = 1$ (and $\psi_{\alpha_m, \alpha_i; p}(n_p) = 1$).

For a real matrix $g \in \mathrm{SL}_n(\mathbb{R})$, we will denote its Iwasawa-decomposition

$$(6.3) \quad g = n_{\infty} a_{\infty} k_{\infty} = \begin{pmatrix} 1 & x_{12} & \cdots & \cdots & x_{1n} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & & \\ & y_2/y_1 & & & \\ & & \ddots & & \\ & & & y_{n-1}/y_{n-2} & \\ & & & & 1/y_{n-1} \end{pmatrix} k_{\infty}.$$

Similarly, for a p -adic matrix $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ we denote it as

$$(6.4) \quad g = n_p a_p k_p = n_p \begin{pmatrix} \eta_{1,p} & & & & \\ & \eta_{2,p}/\eta_{1,p} & & & \\ & & \ddots & & \\ & & & \eta_{n-1,p}/\eta_{n-2,p} & \\ & & & & 1/\eta_{n-1,p} \end{pmatrix} k_p.$$

Appendix B contains closed formulae for the x 's and the y 's, as well as a closed formula for the p -adic norm $|\eta_{i,p}|_p$ of the η 's.

In what follows, we will make use of all formulae that are derived or stated in appendices A, B and C along with the following notation

- A prime on a variable, eg. x' , generally denotes $x' \neq 0$.
- For sums we write $\sum_x \equiv \sum_{x \in \mathbb{Q}}$.
- We write $\sum_{x'} f(x) \equiv \sum_{x \in \mathbb{Q} \setminus \{0\}} f(x)$ and $\sum_{x' \in \mathbb{Z}} f(x) \equiv \sum_{x \in \mathbb{Z} \setminus \{0\}} f(x)$. Note that the prime is used to indicate whether or not zero is included in the sum but the prime is omitted in the summand.
- For products we write $\prod_p \equiv \prod_{p \text{ prime}}$. Writing $\prod_{p \leq \infty}$ denotes the product over all primes p (the non-archimedean places) as well as the element $p = \infty$ (the archimedean place).
- For $x \in \mathbb{R}$ we denote $\mathbf{e}(x) \equiv e^{2\pi i x}$.

w_c	$\langle w_c^{-1} \lambda + \rho \alpha_4 \rangle$	$M(w_c^{-1}, \lambda)$	$(w_c w'_0)^{-1} \lambda + \rho$
Id	0	1	...
w_1	0	...	...
w_{12}	0	...	...
* w_{123}	$2(s - \frac{3}{2})$	$\frac{\xi(2s-3)}{\xi(2s)}$	$[0, 0, 0, 5 - 2s]$

TABLE 2. Data for the reduction formula (6.2) to evaluate $W_{\alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_1 - \rho$. The star in the first column indicates the one and only row that contributes in the sum over Weyl words. The second column is used to compute the projected weight for the G' Whittaker coefficient, the third column gives the intertwiner, and the last column gives the weight for first factor of (6.2) as a vector with the fundamental weights as basis.

6.2. Example: Rank-1 coefficient of $\pi_{\min}$ on $P_{\alpha_4} \subset \mathrm{SL}_5$. Here, we will calculate the real rank-1 Fourier coefficient (6.1) for the Eisenstein series $E(\lambda; g)$ with $\lambda = 2s\Lambda_1 - \rho$ attached to a minimal automorphic representation, with respect to the unipotent radical of the maximal parabolic

$$P_{\alpha_4} = \mathrm{GL}_4 \times \mathrm{GL}_1 \times U_{\alpha_4} \subset \mathrm{SL}_5 \quad \text{subject to} \quad \det(\mathrm{GL}_4 \times \mathrm{GL}_1) = 1,$$

obtained by removing the “last” node in the Dynkin diagram of SL_5 . The unipotent radical is

$$U(\mathbb{R}) = U_{\alpha_4}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}.$$

Theorem B gives for the unramified character $\psi_{y(Y_1)}$ that

$$\mathcal{F}^{\mathbb{A}}(E(2s\Lambda_1 - \rho), \psi_{y(Y_1)}; g) = W_{\alpha_4}(g).$$

The Whittaker function is found by the reduction formula with data given in table 2. In this case, there is no diagonally embedded rational matrix l , or equivalently $l = I_5$, in the general procedure and hence we have $|\eta_{1,p}|_p = |\eta_{2,p}|_p = |\eta_{3,4}|_p = |\eta_{4,p}|_p = 1$. Using (6.2) and (A.5) with $s' = s - 3/2$, we get

$$\begin{aligned} & W_{\alpha_4}(\lambda; (g_\infty, I_5, I_5, \dots)) = \\ &= \mathbf{e}(x_{45}) \left(y_4^{5-2s} \frac{\xi(2s-3)}{\xi(2s)} \prod_{p < \infty} |\eta_{4,p}|_p^{5-2s} \right) \mathcal{K}_{s-3/2} \left(\frac{y_4^2}{y_3}, 1 \right) \\ & \quad \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{4,p}^2}{\eta_{3,p}} \right) \left(1 - p^{-2(s-3/2)} \right) \frac{1 - p^{-2(s-3/2)+1} \left| \frac{\eta_{4,p}^2}{\eta_{3,p}} \right|_p^{2(s-3/2)-1}}{1 - p^{-2(s-3/2)+1}} \\ &= \mathbf{e}(x_{45}) y_4^{5-2s} \frac{1}{\xi(2s)} 2 \left| \frac{y_4^2}{y_3} \right|_\infty^{s-2} K_{s-2} \left(2\pi \left| \frac{y_4^2}{y_3} \right|_\infty \right) \\ &= 2\mathbf{e}(x_{45}) y_3^{2-s} y_4 \frac{1}{\xi(2s)} K_{s-2} \left(2\pi \left| \frac{y_4^2}{y_3} \right|_\infty \right) = \mathcal{F}^{\mathbb{R}}(E(2s\Lambda_1 - \rho), \psi_{y(Y_1)}; g_\infty). \end{aligned}$$

The x 's and y 's are the Iwasawa coordinates for the matrix g_∞ as in (6.3). The function $\mathcal{K}_s$ that appears is a more compact way of writing the SL_2 Whittaker vector defined explicitly in (A.3).

Parameterizing g_∞ as

$$g_\infty = ue = \begin{pmatrix} I_4 & Q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1/4} e_4 & 0 \\ 0 & r \end{pmatrix} \quad \text{where } e_4 \in \mathrm{SL}_4(\mathbb{R}),$$

we get in particular that

$$y_3 = r^{-3/4} \|Ne_4\| \quad \text{and} \quad y_4 = r^{-1},$$

where $N = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ so that Ne_4 is equal to the last row in e_4 . This is obtained using the formula (B.1). We get in particular that

$$y_3^{2-s} y_4 = r^{2s-5} \left(r^{-5/4} \|Ne_4\| \right)^{s-2} \quad \text{and} \quad \frac{y_4^2}{y_3} = r^{-5/4} \|Ne_4\|.$$

One can show that the more general (real) Fourier coefficient with a character ψ on U_{α_4} parametrized by integers $N = \begin{pmatrix} m_1 & m_2 & m_3 & m'_4 \end{pmatrix}$, where m'_4 is non-zero, has the expression

(6.5)

$$\begin{aligned} & \int E \left(2s\Lambda_1 - \rho; \begin{pmatrix} 1 & & u_1 \\ & 1 & u_2 \\ & & 1 & u_3 \\ & & & 1 & u_4 \\ & & & & 1 \end{pmatrix} g_\infty \right) \overline{\mathbf{e}(m_1 u_1 + m_2 u_2 + m_3 u_3 + m'_4 u_4)} d^4 u = \\ & = \mathbf{e}(NQ) r^{2s-5} \frac{2}{\xi(2s)} \sigma_{4-2s}(k) \left(r^{-5/4} \|Ne_4\| \right)^{s-2} K_{s-2} \left(2\pi r^{-5/4} \|Ne_4\| \right) \\ & = \mathbf{e}(NQ) r^{\frac{3s}{4} - \frac{5}{2}} \frac{2}{\xi(2s)} \frac{\sigma_{2s-4}(k)}{|k|^{s-2}} \|\tilde{N}e_4\|^{s-2} K_{s-2} \left(2\pi |k| r^{-5/4} \|\tilde{N}e_4\| \right) \end{aligned}$$

and that, for non-integer rational m 's, it vanishes. Here g_∞ has been parametrized as above, $N = k\tilde{N}$ and $k = \gcd(N)$. This expression can be found by starting from $\psi_{y(kY_1)}$ for the standard Fourier coefficient, which corresponds to $N = \begin{pmatrix} 0 & 0 & 0 & k \end{pmatrix}$, and compute its $L(\mathbb{Z})$ orbit using (1.5). Formula (6.5) agrees with [GMV15, Eq. (H.37)] where the Fourier coefficients were computed by Poisson resummation techniques, after a translation of conventions.

6.3. Example: Rank-1 coefficient of π_{ntm} on $P_{\alpha_4} \subset \mathrm{SL}_5$. Here, we will calculate the real rank-1¹ Fourier coefficient (6.1) for the Eisenstein series $E(\lambda; g)$ with $\lambda = 2s\Lambda_2 - \rho$ attached to a next-to-minimal automorphic representation, with respect to the unipotent radical of the maximal parabolic

$$P_{\alpha_4} = \mathrm{GL}_4 \times \mathrm{GL}_1 \times U_{\alpha_4} \subset \mathrm{SL}_5 \quad \text{subject to} \quad \det(\mathrm{GL}_4 \times \mathrm{GL}_1) = 1$$

obtained by removing the “last” node in the Dynkin diagram of SL_5 . The unipotent radical is

$$U(\mathbb{R}) = U_{\alpha_4}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\}.$$

¹There is no rank-2 character for this parabolic.

w_c	$\langle w_c^{-1} \lambda + \rho \alpha_4 \rangle$	$M(w_c^{-1}, \lambda)$	$(w_c w'_0)^{-1} \lambda + \rho$
Id	0	1	...
w_2	0	...	...
w_{21}	0	...	...
* w_{23}	$2(s-1)$	$\frac{\xi(2s-2)}{\xi(2s)}$	$[2s-1, 0, 0, 4-2s]$
* w_{213}	$2(s-1)$	$\frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)}$	$[3-2s, 2s-2, 0, 4-2s]$
* w_{2132}	$2(s-1)$	$\frac{\xi(2s-3)\xi(2s-2)}{\xi(2s)\xi(2s-1)}$	$[0, 4-2s, 2s-3, 4-2s]$
w_{213243}	0	...	...

TABLE 3. Data for the reduction formula (6.2) to evaluate $W_{\alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_2 - \rho$. The stars indicate which rows contribute in the sum over Weyl words.

Theorem B gives

$$\begin{aligned}
& \mathcal{F}^{\mathbb{A}}(E(2s\Lambda_2 - \rho), \psi_{y(Y_1)}; g) = \\
& = W_{\alpha_4}(g) + \sum_{\lambda_1 \in \Lambda_1(\psi_{\alpha_4})} W_{\alpha_1, \alpha_4}(\hat{l}(\lambda_1)g) + \sum_{\lambda_2 \in \Lambda_2} W_{\alpha_2, \alpha_4}(\hat{l}(\lambda_2)g) \\
& = W_{\alpha_4}(g) \\
& + \sum_{z'} W_{\alpha_1, \alpha_4} \left(\underbrace{\begin{pmatrix} z & & & \\ & 1 & & \\ & & 1/z & \\ & & & 1 \end{pmatrix}}_{l_z} g \right) \\
& + \sum_{x', y} W_{\alpha_2, \alpha_4} \left(\underbrace{\begin{pmatrix} x^{-1} & & & \\ & y & x & \\ & & & I_3 \end{pmatrix}}_{l_{xy}} g \right) \\
& + \sum_{x'} W_{\alpha_2, \alpha_4} \left(\underbrace{\begin{pmatrix} 0 & -x^{-1} & & \\ & x & 0 & \\ & & & I_3 \end{pmatrix}}_{l_x} g \right),
\end{aligned} \tag{6.6}$$

using the representatives derived in appendix C.

The first Whittaker function is found by the reduction formula with the data of table 3. In this case, there is no diagonally embedded rational matrix l , or equivalently $l = I_5$, and hence we have $|\eta_{1,p}|_p = |\eta_{2,p}|_p = |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1$. We get

$$\begin{aligned}
& (6.7) \\
& W_{\alpha_4}(\lambda; (g_\infty, I_n, I_n, \dots)) = \\
& = \mathbf{e}(x_{45}) \mathcal{K}_{s-1} \left(\frac{y_4^2}{y_3}, 1 \right) \left(y_1^{2s-1} y_4^{4-2s} \frac{\xi(2s-2)}{\xi(2s)} \prod_{p < \infty} |\eta_{1,p}|_p^{2s-1} |\eta_{4,p}|_p^{4-2s} \right. \\
& \quad \left. + y_1^{3-2s} y_2^{2s-2} y_4^{4-2s} \frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)} \prod_{p < \infty} |\eta_{1,p}|_p^{3-2s} |\eta_{2,p}|_p^{2s-2} |\eta_{4,p}|_p^{4-2s} + \right.
\end{aligned}$$

w_c	$\langle w_c^{-1}\lambda + \rho \alpha_1 \rangle$	$\langle w_c^{-1}\lambda + \rho \alpha_4 \rangle$	$M(w_c^{-1}, \lambda)$	$(w_c w'_0)^{-1} \lambda + \rho$
Id	0	0	1	...
w_2	$2(s - \frac{1}{2})$	0	...	...
* w_{23}	$2(s - \frac{1}{2})$	$2(s - 1)$	$\frac{\xi(2s-2)}{\xi(2s)}$	v
w_{2132}	0	$2(s - 1)$	...	...
w_{213243}	0	0	...	...

TABLE 4. Data for the reduction formula (6.2) to evaluate $W_{\alpha_1, \alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_2 - \rho$. The stars indicate which Weyl words contribute to the reduction formula. We wrote $v = [3 - 2s, 2s - 2, 0, 4 - 2s]$ here to conserve space.

$$\begin{aligned}
& + y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{\xi(2s-3)\xi(2s-2)}{\xi(2s)\xi(2s-1)} \prod_{p<\infty} |\eta_{2,p}|_p^{4-2s} |\eta_{3,p}|_p^{2s-3} |\eta_{4,p}|_p^{4-2s} \Bigg) \\
& \prod_{p<\infty} \gamma_p \left(\frac{\eta_{4,p}^2}{\eta_{3,p}} \right) \left(1 - p^{-2(s-1)} \right) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{4,p}^2}{\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& = \mathbf{e}(x_{45}) \left(y_1^{2s-1} y_4^{4-2s} \frac{1}{\xi(2s)} + y_1^{3-2s} y_2^{2s-2} y_4^{4-2s} \frac{\xi(2s-2)}{\xi(2s)\xi(2s-1)} \right. \\
& \quad \left. + y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{\xi(2s-3)}{\xi(2s)\xi(2s-1)} \right) 2 \left| \frac{y_4^2}{y_3} \right|_\infty^{s-3/2} K_{s-3/2} \left(2\pi \left| \frac{y_4^2}{y_3} \right|_\infty \right) \\
& = 2\mathbf{e}(x_{45}) \left(y_1^{2s-1} y_3^{3/2-s} y_4 \frac{1}{\xi(2s)} + y_1^{3-2s} y_2^{2s-2} y_3^{3/2-s} y_4 \frac{\xi(2s-2)}{\xi(2s)\xi(2s-1)} \right. \\
& \quad \left. + y_2^{4-2s} y_3^{s-3/2} y_4 \frac{\xi(2s-3)}{\xi(2s)\xi(2s-1)} \right) K_{s-3/2} \left(2\pi \left| \frac{y_4^2}{y_3} \right|_\infty \right).
\end{aligned}$$

The x 's and y 's are the Iwasawa coordinates for the matrix g_∞ as in (6.3).

The second Whittaker function is found by the reduction formula with the data given in table 4. The p -adic Iwasawa-decomposition of l_z has

$$|\eta_{1,p}|_p = |\eta_{2,p}|_p = |z|_p \quad \text{and} \quad |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1.$$

We get

(6.8)

$$\begin{aligned}
& \sum_{z'} W_{\alpha_1, \alpha_4}(\lambda; l_z(g_\infty, I_n, I_n, \dots)) = \\
& = \sum_{z'} \mathbf{e}(\tilde{x}_{12} + \tilde{x}_{45}) \tilde{y}_1^{3-2s} \tilde{y}_2^{2s-2} \tilde{y}_4^{4-2s} \frac{\xi(2s-2)}{\xi(2s)} \\
& \quad \times \mathcal{K}_{s-1/2} \left(\frac{\tilde{y}_1^2}{\tilde{y}_2}, 1 \right) \mathcal{K}_{s-1} \left(\frac{\tilde{y}_4^2}{\tilde{y}_3}, 1 \right) \prod_{p<\infty} |\eta_{1,p}|_p^{3-2s} |\eta_{2,p}|_p^{2s-2} |\eta_{4,p}|_p^{4-2s} \\
& \quad \times \prod_{p<\infty} \gamma_p \left(\frac{\eta_{1,p}^2}{\eta_{2,p}} \right) \left(1 - p^{-2(s-1/2)} \right) \frac{1 - p^{-2(s-1/2)+1} \left| \frac{\eta_{1,p}^2}{\eta_{2,p}} \right|_p^{2(s-1/2)-1}}{1 - p^{-2(s-1/2)+1}}
\end{aligned}$$

w_c	$\langle w_c^{-1}\lambda + \rho \alpha_2 \rangle$	$\langle w_c^{-1}\lambda + \rho \alpha_4 \rangle$	$M(w_c^{-1}, \lambda)$	$(w_c w'_0)^{-1}\lambda + \rho$
Id	$2s$	0	1	$\dots$
w_{21}	0	0	$\dots$	$\dots$
w_{23}	0	$2(s-1)$	$\dots$	$\dots$
$* w_{213}$	$2(s-1)$	$2(s-1)$	$\frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)}$	v
w_{213243}	0	0	$\dots$	$\dots$

TABLE 5. Data for the reduction formula (6.2) to evaluate $W_{\alpha_2, \alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_2 - \rho$. The star indicates the Weyl word that contributes to the reduction formula. We wrote $v = [0, 4 - 2s, 2s - 3, 4 - 2s]$ to save space.

$$\begin{aligned}
& \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{4,p}^2}{\eta_{3,p}} \right) \left(1 - p^{-2(s-1)} \right) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{4,p}^2}{\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& = \sum_{z' \in \mathbb{Z}} \mathbf{e}(\tilde{x}_{12} + \tilde{x}_{45}) \tilde{y}_1^{3-2s} \tilde{y}_2^{2s-2} \tilde{y}_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} \prod_{p < \infty} |z|_p^1 \\
& \quad \times 4 \left(\frac{\tilde{y}_1^2}{\tilde{y}_2} \right)^{s-1} \left(\frac{\tilde{y}_4^2}{\tilde{y}_3} \right)^{s-3/2} K_{s-1} \left(2\pi \frac{\tilde{y}_1^2}{\tilde{y}_2} \right) K_{s-3/2} \left(2\pi \frac{\tilde{y}_4^2}{\tilde{y}_3} \right) \\
& \quad \times \sigma_{-2(s-1/2)+1}(|z|_\infty) \\
& = \sum_{z' \in \mathbb{Z}} 4\mathbf{e}(\tilde{x}_{12} + \tilde{x}_{45}) \tilde{y}_1 \tilde{y}_2^{s-1} \tilde{y}_3^{3/2-s} \tilde{y}_4 \frac{1}{\xi(2s)\xi(2s-1)} |z|_\infty^{-1} \\
& \quad \times K_{s-1} \left(2\pi \frac{\tilde{y}_1^2}{\tilde{y}_2} \right) K_{s-3/2} \left(2\pi \frac{\tilde{y}_4^2}{\tilde{y}_3} \right) \sigma_{2-2s}(|z|_\infty).
\end{aligned}$$

The $\tilde{x}$'s and $\tilde{y}$'s are the Iwasawa coordinates for the matrix $l_z g_\infty$. Writing g_∞ in the coordinates (6.3) we find explicitly for the archimedean components

$$\tilde{x}_{12} = z x_{12}, \quad \tilde{x}_{45} = x_{45}, \quad \tilde{y}_1 = |z|_\infty y_1, \quad \tilde{y}_2 = |z|_\infty y_2, \quad \tilde{y}_3 = y_3, \quad \tilde{y}_4 = y_4.$$

The third and fourth Whittaker functions are found by the reduction formula with the data from table 5. The p -adic Iwasawa-decomposition of l_{xy} has

$$|\eta_{1,p}|_p^{-1} = \max\{|y|_p, |x|_p\} \quad \text{and} \quad |\eta_{2,p}|_p = |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1.$$

We get

$$\begin{aligned}
(6.9) \quad & \sum_{x', y} W_{\alpha_2, \alpha_4}(\lambda; l_{xy}(g_\infty, I_n, I_n, \dots)) = \\
& = \sum_{x', y} \mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_2^{4-2s} \tilde{y}_3^{2s-3} \tilde{y}_4^{4-2s} \frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)} \\
& \quad \times \mathcal{K}_{s-1} \left(\frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3}, 1 \right) \mathcal{K}_{s-1} \left(\frac{\tilde{y}_4^2}{\tilde{y}_3}, 1 \right) \prod_{p < \infty} |\eta_{2,p}|_p^{4-2s} |\eta_{3,p}|_p^{2s-3} |\eta_{4,p}|_p^{4-2s}
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}} \right) (1 - p^{-2(s-1)}) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{4,p}^2}{\eta_{3,p}} \right) (1 - p^{-2(s-1)}) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{4,p}^2}{\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& = \sum_{x', y \in \mathbb{Z}} \mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_2^{4-2s} \tilde{y}_3^{2s-3} \tilde{y}_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} 4 \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty^{s-3/2} \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty^{s-3/2} \\
& \quad \times K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty \right) K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty \right) \sigma_{-2(s-1)+1}(k) \\
& = \sum_{x', y \in \mathbb{Z}} 4 \mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_1^{3/2-s} \tilde{y}_2 \tilde{y}_4 \frac{1}{\xi(2s)\xi(2s-1)} \\
& \quad \times K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty \right) K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty \right) \sigma_{3-2s}(k),
\end{aligned}$$

where $k = \gcd(|y|, |x|)$. Here, the $\tilde{x}$'s and $\tilde{y}$'s are the Iwasawa coordinates for the matrix $l_{xy}g_\infty$. Explicitly, we have for the archimedean components

$$\begin{aligned}
\tilde{x}_{23} &= xx_{23} + yx_{13}, & \tilde{x}_{45} &= x_{45}, \\
\tilde{y}_1 &= \frac{y_1}{\sqrt{(x + yx_{12})^2 + (yy_1^2 y_2^{-1})^2}}, & \tilde{y}_2 &= y_2, & \tilde{y}_3 &= y_3, & \tilde{y}_4 &= y_4.
\end{aligned}$$

Turning to the last piece in (6.6), the p -adic Iwasawa-decomposition of l_x has

$$|\eta_{1,p}|_p^{-1} = \max\{|0|_p, |x|_p\} = |x|_p \quad \text{and} \quad |\eta_{2,p}|_p = |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1.$$

We get therefore

$$\begin{aligned}
& \sum_{x'} W_{\alpha_2, \alpha_4}(\lambda; l_x(g_\infty, I_n, I_n, \dots)) = \\
& = \sum_{x'} \mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_2^{4-2s} \tilde{y}_3^{2s-3} \tilde{y}_4^{4-2s} \frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)} \\
& \quad \times \mathcal{K}_{s-1} \left(\frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3}, 1 \right) \mathcal{K}_{s-1} \left(\frac{\tilde{y}_4^2}{\tilde{y}_3}, 1 \right) \prod_{p < \infty} |\eta_{2,p}|_p^{4-2s} |\eta_{3,p}|_p^{2s-3} |\eta_{4,p}|_p^{4-2s} \\
& \quad \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}} \right) (1 - p^{-2(s-1)}) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& \quad \times \prod_{p < \infty} \gamma_p \left(\frac{\eta_{4,p}^2}{\eta_{3,p}} \right) (1 - p^{-2(s-1)}) \frac{1 - p^{-2(s-1)+1} \left| \frac{\eta_{4,p}^2}{\eta_{3,p}} \right|_p^{2(s-1)-1}}{1 - p^{-2(s-1)+1}} \\
& = \sum_{x' \in \mathbb{Z}} \mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_2^{4-2s} \tilde{y}_3^{2s-3} \tilde{y}_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} 4 \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty^{s-3/2} \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty^{s-3/2}
\end{aligned}$$

$$\begin{aligned}
& \times K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty \right) K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty \right) \sigma_{-2(s-1)+1}(|x|_\infty) \\
& = \sum_{x' \in \mathbb{Z}} 4\mathbf{e}(\tilde{x}_{23} + \tilde{x}_{45}) \tilde{y}_1^{3/2-s} \tilde{y}_2 \tilde{y}_4 \frac{1}{\xi(2s)\xi(2s-1)} \\
(6.10) \quad & \times K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_2^2}{\tilde{y}_1 \tilde{y}_3} \right|_\infty \right) K_{s-3/2} \left(2\pi \left| \frac{\tilde{y}_4^2}{\tilde{y}_3} \right|_\infty \right) \sigma_{3-2s}(|x|_\infty).
\end{aligned}$$

The $\tilde{x}$'s and $\tilde{y}$'s are the Iwasawa coordinates for the matrix $l_x g_\infty$. For the archimedean part we find explicitly

$$\begin{aligned}
\tilde{x}_{23} &= x x_{13}, & \tilde{x}_{45} &= x_{45}, \\
\tilde{y}_1 &= \frac{y_1}{\sqrt{(x x_{12})^2 + (x y_1^2 y_2^{-1})^2}}, & \tilde{y}_2 &= y_2, & \tilde{y}_3 &= y_3, & \tilde{y}_4 &= y_4.
\end{aligned}$$

The complete Fourier coefficient $\mathcal{F}^{\mathbb{R}}(E(2s\Lambda_2 - \rho), \psi_{y(Y_1)}; g_\infty)$ is then given by the combination of (6.7), (6.8), (6.9) and (6.10). We note that our final result differs formally from the one given in [GMV15, Eq. (H.52)] where the result is given as a convoluted integral over two Bessel functions whereas we do not have any remaining integral. The two results need not be in actual disagreement as there are many non-trivial relations involving infinite sums or integrals of Bessel functions.

The automorphic form

$$\lim_{s \rightarrow 1/2} \frac{2\zeta(3)\xi(2s-3)}{\xi(2s)} E(2s\Lambda_2 - \rho; g) = 2\zeta(3)E(3\Lambda_4 - \rho)$$

lies in a minimal automorphic representation and controls the first non-trivial corrections that string theory predicts to the four-graviton scattering amplitude beyond standard general relativity [GMRV10, Pio10]. The Fourier coefficients that we computed above can then be used to extract so-called 1/2 BPS instanton contributions in the string perturbation limit of the amplitude. More precisely, they represent non-perturbative corrections to the scattering amplitude that, albeit smooth, are not analytic in the string coupling constant around vanishing coupling. They are therefore not visible in standard perturbation theory for small coupling but represent important corrections nonetheless. Their interpretation is in terms of specific Dp -branes ($p \leq 2$) that are extended $(p+1)$ -dimensional objects that can wrap non-trivial cycles of the torus T^3 that is present when SL_5 is the duality group. The detailed structure of the Fourier coefficient, in particular the arithmetic divisor sums appearing, can shed some light on the combinatorics of these D-branes similar to what is happening in the SL_2 case [Yi97, SS98, MNS00].

For the next non-trivial correction to the four-graviton scattering amplitude one requires an automorphic form in a next-to-minimal automorphic representation [GMRV10, Pio10, GMV15]. This function is not a single Eisenstein series of the type we have analysed above but a very special combination of two formally divergent Eisenstein series with some Fourier coefficients computed using the Mellin transform of a theta lift in [GMV15].

APPENDIX A. EULER PRODUCTS AND TWISTED CHARACTERS

This appendix contains details and explanations for section 6, which is why we restrict to the field $F = \mathbb{Q}$ with the corresponding ring of adeles $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$.

An Euler product is a product over the primes. The p -adic norm is denoted $|\cdot|_p$ and is defined for the p -adic numbers $\mathbb{Q}_p$. The absolute value norm or “infinity norm” is denoted $|\cdot|_{\infty}$ and is defined for real numbers $\mathbb{R} = \mathbb{Q}_{\infty}$. The p -adic numbers as well as the real numbers (being completions of the rational numbers) all contain the rational numbers: $\mathbb{Q} \subset \mathbb{Q}_p$ for all p prime. The norm of an adele $x = (x_{\infty}, x_2, x_3, x_5, \dots) \in \mathbb{A}$ is denoted $|\cdot|$ (without ornaments) and is the product of norms at the local places

$$|x| = \prod_{p \leq \infty} |x_p|_p.$$

The rational numbers $\mathbb{Q}$ are diagonally embedded into the adeles $\mathbb{A}$

$$\mathbb{Q} \subset \mathbb{A} \quad \text{in the sense that} \quad (q, q, q, q, \dots) \in \mathbb{A} \quad \text{for} \quad q \in \mathbb{Q}.$$

Product of norms. For a rational number $x \in \mathbb{Q}$ with a decomposition into primes as

$$x = \pm \prod_p p^{m^{(p)}},$$

we get a particularly simple result for the adelic norm of x , namely

$$\prod_{p \leq \infty} |x|_p = |x|_{\infty} \prod_p p^{-m^{(p)}} = |x|_{\infty} |x|_{\infty}^{-1} = 1.$$

This is most often used as

$$x \in \mathbb{Q} \quad \Rightarrow \quad \prod_p |x|_p = |x|_{\infty}^{-1}.$$

Greatest common divisor. For a set of natural numbers $\{x_i\}$ where each x_i has a decomposition into primes as

$$x_i = \prod_p p^{m_i^{(p)}},$$

one can express the greatest common divisor k as

$$k \equiv \gcd(\{x_i\}) = \prod_p p^{\min_i \{m_i^{(p)}\}}.$$

Together with

$$|x_i|_p = p^{-m_i^{(p)}},$$

we are led to the expression

$$k = \prod_p p^{\min_i \{m_i^{(p)}\}} = \prod_p \min_i \left\{ p^{m_i^{(p)}} \right\} = \prod_p \min_i \left\{ |x_i|_p^{-1} \right\} = \prod_p \left(\max_i \{ |x_i|_p \} \right)^{-1}.$$

We also have the formula

$$|k|_p = \max_i \{ |x_i|_p \}.$$

Note that

$$\gcd(x_1, \dots, x_n, 0) = \gcd(x_1, \dots, x_n),$$

since every (nonzero) integer divides 0. Additionally, we define

$$\gcd(x) = x \quad \forall x \in \mathbb{Z},$$

including $x = 0$.

Divisor sum. We have the identity

$$\prod_p \frac{1 - p^{-s} |m|_p^s}{1 - p^{-s}} = \sum_{d|m} d^{-s} \equiv \sigma_{-s}(m),$$

for $s \in \mathbb{C}$ and $m \in \mathbb{Z}$.

The completed Riemann zeta function. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re}(s) > 1$$

can be written as an Euler product as

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad \operatorname{Re}(s) > 1$$

and can be analytically continued to the whole complex plane except at $s = 0$ and $s = 1$ where it has simple poles. This is done by defining the completed Riemann zeta function

$$\xi(s) \equiv \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

which obeys the functional relation

$$\xi(s) = \xi(1 - s)$$

as shown by Riemann.

p -adic gaussian. The p -adic gaussian $\gamma_p : \mathbb{Q}_p \rightarrow \{0, 1\}$ is defined as

$$\gamma_p(x) = \begin{cases} 1, & |x|_p \leq 1 \\ 0, & |x|_p > 1 \end{cases} = \begin{cases} 1, & x \in \mathbb{Z}_p \\ 0, & x \notin \mathbb{Z}_p. \end{cases}$$

For a rational number x we then get

$$\prod_p \gamma_p(x) = \begin{cases} 1, & x \in \mathbb{Z}_p \forall p \\ 0, & \text{else} \end{cases} = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & \text{else.} \end{cases}$$

Notice also that for rational numbers $x_1, \dots, x_n \in \mathbb{Q}$ and picking an $x \in \mathbb{Q}$ such that for all primes p

$$(A.1) \quad |x|_p = \max\{|x_1|_p, \dots, |x_n|_p\},$$

we have

$$\gamma_p(x) = \prod_{i=1}^n \gamma_p(x_i).$$

A consequence of this is that for an eulerian function depending only on the p -adic norms of its argument

$$f(x) = \prod_p f_p(|x|_p),$$

then with x as in (A.1), we have

$$f(x) \prod_p \gamma_p(x) = \prod_p f_p(|x|_p) \gamma_p(x) = \prod_p f_p(|k|_p) \gamma_p(x) = f(k) \prod_p \gamma_p(x),$$

where

$$k = \gcd(|x_1|_\infty, \dots, |x_n|_\infty).$$

This equation makes sense as $\prod_p \gamma_p(x)$ ensures that the left- and right hand sides are nonzero only when each x_i is integer for which k is well defined. We now see how a sum over rationals with x as in (A.1) can collapse to a sum over integers due to the p -adic gaussian

$$\sum_{x_1, \dots, x_n} f(x) \prod_p \gamma_p(x) = \sum_{x_1, \dots, x_n} f(k) \prod_p \gamma_p(x) = \sum_{x_1, \dots, x_n \in \mathbb{Z}} f(k).$$

SL₂ Whittaker function. The ramified (meaning m not necessarily unity but rational) SL₂($\mathbb{A}$) Whittaker function evaluated at

$$g = (g_\infty, \mathbf{I}, \mathbf{I}, \dots) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix} k, \mathbf{I}, \mathbf{I}, \dots \right)$$

written as an Euler product reads

$$\begin{aligned} W_\alpha(2s\Lambda - \rho, m; g) &= W_\alpha\left(2s\Lambda - \rho, m; \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix} k, \mathbf{I}, \mathbf{I}, \dots\right)\right) = \\ (A.2) \quad &= \mathbf{e}(mx) y^{2(1-s)} \mathcal{K}_s(m, y^2) \prod_p \gamma_p(m) (1 - p^{-2s}) \frac{1 - p^{-2s+1} |m|_p^{2s-1}}{1 - p^{-2s+1}}, \end{aligned}$$

where α is the simple root and Λ is the fundamental weight. Here

$$(A.3) \quad \mathcal{K}_s(m, y) \equiv \frac{2\pi^s}{\Gamma(s)} y^{s-1/2} |m|_\infty^{s-1/2} K_{s-1/2}(2\pi |m|_\infty y)$$

should be seen as the archimedean SL₂-Whittaker function and each factor in the Euler product in (A.2) as the non-archimedean Whittaker functions. The product

$$\prod_p \gamma_p(m)$$

restricts to $m \in \mathbb{Z}$ as explained above. The expression can then be written as

$$W_\alpha(2s\Lambda - \rho, m; g) = \mathbf{e}(mx) \frac{2}{\xi(2s)} y |m|_\infty^{s-1/2} \sigma_{1-2s}(|m|_\infty) K_{s-1/2}(2\pi |m|_\infty y^2).$$

Notice how the factors of the Eulerian expression for the Riemann zeta function in the non-archimedean part combines with $\pi^s/\Gamma(s)$ in the archimedean part to form a completed Riemann zeta function.

Twisted character. Let $m \in \mathbb{Q}$ and $\psi_{p,m}$ be an additive character on $\mathbb{Q}_p$ defined as

$$\begin{aligned}\psi_{\infty,m}(x) &= e^{2\pi i m x}; & m, x \in \mathbb{R} & \text{ for real numbers} \\ \psi_{p,m}(x) &= e^{-2\pi i [mx]_p}; & m, x \in \mathbb{Q}_p & \text{ for } p\text{-adic numbers.}\end{aligned}$$

A unitary multiplicative character on the unipotent radical $N(\mathbb{A})$ of the Borel subgroup of $\mathrm{SL}_n(\mathbb{A})$ can then be parametrized by $m_1, \dots, m_{n-1} \in \mathbb{Q}$ as

$$\psi(n) = \psi(e^{\sum_{\alpha \in \Delta_+} u_\alpha E_\alpha}) = \psi(e^{\sum_{\alpha \in \Pi} u_\alpha E_\alpha}) = \prod_{p \leq \infty} \prod_{i=1}^n \psi_{p,m_i}((u_{\alpha_i})_p),$$

where Δ_+ is the set of positive roots and $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\} \subset \Delta_+$ is the set of simple roots. The second equality is due to the fact that the additive character is only sensitive to the abelianization of $N(\mathbb{A})$. In the final equality, $(x_{\alpha_i})_p$ denotes the p -adic (or real) component of the adelic coordinate x_{α_i} .

For an element $a \in A(\mathbb{A})$, we would like to evaluate the twisted character

$$\psi^a(n) \equiv \psi(ana^{-1}).$$

We can interpret the transformation $\psi \rightarrow \psi^a$ as that the parameters m_i transform according to

$$(A.4) \quad m_i \rightarrow m'_i = \left(\frac{y_i^2}{y_{i-1}y_{i+1}} \right) m_i, \quad i = 1, \dots, n-1,$$

where we have defined $y_0 = y_n = 1$. Note that starting with rational parameters m_i , the transformed parameters m'_i are no longer necessarily rational, depending on what the y_i are. The Whittaker coefficients $W'_{\psi^a}(w_c^{-1}\lambda, 1)$ of (6.2) should in this case be interpreted as the Jacquet integral over $N'(\mathbb{A})$ as detailed in [FGKP18].

The Whittaker coefficients $W'_{\psi^a}(w_c^{-1}\lambda, 1)$ from the examples in section 6 all have $G' = \mathrm{SL}_2$ or $G' = \mathrm{SL}_2 \times \mathrm{SL}_2$, and since the Whittaker coefficients for the latter can be reduced to a product of two Whittaker coefficients for the former, we only need to consider $G' = \mathrm{SL}_2$ corresponding to some root α_i of G . In the examples, the character ψ , which is supported only on α_i , is parametrized by $m_i = 1$ which, by the twist, is transformed to $m'_i = (m'_{i,\infty}, m'_{i,2}, m'_{i,3}, \dots) \in \mathbb{A}$ according to (A.4). Let us denote the projection of $w_c^{-1}\lambda$ onto the G' weight space as $\lambda' = 2s'\Lambda' - \rho' = (2s' - 1)\Lambda'$ where Λ' and ρ' are the SL_2 fundamental weight and Weyl vector respectively. Parametrizing the real and p -adic parts of a as in (6.3) and (6.4) using $\tilde{y}_i \in \mathbb{R}$ and $\eta_i \in \mathbb{Q}_p$ with $\tilde{y}_0 = \tilde{y}_n = \eta_0 = \eta_n = 1$ one can show that

$$(A.5) \quad \begin{aligned} W'_{\psi^a}(\lambda', 1) &= \mathcal{K}_{s'}\left(\frac{\tilde{y}_i^2}{\tilde{y}_{i-1}\tilde{y}_{i+1}}, 1\right) \\ &\times \prod_{p < \infty} \gamma_p\left(\frac{\eta_{i,p}^2}{\eta_{i-1,p}\eta_{i+1,p}}\right) (1 - p^{-2s'}) \frac{1 - p^{-2s'+1} \left| \frac{\eta_{i,p}^2}{\eta_{i-1,p}\eta_{i+1,p}} \right|_p^{2s'-1}}{1 - p^{-2s'+1}} \end{aligned}$$

by arguing that each local factor ($p \leq \infty$) of the Jacquet integral only depends on the norm $|m'_{i,p}|_p \in \mathbb{Q}$ of m'_p , which means that it can be identified with the same local factor of an SL_2 Whittaker coefficient with character parametrized by this rational number.

APPENDIX B. IWASAWA-DECOMPOSITION

Proof of the following results can be found in [Ahl16]. For a real matrix $g \in \mathrm{SL}_n(\mathbb{R})$ written in Iwasawa form

$$g = n_\infty a_\infty k_\infty = \begin{pmatrix} 1 & x_{12} & \cdots & \cdots & x_{1n} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1 \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & & \\ & y_2/y_1 & & & \\ & & \ddots & & \\ & & & y_{n-1}/y_{n-2} & \\ & & & & 1/y_{n-1} \end{pmatrix} k_\infty,$$

we have

$$(B.1) \quad \begin{aligned} x_{\mu\nu} &= y_{\nu-1}^2 \epsilon(V_\mu, V_{\nu+1}, \dots, V_n; V_\nu, V_{\nu+1}, \dots, V_n), \quad \mu < \nu, \quad \text{and} \\ y_\mu^{-2} &= \epsilon(V_{\mu+1}, \dots, V_n; V_{\mu+1}, \dots, V_n), \end{aligned}$$

where V_μ is the μ^{th} row of g (regarded as an n -vector). Furthermore, ϵ denotes the totally antisymmetric product

$$\epsilon(A_1, \dots, A_m; B_1, \dots, B_m) = \delta_{a_1 \dots a_m}^{i_1 \dots i_m} (A_1)^{a_1} \dots (A_m)^{a_m} (B_1)_{i_1} \dots (B_m)_{i_m},$$

where the A 's and B 's are n -vectors and

$$\delta_{a_1 \dots a_m}^{i_1 \dots i_m} = m! \delta_{[a_1}^{i_1} \dots \delta_{a_m]}^{i_m} = \frac{1}{(n-m)!} \epsilon_{a_1 \dots a_m \alpha_{m+1} \dots \alpha_n} \epsilon^{i_1 \dots i_m \alpha_{m+1} \dots \alpha_n}$$

denotes the generalized Kronecker delta. Put in words, ϵ takes two sets of vectors and returns the sum of every possible product of scalar products between the two sets weighted by the signs of the given permutations. For example

$$\epsilon(A_1, A_2; B_1, B_2) = (A_1 \cdot B_1)(A_2 \cdot B_2) - (A_1 \cdot B_2)(A_2 \cdot B_1).$$

For a p -adic matrix $g \in \mathrm{SL}_n(\mathbb{Q}_p)$, the Iwasawa-decomposition

$$g = n_p a_p k_p = n_p \begin{pmatrix} \eta_{1,p} & & & \\ & \eta_{2,p}/\eta_{1,p} & & \\ & & \ddots & \\ & & & \eta_{n-1,p}/\eta_{n-2,p} \\ & & & & 1/\eta_{n-1,p} \end{pmatrix} k_p$$

is no longer unique. The p -adic norms of the η 's however are constant across the family of decompositions and are given by

$$|\eta_{n-k}|_p = \left(\max_{\sigma \in \Theta_k^n} \left\{ |g \begin{pmatrix} n-k+1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(k) \end{pmatrix}|_p \right\} \right)^{-1}, \quad \text{where } k \in \{1, \dots, n-1\},$$

and Θ_k^n denotes the set of all ordered subsets of $\{1, \dots, n\}$ of order k . Here, $g \begin{pmatrix} r_1 & \cdots & r_k \\ c_1 & \cdots & c_k \end{pmatrix}$ denotes a minor of order k , given as the determinant of the submatrix of g obtained by only picking the k rows $\{r_i\}$ and k columns $\{c_i\}$. For example, the matrix

$$\begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \end{pmatrix},$$

has

$$\begin{aligned}
|\eta_2|_3 &= \max \left\{ \left\| \begin{pmatrix} \frac{1}{9} & \frac{1}{10} \\ \frac{1}{13} & \frac{1}{14} \end{pmatrix} \right\|_3, \left\| \begin{pmatrix} \frac{1}{9} & \frac{1}{11} \\ \frac{1}{13} & \frac{1}{15} \end{pmatrix} \right\|_3, \left\| \begin{pmatrix} \frac{1}{9} & \frac{1}{12} \\ \frac{1}{13} & \frac{1}{16} \end{pmatrix} \right\|_3, \left\| \begin{pmatrix} \frac{1}{10} & \frac{1}{11} \\ \frac{1}{14} & \frac{1}{15} \end{pmatrix} \right\|_3, \left\| \begin{pmatrix} \frac{1}{10} & \frac{1}{12} \\ \frac{1}{14} & \frac{1}{16} \end{pmatrix} \right\|_3, \left\| \begin{pmatrix} \frac{1}{11} & \frac{1}{12} \\ \frac{1}{15} & \frac{1}{16} \end{pmatrix} \right\|_3 \right\} = \\
&= \max \left\{ \left| \frac{1}{4095} \right|_3, \left| \frac{8}{19305} \right|_3, \left| \frac{1}{1872} \right|_3, \left| \frac{1}{5775} \right|_3, \left| \frac{1}{3360} \right|_3, \left| \frac{1}{7920} \right|_3 \right\} \\
&= \max \{ 3^2, 3^3, 3^2, 3^1, 3^1, 3^2 \} = 3^3 = 9.
\end{aligned}$$

APPENDIX C. PARAMETRIZING Γ_i AND Λ_j

Recall the definitions

$$\Gamma_i(\psi_0) := \begin{cases} (\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) & 1 \leq i \leq n-2 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_0} & i = n-1, \end{cases}$$

where

$$(\mathrm{SL}_{n-i}(F))_{\hat{Y}} = \left\{ \begin{pmatrix} 1 & \xi^T \\ 0 & h \end{pmatrix} : h \in \mathrm{SL}_{n-i-1}(F), \xi \in F^{n-i-1} \right\};$$

and

$$\Lambda_j(\psi_0) := \begin{cases} (\mathrm{SL}_j(F))_{\hat{X}} \backslash \mathrm{SL}_j(F) & 2 \leq j \leq n-1 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_0} & j = 1, \end{cases}$$

where

$$(\mathrm{SL}_j(F))_{\hat{X}} = \left\{ \begin{pmatrix} h & \xi \\ 0 & 1 \end{pmatrix} : h \in \mathrm{SL}_{j-1}(F), \xi \in F^{j-1} \right\}.$$

T denotes the diagonal matrices in $\mathrm{SL}_n(F)$ and T_ψ denotes the stabilizer of a character ψ in T . In this appendix, we will find convenient representatives for these coset spaces. We begin with a lemma.

Lemma C.1. *Let $S_k(F)$ denote the set of all $k \times k$ matrices m over the field F satisfying $\dim \ker m = 1$. $\mathrm{GL}_k(F)$ acts on $S_k(F)$ by left multiplication and the quotient this the action $\mathrm{GL}_k(F) \backslash S_k(F)$ can then be parametrized as*

$$\mathrm{GL}_k(F) \backslash S_k = \bigcup_{a=0}^{k-1} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & v & I_{k-a-1} \end{pmatrix} : v \in F^{k-a-1} \right\}.$$

Proof. We will use induction to prove the lemma. Assume that the result holds up to and including matrices of size $k \times k$ and consider the quotient $\mathrm{GL}_{k+1}(F) \backslash S_{k+1}(F)$. Start with a matrix $m_{k+1} \in S_{k+1}(F)$. Left action of the group $\mathrm{GL}_{k+1}(F) \ni h_{k+1}$ taking $m_{k+1} \rightarrow h_{k+1} m_{k+1}$ is equivalent to performing Gauss elimination among the rows of m_{k+1} . Since we have $\dim \ker m_{k+1} = 1$ we can bring m_{k+1} to the form

$$m_{k+1} \rightarrow \begin{pmatrix} 0 & 0 \\ v & m \end{pmatrix},$$

where $v \in F^k$ and m is a $k \times k$ matrix satisfying $\dim \ker m \leq 1$. We cannot have $\dim \ker m \geq 2$ as we could then perform additional row manipulations to produce two zero rows in m and hence another zero row in m_{k+1} which violates $m_{k+1} \in S_{k+1}(F)$.

Case 1: $\dim \ker m = 0$. Here m is invertible and we can bring m_{k+1} to the form

$$\begin{pmatrix} 0 & 0 \\ v & m \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ v & I_k \end{pmatrix}$$

having relabelled v . This is the contribution

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & v & I_{k+1-a-1} \end{pmatrix} : v \in F^{k+1-a-1} \right\} \Big|_{a=0}.$$

Case 2: $\dim \ker m = 1$. We now have $m = m_k$ for some $m_k \in S_k$ and we can apply the induction assumption which leads us to consider matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ v^{(1)} & 0 & 0 & 0 \\ v^{(2)} & I_a & 0 & 0 \\ v^{(3)} & 0 & u & I_{k-a-1} \end{pmatrix}, \quad \text{where } a \in [0, k-2] \cap \mathbb{Z}.$$

We see that we must have $v^{(1)} \neq 0$ and with further row manipulations we can thus bring this to the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ v^{(1)} & 0 & 0 & 0 \\ v^{(2)} & I_a & 0 & 0 \\ v^{(3)} & 0 & u & I_{k-a-1} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & u & I_{k-a-1} \end{pmatrix}.$$

We get the contributions

$$\bigcup_{a=0}^{k-2} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-1} \end{pmatrix} : v \in F^{k-a-1} \right\} = \bigcup_{a=1}^{k-1} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ I_a & 0 & 0 & 0 \\ 0 & v & I_{k+1-a-1} & 0 \end{pmatrix} : v \in F^{k+1-a-1} \right\}.$$

This combines with the contribution from case 1 to give the form stated in the lemma.

That the base case $k = 1$ has the correct form is trivial. Peano's axiom of induction now establishes the lemma. $\square$

Lemma C.2. *The coset space*

$$(\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) \quad 1 \leq i \leq n-2$$

can be parametrized as

$$\begin{aligned} & (\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) = \\ & = \left\{ \begin{pmatrix} x'^{-1} & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{n-i-2} \end{pmatrix} : x' \in F^\times, y \in F, v \in F^{n-i-2} \right\} \\ & \cup \bigcup_{a=0}^{n-i-2} \left\{ \begin{pmatrix} 0 & 0 & (-1)^{a+1} x'^{-1} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{n-i-a-2} \end{pmatrix} : x' \in F^\times, v \in F^{n-i-a-2} \right\}. \end{aligned}$$

Proof. Denote $k \equiv n - i$. Consider a matrix

$$G = \begin{pmatrix} s & T^T \\ B & m \end{pmatrix} \in \mathrm{SL}_k(F),$$

where s is a scalar, m is a $(k-1) \times (k-1)$ -matrix and T and B (for “top” and “bottom”) are $(k-1)$ -column vectors. The action of an element

$$M = \begin{pmatrix} 1 & \xi^T \\ 0 & h \end{pmatrix} \in (\mathrm{SL}_k(F))_{\hat{Y}}$$

on G is

$$G \rightarrow MG = \begin{pmatrix} s + \xi^T B & T^T + \xi^T m \\ hB & hm \end{pmatrix}.$$

Parametrizing the coset space $(\mathrm{SL}_k(F))_{\hat{Y}} \backslash \mathrm{SL}_k(F)$ amounts to choosing $\xi \in F^{k-1}$ and $h \in \mathrm{SL}_{k-1}(F)$ such that the product MG takes a particularly nice form, manifestly with at most k degrees of freedom which is the dimension of this coset space.

Even though $h \in \mathrm{SL}_{k-1}(F)$ we will proceed with $h \in \mathrm{GL}_{k-1}(F)$ and restore the unit determinant of h at the end by left multiplication of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix}$ where $0 \neq x' = (\det h)^{-1}$. By having $h \in \mathrm{GL}_{k-1}(F)$ we are free to perform Gauss elimination among the bottom $k-1$ rows in G .

We consider the two cases $\dim \ker m = 0$ and $\dim \ker m = 1$. Note that the cases $\dim \ker m \geq 2$ do not arise as with row elimination it would then be possible to produce two zero-rows in m and hence a zero-row in G which violates $G \in \mathrm{SL}_k(F)$.

Case 1: $\dim \ker m = 0$. We choose $h = m^{-1}$ and $\xi^T = -T^T m^{-1}$. Since h has full rank, we can redefine $hB \rightarrow B$ without loss of generality and redefine $s + \xi^T B \rightarrow s$. This leads to the representative

$$G \rightarrow \begin{pmatrix} s & 0 \\ B & I_{k-1} \end{pmatrix}.$$

We now restore the unit determinant to h

$$G \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix} \begin{pmatrix} s & 0 \\ B & I_{k-1} \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{k-2} \end{pmatrix},$$

where we have split the $(k-1)$ -vector into a scalar y and a $(k-2)$ -vector v . The condition $\det G = 1$ now sets $s = x'^{-1}$ leading to

$$G = \begin{pmatrix} x'^{-1} & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{k-2} \end{pmatrix}.$$

This is a nice form of the representative G which manifestly has k degrees of freedom.

Case 2: $\dim \ker m = 1$. We can no longer choose $h = m^{-1}$. Having promoted h to be an element of $\mathrm{GL}_{k-1}(F)$, we can make use of lemma C.1 which leads us to consider representatives of the form

$$G \rightarrow \begin{pmatrix} s & T^{(1)T} & T^{(2)} & T^{(3)T} \\ B^{(1)} & 0 & 0 & 0 \\ B^{(2)} & I_a & 0 & 0 \\ B^{(3)} & 0 & v & I_{k-a-2} \end{pmatrix} \quad \text{for } a \in [0, k-2] \cap \mathbb{N} \quad \text{and } v \in F^{k-a-2}.$$

We see that $B^{(1)} \neq 0$ in order for G to remain non-singular. With further row elimination we can therefore bring this to the form

$$\begin{pmatrix} s & T^{(1)T} & T^{(2)} & T^{(3)T} \\ B^{(1)} & 0 & 0 & 0 \\ B^{(2)} & I_a & 0 & 0 \\ B^{(3)} & 0 & v & I_{k-a-2} \end{pmatrix} \rightarrow \begin{pmatrix} s & T^{(1)T} & T^{(2)} & T^{(3)T} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}.$$

Next, using the ξ -freedom we can bring this to the form

$$\begin{aligned} & \begin{pmatrix} s & T^{(1)T} & T^{(2)} & T^{(3)T} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \xi^{(1)T} & \xi^{(2)T} & \xi^{(3)T} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & I_{k-a-2} \end{pmatrix} \begin{pmatrix} s & T^{(1)T} & T^{(2)} & T^{(3)T} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \\ & = \begin{pmatrix} s+\xi^{(1)T} & T^{(1)T}+\xi^{(2)T} & T^{(2)}+\xi^{(3)T}v & T^{(3)T}+\xi^{(3)T} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}, \end{aligned}$$

with a suitable choice of ξ and having redefined $T^{(2)}$. We now restore the unit determinant to h

$$\begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x' & 0 & 0 \\ 0 & 0 & I_{k-2} & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}.$$

The condition $\det G = 1$ now sets $T^{(2)} = (-1)^{a+1}x'^{-1}$, leading to the representative

$$\begin{pmatrix} 0 & 0 & (-1)^{a+1}x'^{-1} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}.$$

□

Lemma C.3. *The coset space*

$$(\mathrm{SL}_j(F))_{\hat{X}} \backslash \mathrm{SL}_j(F) \quad 2 \leq j \leq n-1$$

can be parametrized as

$$\begin{aligned} & (\mathrm{SL}_j(F))_{\hat{X}} \backslash \mathrm{SL}_j(F) = \\ & = \left\{ \begin{pmatrix} I_{j-2} & 0 & 0 \\ 0 & x' & 0 \\ v^T & y & x'^{-1} \end{pmatrix} : x' \in F^\times, y \in F, v \in F^{j-2} \right\} \\ & \cup \bigcup_{a=0}^{j-2} \left\{ \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{j-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & (-1)^{j+a+1}x'^{-1} & v^T & 0 \end{pmatrix} : x' \in F^\times, v \in F^{j-a-2} \right\}. \end{aligned}$$

Proof. Denote $k \equiv j$. Consider a matrix

$$G = \begin{pmatrix} m & T \\ B^T & s \end{pmatrix} \in \mathrm{SL}_k(F),$$

where s is a scalar, m is a $(k-1) \times (k-1)$ -matrix and T and B (for “top” and “bottom”) are $(k-1)$ -column vectors. The action of an element

$$M = \begin{pmatrix} h & h\xi \\ 0 & 1 \end{pmatrix} \in (\mathrm{SL}_k(F))_{\hat{X}}$$

on G is

$$G \rightarrow MG = \begin{pmatrix} h(m+\xi B^T) & h(T+s\xi) \\ B^T & s \end{pmatrix}.$$

Parametrizing the coset space $(\mathrm{SL}_k(F))_{\hat{X}} \backslash \mathrm{SL}_k(F)$ amounts to choosing $\xi \in F^{k-1}$ and $h \in \mathrm{SL}_{k-1}(F)$ such that the product MG takes a particularly nice form, manifestly with at most k degrees of freedom which is the dimension of this coset space.

Even though $h \in \mathrm{SL}_{k-1}(F)$ we will proceed with $h \in \mathrm{GL}_{k-1}(F)$ and restore the unit determinant of h at the end by left multiplication of the matrix $\begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $0 \neq x' = (\det h)^{-1}$. By having $h \in \mathrm{GL}_{k-1}(F)$ we are free to perform Gauss elimination among the top $k-1$ rows in G .

We consider the two cases $s \neq 0$ and $s = 0$.

Case 1: $s = s' \neq 0$. We choose $\xi = \frac{-1}{s'}T$. This leads to the representative

$$G \rightarrow \begin{pmatrix} m - \frac{1}{s'}TB^T & 0 \\ B^T & s' \end{pmatrix}.$$

From the condition

$$1 = \det G = \det \left(m - \frac{1}{s'}TB^T \right) s'$$

we get that

$$\det \left(m - \frac{1}{s'}TB^T \right) \neq 0,$$

and hence the matrix $m - \frac{1}{s'}TB^T$ can be inverted using our h -freedom which leads to the representative

$$\begin{pmatrix} m - \frac{1}{s'}TB^T & 0 \\ B^T & s' \end{pmatrix} \rightarrow \begin{pmatrix} I_{k-1} & 0 \\ B^T & s' \end{pmatrix}.$$

We now restore the unit determinant to h

$$G \rightarrow \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{k-1} & 0 \\ B^T & s' \end{pmatrix} = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ v^T & y & s' \end{pmatrix},$$

where we have split B into a scalar y and a $(k-2)$ -vector v . The condition $\det G = 1$ now sets $s' = x'^{-1}$ leading to

$$G = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ v^T & y & x'^{-1} \end{pmatrix}.$$

This is a nice form of the representative G which manifestly has k degrees of freedom.

Case 2: $s = 0$. We can no longer eliminate T with our ξ -freedom. The group element G takes the form

$$G = \begin{pmatrix} m & T' \\ B'^T & 0 \end{pmatrix},$$

where the vectors T' and B' must be non-zero (as indicated by the primes) in order for G to be non-singular. A ξ -transformation takes the form

$$\begin{pmatrix} m & T' \\ B'^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} m + \xi B'^T & T' \\ B'^T & 0 \end{pmatrix}.$$

We now consider the $k-1$ distinct cases labelled by $a \in [0, k-2] \cap \mathbb{N}$ defined by that B'^T takes the form $B'^T = (0_{1 \times a} \ b' \ v)$, where v is a $k-a-2$ -vector and $0 \neq b' \in F$. The ξ -transformation then lets us eliminate the $(a+1)^{\text{th}}$ column of m . This works since the $(a+1)^{\text{th}}$ column of the matrix $\xi B'^T$ is $b'\xi$ where $b' \neq 0$ by assumption. We are led to the representative

$$\begin{pmatrix} m & T' \\ B'^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} m_1 & 0 & m_2 & T' \\ 0 & b' & v^T & 0 \end{pmatrix},$$

where m_1 is a $(k-1) \times a$ -matrix and m_2 is a $(k-1) \times (k-a-2)$ -matrix.

The $(k-1) \times (k-1)$ matrix $\begin{pmatrix} m_1 & 0 & m_2 \end{pmatrix}$ clearly has column-rank at most $k-2$. Since column-rank and row-rank for matrices are equal, we know that the row-rank is also at most $k-2$ and with row manipulations we can thus produce a zero row

$$\begin{pmatrix} m_1 & 0 & m_2 & T' \\ 0 & b' & v^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & t' \\ m_{21} & 0 & m_{22} & T^- \\ 0 & b' & v^T & 0 \end{pmatrix},$$

where $0 \neq t' \in F$ in order for G to be non-singular. With further row manipulations we can then eliminate the vector T^- and bring this to the form

$$\begin{pmatrix} 0 & 0 & 0 & t' \\ m_{21} & 0 & m_{22} & T^- \\ 0 & b' & v^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ m_{21} & 0 & m_{22} & 0 \\ 0 & b' & v^T & 0 \end{pmatrix}.$$

The $(k-2) \times (k-2)$ -matrix $\begin{pmatrix} m_{21} & m_{22} \end{pmatrix}$ must have full rank in order for G to be non-singular and can thus be inverted, leading to the representative

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ m_{21} & 0 & m_{22} & 0 \\ 0 & b' & v^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & b' & v^T & 0 \end{pmatrix}.$$

We permute the first $k-1$ rows to get

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & b' & v^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^T & 0 \end{pmatrix}.$$

Lastly, we restore the unit determinant to h

$$\begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^T & 0 \end{pmatrix} \rightarrow \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^T & 0 \end{pmatrix} = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & b' & v^T & 0 \end{pmatrix}.$$

The condition $\det G = 1$ now sets $b' = (-1)^{k+a+1}x'^{-1}$, leading to the representative

$$\begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & (-1)^{k+a+1}x'^{-1} & v^T & 0 \end{pmatrix}.$$

□

Remark C.4. Another way of parametrizing the coset $(\mathrm{SL}_k(F))_{\hat{X}} \backslash \mathrm{SL}_k(F)$ is to parametrize the coset $\mathrm{SL}_k(F) / (\mathrm{SL}_k(F))_{\hat{X}}$ which works completely analogously to the how the coset $(\mathrm{SL}_k(F))_{\hat{Y}} \backslash \mathrm{SL}_k(F)$ was parametrized in lemma C.2 and then invert the resulting matrices.

We lastly provide parametrizations of $\Gamma_{n-1}(\psi_{\alpha_i})$ for the cases $1 \leq i \leq n-3$ and of $\Lambda_1(\psi_{\alpha_j})$ for the cases $3 \leq j \leq n-1$.

Lemma C.5. *The coset space $\Gamma_{n-1}(\psi_{\alpha_i}) = (T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_i}}$ can be parametrized as*

$$(T_{\psi_{\alpha_1}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_1}} = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & z' & \\ & & & 1 \\ & & & & \ddots \\ & & & & & 1 \\ & & & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}$$

for $i = 1$ while for $2 \leq i \leq n-3$ it can be parametrized as

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}.$$

Further possible parametrizations are discussed in the proof below.

Proof. For any α_k , the stabilizer $T_{\psi_{\alpha_k}}$ is given by

$$T_{\psi_{\alpha_k}} = \left\{ \begin{pmatrix} t_1 & & & \\ & y_k & & \\ & & y_k & \\ & & & t_2 \end{pmatrix} \leftarrow \text{row } k : y_k \in F^\times, \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T'(y_k^{-2}) \right\}, \quad 1 \leq k \leq n-1$$

where $T'(y_k^{-2})$ are the diagonal matrices in $\mathrm{GL}_{n-2}(F)$ with determinant y_k^{-2} . This is seen by the group action of a diagonal element $h \in T_{\alpha_k}$ on an upper triangular element $n \in N$,

$$hnh^{-1} = \begin{pmatrix} * & y_k & & \\ & y_{k+1} & & \\ & & * & \\ & & & * \end{pmatrix} \begin{pmatrix} * & 1 & x_{i,i+1} & * \\ & 1 & & * \\ & & 1 & \\ & & & * \end{pmatrix} \begin{pmatrix} * & y_k^{-1} & & \\ & y_{k+1}^{-1} & & \\ & & * & \\ & & & * \end{pmatrix} = \begin{pmatrix} * & 1 & \frac{y_k}{y_{k+1}} x_{i,i+1} & * \\ & 1 & & * \\ & & 1 & \\ & & & * \end{pmatrix}.$$

Since the expression $\psi_{\alpha_k}(n)$ only depends on $x_{i,i+1}$ we require $y_k = y_{k+1}$ in order to have $\psi(n) = \psi(hnh^{-1})$. We get that the intersection $T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}$ can be written

$$T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}} = \left\{ \begin{pmatrix} t_1 & & & & \\ & y_i & & & \\ & & y_i & & \\ & & & t_2 & \\ & & & & y_{n-1} \\ & & & & & y_{n-1} \end{pmatrix} : y_i, y_{n-1} \in F^\times; \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T''(y_i^{-2} y_{n-1}^{-2}) \right\}$$

where $T''(y_i^{-2} y_{n-1}^{-2})$ are the diagonal matrices in $\mathrm{GL}_{n-4}(F)$ with determinant $y_i^{-2} y_{n-1}^{-2}$. Looking at the left action of an element of this intersection on an element of the stabilizer T_{α_i} gives

$$\begin{aligned} & \begin{pmatrix} t_1 & & & & \\ & y_i & & & \\ & & y_i & & \\ & & & t_2 & \\ & & & & y_{n-1} \\ & & & & & y_{n-1} \end{pmatrix} \begin{pmatrix} u_1 & \alpha_i & & & \\ & \alpha_i & \alpha_i & & \\ & & u_2 & & \\ & & & \alpha_{n-1} & \\ & & & & \alpha_n \end{pmatrix} = \\ & = \begin{pmatrix} t_1 u_1 & & & & \\ & y_i \alpha_i & & & \\ & & y_i \alpha_i & & \\ & & & t_2 u_2 & \\ & & & & y_{n-1} \alpha_{n-1} \\ & & & & & y_{n-1} \alpha_n \end{pmatrix}. \end{aligned}$$

For finding representatives, the task is to choose t_1, y_i, t_2 and y_{n-1} to simplify the form of the matrix. We pick $y_i = \alpha_i^{-1}$ and with y_{n-1} we can eliminate either α_{n-1} or α_n but not both. After fixing t_1 and t_2 as well we can arrive at a particular class of convenient final forms

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} I_a & \\ & z' \\ & & I_{n-3-a} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} z'^{-1} & \\ & 1 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 1 & \\ & z'^{-1} \end{pmatrix}$$

where the allowed choices for a are $a = 0, \dots, i-2, i+1, \dots, n-3$.

For $i = 1$ we can choose

$$(T_{\psi_{\alpha_1}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_1}} = \left\{ \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & z' & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \\ & & & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}$$

while for $2 \leq i \leq n-3$ we can choose

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}, \quad 2 \leq i \leq n-3.$$

□

Lemma C.6. *The coset space $\Lambda_1(\psi_{\alpha_i}) = (T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_{\alpha_i}}$ can be parametrized as*

$$(T_{\psi_{\alpha_{n-1}}} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_{\alpha_{n-1}}} = \left\{ \begin{pmatrix} z' & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z'^{-1} \\ & & & & & 1 \\ & & & & & & 1 \end{pmatrix} : z' \in F^\times \right\}$$

for $i = n - 1$ while for $3 \leq i \leq n - 2$ it can be parametrized as

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}.$$

Further possible parametrizations can be deduced from the proof of lemma C.5.

Proof. Analogous to the proof of lemma C.5. $\square$

Remark C.7. The lemmas can be extended to include the cosets $\Gamma_{n-1}(\psi_{\alpha_{n-2}})$ and the trivial $\Gamma_{n-1}(\psi_{\alpha_{n-1}})$ (as well as the corresponding Λ -cosets). They are, however, never needed in theorem A or B.

APPENDIX D. LEVI ORBITS

Let P_m for $1 \leq m \leq n - 1$ be the maximal parabolic subgroup of SL_n associated to the simple root α_m with Levi decomposition LU_m (where we drop the subscript for L for convenience) and let $y \in {}^t\mathfrak{u}_m(F)$ be parametrised by a matrix $Y \in \mathrm{Mat}_{(n-m) \times m}(F)$ as in (4.1)

$$y(Y) = \begin{pmatrix} 0_m & 0 \\ Y & 0_{n-m} \end{pmatrix}.$$

We will now study the $L(F)$ -orbits of elements $y(Y)$. We parametrise an element $l \in L(F)$ by the two matrices $A \in \mathrm{Mat}_{m \times m}(F)$ and $B \in \mathrm{Mat}_{(n-m) \times (n-m)}(F)$ with $\det(A) = \det(B) \neq 0$ as $l = \mathrm{diag}(A^{-1}, B)$. This element acts by conjugation on $y(Y)$ as $ly(Y)l^{-1} = y(BYA)$. Using unit determinant matrices A and B we may perform standard row and column additions to put Y on a form which has zero elements everywhere except for an anti-diagonal $r \times r$ matrix with non-vanishing determinant in the upper right corner of Y where $0 \leq r \leq \min(m, n - m)$ is the rank of Y . This can be seen as follows.

The case $\mathrm{rank} Y = 0$ is trivial so we may assume that Y has non-zero elements. If the top right element of Y is zero, pick any non-zero element whose row-column position we denote (i, j) and add multiples of row i to the first row, and column j to the last column to make the upper right element non-zero. Then, use the non-zero upper right element to cancel all remaining non-zero elements on the first row and last column by further row and column additions. Repeat the procedure for the matrix obtained by removing the first row and last column. The iteration terminates when we run out of rows or columns, or when the remaining elements are all zero. This proves the above statement.

We will now rescale the anti-diagonal elements by conjugating y with diagonal matrices l , meaning that the i th diagonal element in l rescales both row i and column i (inversely). Each rescaling of an element in the anti-diagonal of the $r \times r$ matrix then leaves two less diagonal elements in l for further rescalings. Since the non-zero elements of Y at this stage do not share any rows or columns (because of the anti-diagonal $r \times r$ submatrix), we may then perform any and all rescalings until we run out of free diagonal elements in l . The number of free diagonal elements in l is $n - 1$ because of the determinant condition which means that we can make $\lfloor \frac{n-1}{2} \rfloor$ independent

rescalings. We have that $1 \leq m \leq n-1$ and $r \leq \min(m, n-m) \leq \lfloor \frac{n}{2} \rfloor$. Thus, it is possible to rescale all anti-diagonal elements to one unless $n = 2r = 2m$, for which there will be one remaining anti-diagonal element d that we may assume is in the lower left corner of Y and therefore also of $y(Y)$. Using conjugations with $l = \text{diag}(a, 1, \dots, 1, 1/a)$, $a \in F^\times$ this d can be shown to be in $F^\times/(F^\times)^2$.

We have now shown that the $L(F)$ -orbits of elements $y(Y) \in {}^t\mathbf{u}_m(F)$ are characterized by matrices $Y = Y_r(d) \in \text{Mat}_{(n-m) \times m}(F)$, where $Y_r(d)$ is non-zero only for an anti-diagonal $r \times r$ matrix in its upper right corner. The anti-diagonal elements of this $r \times r$ matrix are all one except the for the element in the lower left corner which is d :

$$Y_r(d) = \begin{pmatrix} 0 & \begin{bmatrix} & & & 1 \\ & & \ddots & \\ & d & & \\ & & & \end{bmatrix} \\ 0 & 0 \end{pmatrix}.$$

For $n = 2r = 2m$, $d \in F^\times/(F^\times)^2$ and otherwise $d = 1$. For convenience we will denote $Y_r(1)$ as Y_r . Thus, the $L_m(F)$ -orbits on ${}^t\mathbf{u}_m(F)$ are characterized by the same data as the $\text{SL}_n(F)$ -orbits $([2^r 1^{n-2r}], d)$ with $d \in F^\times/(F^\times)^k$, $k = \gcd([2^r 1^{n-2r}])$ and $0 \leq r \leq \min(m, n-m)$.

By conjugating with a Weyl element $w \in \text{SL}_n(F)$ that maps torus elements $(t_1, t_2, \dots, t_n) \mapsto (\underline{t_m, t_{m+1}, t_{m-1}, t_{m+2}, \dots, t_{m-r+1}, t_{m+r}}, \underline{t_1, \dots, t_{m-r}, t_{m+r+1}, \dots, t_n})$, where we have underlined different groups for readability, we see that $y(Y_r(d))$ is put on the form of the standard representative for the $\text{SL}_n(F)$ -orbit $([2^r 1^{n-2r}], d)$ shown in proposition 2.2.

In this paper we will always be able to find a representative on the form $Y_r(1)$, that is, to rescale all elements, since we consider $n \geq 5$ and $r \leq 2$ where the latter restriction comes from the fact that higher rank elements have vanishing associated Fourier coefficients in a next-to-minimal or minimal automorphic representation according to theorem 2.1.

Lastly, we note that if we instead consider $L(\overline{F})$ -orbits, the last remaining rescaling in the maximal rank $n = 2r$ case would be possible by conjugation with $l = \text{diag}(\sqrt{d}, 1, \dots, 1, 1/\sqrt{d})$.

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