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Derived Hecke Algebra for Weight One Forms

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ABSTRACT
We study the action of the derived Hecke algebra on the space of weight one forms. By analogy with the topological case, we formulate a conjecture relating this to a certain Stark unit. We verify the truth of the conjecture numerically, for the weight one forms of level 23 and 31, and many derived Hecke operators at primes less than 200. Our computation depends in an essential way on Merel's evaluation of the pairing between the Shimura and cuspidal subgroups of $J_0(q)$.

KEYWORDS
Modular forms; number theory

1. Introduction

Let G be an algebraic group over $\mathbf{Q}$. In [Venkatesh nd], the second-named author studied the action of a derived version of the Hecke algebra on the singular cohomology of the locally symmetric space attached to G. One expects that this action transports Hecke eigenclasses between cohomological degrees and moreover (see again [Venkatesh nd]) is related to a "hidden" action of a motivic cohomology group.

It is also possible for a Hecke eigensystem on *coherent* cohomology to occur in multiple degrees. The simplest situation is weight one forms for the modular curve. We study this case, explicating the action of the derived Hecke algebra and formulating a conjectural relationship with motivic cohomology.

The motivic cohomology is particularly concrete: a weight one eigenform f is attached to a two-dimensional Artin representation ρ_f of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ and the motivic cohomology group in question is generated by a certain unit in the splitting field of the adjoint of ρ_f. This makes the conjecture particularly amenable to numerical testing. We carry this out for the forms of conductor 23 and 31, and the first few derived Hecke operators; the numerics all support the conjecture.

We note that this story is also related to the Taylor–Wiles method for coherent cohomology and its obstructed version due to Calegari–Geraghty (see [Harris 13], and the detailed discussion of weight one forms in [Calegari and Geraghty nd]). This is used implicitly in the discussion in Section 4 of the current paper.

Now we outline the contents of the paper. After giving notation in Section 2, we describe the derived Hecke algebra and the main conjecture in Section 3. Although the discussion to this point is self-contained, we postpone the comparison with the results of [Venkatesh nd] until the final section. We translate the Conjecture to an explicitly computable form in Section 5; see in particular Proposition 5.1. Finally, in Section 5.4, we make the conjecture even more explicit in the case of a form associated to a cubic field K, and check it numerically in the case of K with discriminant -23 and -31.

It would be most desirable to extend our computations to the case of an "exotic" weight one form, that is to say, a weight one form whose associated projective Galois representation has non-dihedral image. The case of a form whose Galois representation is induced from a mixed signature character of a *real* quadratic field is also of interest. The minimal level of such a form is 124, but our code becomes very slow once the product qN of the level N and the prime q indexing the derived Hecke operator becomes large: it seems unlikely we could, with our current code, compute enough Hecke operators to shed any real light on the conjecture.

Our numerical computation depends, in a crucial way, on the evaluation of a certain pairing in coherent cohomology on the mod p fiber of a modular curve; this evaluation is postponed to Section 6, where we do it by relating it to Merel's remarkable computation [Merel 96]. It is worth emphasizing how important Merel's computation is for us: it seemed almost impossible to carry through our computation until we learned about Merel's results. Indeed, the role that Merel's computation plays here suggests that it would be worthwhile to understand how it might generalize to Hilbert modular surfaces.

The expression of the derived Hecke algebra action as a cohomological cup product (see 5–2) strongly sug-

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gests a relation with special values of the *triple product L-function*. In the forthcoming work with Henri Darmon, Michael Harris, and Victor Rotger, we will study this further; in particular we are able to prove some special cases of the Conjecture.

2. Notation

2.1. Modular curves

Fix an integer N. We will generically use the letter R for a $\mathbb{Z}[1/N]$-algebra. For $R \to R'$ a morphism of $\mathbb{Z}[1/N]$-algebras and Y an R-scheme, we denote by $Y_{R'}$ the base extension of Y to R'.

Let $X = X_1(N)$ be the compactification [Deligne and Rapoport 73] of the modular curve parametrizing elliptic curves with with an N-torsion point. We may construct X as a smooth proper relative curve over Spec $\mathbb{Z}[\frac{1}{N}]$, and the cusps give rise to a relative divisor $D \subset X$. In particular, we obtain an R-scheme X_R for each $\mathbb{Z}[1/N]$-algebra R. We denote the universal generalized elliptic curve over X by $\mathcal{A} \longrightarrow X$.

We will denote by

$$X_{01}(qN), \quad X_1(qN),$$

the modular curves that correspond to adding $X_0(q)$ and $X_1(q)$ level structure to X.

Let ω be the line bundle over X whose sections are given by weight one forms. More precisely, when X is the modular curve, let $\Omega_{\mathcal{A}/X}$ denote the relative cotangent bundle of $\mathcal{A}/X$, pulled back to X via the identity section. We define ω as the pullback of $\Omega_{\mathcal{A}/X}$ via the zero section $X \to \mathcal{A}$. Therefore, the sections of ω correspond to weight one forms, whereas sections of $\omega(-D)$ corresponds to weight one *cusp* forms. Moreover, there is an isomorphism of line bundles [Katz 73, (1.5), A (1.3.17)]:

$$\omega \otimes \omega(-D) \simeq \Omega^1_{X \to \mathbb{Z}[\frac{1}{N}]}, \tag{2–1}$$

which says that "the product of a weight one form and a cuspidal weight one form is a cusp form of weight two."

Let $\pi : X_R \to \mathrm{Spec} R$ be the structure morphism and consider the space of weight one forms over R, in cohomological degree i – formally:

$$\Gamma(\mathrm{Spec}(R), \; R^i\pi_*\omega).$$

We will denote this space, for short, by $H^i(X_R, \omega)$, and use similar notation for $\omega(-D)$. Therefore, $H^0(X_R, \omega)$ (respectively $H^0(X_R, \omega(-D))$) is the usual space of weight one modular forms (respectively cusp forms) with coefficients in R.

2.2. The residue pairing

The pairing 2–1 induces

$$\pi_*\omega \otimes \mathrm{R}^1\pi_*\omega(-D) \longrightarrow \mathrm{R}^1\pi_*\Omega^1_{X_R/R}.$$

Since X_R is a projective smooth curve over R, there is a canonical identification of the last factor with the trivial line bundle; thus we get a pairing

$$H^0(X_R, \omega) \times H^1(X_R, \omega(-D)) \to R,$$

which we denote as $[-,-]_{\mathrm{res},R}$. (Here, res stands for "residue.") This pairing is compatible with change of ring, and if R is a field, it is a perfect pairing.

2.3. The fixed weight one form g

We want to localize our story throughout at a single weight one form g. Therefore, fix $g = \sum a_n q^n$ a Hecke newform of level N and Nebentypus χ, normalized so that $a_1 = 1$. Here, χ is a Dirichlet character of level N.

We regard the a_n as lying in some number field E, and indeed in the integer ring $\mathcal{O}$ of E. Thus, g extends to a section:

$$g \in H^0(X_{\mathcal{O}[\frac{1}{N}]}, \omega(-D)).$$

We shall denote by $H^*(X_{\mathcal{O}[\frac{1}{N}]}, \omega)[g]$, the part of the cohomology that transforms under the Hecke operators in the same way as g, i.e. the common kernel of all $(T_\ell - a_\ell)$ over all primes ℓ not dividing N.

Extending E if necessary, we may suppose that one can attach to g a Galois representation, unramified away from N [Deligne and Serre 74]:

$$\rho : \mathrm{Gal}(L/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathcal{O}), \tag{2–2}$$

where L is a Galois extension of $\mathbf{Q}$. Here, the Frobenius trace of ρ at ℓ coincides with a_ℓ, and the image of complex conjugation c under ρ is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. (In the body of the text, we will primarily use the field cut out by the *adjoint* of ρ, and we could replace L by this smaller field).

We emphasize the distinction between E and L:

> E is a coefficient field for the weight one form g, and L is the splitting field for the Galois representation of g.

In our numerical examples, we will have $E = \mathbf{Q}$ and L the Galois closure of a cubic field.

It will be convenient to denote by $\mathrm{Ad}^0\rho$ the trace-free adjoint of ρ, i.e. the associated action of $\mathrm{Gal}(L/\mathbf{Q})$ on 2×2 matrices of trace zero and entries in $\mathcal{O}$. We denote by $\mathrm{Ad}^*\rho$ the $\mathcal{O}$-linear dual to $\mathrm{Ad}^0\rho$, i.e.

$$\mathrm{Ad}^*\rho = \mathrm{Hom}(\mathrm{Ad}^0\rho, \mathcal{O}),$$

a locally free $\mathcal{O}$-module endowed with an action of $\mathrm{Gal}(L/\mathbf{Q})$.[1]

For later use, it is convenient to choose a dual form g' that will be paired with g eventually. In order that a Hecke equivariant pairing between g and g' be non-zero, we should take g' to be the form corresponding to the contragredient automorphic representation, i.e.

$$g' := \sum \overline{a_n} q^n \in H^0(X_{\mathcal{O}[\frac{1}{N}]}, \omega(-D)),$$

where $\alpha \mapsto \overline{\alpha}$ is the complex conjugation in the CM field E. (In our examples, $E = \mathbf{Q}$, and therefore $g' = g$).

2.4. The prime $\mathfrak{p}$

Let $\mathfrak{p}$ be a prime of E, above the rational prime p. We make the following assumptions:

- All weight one forms in characteristic p lift to characteristic zero, i.e. the natural map
$$H^0(X_{\mathbb{Z}_p}, \omega) \to H^0(X_{\mathbb{F}_p}, \omega)$$
is surjective.
- $p \geqslant 5$.
- p is unramified inside E.
- There are no pth-roots of unity inside L.
- p does not divide the order $[L : \mathbf{Q}]$.

The representation ρ may be reduced modulo $\mathfrak{p}$, obtaining $\overline{\rho} : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{F}_{\mathfrak{p}})$, where $\mathbf{F}_{\mathfrak{p}} = \mathcal{O}/\mathfrak{p}$ is the residue field at $\mathfrak{p}$. As before, we may define the trace-free adjoint $\mathrm{Ad}^0\overline{\rho}$ and its dual $\mathrm{Ad}^*\overline{\rho}$.

2.5. Taylor–Wiles primes

A *Taylor–Wiles prime q of level $n \geq 1$* for g, or more precisely relative to the pair $(g, \mathfrak{p})$, will be, by definition:

- a rational prime $q \equiv 1$ modulo p^n, relatively prime to N;
- the data of $(\alpha, \beta) \in \mathbf{F}_{\mathfrak{p}}$ with $\alpha \neq \beta$, such that $\overline{\rho}(\mathrm{Frob}_q)$ is conjugate to $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$.

Thus, whenever we refer to Taylor–Wiles primes, we always regard the ordered pair (α, β) as part of the data: this amounts to an ordering of the eigenvalues of Frobenius.

If $\mathfrak{p}, n, q$ have been fixed, where q is a Taylor–Wiles prime of level n and $\mathfrak{p}$ is a prime of $\mathcal{O}$ as above, it is convenient to use the following shorthand notation:

- Write $k = \mathcal{O}/\mathfrak{p}^n$.
- Write $(\mathbb{Z}/q)_p^*$ for the quotient of $(\mathbb{Z}/q)^*$ of size p^n, so that there is a noncanonical isomorphism $(\mathbb{Z}/q)_p^* \cong \mathbb{Z}/p^n$.
- Write
$$k\langle 1\rangle = k \otimes (\mathbb{Z}/q)_p^*, \quad k\langle -1\rangle = \mathrm{Hom}((\mathbb{Z}/q)_p^*, k).$$
These are isomorphic as abelian groups to k, but not canonically so.
- Similarly for a $\mathbb{Z}$-module M we shall write
$$M\langle n\rangle = M \otimes_{\mathbb{Z}} k\langle n\rangle.$$
Thus, for example, $\mathbb{F}_p\langle 1\rangle$ is canonically identified with the quotient of $(\mathbb{Z}/q)^*$ of size p.

These notations clearly depends on $\mathfrak{p}, n, q$; however , we do not explicitly indicate this dependence.

2.6. The Stark unit group

Let U_L be the group of units of the integer ring of L.

The key group of "Stark units" that we shall consider is the following $\mathcal{O}$-module:

$$\mathrm{U}_g := \left(\mathrm{U}_L \otimes_{\mathbb{Z}} \mathrm{Ad}^*\rho\right)^{G_{L/\mathbf{Q}}} = \left(\mathrm{U}_L \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}}(\mathrm{Ad}^0\rho, \mathcal{O})\right)^{G_{L/\mathbf{Q}}} \tag{2–3}$$

$$\tilde{\rightarrow} \mathrm{Hom}_{\mathcal{O}[G_{L/\mathbf{Q}}]}(\mathrm{Ad}^0\rho, \mathrm{U}_L \otimes \mathcal{O}), \tag{2–4}$$

where Ad^0 is the conjugation action of the Galois group on trace-free matrices in $M_2(\mathcal{O})$.

For instance, in the examples of modular forms attached to cubic fields, the group U_L will amount to (essentially) the unit group of that cubic field.

Lemma 2.1. $\mathrm{U}_g \otimes_{\mathbb{Z}} \mathbf{Q}$ *is an E-vector space of rank 1 and* $\mathrm{U}_g \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *is a free $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$-module of rank* 1.

Proof. Fix any embedding $\iota : E \hookrightarrow \mathbb{C}$; the E-dimension of $\mathrm{U}_g \otimes_{\mathbb{Z}} \mathbf{Q}$ then coincides with the complex dimension of Galois invariants on $\mathrm{U}_L \otimes_{\mathbf{Q}} (\mathrm{Ad}^*\rho)^{\iota}$.

In general, for any number field L, Galois over $\mathbf{Q}$, and any $\eta : G_{L/\mathbf{Q}} \to \mathrm{GL}_m(\mathbb{C})$ without a trivial subrepresentation, the dimension of $(\mathrm{U}_L \otimes \eta)^{G_{L/\mathbf{Q}}}$ is known to be the dimension of invariants for complex conjugation in η. (In fact, this is a straightforward consequence of the unit theorem). It follows, therefore, that the E-dimension of $\mathrm{U}_g \otimes_{\mathbb{Z}} \mathbf{Q}$ is 1 as claimed.

The final claim now follows since our assumption on p means that U_L is free of p-torsion. □

Fix a non-zero element $u \in \mathrm{U}_g$ in such a way that $[\mathrm{U}_g : \mathcal{O}.u]$ is relatively prime to p (e.g. a generator, if U_g is a free $\mathcal{O}$-module). Later, we will also work with the $\mathcal{O}$-dual

$$\mathrm{U}_g^{\vee} := \mathrm{Hom}_{\mathcal{O}}(\mathrm{U}_g, \mathcal{O}), \tag{2–5}$$

[1] We apologize for the perhaps pedantic distinction between $\mathrm{Ad}^*\rho$ and $\mathrm{Ad}^0\rho$. Since we will shortly be localizing at a prime larger than 2, one could identify them by means of the pairing $\mathrm{trace}(AB)$. However, when working in a general setting, one really needs to use Ad^*, and following this convention makes it easier to compare with [Venkatesh nd].

and denote by $u^* \in \mathrm{U}_g^\vee$ a non-zero element, chosen so that $\langle u, u^*\rangle \in \mathcal{O}$ is not divisible by any prime above p.

2.7. Aside: comparison of U_g with the motivic cohomology group from [Venkatesh nd]

This section is not used in the remainder of this paper. It serves to connect the previous construction with the discussion in [Venkatesh nd]:

We may construct a three-dimensional Chow motive $\mathrm{Ad}^0\mathrm{M}_g$, with coefficients in E, attached to the trace-free adjoint $\mathrm{Ad}^*\rho_g$—in other words, the étale cohomology of $\mathrm{Ad}^0\mathrm{M}_g$ is concentrated in degree zero and identified, as a Galois representation, with $\mathrm{Ad}^*\rho_g$.

Now consider the motivic cohomology $H^1_{\mathrm{mot}}(\mathbf{Q}, \mathrm{M}_g(1))$, or more precisely the subspace of integral classes $(-)_{\mathrm{int}}$ described by [Scholl 00].

The general conjectures of [Venkatesh nd, Prasanna and Venkatesh nd], transposed to the current (coherent) situation, predict that the dual of $H^1_{\mathrm{mot}}(\mathbf{Q}, \mathrm{M}_g(1))_{\mathrm{int}}$ should act on $H^*(X_E, \omega)[g]$.

There is a natural map

$$H^1_{\mathrm{mot}}(\mathbf{Q}, \mathrm{M}_g(1))_{\mathrm{int}} \longrightarrow H^1_{\mathrm{mot}}(L, \mathrm{M}_g(1))_{\mathrm{int}}^{G_{L/\mathbf{Q}}} \quad (2\text{–}6)$$

$$= (\mathrm{U}_L \otimes \mathrm{Ad}^*\rho \otimes \mathbf{Q})^{G_{L/\mathbf{Q}}} = \mathrm{U}_g \otimes \mathbf{Q}. \quad (2\text{–}7)$$

Although we did not check it, this map is presumably an isomorphism. In the present paper, we will never directly refer to the motivic cohomology group. Rather, we work with the right-hand side (or its integral form U_g) as a concrete substitute for the motivic cohomology group.

2.8. Reduction of a Stark unit at a Taylor–Wiles prime q

Let q be a Taylor–Wiles prime (as in Section 2.5); we shall define a canonical reduction map

$$\theta_q : \mathrm{U}_g \longrightarrow k\langle 1\rangle.$$

For example, in the examples of modular forms attached to cubic fields, this will amount to the reduction of a unit in the cubic field at a degree one prime above q. Although explicit, the general definition is unfortunately opaque (the motivation comes from computations in [Venkatesh nd]).

For any prime $\mathfrak{q}$ of L above q, with associated Frobenius element $\mathrm{Frob}_\mathfrak{q}$, let $D_\mathfrak{q} = \langle \mathrm{Frob}_\mathfrak{q}\rangle \subset \mathrm{Gal}(L/\mathbf{Q})$ be the associated decomposition group, the stabilizer of $\mathfrak{q}$. We may construct a $D_\mathfrak{q}$-invariant element

$$\mathsf{e}_\mathfrak{q} = 2\rho\left(\mathrm{Frob}_\mathfrak{q}\right) - \mathrm{trace}\,\rho\left(\mathrm{Frob}_\mathfrak{q}\right) \in \mathrm{Ad}^0\rho, \quad (2\text{–}8)$$

where we regard the middle quantity as a 2×2 matrix with coefficients in $\mathcal{O}$ and trace zero, thus belonging to $\mathrm{Ad}^0\rho$. Pairing with $\mathsf{e}_\mathfrak{q}$ and reduction mod $\mathfrak{p}^n$ induces

$$\mathsf{e}_\mathfrak{q} : \mathrm{Ad}^*\rho \longrightarrow \mathcal{O} \to k,$$

equivariantly for the Galois group of $\mathbf{Q}_q$. Also, for $g \in \mathrm{Gal}(L/\mathbf{Q})$ we have

$$\mathsf{e}_{g\mathfrak{q}} = \mathrm{Ad}(\rho(g))e_\mathfrak{q}. \quad (2\text{–}9)$$

Write $L_q = (L \otimes \mathbf{Q}_q)$ and let $\mathcal{O}_{L_q}$ be the integer subring thereof. Thus, $\mathcal{O}_{L_q}/q \simeq \prod_{\mathfrak{q}|q} \mathbf{F}_\mathfrak{q}$. Fix a prime $\mathfrak{q}_0$ of L above q. The inclusion of units for the number field L into local units $\mathcal{O}_{L_q}^*$ induces

$$(\mathrm{U}_g) \to \left(\prod_{\mathfrak{q}|q} \mathbf{F}_\mathfrak{q}^* \otimes \mathrm{Ad}^*\rho\right)^{G_{L/\mathbf{Q}}} \xrightarrow{\sim} \left(\mathbf{F}_{\mathfrak{q}_0}^* \otimes \mathrm{Ad}^*\rho\right)^{D_{\mathfrak{q}_0}}$$

$$\xrightarrow{\mathsf{e}_{\mathfrak{q}_0}} (\mathbf{F}_{\mathfrak{q}_0}^* \otimes k)^{D_{\mathfrak{q}_0}} \to k\langle 1\rangle, \quad (2\text{–}10)$$

where the second map is projection onto the factor corresponding to $\mathfrak{q}_0$.

The resulting composite is independent of the choice of $\mathfrak{q}_0$ because of (2–9). We call it θ, or θ_q when we want to emphasize the dependence on the Taylor–Wiles prime q:

$$\theta \text{ or } \theta_q : \mathrm{U}_g \to k\langle 1\rangle.$$

3. Derived Hecke operators and the main conjecture

We follow the notation of Section 2; in particular,

- g is a modular form with coefficients in the integer ring $\mathcal{O}$; we have associated to it a $\mathcal{O}$-module U_g of “Stark units” of rank 1.
- Fixing a prime $\mathfrak{p}$ of $\mathcal{O}$, we will work with the coefficient ring $k = \mathcal{O}/\mathfrak{p}^n$ with residue field $\mathbb{F}_\mathfrak{p}$ of characteristic p.

In this section, we define derived Hecke operators and formulate the main conjecture concerning their relationship to U_g. This discussion is obtained by transcribing the theory of [Venkatesh nd] to the present context; in this section, we just describe the conclusions of this process.

For each $q \equiv 1$ modulo p^n and each $z \in k\langle -1\rangle$, we will produce an operator

$$T_{q,z} : H^0(X_k, \omega) \to H^1(X_k, \omega).$$

Note that q need not be a Taylor–Wiles prime (in the sense of Section 2.4) for the definition of $T_{q,z}$—in other words, we do not use the assumption on the Frobenius element. However, our conjecture pins down the action of $T_{q,z}$ only at Taylor–Wiles primes.

3.1. The Shimura class

Start with the Shimura covering $X_1(q) \to X_0(q)$, and pass to the unique subcovering with Galois group $(\mathbb{Z}/q)^*_p$; call this $X_1(q)^\Delta \to X_0(q)$. By Corollary 2.3 of [Mazur 77, Chapter 2], it extends to an étale covering of schemes over $\mathbb{Z}[\frac{1}{qN}]$, and in particular induces an étale cover $X_1(q)^\Delta_k \to X_0(q)_k$. It, therefore, gives rise to a class in the etale H^1, i.e.

$$\mathfrak{S} \in H^1_{\text{et}}(X_0(q)_k, k\langle 1\rangle).$$

In the category of étale sheaves over $X_0(q)_k$, there is a natural map $k \to \mathbb{G}_a$. Then a class in $H^1_{\text{et}}(X_0(q)_k, k\langle 1\rangle)$ defines a class in

$$H^1_{\text{et}}(X_0(q)_k, \quad \mathbb{G}_a\langle 1\rangle) \simeq H^1_{\text{Zar}}(X_0(q)_k, \quad \mathcal{O}\langle 1\rangle)$$

because of the coincidence of the étale and Zariski cohomologies with coefficients in a quasi-coherent sheaf. This construction has thus given a class, associated to the Shimura cover, but now in Zariski cohomology:

$$\mathfrak{S} \in H^1_{\text{Zar}}(X_0(q)_k, \mathcal{O}\langle 1\rangle),$$

which we shall sometimes call the Shimura class.

It is reassuring to note that $\mathfrak{S}$ is in fact non-zero, even modulo the maximal ideal $\mathfrak{p}$, as one sees by computing with the Artin–Schreier sequence

$$0 \to \mathbb{F}_{\mathfrak{p}} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$$

over $k/(\mathfrak{p})$.

3.2. Construction of the derived Hecke operator

The class $\mathfrak{S}$ just defined can be pulled back to $H^1_{\text{Zar}}(X_{01}(qN)_k, \mathcal{O}\langle 1\rangle)$. We denote this class by $\mathfrak{S}_X$, to distinguish it from $\mathfrak{S}$ at level q.

Thus, cup product with $\mathfrak{S}_X$ gives a mapping

$$H^0(X_{01}(qN)_k, \omega) \xrightarrow{\cup\mathfrak{S}_X} H^1(X_{01}(qN)_k, \omega\langle 1\rangle).$$

Finally, to obtain the derived Hecke operator we add a push–pull as in the usual Hecke operator definition:

$$\begin{aligned} H^0(X_k, \omega) &\xrightarrow{\pi_1^*} H^0(X_{01}(qN)_k, \omega) \xrightarrow{\cup\mathfrak{S}_X} H^1(X_{01}(qN)_k, \omega)\langle 1\rangle \\ &\xrightarrow{\pi_{2*}} H^1(X_k, \omega)\langle 1\rangle, \end{aligned} \quad (3\text{–}1)$$

where $\pi_1, \pi_2 : X_{01}(qN) \to X$ are the two natural degeneracy maps (at the level of the upper half-plane, we understand π_1 to be $z \mapsto z$, and π_2 to be $z \mapsto qz$). Observe that without the middle $\cup\mathfrak{S}_X$ this would be the usual Hecke operator at q. In other words, we have constructed a map

$$H^0(X_k, \omega) \to H^1(X_k, \omega)\langle 1\rangle, \quad (3\text{–}2)$$

and correspondingly for $z \in k\langle -1\rangle$, we will denote by $T_{q,z}$ the corresponding "derived Hecke operator"

$$T_{q,z} : H^0(X_k, \omega) \to H^1(X_k, \omega) \quad (3\text{–}3)$$

obtained by multiplying (3–2) by z.

Although by presenting the bare definition the construction may seem a little *ad hoc*, this definition is really a specialization of the general theory of [Venkatesh nd], and is indeed very natural. We explain this in more detail in Section 7.

3.3. The conjecture

We now formulate the main conjecture. It asserts that the various operators $T_{q,z}$ all fit together into a single action of $\mathrm{U}_g^\vee$ on the g-part of cohomology. As formulated in [Venkatesh nd], the conjecture is ambiguous up to a rational factor, and we will not attempt to remove this ambiguity here (although our computations suggest that this factor might have a simple description).

Terminology:

- Suppose that $\alpha \in E$ and V is a k-module. For $x, y \in V$, we will write
  $$x = \alpha y, \quad (3\text{–}4)$$
  if we may write $\alpha = A/B$, where $A, B \in \mathcal{O}$ are not both divisible by $\mathfrak{p}$, in such a way that $\bar{B}x = \bar{A}y$. (Here $\bar{A}, \bar{B}$ are the reductions of A, B under $\mathcal{O} \to k$).
  In particular, if V is a k-line, this has the following meaning:
  - if $x = y = 0$, then 3–4 is understood to always be true.
  - Otherwise, we can make sense of $[x : y] \in \mathbb{P}^1(k)$, and 3–4 means that the reduction of $\alpha \in \mathbb{P}^1(E) \to \mathbb{P}^1(k)$ equals $[x : y]$.
- For $h \in H^*(X_{\mathcal{O}[\frac{1}{N}]}, \omega)$, we write $\bar{h}$ for the reduction of h to $H^*(X_k, \omega)$.
- Recall that we defined a reduction map $\theta_q : \mathrm{U}_g \to k\langle 1\rangle$. Also, the pairing between U_g and $\mathrm{U}_g^\vee$, which is perfect after localization at p, descends to a perfect pairing on $\mathrm{U}_g \otimes k$ and $\mathrm{U}_g^\vee \otimes k$. With respect to this pairing, the map θ_q has an adjoint:
  $$\theta_q^\vee : k\langle -1\rangle \to \mathrm{U}_g^\vee \otimes k.$$
  Explicitly for $z \in k\langle -1\rangle$,
  $$\theta_q^\vee(z) = u^* \otimes \frac{\langle z, \theta_q(u)\rangle}{\langle u^*, u\rangle}, \quad (3\text{–}5)$$
  where $u \in \mathrm{U}_g$, $u^* \in \mathrm{U}_g^\vee$ are as defined around (2–5).

Conjecture 3.1. There is an action $\star$ of $\mathrm{U}_g^\vee$ on $H^*(X_{\mathcal{O}[\frac{1}{N}]}, \omega)[g]$, and $\alpha \in E$ such that for every $(\mathfrak{p}, n, q, z)$, with

- $\mathfrak{p}$, a prime of E satisfying the conditions of Section 2.4;
- $n \geqslant 1$ an integer;
- q a Taylor–Wiles prime of level n, in particular $q \equiv 1(p^n)$.
- $z \in (\mathcal{O}/\mathfrak{p}^n)\langle -1\rangle$,

we have the following equality:

$$T_{q,z}\bar{g} = \alpha\overline{(\theta_q^\vee(z)^\sim \star g)}. \tag{3–6}$$

On the right-hand side, $\theta_q^\vee(z)^\sim$ means that we choose an arbitrary lift of $\theta_q^\vee(z) \in \mathrm{U}_g^\vee \otimes k$ to $\mathrm{U}_g^\vee$, and the bar refers to reduction mod $\mathfrak{p}^n$.

In what follows, we will write (3–6) in the abridged form

$$T_{q,z}g \propto \overline{\theta_q^\vee(z) \star g}. \tag{3–7}$$

The meaning here is that equality holds, in the sense described above, for some fixed coefficient of proportionality $\alpha \in E$. (Note that we have suppressed explicit mention of the lift $\theta_q^\vee(z)^\sim$ from the notation; in any case the right-hand side is independent of this choice of lift).

4. Relationship to Galois deformation theory

In this section—*which is not used in the rest of the paper*—we shall sketch a proof that, in the case $n = 1$,

$$\text{vanishing of } T_{q,z}\bar{g} \Longrightarrow \text{vanishing of } \theta_q : \mathrm{U}_g \to k\langle 1\rangle, \tag{4–1}$$

assuming an "$R = T$" theorem for weight one forms at the level of g, as well as further technical conditions. Such a theorem is known in some generality by the work of [Calegari 17].

This result (and its proof) is in line with results and proofs from [Venkatesh nd]. Indeed, our methods would show that (4–1) is an equivalence, *if* we knew an "$R = T$" theorem for weight one forms with (Taylor–Wiles) auxiliary level.

4.1. Setup

Let q be a prime such that the eigenvalues of $\overline{\rho}$ on the Frobenius at q are distinct elements of $\mathbf{F}_\mathfrak{p}$, say α and β. Let $\mathfrak{m}$ be the ideal of the Hecke algebra associated to the Galois representation $\overline{\rho}$.

In addition to the conditions from Section 2.4, we assume that :

(i) $n = 1$ so that $k = \mathcal{O}/\mathfrak{p}$ is a field.
(ii) For each prime ν dividing N, the residual representation $\overline{\rho}$ is of the form $\chi_1 \oplus \chi_2$, where χ_1 is ramified and χ_2 is unramified.
(iii) p does not divide $\nu - 1$, for each ν as above.
(iv) p does not divide $[L : \mathbf{Q}]$, and does not divide the order of the class group of L.
(v) The $\mathfrak{m}$-completion of the space of modular forms at level $\Gamma_1(N)$, with coefficients in $\mathcal{O}$, is free rank 1 over $\mathcal{O}_\mathfrak{p}$. (In particular, there are no congruences modulo $\mathfrak{p}$ between g and other weight one forms, either in characteristic zero or characteristic p).

Let $\mathfrak{m}_\alpha$ be the maximal ideal of the Hecke algebra for $X_{01}(qN)$ obtained by adjoining $U_q - \alpha$ to the ideal $\mathfrak{m}$; similarly, we define $\mathfrak{m}_\beta$. These ideals also have evident analogues where we add $\Gamma_1(q)$ level to X, rather than just $\Gamma_0(q)$ level, and we denote these analogues by the same letters.

Our assumption (v), and the assumption of torsion-freeness from Section 2.4, means that

$$\begin{aligned}\dim H^0(X_k, \omega)_\mathfrak{m} &= \dim H^0(X_{01}(qN)_k, \omega)_{\mathfrak{m}_\alpha} \\ &= \dim H^0(X_{01}(qN)_k, \omega)_{\mathfrak{m}_\beta} = 1,\end{aligned} \tag{4–2}$$

i.e. all three spaces above are k-lines; the same statement is true for $H^1(-)$.

Let g_α and g_β, respectively, span the second and third spaces in the line above. Therefore, $U_q g_\alpha = \alpha g_\alpha$ and $U_q g_\beta = \beta g_\beta$; we normalize these so that $\pi_1^* g = g_\alpha + g_\beta$.

Since the pushforward π_{1*} via the natural projection $\pi_1 : X_{01}(qN) \to X$ induces an isomorphism on each of the U_q-eigenspaces, $g_\alpha \cup \mathfrak{S}_X$ vanishes if and only if $\pi_{1*}(g_\alpha \cup \mathfrak{S}_X)$ vanishes. Observe $(U_q - \beta)\pi_1^* g = (\alpha - \beta) g_\alpha$. We are assuming $\alpha \neq \beta$ and therefore,

$$g_\alpha \cup \mathfrak{S}_X = 0 \iff \pi_{1*}((U_q - \beta)\pi_1^* g \cup \mathfrak{S}_X) = 0. \tag{4–3}$$

Now, $\pi_{1*}(\pi_1^* g \cup \mathfrak{S}_X) = g \cup (\pi_{1*}\mathfrak{S}_X) = 0$ and $\pi_{1*}\mathfrak{S}_X$ is trivial.

Lemma 4.1. *The pushforward of $\mathfrak{S}_X$ by the natural projection $\pi : X_{01}(qN) \to X$ is trivial.*

Proof. The existence of the trace map [Artin et al. 72, Expose 17, Section 6.2] gives a map $\pi_*(\mathbb{Z}/p) \to \mathbb{Z}/p$ of étale sheaves, compatible with the usual trace $\pi_*\mathbb{G}_a \to \mathbb{G}_a$. For this reason, it is sufficient to show that the (trace-induced) map

$$H^1_{\text{et}}(X_{01}(qN)_k, (\mathbb{Z}/q)^*_p) \to H^1_{\text{et}}(X_k, (\mathbb{Z}/q)^*_p)$$

pushes the Shimura class forward to the trivial class.

If ι is the inclusion of an open curve into a complete curve induces then ι^* is an injection on H^1. Therefore, it suffices to show a similar statement for the open modular curves; restricted to these, the map π is étale.

Define finite groups

$$G = \mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z}) \supset B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Then $X_{01}(qN)$ and X_k are quotients of a suitable modular curve by B and G, respectively. This allows to reduce to verifying the triviality of the transfer in group cohomology, from B to G, of $\alpha \in H^1(B, (\mathbb{Z}/q)^*_p)$, defined via

$$\alpha : \begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \mapsto (a/d) \in (\mathbb{Z}/q)^*_p.$$

This is a straightforward computation. □

Continuing from (4–3), we find

$$g_\alpha \cup \mathfrak{S}_X = 0 \iff \pi_{1*}(U_q\pi_1^* g \cup \mathfrak{S}_X) = 0. \tag{4–4}$$

The final expression can be verified to be an invertible multiple of $T_{q,z}g$ for some non-vanishing $z \in \mathcal{O}/\mathfrak{p}\langle -1\rangle$. Therefore,

$$g_\alpha \cup \mathfrak{S}_X = 0 \iff T_{q,z}\bar{g} = 0. \tag{4–5}$$

Write $\Delta = (\mathbb{Z}/q)^*_p$; since we are assuming that $n = 1$, the group Δ is cyclic of order p and we have an isomorphism $k[\Delta] \simeq k[T]/T^p$, whose inverse sends T to $\delta - 1$, for any generator δ of Δ. Let $X_1(Nq)^\Delta$ be the subcovering of $X_1(Nq) \to X_{01}(qN)$ that corresponds to the quotient $(\mathbb{Z}/q)^* \to (\mathbb{Z}/q)^*_p$ of deck transformation groups.

Lemma 4.2. *The cup product $\cup\mathfrak{S}_X$ is non-zero as a map on $H^*(X_{01}(qN)_k, \omega)_{\mathfrak{m}_\alpha}$ if and only if*

$$\dim H^0(X_1(Nq)^\Delta_k, \omega)_{\mathfrak{m}_\alpha} = 1. \tag{4–6}$$

The usual Taylor–Wiles method, for classical modular forms on GL_2, relies crucially on producing "more" modular forms when adding "$\Gamma_1(q)^\Delta$ level" at auxiliary primes q. Thus, the Lemma says: the derived Hecke operator is non-trivial precisely when this *fails*, a failure that is rectified in the Calegari–Geraghty approach [Calegari and Geraghty nd].

Proof. By the methods of [Calegari and Geraghty nd], we may obtain a complex C of free $k[\Delta]$-modules (with degree-decreasing differential) together with isomorphisms:

$$H^i\mathrm{Hom}_{k[\Delta]}(C, k) \simeq H^i(X_{01}(qN)_k, \omega)_{\mathfrak{m}_\alpha}. \tag{4–7}$$

With reference to the latter isomorphism, cup product with $\mathfrak{S}_X$ on the right is represented by the natural action of a non-trivial class in $\mathrm{Ext}^1_{k[\Delta]}(k, k)$ on the left-hand side. (Note that $H^i\mathrm{Hom}_{k[\Delta]}(C, k)$ is identified with homomorphisms from C to $k[i]$ in the derived category).

Replacing C by a minimal free resolution, we may assume that C is the complex given by

$$k[\Delta] \overset{A}{\leftarrow} k[\Delta],$$

where $A \in k[\Delta]$ belongs to the augmentation ideal. Under the identification of $k[\Delta]$ with $k[T]/T^p$, the element A corresponds to an invertible multiple of T^i, for some $0 \le i \le p-1$, and then (4–7) implies

$$\dim H^0(X_1(Nq)^\Delta_k, \omega)_{\mathfrak{m}_\alpha} = i.$$

We shall show that cup product with $\mathfrak{S}_X$ is non-trivial if and only if $i = 1$. To compute the action of $\mathrm{Ext}^1_{k[\Delta]}(k, k)$, we may consider the following diagram:

$$\begin{array}{lccccccccc}
C & 0 & \leftarrow & k[\Delta] & \xleftarrow{T^i} & k[\Delta] & \leftarrow & 0 & & \\
 & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle T^{i-1}} & & \downarrow & & \\
k & 0 & \leftarrow & k[\Delta] & \xleftarrow{T} & k[\Delta] & \xleftarrow{T^{p-1}} & k[\Delta] & \xleftarrow{T} & k[\Delta] \leftarrow \dots \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle T^{p-2}} & & \downarrow{\scriptstyle 1} \\
k[1] & 0 & \leftarrow & 0 & \leftarrow & k[\Delta] & \xleftarrow{T} & k[\Delta] & \xleftarrow{T^{p-1}} & k[\Delta] \leftarrow \dots
\end{array} \tag{4-8}$$

The horizontal complexes are, respectively, C, a projective resolution of k, and a projective resolution of $k[1]$. Continuing to take Hom in the derived category of $k[\Delta]$-modules, the top vertical map of complexes represents a generator for $\mathrm{Hom}(C, k)$ and the bottom vertical map of complexes represents a non-trivial class in $\mathrm{Ext}^1_{k[\Delta]}(k, k) = \mathrm{Hom}(k, k[1])$. Therefore, the composite map in $\mathrm{Hom}(C, k[1])$ is represented by the diagram

$$\begin{array}{ccccccc}
0 & \leftarrow & k[\Delta] & \xleftarrow{T^i} & k[\Delta] & \leftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle T^{i-1}} & & \downarrow \\
0 & \leftarrow & 0 & \leftarrow & k[\Delta] & \xleftarrow{T} & k[\Delta]
\end{array} \tag{4-9}$$

This is nullhomotopic exactly when T^{i-1} is divisible by T, i.e. $i \ge 2$. □

Taking (4–5) together with the Lemma, we see

$$T_{q,z}\bar{g} \neq 0 \iff \dim H^0(X_1(Nq)^\Delta_k, \omega)_{\mathfrak{m}_\alpha} = 1.$$

Consider the map

$$f : \mathrm{R}' \otimes k \to \mathrm{R} \otimes k,$$

where R (resp. R′) are the weight one, determinant χ, deformation rings for $\overline{\rho}$ at level $\Gamma_1(N)$ and with level $\Gamma_1(Nq)$, respectively. The local conditions $\mathcal{L}$ for R and $\mathcal{L}'$ for R' are as follows:

- At p, we require that deformation remains unramified.
- At q, we impose unramified for R and no condition for R'.
- For primes v dividing N, we do not need to impose any condition: We have assumed that $\overline{\rho}$ is a direct sum $\chi_1 \oplus \chi_2$ of two characters, with χ_1 ramified and χ_2 unramified. In particular, $H^1(\mathbf{Q}_v, \mathrm{Ad}^0\overline{\rho})$ is one-dimensional, corresponding to deforming $\chi_1 \leftarrow \chi_1\psi$, $\chi_2 \leftarrow \chi_2\psi^{-1}$ for a character ψ with trivial reduction. In [Calegari 17], the assumption is imposed that in fact $\chi_2\psi^{-1}$ remains unramified, but we do not need to explicitly impose this because we

assumed that p is relatively prime to $\nu - 1$—thus, the character ψ is automatically unramified at ν. In particular, we have automatically

$$H^1(\mathbf{Q}_\nu, \mathrm{Ad}^0\overline{\rho}) = H^1_{\mathrm{ur}}(\mathbf{Q}_\nu, \mathrm{Ad}^0\overline{\rho}),$$

where we recall that for a module M under the Galois group of $\mathbf{Q}_\ell$, the "unramified" classes $H^1_{\mathrm{ur}} \subset H^1$ are defined to be those that arise from inflation from the Galois cohomology of $\mathbf{F}_\ell$ acting on inertial invariants on M.

Assuming an $R = T$ theorem for g, we have $R \otimes k = k$. The map on tangent spaces induced by f, call it f^*, fits into the following diagram, with reference to the usual identification of tangent spaces with Galois cohomology:

$$H^1_{\mathcal{L}}(\mathbf{Q}, \mathrm{Ad}^0\overline{\rho}) \overset{f^*}{\hookrightarrow} H^1_{\mathcal{L}'}(\mathbf{Q}, \mathrm{Ad}^0\overline{\rho}) \to \frac{H^1(\mathbb{Q}_q, \mathrm{Ad}^0\overline{\rho})}{H^1(\mathbb{Z}_q, \mathrm{Ad}^0\overline{\rho})} \overset{j}{\to} H^2_{\mathcal{L}}(\mathbf{Q}, \mathrm{Ad}^0\overline{\rho}) \to H^2_{\mathcal{L}'}(\mathbf{Q}, \mathrm{Ad}^0\overline{\rho}),$$

f^* is surjective exactly when j is injective. Since the middle group in the exact sequence is one-dimensional, injectivity of j is the same as non-vanishing of j. Under Tate global duality, the map j is dual to

$$H^1_{\mathcal{L}'^\vee}(\mathbf{Q}, \mathrm{Ad}^*\overline{\rho}(1)) \overset{j^\vee}{\to} H^1(\mathbf{F}_q, \mathrm{Ad}^*\overline{\rho}(1)), \qquad (4\text{–}10)$$

where $\mathcal{L}'^\vee$ is the dual condition to $\mathcal{L}'$: it refers to classes that are unramified at primes not dividing pN, unramified (equivalently: trivial) at primes dividing N, and at p belong to the Bloch-Kato f-cohomology (a more concrete description is given below).

We will show in the next subsection that:

$$\text{the map } j^\vee \text{ vanishes exactly when } \theta_q : \mathrm{U}_g \to k\langle 1\rangle \text{ does.} \qquad (4\text{–}11)$$

Therefore, the non-vanishing of θ_q implies the injectivity of j, which implies the surjectivity of f^*, which implies $R' \otimes k = k$, which implies (4–6) by a multiplicity one argument. Then (4–5) and Lemma 4.2 show that $T_{q,z}\overline{g} \neq 0$ as desired.

That concludes our proof for 4–1; note finally that if we had a theorem $R' = T'$ all this reasoning would be reversible and we get an equivalence in (4–1).

4.2. Relation of U_g to Galois cohomology

To conclude we must relate $j^\vee$ and θ_q, and thereby prove (4–11).

As in (2–2), ρ is a representation into $\mathrm{GL}_2(\mathcal{O})$; let $\rho_{\mathfrak{p}}$ be the same representation, but now considered as valued in $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$; thus,

$$\mathrm{Ad}^*\rho_{\mathfrak{p}} = \mathrm{Ad}^*\rho \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}.$$

We write $(\mathrm{U}_g)_{\mathfrak{p}}$ for $\mathrm{U}_g \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$.

Consider

$$H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)),$$

where the subscript ur means that we consider classes that are unramified at primes away from p, and, at p, belong to the Bloch-Kato f-space. (What this means is made explicit in the computation of $H^1_{\mathrm{ur}}(L, \mathcal{O}_{\mathfrak{p}}(1))$ in the diagram below).

Restriction to L gives horizontal maps in the following diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) & \overset{i}{\longrightarrow} & [\mathrm{Ad}^*\rho_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} H^1_{\mathrm{ur}}(L, \mathcal{O}_{\mathfrak{p}}(1))]^{G_{L/\mathbf{Q}}} = (\mathrm{U}_g)_{\mathfrak{p}} \\
\downarrow & & \downarrow \\
H^1(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) & \overset{j}{\longrightarrow} & [\mathrm{Ad}^*\rho_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} H^1(L, \mathcal{O}_{\mathfrak{p}}(1))]^{G_{L/\mathbf{Q}}} \\
\downarrow & & \downarrow \\
\bigoplus_v \frac{H^1(\mathbf{Q}_v, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}{H^1_{\mathrm{ur}}(\mathbf{Q}_v, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))} & \overset{k}{\longrightarrow} & \left[\bigoplus_w \frac{H^1(L_w, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}{H^1_{\mathrm{ur}}(L_w, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}\right]^{G_{L/\mathbf{Q}}}.
\end{array}$$

The vertical columns are exact at top and middle, and, in the bottom row, the sum is taken over all places v of $\mathbf{Q}$, and then over all places w of L.

Lemma 4.3. *i induces an isomorphism $H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) \simeq (\mathrm{U}_g)_{\mathfrak{p}}$. Also, as long as the class group of L is prime to p, the reduction modulo $\mathfrak{p}$ map $H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) \to H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\overline{\rho}(1))$ is surjective.*

Proof. The map j is an isomorphism by considering the inflation-restriction sequence: the group $G_{L/\mathbf{Q}}$ has order prime to p.

This means i is injective. i will be surjective if k is injective. In fact, for a place $\mathfrak{q}$ of L above v, the map

$$\frac{H^1(\mathbf{Q}_v, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}{H^1_{\mathrm{ur}}(\mathbf{Q}_v, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))} \to \left[\frac{H^1(L_{\mathfrak{q}}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}{H^1_{\mathrm{ur}}(L_{\mathfrak{q}}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1))}\right] \qquad (4\text{–}12)$$

is split, up to multiplication by $[L_{\mathfrak{q}} : \mathbf{Q}_v]$, by corestriction, and $[L_{\mathfrak{q}} : \mathbf{Q}_v]$ is invertible on $\mathcal{O}_{\mathfrak{p}}$.

This proves the first assertion about i. For the second assertion, note that the assumption about class groups means that $H^1_{\mathrm{ur}}(L, \mathbb{F}_{\mathfrak{p}}(1))$ coincides with $\mathrm{U}_L \otimes \mathbb{F}_{\mathfrak{p}}$. The same analysis as above means that the rank of $H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\overline{\rho}(1))$ over $\mathbb{F}_{\mathfrak{p}}$ is bounded above by the dimension of

$$\left(\mathrm{Ad}^*\overline{\rho} \otimes \mathrm{U}_L\right)^{G_{L/\mathbf{Q}}},$$

and (again because $G_{L/\mathbf{Q}}$ has no Galois cohomology in characteristic p) this dimension coincides with the $\mathcal{O}_{\mathfrak{p}}$-rank of $\mathrm{U}_g \otimes \mathcal{O}_{\mathfrak{p}}$ (which is exactly 1). The surjectivity now follows. □

Now, under the identification $i : H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) \simeq (\mathrm{U}_g)_{\mathfrak{p}}$, the composite

$$H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\rho_{\mathfrak{p}}(1)) \to H^1_{\mathrm{ur}}(\mathbf{Q}, \mathrm{Ad}^*\overline{\rho}(1)) \xrightarrow{j^\vee} H^1(\mathbf{F}_q, \mathrm{Ad}^*\overline{\rho}(1)) \xrightarrow{\sim} H^1(\mathbf{F}_q, \mathbb{F}_{\mathfrak{p}}(1)) \simeq \mathbf{F}_q^* \otimes (\mathcal{O}/\mathfrak{p})$$

is identified with the map θ_q described in (2–10). Here we have made use of a map $\mathrm{Ad}^*\overline{\rho} \to \mathbb{F}_{\mathfrak{p}}$, which comes from pairing with the element defined in (2–8). In particular, θ_q vanishes if and only if $j^\vee$ does, as required.

5. Explication

Our main Conjecture 3.1, as formulated, involves a cup product in coherent cohomology on the special fiber of a modular curve. We want to translate it to a readily computable form, i.e. one that can be carried out just by using manipulations with q-series. We will achieve this in this section, at least in the case $n = 1$ and under modest assumptions on q, and then test the conjecture numerically.

5.1. *Pairing with g'*

Recall (Section 2.3) that we have fixed another weight one modular form g' that is contragredient to g. To extract numbers from the Conjecture, we pair both sides of (3–6) with g', using the residue pairing (Section 2.2). Pairing 3–6 with g', and using $\theta_q^\vee(z) = u^* \otimes \frac{\langle z, \theta_q(u)\rangle}{\langle u^*, u\rangle}$ from (3–5), we arrive at:

$$[T_{q,z}\bar{g}, \bar{g}']_{\mathrm{res},k} = \langle z, \theta_q(u)\rangle \cdot \overline{\left[\frac{\alpha[\theta_q^\vee(u) \star g, g']_{\mathrm{res},\mathcal{O}}}{\langle u, u^*\rangle}\right]}, \tag{5–1}$$

where both sides lie in k; and we recall again that we have written $\bar{g}$ for the reduction of g to a modular form with k coefficients.

Now the square-bracketed quantity on the right-hand side is an element of E, integral at $\mathfrak{p}$, and independent of choice of $(\mathfrak{p}, n, q, z)$. We abridge (5–1) to

$$[T_{q,z}\bar{g}, \bar{g}']_{\mathrm{res},k} \propto \langle z, \theta_q(u)\rangle.$$

This should hold true for any $(\mathfrak{p}, n, q, z)$.

Unwinding the definition of the derived Hecke operator,

$$[T_{q,z}\bar{g}, \bar{g}']_{\mathrm{res},k} = [\pi_1^*\bar{g} \cup z\mathfrak{S}_X, \pi_2^*\overline{g'}]_{\mathrm{res},k}, \tag{5–2}$$

where the residue pairing is now taken on $X_{01}(qN)_k$, π_1, π_2 are the two projections $X_{01}(qN) \to X$, and

$$z\mathfrak{S}_X \in H^1(X_{01}(qN)_k, \mathcal{O}).$$

(Recall that $\mathfrak{S}_X \in H^1(X_{01}(qN)_k, \mathcal{O}\langle -1\rangle)$, so its product with $z \in k\langle 1\rangle$ lies in the right-hand group above). To simplify notation, define the weight two form

$$G = \pi_1^* g \cdot \pi_2^* g' \in H^0(X_{01}(qN)_k, \Omega^1). \tag{5–3}$$

In terms of classical modular forms, G would be the form "$z \mapsto g(z)g'(qz)$." Then the right-hand side of 5–2 is simply the (Serre duality) pairing of $G \in H^0(\Omega^1)$ and $z\mathfrak{S}_X \in H^1(\mathcal{O})$ in the coherent cohomology of $X_{01}(qN)$. Therefore, the conjecture implies that $\langle z\mathfrak{S}_X, G\rangle \propto \langle \theta_q(u), z\rangle$; and here we may as well cancel the zs from both sides:

$$\langle \mathfrak{S}_X, G\rangle \propto \theta_q(u). \tag{5–4}$$

Here, both sides lie in $k\langle 1\rangle$, that is to say, in $(\mathbb{Z}/q)^* \otimes k$.

Now the class $\mathfrak{S}_X$ is pulled back from a class $\mathfrak{S}$ on $X_0(q)$, and correspondingly the pairing on the left-hand side can be pushed down to $X_0(q)$. Writing

$$G^{\mathrm{proj}} = \text{ projection of } G \text{ to level } q \in H^0(X_0(q)_k, \Omega^1),$$

we have $\langle \mathfrak{S}_X, G\rangle = \langle \mathfrak{S}, G^{\mathrm{proj}}\rangle$.

Thus, our conjecture implies that

$$\langle \mathfrak{S}, G^{\mathrm{proj}}\rangle \propto \theta_q(u), \text{ equality in } (\mathbb{Z}/q)^* \otimes k, \tag{5–5}$$

where we recall that:

- $\mathfrak{S} \in H^1(X_0(q)_k, \mathcal{O} \otimes (\mathbb{Z}/q)^*)$ is constructed from the covering $X_1(q) \to X_0(q)$;
- $G^{\mathrm{proj}} \in H^0(X_0(q)_k, \Omega^1)$ is the pushforward of the form "$z \mapsto g(z)g'(qz)$" from level $X_{01}(qN)$ to level $X_0(q)$; it is a weight two cusp form.
- $\langle -, -\rangle$ is the pairing of Serre duality.
- The symbol $\propto$ is interpreted as in (3–7).

5.2. *Localization at the Eisenstein ideal*

To translate (5–5) to a computable form, we will use computations of Merel and Mazur. Let

$$\mathrm{E} \in H^0(X_0(q)_k, \Omega^1)$$

be the "Eisenstein" cusp form with k coefficients, in other words, the unique element whose q-expansion coincides with the reduction modulo p^n of the weight two Eisenstein series; the condition that $q \equiv 1$ modulo p^n means that this weight two Eisenstein series indeed has cuspidal reduction in k. The pairing

$$\langle \mathfrak{S}, \mathrm{E}\rangle \in (\mathbb{Z}/q)_p^*$$

was considered by ([Mazur 77, p. 103], discussion of the element u) and was computed in a remarkable paper of [Merel 96]. We will carefully translate Merel's computation into our setting in the next section; unfortunately, in doing so, we will have to impose the restriction $n = 1$, i.e. we can only compute things modulo p and not higher powers of p.

Lemma 5.1. *(Merel; see Section 6 for details of the translation from Merel's framework to this one).*

$$\langle \mathfrak{S}, \mathrm{E}\rangle = \varpi_{\mathrm{Merel}} \text{ mod } p, \tag{5–6}$$

where mod p means that the two sides have the same projection to $\mathbb{F}_p\langle 1\rangle$.[2]

Here, the Merel unit $\varpi_{\mathrm{Merel}} \in (\mathbb{Z}/q)^*$ is the element

$$\varpi_{\mathrm{Merel}} = \zeta^2 \prod_{i=1}^{(q-1)/2} i^{-8i}, \quad \zeta = \begin{cases} 1, q \equiv 2(3), \\ 2^{(q-1)/3}, \ else \end{cases}. \tag{5–7}$$

In the remainder of this section, we will compute $\langle \mathfrak{S}, G^{\mathrm{proj}}\rangle$ (the left-hand side of (5–5)) using Lemma 5.1.

Let $\mathbb{T}$ be the Hecke algebra for cusp froms on $X_0(q)_{\mathbb{Z}_p}$, i.e. the algebra of endomorphisms of $S_2(q) := H^0(X_0(q)_{\mathbb{Z}_p}, \Omega^1)$ generated by T_ℓ for all $\ell \neq q$. Let $\mathfrak{I} \leqslant \mathbb{T}$ be the Eisenstein ideal, i.e. the kernel of the character

$$\mathbb{T} \to \mathbb{Z}/p\mathbb{Z}, \quad T_\ell \mapsto (\ell + 1),$$

by which $\mathbb{T}$ acts on the modulo p reduction of E. In particular it's a maximal ideal.

Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be all other maximal ideals of $\mathbb{T}$. Then the natural map

$$\mathbb{T} \longrightarrow \mathbb{T}_{\mathfrak{I}} \times \prod_{s=1}^{r} \mathbb{T}_{\mathfrak{m}_s}$$

is an isomorphism (here, $\mathbb{T}_{\mathfrak{I}}$ means the completion, and similarly for $\mathfrak{m}_s$). Let $e_{\mathfrak{I}}$ be the idempotent of $\mathbb{T}$ corresponding to the first factor; the splitting

$$1 = e_{\mathfrak{I}} + \underbrace{(1 - e_{\mathfrak{I}})}_{=e'_{\mathfrak{I}}}$$

gives rise to a splitting

$$S_2(q) = S_2(q)_{\mathfrak{I}} \oplus S_2(q)'_{\mathfrak{I}}, \tag{5–8}$$

where $S_2(q)_{\mathfrak{I}}$ is the image of the idempotent $e_{\mathfrak{I}}$, and the complementary subspace is the image of $1 - e_{\mathfrak{I}}$. Therefore, if $T \in \mathfrak{I}$ is chosen so that $T \notin \bigcup_{i=1}^{s} \mathfrak{m}_i$, then T acts invertibly on the second factor.

Decompose G^{proj} as

$$G^{\mathrm{proj}} = G^{\mathrm{proj}}_{\mathfrak{I}} + (G^{\mathrm{proj}})'$$

according to the splitting above. The Shimura class $\mathfrak{S}$ is annihilated by $\mathfrak{I}$ (see for example [Mazur 77], Lemma 18.7). Choose as above $T \in \mathfrak{I}$ that acts invertibly on the second factor of (5–8). We may write

$$\langle \mathfrak{S}, (G^{\mathrm{proj}})'\rangle = \langle \mathfrak{S}, TT^{-1}(G^{\mathrm{proj}})'\rangle = \langle T\mathfrak{S}, T^{-1}(G^{\mathrm{proj}})'\rangle = 0,$$

and so

$$\langle \mathfrak{S}, G^{\mathrm{proj}}\rangle = \langle \mathfrak{S}, G^{\mathrm{proj}}_{\mathfrak{I}}\rangle,$$

where as before the pairings come from Serre duality.

Next, Mazur proves [Mazur 77, Proposition 19.2] that

$$\varpi_{\mathrm{Merel}} \text{ is non-zero} \textit{modulo } p \iff S_2(q)_{\mathfrak{I}} \text{ is of rank 1 over } \mathbb{Z}_p. \tag{5–9}$$

We will complete our computation only in this case.[3] Since E is annihilated by $\mathfrak{I}$, we have in fact $\mathrm{E} \in S_2(q)_{\mathfrak{I}}$, and since the first Fourier coefficient of E is 1, we have (under the assumption of (5–9)) $S_2(q)_{\mathfrak{I}} = \mathbb{Z}_p.E$. Thus, after extending scalars to $\mathcal{O}$, we find

$$G^{\mathrm{proj}}_{\mathfrak{I}} = a_1(G^{\mathrm{proj}}_{\mathfrak{I}}) \cdot \mathrm{E},$$

where $a_1(G^{\mathrm{proj}}_{\mathfrak{I}}) \in \mathcal{O} \otimes \mathbb{Z}_p$ denotes the first coefficient in the q-expansion. Putting this together with our prior discussion, we have shown the following proposition.

Proposition 5.1. *Conjecture 3.1 implies that there exists $\alpha \in E$ such that*

$$a_1\left(G^{\mathrm{proj}}_{\mathfrak{I}}\right) \otimes (\varpi_{\mathrm{Merel}})_p \equiv \alpha \cdot \theta_q(u) \text{ modulo } p \cdot (\mathcal{O} \otimes (\mathbb{Z}/q)^*_p) \tag{5–10}$$

for any $(\mathfrak{p}, n, q)$ as in Section 2.4 with the additional property that $(\varpi_{\mathrm{Merel}}) \in (\mathbb{Z}/q)^*_p$ is non-trivial modulo p.[4] *Other conventions are as follows:*

- *$\varpi_{\mathrm{Merel}} \in (\mathbb{Z}/q)^*_p$ is the Merel unit, see (5–6).*
- *$a_1(G^{\mathrm{proj}}_{\mathfrak{I}}) \in \mathcal{O} \otimes \mathbb{Z}_p$ is the first Fourier coefficient of $G = (\pi_1^* g)(\pi_2^* g')$, after taking projection G^{proj} to level $X_0(q)$ and then projection $G^{\mathrm{proj}}_{\mathfrak{I}}$ to the localization at the Eisenstein ideal.*
- *$\theta_q(u) \in k\langle 1\rangle = \mathcal{O} \otimes (\mathbb{Z}/q)^*_p$ is the reduction of the Stark unit.*

5.3. Some philosophical worries

Let us take to examine some consequences of an inadequacy of our conjecture, namely, it is only formulated "up to E^*."

For each $(\mathfrak{p}, n, q)$ as in Section 2.4, we can compute both $a_1(G^{\mathrm{proj}}_{\mathfrak{I}}) \otimes (\varpi_{\mathrm{Merel}})_p$ and $\theta_q(u)$ and compare them. Let us also restrict to $(\mathfrak{p}, n, q)$ for which $\theta_q(u) \neq 0$; there are infinitely many such $\mathfrak{p}$. Therefore, (5–10) specifies the reduction $\alpha \in E$ to $\mathbb{P}^1(\mathbf{F}_{\mathfrak{p}})$, for an infinite collection of $\mathfrak{p}$. This uniquely specifies α if it exists.

The conjecture is numerically falsifiable to some extent. For example, if we find two different pairs $(\mathfrak{p}, n, q)$

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[2] It seems likely that the two sides are actually equal in $(\mathbb{Z}/q)^*_p$ but we do not prove this.

[3] Recently, Lecouturier has proposed a very interesting generalization of the conjectural equality (5–10) to the case when ϖ_{Merel} is zero modulo p and has verified it numerically in some cases.

[4] To be absolutely clear, we write out the meaning of this statement. We understand

$$L := a_1(G^{\mathrm{proj}}_{\mathfrak{I}}) \otimes (\varpi_{\mathrm{Merel}})_p, \ R := \theta_q(u)$$

as elements of $\mathcal{O} \otimes (\mathbb{Z}/q)^*_p$; and the statement above means that if we reduce $\bar{L}, \bar{R}$ to $\mathcal{O}/p \otimes (\mathbb{Z}/q)^*_p$, then $\bar{L} = \alpha\bar{R}$, in the sense of (3–4).

and $(\mathfrak{p}, n', q')$ for which the predicted reductions of α mod $\mathfrak{p}$ differ, this clearly contradicts the conjecture. Indeed, the fact that this did not occur in our numerical computations was very encouraging to us.

However, if this type of clash does not occur, no amount of computation can falsify the conjecture: we can, of course, produce an $\alpha \in E$ with any specified reduction at any number of places. Nonetheless, this proves to be largely a theoretical worry. In our examples, we shall find an α of very low height for which (5–10) holds for many $(\mathfrak{p}, n, q, z)$. Our sense is that this should be taken as a satisfactory indication that the Conjecture, or something very close to it at least, is valid.

As a final excuse, we may note that the conjectures about special values of L-functions were initially phrased with a $\mathbf{Q}^*$ ambiguity that is similarly unfalsifiable.

Eventually, we hope that these issues will be solved by formulating an integral form of the conjecture; this could perhaps be done using the theory of derived deformation rings.

5.4. Forms associated to cubic fields

We now make the foregoing discussion even more explicit for the form g associated to a cubic field K; write L for the Galois closure of K. (This will coincide with our previously defined L in a moment).

Such a field K defines a representation $\mathrm{Gal}(L/\mathbf{Q}) \to S_3$; if we regard S_3 as acting on $M = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : \sum x_i = 0\}$ by permuting the coordinate axes, we may regard ρ as a rank 2 Galois representation:

$$\rho : \mathrm{Gal}_{\mathbf{Q}} \twoheadrightarrow S_3 \to \mathrm{GL}_2(M). \tag{5–11}$$

Under the representation (5–11), there is a basis for M such that the transposition $\sigma = (12) \in S_3$ is sent to $S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, whereas a 3-cycle $\tau = (123) \in S_3$ is sent to $T := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. We may set things up so that the fixed field of $(12) \in S_3$ is equal to K.

In our previous notation, take

- L as above, namely, the Galois closure of the cubic field K.
- $E = \mathbf{Q}$ and $\mathcal{O} = \mathbb{Z}$.
- $\mathfrak{p} = p \geqslant 5$ to be a rational prime of $\mathbf{Q}$.
- $n = 1$ (thus we work only modulo p rather than p^n).
- $q \equiv 1(p)$ to be a prime such that the qth Hecke eigenvalue $a_q(g) = 0$. In this case, the Frobenius is a transposition[5] in S_3. Thus q is a Taylor–Wiles prime with eigenvalues $(1, -1)$.
- Therefore, in this case, $k\langle 1\rangle = \mathbb{F}_p\langle 1\rangle$ is just the unique quotient of $(\mathbb{Z}/q)^*$ of order p.
- We also fix a prime $\mathfrak{q}_0$ of L over q such that the image of the Frobenius for $\mathfrak{q}_0$ is equal to S. In particular, this fixes K, so the prime $\tilde{q}$ of K below $\mathfrak{q}_0$ is of degree 1 over q.

Lemma 5.2. *Consider the isomorphism $\mathrm{U}_g \simeq \mathrm{Hom}_{G_{L/\mathbf{Q}}}(\mathrm{Ad}^0\rho, \mathrm{U}_L)$ of (2–4). (Recall that U_L is the unit group of L). Then computing the image of $S \in \mathrm{Ad}^0\rho$ gives rise to an isomorphism*

$$\mathrm{U}_g \otimes \mathbb{Z}\left[\frac{1}{6}\right] \simeq \mathcal{O}_K^{(1)} \otimes \mathbb{Z}\left[\frac{1}{6}\right], \tag{5–12}$$

where $\mathcal{O}_K^{(1)}$ is the group of norm one units of K.

Moreover, for $p \geq 5$ the reduction map $\theta_q : \mathrm{U}_g \to \mathbb{F}_p\langle 1\rangle$ described in (2–10) becomes identified with the composite

$$\mathcal{O}_K^* \to (\mathcal{O}_K/\tilde{q})^* = (\mathbb{Z}/q)^* \to \mathbb{F}_p\langle 1\rangle,$$

where $\tilde{q}$ is the unique degree one prime of K above p.

Proof. Indeed we may split

$$\mathrm{Ad}^0\, \rho \otimes \mathbb{Z}\left[\frac{1}{6}\right] = \mathrm{Hom}^0(M, M) \otimes \mathbb{Z}\left[\frac{1}{6}\right] = \mathbb{Z}\left[\frac{1}{6}\right] e \oplus \mathsf{W},$$

where e is the projection of $T \in \mathrm{Hom}(M, M)$ to the trace zero subspace Hom^0, and W is the $\mathbb{Z}[\frac{1}{6}]$-submodule of $\mathrm{Hom}(M, M) \otimes \mathbb{Z}[\frac{1}{6}]$ spanned by the images of $(12), (13), (23)$ under ρ.

Therefore S_3 acts on e by the sign character, whereas for any S_3-module V, the space of homomorphisms $\mathrm{Hom}_{S_3}(\mathsf{W}, V)$ is identified with the subspace of

$$v \in V^{(12)} = (12)\text{-fixed vectors in } V$$

such that $v + (123)v + (132)v = 0$.

Using the definition of U_g and the splitting above, we find that evaluation at S induces an isomorphism

$$\mathrm{U}_g \otimes \mathbb{Z}\left[\frac{1}{6}\right] \simeq \left(\mathrm{U}_L^{(\mathrm{sign})} \oplus \mathcal{O}_K^{(1)}\right) \otimes \mathbb{Z}\left[\frac{1}{6}\right].$$

The first factor corresponds to units in the imaginary quadratic field $\mathbf{Q}(\sqrt{\mathrm{disc}(L)})$, and is thus trivial upon inverting 6. This proves (5–12).

Now let u be a norm one unit in K; we may now identify it with an element of $\mathrm{U}_g \otimes \mathbb{Z}[1/6]$. We will compute its image under the reduction map. Let $\mathbf{u} \in \mathrm{Hom}(\mathrm{Ad}^0\overline{\rho}, \mathrm{U}_L)$ be the element associated to u. By definition $\mathbf{u}(S) = u$. Let $\mathfrak{q}_0$ be the prime of L above $\tilde{q}$, as before; to compute $\theta_q(u)$ we must, by definition, compute the image of $\mathbf{u}$ under the sequence (2–10):

$$\mathrm{Hom}_{G_{L/\mathbf{Q}}}\left(\mathrm{Ad}^0\overline{\rho}, \prod_{\mathfrak{q}|q} \mathbf{F}_{\mathfrak{q}}^*\right) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Ad}^0\overline{\rho}, \mathbf{F}_{\mathfrak{q}_0}^*)^{D_{\mathfrak{q}_0}} \xrightarrow{\mathsf{e}_q} \mathbb{F}_p\langle 1\rangle,$$

where we phrased the previous definition dually. The element e_q from (2–8) is identified here with S, so that the

[5] The primes q for which $\rho(\mathrm{Frob}_q)$ is a 3-cycle also are Taylor–Wiles primes, but it is then easy to see that $T_{q,z}g = 0$ for such q. To verify this, one can use the fact—notation as in (5–3—that the Atkin–Lehner involution at q for $X_{01}(qN)$ acts by -1 on $\mathfrak{S}_X$, but it acts by $\chi(q)$ on G, where χ is the quadratic Nebentypus character for g.

Table 1. Data for the weight one form associated to the cubic field with discriminant −23; in all cases the ratio is −1/72 modulo p. All allowable $p \le 100$ and $q \le 150$ shown.

p	q	$\log(\bar{u})/\log(\varpi_{\text{Merel}}) \in \mathbb{Z}/p$	$\eta \in \mathbb{Z}/p$	Ratio
5	11	3 (5)	4(5)	2(5)
5	61	1(5)	3(5)	2(5)
7	43	3(7)	1(7)	3(7)
7	113	1(7)	5(7)	3(7)
11	67	6(11)	8(11)	−2(11)
11	89	1(11)	5(11)	−2(11)
13	53	6(13)	10(13)	−2(13)
13	79	5(13)	4(13)	−2(13)
17	137	5(17)	14(17)	4(17)
37	149	20(37)	3(37)	19(37)
41	83	12(41)	38(41)	−4(41)
53	107	30(53)	13(53)	39(53)

Table 2. Data for the weight one form associated to the cubic field with discriminant −31; in all cases the ratio is 1/72 modulo p. All allowable $p \le 100$ and $q \le 150$ shown – means undefined.

p	q	$\log(\bar{u}/\varpi_{\text{Merel}}) \in \mathbb{Z}/p$	$\eta \in \mathbb{Z}/p$	Ratio
5	11	2(5)	4(5)	3(5)
5	61	2(5)	4(5	3(5)
7	29	1(7)	2(7)	4(7)
7	43	4(7)	1(7)	4(7)
7	127	∞	—	—
11	23	3(11)	7(11)	2(11)
11	89	7(11)	9(11)	2(11)
13	53	2(13)	1(13)	2(13)
13	79	3(13)	8(13)	2(13)
17	137	4(17)	16(17)	13(17)
23	139	4(23)	12(23)	8(23)
41	83	28(41)	7(41)	4(41)

last map is evaluation at S. It follows that this map is simply the reduction of u at $\tilde{q}$. □

It follows from this discussion and Proposition 5.1 that we can rephrase our conjecture in the following way:

Conjecture 5.1. Let K be a cubic extension with negative discriminant $-D$, with sextic Galois closure L. Let g be the associated weight one form of level D. Let $u \in \mathcal{O}_K^*$ be a unit. Let $q \equiv 1$ modulo p be as above; suppose that $(\frac{-D}{q}) = -1$, and $p \ge 5$, and finally $\varpi_{\text{Merel}} \in (\mathbb{Z}/q)^*$ (see (5–7) for definition) is non-zero modulo p, i.e. upon projection to the quotient $\mathbb{F}_p\langle 1\rangle$.

Then there exist $A, B \in \mathbf{Z}$ such that, for all such q we have

$$\varpi_{\text{Merel}}^{A\cdot\eta} = \bar{u}^B \text{ in } \mathbb{F}_p\langle 1\rangle, \tag{5–13}$$

where:

- $\eta \in \mathbb{Z}$ is the first Fourier coefficient of the Eisenstein component of G_3^{proj}, the projection of $g(z)g(qz)$ to the Eisenstein component at level q. (This is well defined modulo the numerator of $\frac{q-1}{12}$, which is sufficient to make sense of the above definition).
- $\bar{u} \in (\mathbb{Z}/q)^*$ is the reduction of u modulo the unique degree one prime of K, above q.

We have tested this conjecture numerically (see data tables) for the fields K of discriminant −23 and −31. In all the cases for discriminant −23, we find $\frac{A}{B} = \frac{-1}{72}$; in all the cases for discriminant −31, we find $\frac{A}{B} = \frac{1}{72}$. The fact that 72 is divisible only by 2 and 3 is striking.

Remark 5.1. Although we are not able to do any computations with exotic weight one forms at present, we comment on how some of the previous identifications change.

First of all, one can consider the case of a weight one form g whose Galois representation is induced from a character of a real quadratic field L_0 of mixed signature at ∞, i.e. taking the value +1 on one complex conjugation and −1 on the other complex conjugation. In this case, the unit group U_g is simply the unit group $\mathcal{O}_{L_0}^* \otimes \mathbf{Q}$. It is possible that this case would be more amenable to theoretical analysis.

The remaining cases correspond to adjoint Galois representation with image A_4, S_4, or A_5. In the A_4 case the description of the unit group is quite straightforward: let L_0 be the fixed field of $A_3 \leqslant A_4$. Then (up to possibly extending coefficients to a larger extension of $\mathbf{Q}$) the unit group U_g is simply $\mathcal{O}_{L_0}^* \otimes \mathbf{Q}$. In the other cases, the description becomes a little more complicated since the unit group must be "cut out" from units in the Galois closure of L_0. In these A_4, S_4, orA_5 cases, the adjoint representation is irreducible, which should mean that the trivial vanishing described in footnote 5 does not occur.

6. Flat cohomology and Merel's computation

We now explain why Merel's computation implies Lemma 5.1. The issue is that Merel's computation is in characteristic zero. To relate it to $\langle \mathrm{E}, \mathfrak{S}\rangle$, which is defined in characteristic p, we will need to do a little setup in flat cohomology.

Let $X = X_0(q)$ regarded now as a proper smooth curve over $\mathbb{Z}_p$; here $q \equiv 1$ modulo p. Let J_p be the p-torsion of the Jacobian of $X_{\overline{\mathbf{Q}_p}}$. We shall define several incarnations of both the Shimura class and the Eisenstein class.

6.1. The (Shimura) class α

The Shimura cover $X_1(q)^\Delta \to X_0(q)$ (from Section 3.1) is a $(\mathbb{Z}/q)_p^*$ torsor for the etale topology. As before it defines a class $\mathfrak{S} \in H^1_{\text{et}}(X, \mathbb{F}_p\langle 1\rangle)$, which can be pulled back to flat cohomology:

$$\alpha \in H^1_{\text{fl}}(X, \mathbb{F}_p\langle 1\rangle).$$

Restricting $\mathfrak{S}$ to the geometric generic fiber $X_{\overline{\mathbf{Q}_p}}$, we get a class in étale cohomology

$$\alpha_{\text{et}} \in H^1_{\text{et}}(X_{\overline{\mathbf{Q}_p}}, \mathbb{F}_p\langle 1\rangle).$$

The inclusion $\mu_p \hookrightarrow \mathbb{G}_m$ induces $H^1_{\text{et}}(X_{\overline{\mathbf{Q}_p}}, \mu_p) \to J_p$, and thus α_{et} gives

$$P_\alpha \in \operatorname{Hom}(\mu_p\langle -1\rangle, J_p), \tag{6–1}$$

we use the notation P_α to suggest that this is a point on the Jacobian.

Finally, we also obtain a Zariski class on the geometric special fiber, using the inclusion $\mathbb{F}_p \hookrightarrow \mathcal{O}$ and the identification of Zariski and etale cohomology for $\mathcal{O}$:

$$\alpha_{\text{Zar}} \in H^1_{\text{Zar}}(X_{\overline{\mathbb{F}_p}}, \mathcal{O}\langle 1\rangle).$$

6.2. The (Eisenstein) class β

Let Δ be the weight twelve cusp form $q\prod(1-q^n)^{24}$, and consider the function $f := \Delta(qz)/\Delta(z)$ on X. Extracting its pth root gives a μ_p-torsor (in the flat topology) on X. Indeed, f is invertible except for the divisors corresponding to 0 and ∞, and along those divisors its valuation is divisible by p. Thus, we get a class

$$\beta \in H^1_{\text{fl}}(X, \mu_p).$$

The μ_p-torsor is étale over the geometric generic fiber $X_{\overline{\mathbf{Q}_p}}$ and we get a corresponding class in étale cohomology

$$\beta_{\text{et}} \in H^1_{\text{et}}(X_{\overline{\mathbf{Q}_p}}, \mathbb{F}_p\langle 1\rangle).$$

There is a corresponding class in the p-torsion of the Jacobian, namely, writing 0 and ∞ for the two cusps of X we may form

$$Q_\beta := \frac{(q-1)}{p}((\infty)-(0)) \in J_p$$

—this is related to our prior discussion because $p.Q_\beta$ is the divisor of f.

Finally, there is also a Zariski class "corresponding" to β on the special fiber. Namely, the logarithmic derivative $\frac{df}{f}$ in fact extends to a global section of Ω^1, i.e. a class

$$\beta_{\text{Zar}} \in H^0(X_{\overline{\mathbb{F}_p}}, \Omega^1).$$

Observe that $\frac{df}{f}$ is the differential form associated to the "Eisenstein cusp form" G of weight two.

With these preliminaries, the main point is to check the following:

Proposition 6.1. *We have an equality in $\mathbb{F}_p\langle 1\rangle$:*

$$\langle P_\alpha, Q_\beta\rangle_{\text{Weil}} = \langle \alpha_{\text{et}}, \beta_{\text{et}}\rangle_{\text{et}} = \langle \alpha_{\text{Zar}}, \beta_{\text{Zar}}\rangle_{\text{Zar}}.$$

Here, $\langle -,-\rangle_{\text{Weil}}$ is the Weil pairing, $\langle -,-\rangle_{\text{et}}$ is the pairing given by Poincaré duality in étale cohomology on the geometric fiber, and $\langle -,-\rangle_{\text{Zar}}$ is the pairing given by Serre duality in coherent cohomology on the special fiber. Keeping track of twists, we see that all these take values in $\mathbb{F}_p\langle 1\rangle$.

Now $\langle \mathrm{E}, \mathfrak{S}\rangle$ is given by $\langle \alpha_{\text{Zar}}, \beta_{\text{Zar}}\rangle_{\text{Zar}}$; the Proposition shows this coincides (in $\mathbb{F}_p\langle 1\rangle$) with $\langle P_\alpha, Q_\beta\rangle_{\text{Weil}}$. The Weil pairing on the right is computed by Merel; we pin down the relation to Merel's computation in Section 6.3. Taken together, the Proposition and this computation prove Lemma 5.1.

Proof. The first equality is straightforward: an explicit representative for $Q_\beta \in J_p \simeq H^1(X, \mu_p)$ is given by the μ_p-torsor associated to $f := \Delta(qz)/\Delta(z)$, because the divisor of f is pQ_β.

We now discuss the second equality. We will compare everything to the cup product in flat cohomology, i.e.

$$\alpha \cup \beta \in H^2_{\text{fl}}(X, \mu_p\langle 1\rangle).$$

There is a degree map $H^2_{\text{fl}}(X, \mu_p) \to \mathbb{F}_p$; let us explicate it. On any scheme, the sequence $\mu_p \to \mathbb{G}_m \to \mathbb{G}_m$ induces an exact sequence of represented sheaves for the flat topology. This identifies the flat cohomology of μ_p with the hypercohomology of $[\mathbb{G}_m \xrightarrow{x\mapsto x^p} \mathbb{G}_m]$.

Let $X_{\overline{\mathbb{Z}_p}}$ be the base change of X to $\overline{\mathbb{Z}_p}$ (the Witt vectors of $\overline{\mathbb{F}_p}$). We obtain an exact sequence

$$\operatorname{Pic}(X_{\overline{\mathbb{Z}_p}})/p \hookrightarrow H^2_{\text{fl}}(X_{\overline{\mathbb{Z}_p}}, \mu_p) \to H^2_{\text{fl}}(X_{\overline{\mathbb{Z}_p}}, \mathbb{G}_m)[p]. \tag{6–2}$$

Flat and étale cohomology of $\mathbb{G}_m$ coincide (see [Milne 80, III, Theorem 3.9]), and the right-hand side is a subgroup of the Brauer group of $X_{\overline{\mathbb{Z}_p}}$, which vanishes [Grothendieck 68, Theorem 3.1]. Accordingly, any class in $H^2_{\text{fl}}(X_{\overline{\mathbb{Z}_p}}, \mu_p)$ is the coboundary of a line bundle, and computing degree gives the desired homomorphism

$$\deg : H^2_{\text{fl}}(X_{\overline{\mathbb{Z}_p}}, \mu_p) \longrightarrow \mathbb{F}_p.$$

We see that $\deg(\alpha \cup \beta) = \langle \alpha_{\text{et}}, \beta_{\text{et}}\rangle_{\text{et}}$ and so it remains to see

$$\deg(\alpha \cup \beta) = \langle \alpha_{\text{Zar}}, \beta_{\text{Zar}}\rangle_{\text{Zar}}.$$

Let π be the morphism from the flat site on $X_{\overline{\mathbb{F}_p}}$ to the étale site. As a reference for what follows, we refer to the paper of [Artin and Milne 76]. We have isomorphisms:

$$R\pi_*(\mathbb{Z}/p\mathbb{Z}) \simeq [\mathcal{O} \xrightarrow{1-\mathrm{F}} \mathcal{O}],$$
$$R\pi_*\mu_p \simeq [\Omega^1 \xrightarrow{1-\mathrm{C}} \Omega^1][1],$$

where F and C are, respectively, the Frobenius and Cartier maps. and Artin–Milne show that the pairing $\mathbb{Z}/p\mathbb{Z} \times \mu_p \to \mu_p$ induces, after push-forward, the "obvious" pairing on the complexes on the right, which can be computed in the Zariski topology, because flat and Zariski cohomology coincide for quasi-coherent sheaves.

For the same reason, the second identification induces an isomorphism

$$H^2_{\mathrm{fl}}(X_{\overline{\mathbb{F}}_p}) \simeq H^1(X_{\overline{\mathbb{F}}_p}, \Omega^1)^{\mathrm{C}} = \mathbb{F}_p,$$

where the map $H^1(\Omega^1) \to \overline{\mathbb{F}_p}$ comes from Serre duality. Moreover, the resulting identification is simply the degree map, alluded to above; this comes down to the fact that the map

$$H^1(X_{\overline{\mathbb{F}}_p}, \mathbb{G}_m) \xrightarrow{d\log} H^1(X_{\overline{\mathbb{F}}_p}, \Omega^1) \to \overline{\mathbb{F}_p}$$

again computes the degree of a line bundle modulo p.

With respect to the resulting identification of $H^1_{\mathrm{fl}}(X, \mu_p) \simeq \mathbb{H}^0(\Omega^1 \stackrel{1-\mathrm{C}}{\to} \Omega^1)$, and the Čech representation of this last hypercohomology, the class β is represented by $\frac{df}{f} \in \check{C}^0(\Omega^1)$, which has zero boundary and is annihilated on the nose by $1 - \mathrm{C}$. Similarly the class α in étale cohomology is represented by a Cech cocycle $c^1 \in \check{C}^1(\mathcal{O})$ together with a class $c^0 \in \check{C}^0(\mathcal{O})$ satisfying $(1 - \mathrm{F})c^1 = dc^0$. The image of the pairing $\alpha \cup \beta \in H^2_{\mathrm{fl}}(\mu_p)$, under the map $H^2_{\mathrm{fl}}(\mu_p) \to H^1(\Omega^1)^{\mathrm{C}}$, is represented by $c^1 \cdot \frac{df}{f} \in \check{C}^1(\Omega^1)$; its image by the trace pairing is the usual Serre duality pairing between the cohomology classes of c^1 and $\frac{df}{f}$. This concludes the proof. □

6.3. Merel's computation

Although routine, we write out the details involving $\langle P_\alpha, Q_\beta\rangle$ to be sure of factors involving $\gcd(q-1, 12)$. In what follows, we understand our modular curves to be considered over an algebraically closed field of characteristic zero.

Recall that P_α is an element of $\mathrm{Hom}(\mu_p\langle -1\rangle, J_p)$. Thus, the Weil pairing $\langle P_\alpha, Q_\beta\rangle \in \mathbb{F}_p\langle 1\rangle$ has the property that

$$\text{Weil pairing of } P_\alpha(u) \text{ and } Q_\beta = u \cdot \langle P_\alpha, Q_\beta\rangle_{\text{Weil}} \quad (u \in \mu_p\langle -1\rangle), \qquad (6\text{–}3)$$

where, on the left-hand side we have the "usual" Weil pairing of two torsion points in J_p.

Following Merel, let ν be the gcd of $q-1$ and 12; let $n = \frac{q-1}{\nu}$. Let $U \subset (\mathbb{Z}/q)^*$ be the subgroup of νth powers; the map $(\mathbb{Z}/q)^* \to \mathbb{F}_p\langle 1\rangle$ factors through the νth power map, and we get a sequence

$$(\mathbb{Z}/q)^* \xrightarrow{x \mapsto x^\nu} U \to \mathbb{F}_p\langle 1\rangle.$$

The Galois group of the covering $X_1(q) \to X_0(q)$ can be identified with U (as in Section 3.3 [Merel 96]). This gives rise to a map

$$\alpha' : \mathrm{Hom}(U, \mu_n) \to J_n.$$

Also $Q' = (\infty) - (0)$ is n-torsion in the divisor class group, thus defining another class in J_n. Then Merel shows that

$$\langle \alpha'(t), Q'\rangle_n = t(\varpi_{\mathrm{Merel}}), \quad t \in \mathrm{Hom}(U, \mu_n), \qquad (6\text{–}4)$$

where the equality is in μ_n and the subscript n means we are using the Weil pairing at the n-torsion level.

We want to compare α' to P_α. Note that if $t \in \mathrm{Hom}(U, \mu_n)$, the power $t^{n/p}$ defines an element of $\mathrm{Hom}(U, \mu_p)$ that, considered as an element of $\mathrm{Hom}((\mathbb{Z}/q)^*, \mu_p)$, factors through $\mathbb{F}_p\langle 1\rangle$. We refer to the resulting element as $\bar{t} \in \mathrm{Hom}(\mathbb{F}_p\langle 1\rangle, \mu_p)$. Explicitly, if $\mu \in (\mathbb{Z}/q)^*$, we have

$$t^{n/p}(\mu^\nu) = \bar{t}(\mu). \qquad (6\text{–}5)$$

Now consider the commutative diagram (where we write $X = X_0(q)$ for short)

$$\begin{array}{ccc}
H^1(X, U) \times \mathrm{Hom}(U, \mu_n) & \longrightarrow & J_n \\
\Big\downarrow{\scriptstyle \mathrm{id} \times t \mapsto \bar{t}} & & \Big\downarrow{\scriptstyle \times n/p} \\
\underbrace{H^1(X, U)}_{\to H^1(X, \mathbb{F}_p\langle 1\rangle)} \times \underbrace{\mathrm{Hom}(\mathbb{F}_p\langle 1\rangle, \mu_p)}_{\simeq \mu_p\langle -1\rangle} & \longrightarrow & J_p
\end{array}$$

When we evaluate at the element of $H^1(X, U)$ corresponding to the cover $X_1(q) \to X_0(q)$, the top horizontal map becomes α' and the bottom map becomes P_α from (6–1). Thus, we have

$$\alpha'(t)^{n/p} = P_\alpha(\bar{t}), \quad t \in \mathrm{Hom}(U, \mu_n) \mapsto \bar{t} \in \mathrm{Hom}(\mathbb{F}_p\langle 1\rangle, \mu_p).$$

Pairing with $Q_\beta = \frac{q-1}{p} Q' \in J_p$ and comparing with (6–3):

$$\underbrace{\bar{t}}_{\mu_p(-1)} \underbrace{\langle P_\alpha, Q_\beta\rangle}_{\mathbb{F}_p\langle 1\rangle} = \langle P_\alpha(\bar{t}), \quad Q_\beta\rangle_p = \langle \alpha'(t)^{n/p}, \quad \frac{q-1}{p} Q'\rangle_p$$
$$= \frac{q-1}{p} \langle \alpha'(t), Q'\rangle_n \in \mu_p$$

and so

$$\underbrace{\bar{t}}_{\mu_p(-1)} \underbrace{\langle P_\alpha, Q_\beta\rangle}_{\mathbb{F}_p\langle 1\rangle} \stackrel{6.4}{=} \frac{q-1}{p} t(\varpi_{\mathrm{Merel}})$$
$$= t^{n/p}(\varpi^\nu_{\mathrm{Merel}}) \stackrel{6.5}{=} \bar{t}(\varpi_{\mathrm{Merel}}),$$

where the equality is once again in μ_p. We conclude that $\langle P_\alpha, Q_\beta\rangle$ is indeed the image of ϖ_{Merel} inside $\mathbb{F}_p\langle 1\rangle$.

7. Comparison with the theory of [Venkatesh nd]

Derived Hecke operators at Taylor–Wiles primes have been defined abstractly for general q-adic groups in [Venkatesh nd]. The purpose of the present section is to identify the operators introduced in (3–1) with those defined in [Venkatesh nd]. (The results of this section are, strictly speaking, not used elsewhere in the paper; however they show that all the constructions we have made are inevitable).

Write $G = \mathrm{GL}_2(\mathbf{Q}_q)$, where $q \equiv 1 \pmod p$, and $K = \mathrm{GL}_2(\mathbb{Z}_q)$. Fix a base ring S that is a $\mathbb{Z}_p$-algebra.

What we will need to do, in order to study the derived Hecke operator at q, is to identify the cohomology of the modular curve with the cohomology of the K-invariants of a complex of G-representations. Unsurprisingly, this is done by adding infinite level at q; we just pin down the details. We need to take a little care because the tower of coverings that one gets by adding infinite q-level is not étale; however, its ramification is prime to p, which will be enough for our purposes.

In particular, we will use[6] Lemma A.10 of Appendix A of [Venkatesh nd], which explicates the action of the abstract derived Hecke algebra in terms of restrictions, corestrictions, and cup products.

7.1. Construction of complexes with an action of $\mathrm{GL}_2(\mathbf{Q}_q)$

Let us fix a level structure away from q for the usual modular curve, i.e. an open compact subgroup $K^{(q)} \subset \mathrm{GL}_2(\mathbb{A}^{(\infty,q)})$. We require that $K^{(q)} = \prod_{v \neq q} K_v$, where K_v is hyperspecial maximal for almost all v.

For $U \subset \mathrm{GL}_2(\mathbf{Q}_q)$, an open compact subgroup, let $X(U)$ be the Deligne–Rapoport compactification of the modular curve with level structure $K^{(q)} \times U$. This again has (Deligne–Rapoport) a smooth proper model over Spec S, denoted $X(U)_S$. We denote again by $\omega_U \to X(U)$ the relative cotangent bundle of the universal elliptic curve; this defines a locally free sheaf over $X(U)_S$.

Let us consider the pro-system of schemes

$$X_\infty : U \mapsto X(U),$$

indexed by the collection of all open compact subgroups of $\mathrm{GL}_2(\mathbf{Q}_q)$; the maps are inclusions $V \subset U$ of open compact subgroups.

The isomorphisms $X(g^{-1}Ug) \xrightarrow{\sim} X(U)$ induce an action of $G = \mathrm{GL}_2(\mathbf{Q}_q)$ on X_∞ (considered as a pro-object in the category of schemes). Let ω_∞ be the "vector bundle" over X_∞ defined by ω: by this we mean that ω_∞ is a pro-scheme over X_∞, which is level-wise a vector bundle.

We will need the following properties:

(i) The action of G on X_∞ lifts to an action on ω_∞.

(ii) Suppose that V is a normal subgroup of U. Then the natural map

$$f_{UV} : X(V)_S \to X(U)_S$$

is finite, and identifies $X(U)_S$ with the quotient of $X(V)_S$ by U/V in the category of schemes. (See [Deligne and Rapoport 73, 3.10]).

Moreover, there is a natural (in S) identification $f_{UV}^* \omega_U \simeq \omega_V$.

(iii) With notation as in (ii), if the order of U/V is a power of p, then the map $X(V)_S \to X(U)_S$ is étale.

Proof. (of (iii) only:) We may suppose that $S = \mathbb{Z}_p$. The map is étale over the interior of the modular curve, so, by purity of the branch locus, it is enough to check that it is étale at the cusps in characteristic zero. The cusps of a modular curve are parameterized by an adelic quotient, but replacing the role of an upper half-plane by $\mathbb{P}^1(\mathbf{Q})$; so we must verify that the map

$$\mathrm{GL}_2(\mathbf{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbf{Q}))/V \longrightarrow \mathrm{GL}_2(\mathbf{Q})\backslash(\mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbf{Q}))/U,$$

considered as a morphism of groupoids, induces isomorphisms on each isotropy group.

Let $\mathbf{B}$ be a Borel subgroup in $\mathrm{GL}_{2,/\mathbf{Q}}$ and $\mathbf{N}$ its unipotent radical. We can identify $\mathbb{P}^1(\mathbf{Q})$ with $\mathrm{GL}_2(\mathbf{Q})/\mathbf{B}(\mathbf{Q})$. The desired result follows, then, if for each $g \in \mathrm{GL}_2(\mathbb{A}_f)$ we have

$$\mathbf{B}(\mathbf{Q}) \cap gUg^{-1} \subset gVg^{-1}.$$

However, the projection of $\mathbf{B}(\mathbf{Q}) \cap gUg^{-1}$ to the toral $\mathbf{Q}^*$ is a finite subgroup of $\mathbf{Q}^*$, thus contained in $\{\pm 1\}$. It follows that an index 2 subgroup of the left-hand side is contained in $\mathbf{N}(\mathbf{Q}) \cap gUg^{-1}$, which is certainly contained in gVg^{-1} because any open compact of $\mathbf{N}(\mathbf{Q}_q)$ is pro-q. □

Lemma 7.1. *Suppose that, as above, V is a normal subgroup of U. Let $f = f_{UV}$ be as in (ii) above. Let $\mathcal{F}$ be any sheaf of $\mathcal{O}_{X(V)}$-modules on $X(V)$, equipped with a compatible action of U/V.*

Then

(i) For each $x \in X(V)$, the higher cohomology of the stabilizer $(U/V)_x$ on $\mathcal{F}_x$ is trivial.

(ii) For each $y \in X(U)$, the higher cohomology of (U/V) acting on $(\pi_* \mathcal{F})_y$ is trivial.

Proof. Note that we can reduce (i) to the case when $(U/V)_x = (U/V)$ by shrinking U. Both (i) and (ii) will follow, then, if we prove that for any U/V-stable affine set $\mathrm{Spec}(A) \subset X(V)$,

$$\text{higher cohomology of } U/V \text{ on } \Gamma(\mathrm{Spec}(A), \mathcal{F}) = 0, \qquad (7\text{–}1)$$

since the stalks appearing in (i) and (ii) are direct limits of such spaces.

Let $\Delta = U_1/V$ be a Sylow p-subgroup of U/V; it is sufficient to make the same verification for the higher cohomology of Δ. Write $B = A^\Delta$. The map $\mathrm{Spec}(A) \to$

[6] with an apology to 21st century readers, see below...

$\mathrm{Spec}(B)$ is finite étale with Galois group Δ, by (iii) above. It is now sufficient to show:

> If M is an A-module, equipped with a Δ-action compatible with its module structure, then $H^q(\Delta, M) = 0$ for $q > 0$.

Let $M' = M \otimes_B A$; define a Δ-action on M' using $g(m \otimes a) = gm \otimes a$ for $g \in \Delta$. Since A is a flat B-module, the natural map $H^q(\Delta, M) \otimes_B A \to H^q(\Delta, M')$ is an isomorphism. We shall show $H^q(\Delta, M') = 0$; the vanishing of $H^q(\Delta, M)$ follows from faithful flatness.

Now M' is a module over $A \otimes_B A \simeq \prod_{\delta \in \Delta} A$, and this module structure is compatible with the Δ-action on $\prod_{\delta \in \Delta} A$, which permutes the factors. Therefore, M' is induced (as a Δ-module) from a representation of the trivial group, and thus has vanishing higher Δ-cohomology by Shapiro's lemma. $\square$

7.2. Godement resolution

Let T be the "Godement functor," which assigns to a sheaf $\mathcal{F}$ the sheaf $U \mapsto \prod_{x \in U} \mathcal{F}_x$ of discontinuous sections. It carries a sheaf of $\mathcal{O}$-modules to another sheaf of $\mathcal{O}$-modules.

We will need to discuss the behavior under images. Suppose given a map $f : X' \to X$ of schemes. There is a map of functors

$$T \to f_* T f^{-1}.$$

For a sheaf $\mathcal{F}$ on X and an open set $V \subset X$, this is given by the natural pullback of discontinuous sections

$$\prod_{x \in V} \mathcal{F}_x \to \prod_{x' \in f^{-1}V} (f^{-1}\mathcal{F})_{x'}.$$

If we are working with sheaves of $\mathcal{O}$-modules, then, composing with the natural $f^{-1} \to f^*$, we get $T \to f_* T f^*$, or, what is the same by adjointness, a natural transformation

$$f^* T \longrightarrow T f^* \text{ and (by iterating) } f^* T^k \to T^k f^*.$$

In particular, for a sheaf $\mathcal{F}$ on X, there is a map

$$\begin{aligned} &f^* \text{ (Godement resolution of } \mathcal{F}) \\ &\quad \longrightarrow \text{Godement resolution of } f^*\mathcal{F}. \end{aligned} \tag{7–2}$$

This gives rise to the pullback map in cohomology $H^*(X, \mathcal{F}) \to H^*(X', f^*\mathcal{F})$.

7.3.

It follows from Lemma 7.1 that (with notations as in that Lemma and) for any sheaf $\mathcal{F}$ of $\mathcal{O}_{X(V)}$-modules,

$$H^p(U/V, \Gamma(X(V), T\mathcal{F})) = 0, \quad p > 0. \tag{7–3}$$

Indeed, group cohomology commutes with products (even infinite ones).

Now let $\mathcal{G}^\bullet(U)$ be the Godement resolution of ω_U. It is a complex of sheaves of $\mathcal{O}_{X(U)}$-modules on $X(U)_S$. Let $M^\bullet(U)$ be the global sections of $\mathcal{G}^\bullet(U)$: this is a complex of S-modules. If $V \subset U$, there is a natural action of U/V on $M^\bullet(V)$. It follows from (7–3) that

Lemma 7.2. *For each degree i, the U/V-cohomology of $M^i(V)$ vanishes, i.e. $H^p(U/V, M^i(V)) = 0$ for $p > 0$.*

The following result is the crucial one for us.

Lemma 7.3. *The map arising from (7–2)*

$$\mathcal{G}^\bullet(U) \to \left(f_*\mathcal{G}^\bullet(V)\right)^{U/V} \tag{7–4}$$

(where U/V denotes invariants) induces on global sections a quasi-isomorphism

$$M^\bullet(U) \longrightarrow M^\bullet(V)^{U/V}. \tag{7–5}$$

Proof. It is enough to verify that (7–4) is a quasi-isomorphism: the sheaves $\mathcal{G}^\bullet(U)$ and $f_*\mathcal{G}^\bullet(V)^{U/V}$ are flasque—the latter follows just by examining the definition of the Godement functor T—and so taking global sections will preserve the quasi-isomorphism.

Consider the following diagram:

$$\begin{array}{ccc} \omega_U & \xrightarrow{\ \sim\ } & \mathcal{G}^\bullet(U) \\ \downarrow{\scriptstyle \sim} & & \downarrow \\ (f_*\omega_V)^{U/V} & \xrightarrow{\ j\ } & (f_*\mathcal{G}^\bullet(V))^{U/V}. \end{array}$$

The left vertical arrow is a quasi-isomorphism: we have an isomorphism $f_*\omega_V \simeq \omega_U \otimes f_*\mathcal{O}_V$, and $(f_*\mathcal{O}_V)^{U/V} = \mathcal{O}_U$. The top horizontal arrow is also a quasi-isomorphism. It then suffices to show that the arrow j is also a quasi-isomorphism.

The complex $f_*\mathcal{G}^\bullet(V)$ is a resolution of $f_*\omega_V$ because f_* has no higher cohomology on the quasi-coherent sheaf ω_V. Next the stalks of $f_*\omega_V$ and $f_*\mathcal{G}^\bullet(V)$ have vanishing U/V-cohomology by Lemma 7.1. Given an acyclic complex of U/V-modules supported in degrees ≥ 0, each of which have no higher U/V-cohomology, the U/V-invariants remain acyclic. This implies that $f_*\mathcal{G}^\bullet(V)^{U/V}$ is a resolution of $(f_*\omega_V)^{U/V}$ as desired. $\square$

7.4. Compatibility with traces

We must also mention the compatibility with trace maps. Suppose, we are given a subgroup U' intermediate between U and V:

$$V \subset U' \subset U.$$

We do not require that U' be normal.

There is a natural trace map

$$H^*(X(U'), \omega_{U'}) \to H^*(X(U), \omega_U).$$

Explicitly the trace $f_*\mathcal{O}_{X(U')} \to O_{X(U)}$ induces

$$\begin{aligned} H^*(X(U'), \omega_{U'}) &= H^*(X(U), f_*\omega_{U'}) \\ &= H^*(X(U), \omega_U \otimes f_*\mathcal{O}_{X(U')}) \\ &\xrightarrow{\text{tr}} H^*(X(U), \omega_U). \end{aligned}$$

With reference to the identifications of the previous lemma, this trace map is induced at the level of cohomology by

$$M^\bullet(U') \to M^\bullet(V)^{U'/V} \xrightarrow{\mathrm{T}} M^\bullet(V)^{U/V} \xleftarrow{\sim} M^\bullet(U),$$

where $\mathrm{T} \in S[U/V]$ is the sum of a set of coset representatives for U/U'.

7.5. Derived invariants and the derived Hecke algebra

As in Section 7.3, $M^\bullet(U)$ is a Godement complex computing the complex of ω_U Now set

$$M_\infty^\bullet = \varinjlim M^\bullet(U),$$

which is now a complex of S-modules equipped with an action of $G = \mathrm{GL}_2(\mathbf{Q}_q)$.

We will argue that the "derived invariants" of U on $M_\infty^\bullet$ give a complex that computes the cohomology of $X(U)_S$. We first recall the notion of derived invariants, and its relationship with the derived Hecke algebra.

Let U be an open compact subgroup of G. Let $U_1 \subset U$ be a normal subgroup with the property that the pro-order of U_1 is relatively prime to p. Let $\mathbf{Q}$ be a projective resolution of S in the category of $S[U/U_1]$-modules; we regard this as a complex with degree-increasing differential concentrated in degrees ≤ 0:

$$\cdots \to Q_{-2} \to Q_{-1} \to Q_0 = S.$$

We may of course regard $\mathbf{Q}$ as a complex of $S[U]$-modules.

Let $\mathbf{P} = \mathrm{ind}_U^G \mathbf{Q}$. This is a projective resolution of the smooth $S[G]$ module $S[G/U]$ (in the category of smooth $S[G]$ modules). For any complex $R^\bullet$ of G-modules, we define the derived U-invariants to be the complex

$$\mathrm{Hom}_{S[G]}(\mathbf{P}, R^\bullet) = \mathrm{Hom}_{S[U]}(\mathbf{Q}, (R^\bullet)^{U_1}).$$

Explicitly, this is a complex whose cohomology computes the hypercohomology $\mathbb{H}^*(U, R^\bullet)$.

In the case above, the derived invariants of U on $M_\infty^\bullet$ compute the cohomology of $X(U)$, in the following sense:

Lemma 7.4. *The natural inclusion of $M^\bullet(U) \hookrightarrow M_\infty^\bullet$ and the augmentation $\mathbf{Q} \to S$ induce a quasi-isomorphism:*

$$M^\bullet(U) \xrightarrow{\sim} \mathrm{Hom}_{S[U]}(\mathbf{Q}, M_\infty^\bullet) = \mathrm{Hom}_{S[G]}(\mathbf{P}, M_\infty^\bullet). \tag{7–6}$$

Proof. Using the remarks after (7–5), we see that

$$M^\bullet(U_1) \xrightarrow{\text{q.i.}} \varinjlim_{U' \subset U_1} M^\bullet(U')^{U_1/U'} \xrightarrow{\sim} U_1\text{-invariants on } M_\infty^\bullet. \tag{7–7}$$

(for the second arrow: since U_1 is prime to p the functor of taking U_1 invariants commutes with taking a direct limit of smooth $S[U_1]$-modules).

The inclusion $M^\bullet(U) \hookrightarrow M^\bullet(U_1)$ and the homomorphism $\mathbf{Q} \to S$ induce

$$M^\bullet(U) \to \mathrm{Hom}_{U/U_1}(S, M^\bullet(U_1)) \to \mathrm{Hom}_{U/U_1}(\mathbf{Q}, M^\bullet(U_1))$$

and it remains to show that this composite is a quasi-isomorphism.

The first map is a quasi-isomorphism by Lemma 7.3. To show that the second map is a quasi-isomorphism, it is enough (by a devissage) to show that for each fixed degree j

$$\mathrm{Hom}(S, M^j(U_1)) \to \mathrm{Hom}(\mathbf{Q}, M^j(U_1))$$

induces a quasi-isomorphism. But the right hand side computes the U/U_1 cohomology of $M^j(U_1)$, and we have seen (Lemma 7.2) that this is concentrated in degree zero, where it is just the U/U_1 invariants, as needed. □

Now we may imitate all the reasoning above, with the role of ω replaced by $\mathcal{O}$. Let $N^\bullet$ be the corresponding complex. Reasoning as in Lemma 7.4, we get a quasi-isomorphism

$$N^\bullet(U) \simeq \mathrm{Hom}_{S[U]}(\mathbf{Q}, N_\infty^\bullet). \tag{7–8}$$

The identification of S with global sections of $\mathcal{O}_{X(V)}$ induce compatible maps $S \to N^\bullet(V)$ for each level V, and so by passage to the limit a map $S \hookrightarrow N_\infty^\bullet$. This induces

$$H^*(U, S) \longrightarrow H^*(X(U), \mathcal{O}). \tag{7–9}$$

For $\alpha \in H^j(U, S)$, write $\langle\alpha\rangle \in H^j(X(U), \mathcal{O})$ for its image under this map. Then we have:

Lemma 7.5. *Under the identification $H^*(X(U), \omega_U)$ with the hypercohomology $\mathbb{H}^*(U, M_\infty^\bullet)$, (as in the prior Lemma), cup product with $\langle\alpha\rangle$ in Zariski cohomology is carried to cup product with α in hypercohomology.*

Proof. The product $\mathcal{O} \otimes \omega_U \to \omega_U$ extends to a map $N^\bullet(U) \otimes M^\bullet(U) \to M^\bullet(U)$ (see [Godement 58, Chapter 6]), which computes on cohomology the cup product. This exists compatibly at every level, and by passage to the direct limit, we arrive at a map $N_\infty^\bullet \otimes M_\infty^\bullet \to M_\infty^\bullet$ (the tensor product can be passed through the direct limit, by [Bourbaki 98, Chapter 2, Prop. 7, Section 6.3]).

Fix a quasi-isomorphism of $S[U]$-modules:

$$q : \mathbf{Q} \to \mathbf{Q} \otimes_S \mathbf{Q}.$$

Consider the following diagram, with commutative squares:

$$\begin{array}{ccccc}
m' \otimes \alpha'' \in H^i(M^\bullet(U)) \otimes H^j(N^\bullet(U)) & \longrightarrow & & \longrightarrow & H^{i+j}(M^\bullet(U)) \\
\downarrow & & & & \downarrow \\
m \otimes \alpha' \in \mathrm{Hom}^i(\mathbf{Q}, M^\bullet_\infty) \otimes \mathrm{Hom}^j(\mathbf{Q}, N^\bullet_\infty) & \xrightarrow{\otimes} & \mathrm{Hom}^{i+j}(\mathbf{Q} \otimes \mathbf{Q}, M^\bullet_\infty \otimes N^\bullet_\infty) & \xrightarrow{q} & \mathrm{Hom}^{i+j}(\mathbf{Q}, M^\bullet_\infty), \\
\uparrow & & \uparrow & & \uparrow \\
m \otimes \alpha \in \mathrm{Hom}^i(\mathbf{Q}, M^\bullet_\infty) \otimes \mathrm{Hom}^j(\mathbf{Q}, S) & \xrightarrow{\otimes} & \mathrm{Hom}^{i+j}(\mathbf{Q} \otimes \mathbf{Q}, M^\bullet_\infty \otimes S) & \xrightarrow{q} & \mathrm{Hom}^{i+j}(\mathbf{Q}, M^\bullet_\infty)
\end{array}$$

where:

- Hom means in every case homomorphisms of chain complexes of $S[U]$ modules, taken modulo chain homotopy.
- $\otimes$ comes from the tensor product, which induces a bifunctor on the homotopy category of chain complexes.
- We fix $m \in \mathrm{Hom}^i(\mathbf{Q}, M^\bullet_\infty)$, and m' is the cohomology class corresponding to m under the quasi-isomorphism (7–6).
- We identify α with a class in $\mathrm{Hom}^j(\mathbf{Q}, S)$ and α' is the image of this class, under $S \to N^\bullet_\infty$. Also α'' is a cohomology class in $H^j(N^\bullet(U))$ that matches with α' under the quasi-isomorphism (7–8).

The image of $m \otimes \alpha$, under the bottom horizontal arrows, computes the cup product of m and α in U-hypercohomology. This corresponds to the image of $m \otimes \alpha'$ in the middle horizontal row. Finally, this corresponds to the image of $m' \otimes \alpha''$ in the top row, which gives the Zariski product. □

7.6. *Derived Hecke algebra*

Let notation be as above, but specialized to the case $U = K$, a maximal compact of $\mathrm{GL}_2(\mathbf{Q}_q)$. We may form the differential graded algebra $\mathrm{End}_{S[G]}(\mathbf{P})$ whose cohomology we understand to be the (graded) derived Hecke algebra for the pair (G, K). There is an isomorphism ([Venkatesh nd], (148))

$$\mathrm{End}_{S[G]}(\mathbf{P}, \mathbf{P}) \simeq \oplus_{x \in K \backslash G/K} \mathrm{Hom}_{K_x}(\mathbf{Q}, \mathbf{Q}_x), \qquad (7\text{–}10)$$

where $\mathbf{Q}_x$ is the complex $\mathbf{Q}$ but with the twisted action of $K_x = K \cap \mathrm{Ad}(g_x)K$ defined by $\kappa * q = (Ad(g_x^{-1})\kappa)q$; here, we have implicitly chosen coset representatives $g_x K$ for each $x \in K \backslash G/K$. Taking cohomology, one finds that, for any i there is an isomorphism ([Venkatesh nd], (149))

$$H^i(\mathrm{End}_{S[G]}(\mathbf{P}, \mathbf{P})) \xrightarrow{\sim} \oplus_{x \in K \backslash G/K} H^i(K_x, S). \qquad (7\text{–}11)$$

Now the differential graded algebra $\mathrm{End}_{S[G]}(\mathbf{P}, \mathbf{P})$ acts on $\mathrm{Hom}_{S[G]}(\mathbf{P}, M^\bullet_\infty)$. Passing to cohomology and applying Lemma 7.4, we get a graded action of the derived Hecke algebra for (G, K) on $H^*(X_K, \omega_K)$. This action is specified by specifying, for each $x = K g_x K \in K \backslash G/K$ as above, the corresponding action of $H^*(K_x, S)$ on coherent cohomology. We can now restate Lemma A.10 of [Venkatesh nd]:

Lemma 7.6. *The action of* $h_x \in H^*(K_x, S)$ *on* $\mathbb{H}^*(K, M^\bullet_\infty)$ *is given explicitly by the following composite:*

$$\begin{array}{c}
\mathbb{H}^*(K, M^\bullet_\infty) \xrightarrow{\mathrm{Ad}(g_x^{-1})^*} \mathbb{H}^*(K_x, M^\bullet_\infty) \xrightarrow{m \mapsto g_x m} \mathbb{H}^*(K_x, M^\bullet_\infty) \xrightarrow{m \cup h_x} \mathbb{H}^*(K_x, M^\bullet_\infty) \\
\hfill \downarrow \mathrm{Cores} \\
\hfill \mathbb{H}^*(K, M^\bullet_\infty)
\end{array}$$

We obtain the derived Hecke operator $T_{q,z}$ described in Section 3, with the coefficient ring $S = \mathcal{O}/\mathfrak{p}^n$, by taking $x = \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and by taking the cohomology class $h_x \in H^1(K_x, S)$ as the composite:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_x \mapsto \langle a/d \bmod q, z \rangle,$$

where $z \in k\langle -1 \rangle$ is regarded as a homomorphism $(\mathbb{Z}/q)^* \to \mathcal{O}/\mathfrak{p}^n$. Indeed, to verify this, it only remains to show that the induced map

$$\mathbb{H}^*(K_x, M^\bullet_\infty) \to \mathbb{H}^*(K_x, M^\bullet_\infty) \qquad (7\text{–}12)$$

given by cupping with the class h_x is identified with

$$H^*(X(K_x), \omega) \xrightarrow{\cup z\mathfrak{S}} H^*(X(K_x), \omega)$$

that is to say the cup product with $z\mathfrak{S}$, i.e. the Shimura class multiplied by z, regarding as a class in the cohomology of $X(K_x)$ with coefficients in $\mathcal{O}/\mathfrak{p}^n$. This follows easily from Lemma 7.5.

The following remark is due entirely to the first-named author (M.H.); the second-named author disclaims both credit and responsibility for it.

Remark 7.1. For the benefit of those millenials who believe that the Godement resolution is one of the founding documents of the United Nations, here is a translation of the above construction into contemporary language. We thank Nick Rozenblyum for his patient guidance. We work in the DG category (or stable ∞-category) C of complexes of quasicoherent sheaves on the scheme X_∞, and consider the object ω_∞, all over $Spec(S)$. This object carries an action by $G = GL_2(\mathbf{Q}_q)$. Therefore, the object $R\Gamma(\omega_\infty)$ in the DG category Mod_S of complexes of S-modules carries an action of G. Everything up through Lemma 7.5 is automatic in this setting. The remaining observations are not strictly necessary to formulate the conjecture; however, they do provide the explicit computation of the derived Hecke operator, as in Lemma 7.6, needed in order to test the conjecture in specific applications.

8. Magma Code

What follows is a sample of Magma code that we used to compute the derived Hecke operator for the modular form of level 31, with $q = 139$ and $p = 23$.

```
N := 31;
Q := 139;
L := 23;
F := FiniteField(L);
M := ModularForms(N*Q);
S := CuspidalSubspace(M);
SQ := BaseExtend(S, RationalField());
SF := BaseExtend(S, F);
V, h := VectorSpace(SF);
time Tq := HeckeOperator(SF,N);
time Wq := AtkinLehnerOperator(SF,N);
Iq := IdentityMatrix(F, Dimension(S));
Qq := Iq +Wq*Tq; /* Qq projects
from level QN back down to level
Q */
Pro := Dimension(S);
Z<q> := PowerSeriesRing(IntegerRing());
QQ<q> := PowerSeriesRing(RationalField());
CUTOFF := Dimension(S)+3;
eps := KroneckerCharacter(-N);
WeightOneSpace := ModularForms(eps, 1);
etatemp := WeightOneSpace.2;
etaprodA := qExpansion(etatemp,
CUTOFF);
etaprodB := Composition(etaprodA, q^Q+O(q^CUTOFF));
g := etaprodA * etaprodB
 + O(q^CUTOFF);
g0 := SF ! g;
W := Vector(F, Inverse(h)(g0));
Wfin := W * Qq;
/*denom := Denominator(Wfin);
print(Factorization(denom)); */
M2 := ModularForms(Q);
S2 := CuspidalSubspace(M2);
S2Q := BaseExtend(S2, RationalField());
S2F := BaseExtend(S2, F);
V2,h2 := VectorSpace(S2F);
CUTOFF2 := Dimension(S2)+3;
projform := S2F ! h(Wfin);
projformcoeff := Vector(F, Inverse(h2)(projform));
normcoeffF := projformcoeff;
randprime := 41;
randT := HeckeOperator(S2F, randprime);
charpoly := CharacteristicPolynomial(randT);
P<u>,h3 := ChangeRing(PolynomialRing(IntegerRing()), F);
Factorization(P! charpoly);
unnormalizedredpoly := charpoly/(u-randprime-1);
redpoly := unnormalizedredpoly/Evaluate(unnormalizedredpoly,
randprime+1);
print(Evaluate(redpoly, randprime+1));
projmatrix := Evaluate(P! redpoly, randT);
finalanswerinbasis := normcoeffF * projmatrix;
print(finalanswerinbasis);
```

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