A REMARK ON A CONVERSE THEOREM OF COGDELL AND PIATETSKI-SHAPIRO

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Abstract. In this paper, we reprove a global converse theorem of Cogdell and Piatetski-Shapiro using purely global methods.

1. Introduction

Let F be a number field or a function field. Denote by $\mathbb{A}$ the ring of adeles of F and by ψ a non-trivial additive character of $F\backslash\mathbb{A}$. Let $n \geq 4$. Let π be an irreducible generic representation of $GL_n(\mathbb{A})$. We assume that the central character ω_π of π is automorphic (condition $\mathcal{A}(n,0)$). We also assume that if τ is a cuspidal automorphic representation of $GL_m(\mathbb{A})$ the complete $L-$function $L(s,\pi\times\tau)$ converges for Res large enough. We denote by $\mathcal{A}(n,m)$ the condition that, for every such τ, the $L-$function $L(s,\pi\times\tau)$ has the standard analytic properties (is nice in the terminology of Cogdell and Piatetski-Shapiro [CPS94, CPS96, CPS99, Cog02], see p. 4 for details).

Following them, for every ξ in the space V_π of π, we let W_ξ be the corresponding element of the Whittaker model $\mathcal{W}(\pi,\psi)$ of π. We denote by U_n the group of upper triangular matrices in GL_n with unit diagonal. We set

$$U_\xi(g) = \sum_{\gamma\in U_{n-1}(F)\backslash GL_{n-1}(F)} W_\xi\left[\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right],$$

$$V_\xi(g) = \sum_{\gamma\in U_{n-1}(F)\backslash GL_{n-1}(F)} W_\xi\left[\begin{pmatrix}1 & 0\\ 0 & \gamma\end{pmatrix} g\right].$$

If π is automorphic cuspidal then $U_\xi = V_\xi$ for all $\xi\in V_\pi$. Conversely, if $U_\xi = V_\xi$ for all $\xi\in V_\pi$ or, what amounts to the same, $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi\in V_\pi$, then π is automorphic.

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Let Z_{U_n} be the center of the group U_n. Cogdell and Piatetski-Shapiro ([PS76], [CPS96], [CPS99]) prove that the conditions $\mathcal{A}(n,m)$ with $0 \leq m \leq n-2$ imply that

$$\int_{Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})} (U_\xi - V_\xi)(z)\theta(z)dz = 0$$

for all non-trivial characters θ of $Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})$. They do not have the same relation for the trivial character which would then imply that $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$. Nonetheless, they prove that π is automorphic by using an ingenious local construction.

Our goal in this paper is to prove that conditions $\mathcal{A}(n,m)$, $0 \leq m \leq n-3$, imply that

$$\int_{Z_{U_n}(F)\backslash Z_{U_n}(\mathbb{A})} (U_\xi - V_\xi)(z)dz = 0\,,\ \forall \xi \in V_\pi\,.$$

This proves directly that the conditions $\mathcal{A}(n,m)$ with $0 \leq m \leq n-2$ imply $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ and, in turn, imply π is automorphic (and cuspidal).

While our result is not needed, it gives a purely global proof of the Theorem of Cogdell and Piatetski-Shapiro. It is also germane to the conjecture that the conditions $\mathcal{A}(n,m)$ with $0 \leq m \leq [\frac{n}{2}]$ imply that π is automorphic (and cuspidal). Of course, the conjecture is true for $n = 2, 3, 4$ (see [JL70], [JPSS79], [PS76] and [CPS96]).

The material is arranged as follows. In the next section, for the convenience of the reader, we review the work of Cogdell and Piatetski-Shapiro and state our result. In section 3 we provide preliminary material of an elementary nature. In section 4 we prove our result.

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2. Preliminaries and the main result

In $G_n = GL_n$ we let U_n be the subgroup of upper triangular matrices with unit diagonal. We let A_n be the group of diagonal matrices and

Z_n the center of G_n. We define a character ψ_{U_n} of $U_n(\mathbb{A})$ which is trivial on $U_n(F)$ by

$$\psi_{U_n}(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}) .$$

We let π be an irreducible generic representation of $G_n(\mathbb{A})$. As usual, this means that π is a restricted tensor product of local irreducible representations π_v. For a finite place v, π_v is an irreducible admissible representation of $G_n(F_v)$ on a complex vector space V_v. We assume that π_v is generic, that is, there is a non-zero linear form

$$\lambda_v : V_v \to \mathbb{C}$$

such that

$$\lambda_v(\pi_v(u)e) = \psi_{U_n,v}(u)\lambda_v(e)$$

for all vectors $e \in V_v$ and all $u \in U_n(F_v)$. We denote by $\mathcal{W}(\pi_v, \psi_{U_n,v})$ the space of functions

$$g \mapsto \lambda_v(\pi_v(g)e) \,, \, e \in V_v$$

on $G_n(F_v)$. It is the Whittaker model of π_v noted $\mathcal{W}(\pi_v, \psi_{U_n,v})$. For all finite v not in a finite set S, the space contains a unique vector $W_{v,0}$ fixed under $G_n(\mathcal{O}_v)$ and taking the value 1 at I_n. The representation π_v is then determined by its Langlands semi-simple conjugacy class $A_v \in G_n(\mathbb{C})$. We assume that there is an integer $m \geq 0$ such that for all finite $v \notin S$, any eigenvalue α of A_v verifies $q_v^{-m} \leq |\alpha| \leq q_v^m$.

For an infinite place v, the representation π_v is really an irreducible admissible Harish-Chandra module. We denote by (π_v, V_v) its canonical completion of slow growth in the sense of Casselman and Wallach. We assume that there is a non-zero continuous linear form

$$\lambda_v : V_v \to \mathbb{C}$$

satisfying the same condition as before. We also define $\mathcal{W}(\pi_v, \psi_{U_n,v})$ as before.

Finally, let ∞ be the set of infinite places of F and

$$G_{n,\infty} = \prod_{v \in \infty} G_n(F_v)$$

We let (π_∞, V_∞) be the topological tensor product of the representations (π_v, V_v), $v \in \infty$. Let λ be the tensor product of the linear forms λ_v, $v \in \infty$. We can define the space $\mathcal{W}(\pi_\infty, \psi_{U_n,\infty})$.

We denote by V the restricted tensor product $\bigotimes'_v V_v$ and π the natural representation of $G_n(\mathbb{A})$ on $V_\pi = V$. The Whittaker model

$\mathcal{W}(\pi, \psi_{U_n})$ of π is the space spanned by the functions

$$W_\infty \prod_{v \notin \infty} W_v$$

with $W_\infty \in \mathcal{W}(\pi_\infty, \psi_{U_n,\infty})$, $W_v \in \mathcal{W}(\pi_v, \psi_{U_n,v})$ and $W_v = W_{v,0}$ for almost all $v \notin S$. For every $\xi \in V_\pi$ we denote by W_ξ the corresponding element of $\mathcal{W}(\pi, \psi_{U_n})$.

We assume that the central character of π is automorphic. It is convenient to refer to this condition as condition $\mathcal{A}(n,0)$. In view of our assumptions, for any cuspidal automorphic representation τ of $G_m(\mathbb{A})$, $1 \leq m \leq n-1$, the complete $L-$function $L(s, \pi \times \tau)$ is defined by a convergent product for $\mathrm{Re}s$ large enough. *Condition* $\mathcal{A}(n,m)$ is that, for any such τ, the function $L(s, \pi \times \tau)$ extends to an entire function of s, bounded in vertical strips and satisfies the functional equation

$$L(s, \pi \times \tau) = \epsilon(s, \pi \times \tau, \psi) L(1-s, \widetilde{\pi} \times \widetilde{\tau}) \,.$$

In G_n let $Y_{n,m}$ be the unipotent radical of the standard parabolic subgroup of type $(m+1, 1, 1, \ldots, 1)$. For instance:

$$Y_{3,1} = \left\{ \begin{pmatrix} 1 & 0 & \bullet \\ 0 & 1 & \bullet \\ 0 & 0 & 1 \end{pmatrix} \right\}, \, Y_{5,2} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \bullet & \bullet \\ 0 & 1 & 0 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

If a function ϕ on $G_n(\mathbb{A})$ is invariant on the left under $Y_{n,m}(F)$ we set

$$\mathbb{P}^n_m(\phi)(g) = \int_{Y_{n,m}(F)\backslash Y_{n,m}(\mathbb{A})} \phi(yg) \overline{\psi}_{U_n}(y) dy \,.$$

Here dy is the Haar measure on $Y_{n,m}(\mathbb{A})$ normalized by the condition that the quotient $Y_{n,m}(F)\backslash Y_{n,m}(\mathbb{A})$ has measure 1. Our notation differs slightly from the notations of Cogdell and Piatetski-Shapiro ([Cog02]). Here $\mathbb{P}^n_m(\phi)$ is a function on $G_n(\mathbb{A})$, while in [Cog02], it is a function on $P_{m+1}(\mathbb{A})$, the mirabolic subgroup of $G_{m+1}(\mathbb{A})$, embedded in $G_n(\mathbb{A})$. Note that $Y_{n,n-1} = \{I_n\}$ and $\mathbb{P}^n_{n-1}$ is the identity.

Suppose π is as above. For each ξ in the space V_π of π we set

$$U_\xi(g) = \sum_{\gamma \in U_n(F)\backslash P_n(F)} W_\xi(\gamma g),$$

where P_n is the subgroup of matrices of G_n whose last row has the form

$$(0, 0, \ldots, 0, 1) \,.$$

This is also

$$U_\xi(g) = \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right] .$$

Then we have the following result.

Lemma 2.1. *With the previous notations,*

$$\mathbb{P}_m^n(U_\xi)(g) = \sum_{\gamma \in U_{m+1}(F)\backslash P_{m+1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m-1} \end{pmatrix} g \right] ,$$

or equivalently

$$\mathbb{P}_m^n(U_\xi)(g) = \sum_{\gamma \in U_m(F)\backslash G_m(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix} g \right] .$$

Likewise, let R_n be the subgroup of matrices of G_n whose first column is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

We can consider the function

$$V_\xi(g) = \sum_{\xi \in U_n(F)\backslash R_n(F)} W_\xi(\gamma g) .$$

Let ω_n be the permutation matrix defined by

$$\omega_1 = 1 \, , \, \omega_n = \begin{pmatrix} 0 & \omega_{n-1} \\ 1 & 0 \end{pmatrix} .$$

Then

$$R_n = \omega_n {}^t P_n^{-1} \omega_n \, , \, U_n = \omega_n {}^t U_n^{-1} \omega_n .$$

Moreover the automorphism $u \mapsto \omega_n {}^t u^{-1} \omega_n$ changes ψ_{U_n} into $\overline{\psi}_{U_n}$.

If π is automorphic cuspidal then for the cusp form ϕ_ξ corresponding to $\xi \in V_\pi$ we have

$$W_\xi(g) = \int_{U_n(F)\backslash U_n(\mathbb{A})} \phi_\xi(ug) \overline{\psi}_{U_n}(u) du$$

and

$$\phi_\xi(g) = U_\xi(g) .$$

By the previous observation relative to R_n, we also have

$$\phi_\xi(g) = V_\xi(g) .$$

Thus we have

$$U_\xi = V_\xi\,,\ \forall \xi \in V_\pi\,.$$

Conversely if π is given and

$$U_\xi = V_\xi\,,\ \forall \xi \in V_\pi\,,$$

then U_ξ is invariant on the left under $Z_n(F), P_n(F), R_n(F)$. Since these groups generate $G_n(F)$, for every $\xi \in V_\pi$ the function U_ξ is invariant on the left under $G_n(F)$ and hence π is automorphic.

In general, U_ξ and V_ξ are invariant on the left under $P_n(F) \cap R_n(F)$ and $A_n(F)$. In other words, they are invariant under $S_n(F)$ where S_n is the standard parabolic subgroup of type $(1, n-2, 1)$. The notations here differ slightly from those of Cogdell and Piatetski-Shapiro ([CPS99]). Moreover, we have, for all $g, h \in G_(\mathbb{A})$,

$$W_{\pi(h)\xi}(g) = W_\xi(gh)$$

and similar formulae for U_ξ and V_ξ. As a consequence, if an identity involving W_ξ, U_ξ or V_ξ is true for all $\xi \in V_\pi$, the identity obtained by translating the function on the right by an arbitrary element of $G_n(\mathbb{A})$ is also true for all $\xi \in V_\pi$, and conversely. For instance, the relation $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ is equivalent to the relation $U_\xi(g) = V_\xi(g)$ for all $\xi \in V_\pi$ and for all $g \in G_n(\mathbb{A})$. We appeal repeatedly to this principle.

Following Cogdell and Piatetski-Shapiro ([Cog02, Section 5.2]), for $1 \le m \le n-2$, we define

$$\alpha_m = \begin{pmatrix} 0 & 1 & 0 \\ I_m & 0 & 0 \\ 0 & 0 & I_{n-m-1} \end{pmatrix}.$$

We note that $\alpha_m \in P_n$. We also define

$$V_\xi^m(g) = V_\xi(\alpha_m g)\,, \forall g \in G_n(\mathbb{A})\,.$$

Thus V_ξ^m is invariant under $Q_m = \alpha_m^{-1} R_n \alpha_m$. This is the subgroup of matrices of G_n whose $(m+1)$-th column has the form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

with 1 in the $(m+1)$-th row. Note that Q_m contains the group $Y_{n,m}$. Thus we may consider $\mathbb{P}^n_m(V^m_\xi)$.

Theorem 2.2 (Cogdell, Piatetski-Shapiro, Section 5 of [Cog02], [CPS99])**.** *Suppose conditions $\mathcal{A}(n,j)$, $0 \le j \le m$, are satisfied. Then, for all $\xi \in V_\pi$,*

$$\mathbb{P}^n_m(U_\xi) = \mathbb{P}^n_m(V^m_\xi).$$

We denote by $\mathcal{E}(n,m)$ the condition that

$$\mathbb{P}^n_m(U_\xi) = \mathbb{P}^n_m(V^m_\xi)\,,\ \forall \xi \in V_\pi\,.$$

This condition can be simplified. Indeed, we have the following result.

Proposition 2.3 (Cogdell, Piatetski-Shapiro, Section 5 of [Cog02], [CPS99])**.** *Let $k, 1 \le k \le n-m$, be an integer. The condition $\mathcal{E}(n,m)$ is equivalent to the condition*

$$\begin{aligned}(2.1)\qquad &\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})} \int U_\xi \left[\begin{pmatrix} I_m & x \\ 0 & u \end{pmatrix} g \right] \overline{\psi}_{U_{n-m}}(u)dxdu \\ &= \int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})} \int V^m_\xi \left[\begin{pmatrix} I_m & x \\ 0 & u \end{pmatrix} g \right] \overline{\psi}_{U_{n-m}}(u)dxdu\,,\end{aligned}$$

for all $\xi \in V_\pi$ and for all $g \in G_n(\mathbb{A})$, where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$ with zero first k columns.

Proof: For the convenience of the reader, we review the proof. For $k = 1$ our conclusion is just the hypothesis. Thus we may assume our assertion true for k, $1 \le k \le n-m-1$ and prove it for $k+1$. In the integral we write $x = (0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots x_{n-m})$ where the x_i are column vectors of length m. We also introduce

$$\gamma_\beta = \begin{pmatrix} I_m & 0 & 0 \\ X_\beta & I_k & 0 \\ 0 & 0 & I_{n-m-k} \end{pmatrix}\,,\ X_\beta = \left.\overbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}}^{m}\right\}k\,,\ \beta \in F^m\,.$$

Since γ_β is in $P_n(F) \cap Q_m(F)$, in the identity, we can conjugate the matrices by γ_β. We note that

$$\gamma_\beta \begin{pmatrix} I_m & x \\ 0 & u \end{pmatrix} \gamma_\beta^{-1} = \begin{pmatrix} I_m & x \\ 0 & u' \end{pmatrix}$$

where

$$u'_{m+k,m+k+1} = u_{m+k,m+k+1} + \beta x_{k+1}\,.$$

Hence the equality becomes

$$\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)\psi(-\beta x_{k+1})dxdu$$
$$= \int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi^m\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)\psi(-\beta x_{k+1})dxdu\,,$$

where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$, with zero first k columns. Summing over all $\beta \in F^m$ and applying the theory of Fourier series, we get

$$\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)dxdu$$
$$= \int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi^m\left[\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix} g\right]\overline{\psi}_{U_{n-m}}(u)dxdu\,,$$

where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$, with zero first $k+1$ columns.

The other direction is obvious, since the if the condition (2.1) holds for $k+1$, then integrating both sides with respect to

$$x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$$

with only $(k+1)$-th column being possibly nonzero, we obtain the condition (2.1) for k, which is equivalent to the condition $\mathcal{E}(n, n-m)$ by induction assumption. This concludes the proof of the proposition. □

Since $\alpha_m \in P_n$ and U_ξ is $P_n(F)$ invariant on the left, we can apply the condition (2.1) with g replaced by α_m^{-1}. Thus the condition (2.1) can be written also as

$$\begin{aligned}&\int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int U_\xi\left[\alpha_m\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix}\alpha_m^{-1}\right]\overline{\psi}_{U_{n-m}}(u)dxdu\\ &= \int_{U_{n-m}(F)\backslash U_{n-m}(\mathbb{A})}\int V_\xi\left[\alpha_m\begin{pmatrix} I_m & x \\ 0 & u\end{pmatrix}\alpha_m^{-1}\right]\overline{\psi}_{U_{n-m}}(u)dxdu\,,\end{aligned}\tag{2.2}$$

for all $\xi \in V_\pi$, where $x \in M_{m\times(n-m)}(F)\backslash M_{m\times(n-m)}(\mathbb{A})$ with zero first k columns. In this paper, we are mainly interested in the case $k = n-m$ of condition (2.2). We make it explicit.

Taking $k = n-m$ and $m = n-2$, then condition (2.2) leads to the condition of Cogdell and Piatetski-Shapiro

$$\int_{F\backslash\mathbb{A}}(U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1\end{pmatrix}\psi(-z)dz = 0$$

for all $\xi \in V_\pi$. Since U_ξ and V_ξ are invariant on the left under $A_n(F)$ and this relation is true for any right translate of U_ξ and V_ξ we can conjugate by an element of $A_n(F)$ and obtain the condition

$$\int_{F\backslash\mathbb{A}}(U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\psi(-\alpha z)dz = 0\,,$$

for all $\xi \in V_\pi$ and all $\alpha \in F^\times$.

Now let us take $k = n - m$ and $m = n - 3$. Let e_0 and f_0 be respectively the following row and column of size $n-2$:

$$e_0 = (0, 0, \dots, 0, 1)\,,\ f_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Condition (2.2) reads

$$\int_{(F\backslash\mathbb{A})^2}\int_{F\backslash\mathbb{A}}(U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix}\psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. Abusing notation, we write the above integral as

$$\int_{(F\backslash\mathbb{A})^3}(U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix}\psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. For instance, for $n = 4$ the condition reads

$$\int_{(F\backslash\mathbb{A})^3}(U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}\psi(-x-y)dzdxdy = 0\,,$$

for all $\xi \in V_\pi$. Again we can conjugate by an element of $A_n(F)$ to obtain

$$\int_{(F\backslash\mathbb{A})^3}(U_\xi - V_\xi)\begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix}\psi(-\alpha x-\beta y)dzdxdy = 0\,, \tag{2.3}$$

for all $\xi \in V_\pi, \alpha \in F^\times, \beta \in F^\times$. Our own contribution is the following.

Theorem 2.4. *Suppose $n > 3$. Then condition $\mathcal{E}(n, n-3)$ is equivalent to the condition*

$$\int_{F\backslash\mathbb{A}}(U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}dz = 0$$

for all $\xi \in V_\pi$.

Combining our Theorem 2.4 with the results of Cogdell and Piatetski-Shapiro ([Cog02]) we arrive at the following result.

Corollary 2.5. *Suppose conditions* $\mathcal{A}(n,i)$, $0 \leq i \leq n-3$, *are satisfied. Then for every* $\xi \in V_\pi$

$$\int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} dz = 0\,.$$

3. Separable functions

Let U and V be vector spaces over F. A function

$$\Phi : U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$$

is said to be (additively) *separable*, if there exist two functions $\Phi_1 : U(F)\backslash U(\mathbb{A}) \to \mathbb{C}$ and $\Phi_2 : V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$ such that, for all $(u,v) \in U(\mathbb{A}) \times V(\mathbb{A})$,

$$\Phi(u,v) = \Phi_1(u) + \Phi_2(v)\,.$$

It amounts to the same to demand that, for all (u,v),

$$\Phi(u,v) = \Phi(u,0) + \Phi(0,v) - \Phi(0,0)\,.$$

In what follows, if $\Phi : U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A}) \to \mathbb{C}$ is a function, an integral

$$\int\int \Phi(u,v)dudv$$

means that the integral is over the product $U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A})$ and the Haar measure du (resp. dv) is normalized by demanding that the quotients have volume 1. This convention remains in force for the rest of this paper.

Lemma 3.1. *Suppose that*

$$\Phi : F\backslash\mathbb{A} \times F\backslash\mathbb{A} \to \mathbb{C}$$

is a smooth function such that, for all $\alpha \in F^\times$, $\beta \in F^\times$,

$$\int\int \Phi(u,v)\psi(\alpha u + \beta v)dudv = 0\,.$$

Then Φ *is separable.*

Proof: We write the Fourier expansion of Φ,

$$\begin{aligned}&\Phi(x,y)\\ =&\sum_{\alpha\in F,\beta\in F}\psi(\alpha x+\beta y)\left(\int\int\Phi(u,v)\psi(-\alpha u-\beta v)dudv\right).\end{aligned}$$

In view of the assumptions, we have

$$\begin{aligned}&\Phi(x,y)\\ =&\int\int\Phi(u,v)dudv\\ +&\sum_{\alpha\in F^\times}\psi(\alpha x)\left(\int\int\Phi(u,v)\psi(-\alpha u)dudv\right)\\ +&\sum_{\beta\in F^\times}\psi(\beta y)\left(\int\int\Phi(u,v)\psi(-\beta v)dudv\right).\end{aligned}$$

The function on the right hand side of the equation is indeed separable.
□

Proposition 3.2. *Suppose U and V are finite dimensional spaces over F in duality by the bi-linear form $(u,v)\mapsto\langle u,v\rangle$. Suppose*

$$\Phi : U(F)\backslash U(\mathbb{A})\times V(F)\backslash V(\mathbb{A})\to\mathbb{C}$$

is a smooth function with the following property: for all pairs $(e,f)\in U(F)\times V(F)$ with $\langle e,f\rangle=1$ we have

$$\int_{(F\backslash\mathbb{A})^2}\Phi(u+xe,v+yf)\psi(\alpha x+\beta y)dxdy=0$$

for all $u\in U(\mathbb{A})$, all $v\in V(\mathbb{A})$, all $\alpha\in F^\times$, and all $\beta\in F^\times$. Then Φ is separable.

Proof: If $\dim(U)=\dim(V)=1$, our assertion follows from the previous lemma. Thus we may assume that $\dim(U)=\dim(V)=n+1$, $n>0$ and our assertion is true for dimension n. Let $e\in U(F), f\in V(F)$ with $\langle e,f\rangle=1$. Let U_1 be the subspace of U orthogonal to f and V_1 the subspace of V orthogonal to e. By Lemma 3.1, for $u\in U_1(\mathbb{A}), v\in V_1(\mathbb{A})$ we have

$$\Phi[u+se,v+tf]=\Phi[u+se,v]+\Phi[u,v+tf]-\Phi[u,v]\,.$$

Each one of the functions

$$(u,v)\mapsto\Phi[u+se,v]\,,\,(u,v)\mapsto\Phi[u,v+tf]\,,\,(u,v)\mapsto\Phi[u,v]$$

satisfies the assumptions of the proposition. By the induction hypothesis, the right hand side is equal to

$$\Phi[u+se,0]+\Phi[se,v]-\Phi[se,0]+\Phi[u,tf]+\Phi[0,v+tf]-\Phi[0,tf]$$
$$-\Phi[u,0]-\Phi[0,v]+\Phi[0,0]\,.$$

Thus it suffices to show that $(u,t)\mapsto\Phi[u,tf]$ and $(s,v)\mapsto\Phi[se,v]$ are separable functions. Let us show this is the case for the first function. Let $e_1,e_2,\dots,e_n$ be a basis of $U_1(F)$. Write

$$u=s_1e_1+s_2e_2+\cdots+s_ne_n\,.$$

Now $\langle e_1+e,f\rangle=1$. Thus,

$$\Phi[s_1e_1+s_2e_2+\cdots s_ne_n+s_1e,v+tf]$$

must be separable as a function of (s_1,t). All the terms on the right hand side (with $s=s_1$) have this property, except possibly the term

$$\Phi[u,tf]\,.$$

Thus this term must have this property as well. Hence

$$\Phi[s_1e_1+s_2e_2+\cdots+s_ne_n,tf]$$

is a separable function of the pair (s_1,t). Likewise it is a separable function of each pair (s_j,t), $1\le j\le n$. By the lemma below it is a separable function of $((s_1,s_2,\dots,s_n),t)$, that is, $\Phi[u,tf]$ is a separable function of (u,t). □

Lemma 3.3. *Suppose*

$$\Phi((s_1,s_2,\dots,s_n),t)$$

is a function with the property that for each index j it is a separable function of (s_j,t). Then it is a separable function of the pair

$$((s_1,s_2,\dots,s_n),t)\,.$$

PROOF: Our assertion is obvious if $n=1$. So we may assume $n>1$ and our assertion true for $n-1$. We have, by separability in (s_1,t),

$$\Phi((s_1,s_2,\dots s_n),t)=$$
$$\Phi((s_1,s_2,\dots s_n),0)+\Phi((0,s_2,\dots s_n),t)-\Phi((0,s_2,\dots s_n),0)\,.$$

By the induction hypothesis the term $\Phi((0,s_2,\dots s_n),t)$ is a separable function of the pair $((s_2,\dots s_n),t)$. Thus the right hand side is a separable function of the pair $((s_1,s_2,\dots s_n),t)$. □

Finally, we have a simple criterion to decide whether a separable function vanishes.

Proposition 3.4. *Suppose Φ is a separable smooth function on*

$$U(F)\backslash U(\mathbb{A}) \times V(F)\backslash V(\mathbb{A})\,,$$

where (U,V) is a pair of vector spaces over F in duality by the bilinear form $(u,v) \mapsto \langle u,v\rangle$. Suppose that

$$\int\int \Phi(u,v)\psi(\langle u,f\rangle + \langle e,v\rangle)dudv = 0\,,$$

if $e \in U(F)$, $f \in V(F)$, and either $e=0$ or $f=0$ (or both $e=0$ and $f=0$). Then $\Phi = 0$.

Proof: By assumption,

$$\Phi(u,v) = \Phi(u,0) + \Phi(0,v) - \Phi(0,0)\,.$$

Taking $e=0$ and $f \neq 0$ we get

$$\int \Phi(u,0)\psi(\langle u,f\rangle)du + \left(\int \Phi(0,v)dv - \Phi(0,0)\right)\int \psi(\langle u,f\rangle)du = 0\,.$$

Since $f \neq 0$, there exists $u_0 \in U(F)\backslash U(\mathbb{A})$, such that $\psi(\langle u_0,f\rangle) \neq 1$. By changing of variables,

$$\int \psi(\langle u,f\rangle)du = \int \psi(\langle u+u_0,f\rangle)du = \psi(\langle u_0,f\rangle)\cdot\int \psi(\langle u,f\rangle)du\,,$$

which implies that $\int \psi(\langle u,f\rangle)du = 0$. Hence we get

$$\int \Phi(u,0)\psi(\langle u,f\rangle)du = 0\,.$$

This shows that $u \mapsto \Phi(u,0)$ is a constant function. Likewise $v \mapsto \Phi(0,v)$ is a constant function. By the above formula from assumption, $(u,v) \mapsto \Phi(u,v)$ is a constant function. Now if we take $e=0$ and $f=0$ we get

$$\int\int \Phi(u,v)dudv = 0\,.$$

But since Φ is constant, this integral is just the constant value of Φ. Hence Φ is 0 as claimed. □

4. Proof of Theorem 2.4

It will be convenient to introduce for every $\xi \in V_\pi$ the function

$$\Phi_\xi(u,v) = \int_{F\backslash\mathbb{A}} (U_\xi - V_\xi)\begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} dz\,.$$

Here u is a row of size $n-2$ and v a column of size $n-2$. The scalar product of u and v is denoted by $\langle u, v\rangle$. Thus Φ_ξ is a smooth function on $(F\backslash\mathbb{A})^{n-2}\times(F\backslash\mathbb{A})^{n-2}$.

Proposition 4.1. *Let A be a rational column of size $n-2$ and B a rational row of size $n-2$. Suppose $A=0$ or $B=0$ (or both $A=0$ and $B=0$). Then, for every $\xi\in V_\pi$, the integral*

$$\int\int \Phi_\xi(u,v)\psi(\langle u,A\rangle+\langle B,v\rangle)dudv$$

is 0.

Proof: It suffices to prove the integral

$$\int\int\int U_\xi\begin{pmatrix}1&u&z\\0&I_{n-2}&v\\0&0&1\end{pmatrix}\psi(\langle u,A\rangle+\langle B,v\rangle)dudvdz$$

has the properties described in the proposition.

Indeed, the automorphism

$$g\mapsto \omega_n{}^tg^{-1}\omega_n$$

changes the function U_ξ relative to the character ψ into the function V_ξ relative to the character ψ^{-1} and leaves invariant the group over which we integrate. Thus the integral

$$\int\int\int V_\xi\begin{pmatrix}1&u&z\\0&I_{n-2}&v\\0&0&1\end{pmatrix}\psi(uA+Bv)dudvdz$$

has the same properties and so does the integral of the difference $U_\xi-V_\xi$. Note that $\langle u,A\rangle=uA$, $\langle B,v\rangle=Bv$.

Now consider the integral for U_ξ. If $B=0$ it suffices to show that, for every ξ,

$$\int\int U_\xi\begin{pmatrix}1&0&z\\0&I_{n-2}&v\\0&0&1\end{pmatrix}dvdz=0\,.$$

Indeed, as we have remarked, if this identity is true for all ξ, then the identity obtained by translating on the right by an element of G_n is still true for every ξ. In particular, then, for all ξ and all u,

$$\int\int U_\xi\left[\begin{pmatrix}1&0&z\\0&I_{n-2}&v\\0&0&1\end{pmatrix}\begin{pmatrix}1&u&0\\0&I_{n-2}&0\\0&0&1\end{pmatrix}\right]dvdz=0\,.$$

Integrating over u with respect to the character $\psi(uA)$ we find

$$\int du \int\int U_\xi \left[\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\psi(uA)du = 0\,,$$

or

$$\int\int\int U_\xi \begin{pmatrix} 1 & u & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \psi(uA)dudvdz = 0\,,$$

as claimed.

Replacing U_ξ by its definition, we find

$$\int\int U_\xi \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} dvdz =$$

$$\int\int \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

Exchanging summation and integration we find

$$\int\int \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

$$= \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} \int\int W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \right] dvdz\,.$$

After a change of variables this becomes

$$= \sum_{\gamma \in U_{n-1}(F)\backslash G_{n-1}(F)} \int\int W_\xi \left[\begin{pmatrix} 1 & 0 & z \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \right] dvdz$$

which is 0.

Now suppose $B \neq 0$ (and thus $A = 0$). To prove that the integral vanishes we may conjugate by a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \gamma \in G_{n-2}(F)\,.$$

This amounts to replacing B by $B\gamma^{-1}$. Thus we may assume $B = (0,0,\ldots,0,-1)$. Then the integral takes the form

$$\int \mathbb{P}^n_{n-2} U_\xi \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} du\,.$$

Replacing $\mathbb{P}^n_{n-2}U_\xi$ by its expression in terms of W_ξ we find

$$\int \sum_{\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & u & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] du\,.$$

Let us write this as

$$\int \left(\int \sum_\gamma W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & u'' & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right] du'' \right) du'\,,$$

with $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$. The inner integral is in fact a multiple of the integral

$$\int \psi \left(\langle \gamma_{n-2}, \begin{pmatrix} u'' \\ 0 \end{pmatrix} \rangle \right) du'\,,$$

where γ_{n-2} is the last row of γ. This integral is then 0 unless the last row has the form

$$(0, \bullet, \bullet, \ldots, \bullet)\,,$$

in which case the integral is 1. Thus our expression is

$$\int \sum_\gamma W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right] du'\,,$$

where $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$ and the last row of γ has the form $(0, \bullet, \bullet, \ldots, \bullet)$.

Now let us write the Bruhat decomposition of $\gamma \in U_{n-2}(F)\backslash G_{n-2}(F)$:

$$\gamma = w\nu\alpha$$

with w a permutation matrix, α a diagonal matrix and $\nu \in U_{n-2}(F)$. The last row of w cannot have the form $(1,0,\ldots,0)$ otherwise the last row of γ would have the form $(x, \bullet, \bullet, \ldots, \bullet)$, $x \neq 0$. Thus the last row of w has the form $(0, \bullet, \bullet, \ldots, \bullet)$. Let us write down the contribution of such a w to the above expression and show it is 0. We introduce the abelian group

$$X = \left\{ \begin{pmatrix} 1 & u' & 0 \\ 0 & I_{n-3} & 0 \\ 0 & 0 & I_2 \end{pmatrix} \right\}\,.$$

The contribution of w has the form

$$\int_{X(F)\backslash X(\mathbb{A})} \sum_{\gamma} W_\xi \left[\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix} x \right] dx \,,$$

where $\gamma = w\nu\alpha$, $\alpha \in A_{n-2}(F)$ and ν is in a set of representatives for the cosets $w^{-1}U_{n-2}(F) \cap U_{n-2}(F)\backslash U_{n-2}(F)$. For a set of representatives, we will take the group

$$S = w^{-1}\overline{U_{n-2}}(F)w \cap U_{n-2}(F) \,,$$

where as usual $\overline{U_{n-2}}$ is the subgroup opposite to U_{n-2} (that is, its transpose).

Now viewed as a subgroup of $G_{n-2}(F)$ the group $X(F)$ is the unipotent radical of the standard parabolic subgroup of type $(1, n-3)$ and $U_{n-2}(F)$ is contained in this parabolic subgroup. In particular, X is normalized by $U_{n-2}(F)$ and by S.

We keep in mind that X is an abelian group. Let us write X as the product X_1X_2 where

$$X_1(F) = w^{-1}U_{n-2}(F)w \cap X(F) \,,\ X_2(F) = w^{-1}\overline{U_{n-2}}(F)w \cap X(F) \,.$$

Thus S contains $X_2(F)$ and is the product of $X_2(F)$ and the group T,

$$T = S \cap \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & I_2 \end{pmatrix} : \mu \in U_{n-3}(F) \right\} .$$

Moreover the group S normalizes X_2. In particular it normalizes the groups

$$X(F) = X_1(F)X_2(F) \,,\ X_2(\mathbb{A}) \,,$$

hence also the closed subgroup

$$X_1(F)X_2(\mathbb{A}) \,.$$

Then our expression becomes

$$\int\int \sum W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \nu\alpha & 0 \\ 0 & I_2 \end{pmatrix} x_1x_2 \right] dx_1dx_2 \,.$$

Here x_1 and x_2 are integrated over $X_1(F)\backslash X_1(\mathbb{A})$ and $X_2(F)\backslash X_2(\mathbb{A})$ respectively. We sum for $\alpha \in A_{n-2}(F)$ and $\nu \in S$. We can take the sum over α outside as follows:

$$\sum_{\alpha} \int\int \sum_{\nu} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & I_2 \end{pmatrix} x_1x_2 \begin{pmatrix} \alpha & 0 \\ 0 & I_2 \end{pmatrix} \right] dx_1dx_2 \,.$$

We now show that each term of the α sum is 0 for all ξ. As usual, we may take $\alpha = I_{n-2}$. Now we write

$$\nu = \tau\sigma$$

with $\sigma \in X_2(F)$ and τ in T. We combine the integration over x_2 and the sum over σ to obtain an integral over $X_2(\mathbb{A})$. We arrive at

$$\int dx_1 \left(\sum_\tau \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_1 x_2 \right] dx_2 \right) .$$

Here the integral is for $x_1 \in X_1(F)\backslash X_1(\mathbb{A})$ and $x_2 \in X_2(\mathbb{A})$. We will show that for every τ and every ξ the following integral is 0:

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 \left(\int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_1 x_2 \right] dx_2 \right) .$$

In order for this expression to even make sense we better show first that, on $X(\mathbb{A})$, the function

$$x \mapsto \int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x x_2 \right] dx_2$$

is invariant under $X_1(F)$. Recall it is invariant under $X_2(\mathbb{A})$. So it amounts to the same to prove it is invariant under $X_1(F)X_2(\mathbb{A})$. Since τ normalizes the groups $X(\mathbb{A})$ and $X_1(F)X_2(\mathbb{A})$ it amounts to the same to prove that on $X(\mathbb{A})$ the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

is invariant under $X_1(F)X_2(\mathbb{A})$. The invariance under $X_2(\mathbb{A})$ being clear we check the invariance under $X_1(F)$. But if $x_1 \in X_1(F)$ we have

$$\int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

$$= \int W_\xi \left[y_1 \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

with

$$y_1 = \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix}^{-1} .$$

Since y_1 is in $U_{n-2}(F)$, this expression does not depend on x_1.

At this point we can reformulate our goal as follows: we have to prove that for every τ and every ξ, the integral of the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x x_2 \right] dx_2$$

over the quotient

$$X_1(F)X_2(\mathbb{A})\backslash X(\mathbb{A})$$

is 0. Conjugation by τ defines an automorphism of this quotient which preserves the Haar measure. Hence it suffices to prove that the integral of the function

$$x \mapsto \int W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2$$

over the same quotient vanishes. Equivalently, we want to prove that

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 \int_{X_2(\mathbb{A})} W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \begin{pmatrix} \tau & 0 \\ 0 & I_2 \end{pmatrix} x_2 \right] dx_2 = 0$$

for all τ and all ξ. At this point we may exchange the order of integration. So it will be enough to prove that

$$\int_{X_1(F)\backslash X_1(\mathbb{A})} dx_1 W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} x_1 \right] g = 0$$

for all ξ and g. Now let Y be the subgroup defined by

$$Y(F) = U_{n-2}(F) \cap w X_1(F) w^{-1} \,.$$

The integral can be written as

$$\int_{Y(F)\backslash Y(\mathbb{A})} \psi_{U_n}(y) dy \; W_\xi \left[\begin{pmatrix} w & 0 \\ 0 & I_2 \end{pmatrix} g \right] .$$

Now we claim that the subgroup Y contains a root subgroup for a positive simple root. Thus the character ψ_{U_n} is non-trivial on the subgroup $Y(\mathbb{A})$ and the integral vanishes which concludes the proof.

It remains to prove the claim. Since the proof requires an inductive argument we state the claim as a separate lemma. □

Lemma 4.2. *Let X be the unipotent radical of the standard parabolic subgroup of type $(1, m-1)$ in GL_m. Let $w \in GL_m$ be a permutation matrix whose last row has the form $(0, \bullet, \bullet, \dots, \bullet)$. Then conjugation by w changes one of the root subgroups of X into the root subgroup associated to a positive simple root.*

Proof: Our assertion is trivial if $m = 2$ because then $w = I_2$. So we may assume $m > 2$ and our assertion true for $m - 1$. The matrix w has the form

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w_2 \end{pmatrix}$$

where w_1 and w_2 are permutation matrices of size $m - 1$. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & w_2 \end{pmatrix}$$

normalizes X, it suffices to prove our assertion for

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & 1 \end{pmatrix} .$$

If the last row of w_1 has the form

$$(1, 0, \dots, 0) ,$$

the root group corresponding to $e_1 - e_m$ is conjugated to the root group corresponding to $e_{m-1} - e_m$. So our assertion is true in this case. If the last row of w_1 has the form

$$(0, \bullet, \dots, \bullet) ,$$

we can apply the induction hypothesis to w_1 and obtain again our assertion. □

Proof of Theorem 2.4: Recall from (2.3) that

$$\int_{(F\backslash \mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe_0 & z \\ 0 & I_{n-2} & yf_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0$$

for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Here e_0 and f_0 are of size $n-2$ and

$$e_0 = (0, 0, \dots, 0, 1) ,\ f_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} .$$

Note that if $\Phi_\xi = 0$ for all $\xi \in V_\pi$, then the condition $\mathcal{E}(n, n-3)$ holds, since the integral over z is just an inner integral of the above integral. Hence, in the following, we assume that the condition $\mathcal{E}(n, n-3)$ holds, that is, the above integral equals to 0.

We can conjugate by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} ,\ \gamma \in G_{n-2}(F)$$

to obtain

$$\int_{(F\backslash \mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe_0\gamma^{-1} & z \\ 0 & I_{n-2} & y\gamma f_0 \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0 ,$$

for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Thus

$$\int_{(F\backslash\mathbb{A})^3} (U_\xi - V_\xi) \begin{pmatrix} 1 & xe & z \\ 0 & I_{n-2} & yf \\ 0 & 0 & 1 \end{pmatrix} \psi(\alpha x + \beta y) dz dx dy = 0\,,$$

for e (resp. f) an F-row (resp. column) of size $n-2$ and $\langle e, f\rangle = 1$ and for all $\xi \in V_\pi$, $\alpha \in F^\times, \beta \in F^\times$. Moreover, right translating by an adelic matrix in the unipotent radical of the parabolic subgroup of type $(1, n-2, 1)$ we obtain

$$\int_{(F\backslash\mathbb{A})^3} \Phi_\xi(u + xe, v + yf)\psi(\alpha x + \beta y) dz dx dy = 0\,,$$

for all $(u, v) \in \mathbb{A}^{n-2} \times \mathbb{A}^{n-2}$, and for all $\xi \in V_\pi$, $\alpha \in F^\times$, $\beta \in F^\times$. Thus by Proposition 3.2, the function Φ_ξ is separable, for all $\xi \in V_\pi$. By Propositions 4.1 and 3.4, it is in fact 0 and we are done. $\square$

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