MUSIC and Ramanujan: MUSIC-like Algorithms for Integer Periods Using Nested-Periodic-Subspaces

Srikanth V. Tenneti and P. P. Vaidyanathan
Department of Electrical Engineering, California Institute of Technology
Pasadena, California 91125
Email: stenneti@caltech.edu; ppvnath@systems.caltech.edu

***Abstract*—Can the MUSIC algorithm be used for period estimation? Prior works in this direction were based on modifying the search over the conventional complex-exponentials based pseudospectrum to look for harmonically spaced peaks. For applications where the period of the discrete signal can be well approximated by integers, this paper proposes much simpler integer valued basis functions. It is shown that this new reformulation of MUSIC not only makes the pseudo-spectrum computation much simpler, but also offers significantly higher accuracy than the conventional techniques, especially for mixtures of periodic signals[1].**

Keywords—Period Estimation, MUltiple SIgnal Classification (MUSIC), Ramanujan Subspaces, Nested Periodic Subspaces, Subspace Methods.

I. Introduction

One of the most well-known family of algorithms for recovering complex-exponentials in noise is the MUltiple SIgnal Classification (MUSIC) algorithm [12] and its extensions. They have widespread applications in many fields, such as in DOA estimation [20], time delay estimation [10], speech processing [5] and so on. Broadly speaking, MUSIC involves two steps: First, a signal's autocorrelation matrix is eigen-decomposed to estimate the noise subspace. The signal frequencies are then estimated by identifying the frequencies (Vandermonde vectors) that are orthogonal to the noise subspace.

Closely related to estimating complex-exponentials, is the problem of period estimation. Formally, a discrete time signal $x(n)$ is said to be periodic if there exists an integer P such that:

$$x(n+P) = x(n) \qquad \forall n \in \mathbb{Z} \tag{1}$$

Here, P is known as a *repetition index*, and the smallest positive repetition index is known as the *period*. Using Fourier series, such a signal can be decomposed as:

$$x(n) = \sum_{k=0}^{P-1} c_k e^{j\frac{2\pi k}{P}n} \tag{2}$$

So in principle, we may use MUSIC to estimate the fundamental frequency $2\pi/P$, from which we can estimate the period. But is this a good strategy?

There are two distinct contexts in which this question can be answered. In conventional applications such as speech, cardiology, EEG analysis etc., a continuous time periodic signal is sampled. For such applications, assuming the period of the sampled signal to be an integer may not be exactly true, but the approximation gets better as the sampling rate increases. However, such signals can still be modeled as a sum of harmonically spaced sinusoids as:

$$x(n) = \sum_{k=0}^{K-1} c_k e^{jk\omega_0 n} \tag{3}$$

where $2\pi/\omega_0$ (possibly not an integer) is usually considered as the 'period'. Now, MUSIC itself does not take into account this harmonic structure between the sinusoids. But in a series of publications by Christensen, et al., [3], [4], [5] the search over the conventional MUSIC pseudospectrum was modified to look for harmonically spaced peaks, with the resulting algorithms offering very accurate estimates of the pitch period.

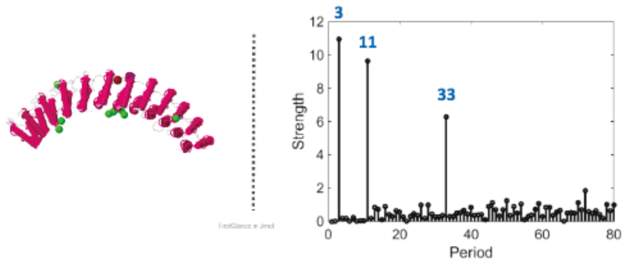

Fig. 1: *Applications with Integer Periodicity:* The protein AnkyrinR (PDB 1n11) that enables red blood cells to resist shear forces. Its period 33 structural repeats can be clearly identified in the plot on the right, produced by the proposed techniques.

However, there is a second context of applications, where the period is naturally an integer. For example, repeating segments in DNA, known as micro-satellites, are used as the primary bio-markers in forensics and kinship analysis [2], [14]. Similarly, periodic three dimensional structures in proteins can be traced back to repeats in the underlying amino acid sequence (Fig. 1). Such repeats in proteins play important roles in a number of diverse contexts, such as in enzyme functioning, intracellular transport, behavioral disorders etc. [1], [7]. For such applications, this paper shows that one can design much more accurate, as well as computationally efficient eigen-space techniques than MUSIC and its prior adaptations.

The MUSIC algorithm and its variants are based on a complex-exponentials based representation for the signal. Consequently, a significant amount of computation is spent on evaluating the MUSIC psuedo-spectrum on a fine grid of uniformly spaced frequencies between 0 and 2π (e.g. [4]). We will show that this approach is not necessarily the best for signals satisfying (1). One in fact needs a non-uniform grid as shown in Fig. 3 later. Further, while prior MUSIC based techniques interpret the period of a complex-exponential

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$e^{j\frac{2\pi k}{P}n}$ as P/k, its correct integer period according to (1) is in fact $P/\gcd(P,k)$. This small change results in a significant improvement in the estimation accuracy. We also show that the psuedo-spectrum can in fact be evaluated on much simpler integer valued basis vectors than complex-exponentials. This is done using the recently proposed Nested Periodic Subspaces (NPSs) [15], [16] (which were originally inspired by the Ramanujan-subspaces of [17]). The NPSs offer simple integer valued basis vectors to span sequences with integer periods. In fact, some of the NPSs have sparse basis vectors consisting of 1's and mostly 0's (Fig. 2). The resulting algorithms are much more accurate, and at the same time, require significantly less computations compared to prior adaptations of MUSIC.

Outline of the paper: Sec. II gives a short summary of MUSIC and its prior adaptations to harmonic spectra. Sec. III presents our new algorithms, and Sec. IV discusses their performance using several simulation experiments.

II. A Summary of Prior Works

Let us start with a brief overview of MUSIC [12] in the context of time domain signals. Let $x(n)$ be as follows:

$$x(n) = \sum_{k=0}^{K-1} c_k e^{j\omega_k n} + e(n) \tag{4}$$

where $e(n)$ is a zero mean WSS noise term with $\mathrm{E}[e^*(r)e(k)] = \sigma^2\delta(r-k)$. The coefficients c_k's are typically modeled as having deterministic amplitudes (i.e., $|c_k|$), but with their phases as independent random variables, uniformly distributed over $[-\pi, \pi)$ [4], [5]. We can rewrite (4) as:

$$\mathbf{x} = \mathbf{Ac} + \mathbf{e} \tag{5}$$

where $\mathbf{x}$ and $\mathbf{e}$ are $N \times 1$ vectors consisting of any $N > K$ consecutive samples of $x(n)$ and $e(n)$ respectively. $\mathbf{A}$ is an $N \times K$ matrix whose columns are the complex-exponentials in (4), and $\mathbf{c}$ consists of the corresponding coefficients. The autocorrelation matrix of $\mathbf{x}$ is then given by:

$$\mathbf{R}_{xx} = \mathbf{ACA}^\dagger + \sigma^2\mathbf{I} \tag{6}$$

where $(\dagger)$ denotes transpose conjugation, and $\mathbf{C} = \mathrm{E}[\mathbf{cc}^\dagger]$ is a diagonal matrix with $|c_k|^2$ being the k^{th} diagonal entry.

In practice, $\mathbf{R}_{xx}$ is usually estimated in the following way [5]. Given $x(n)$ over a long enough duration, say L samples, we can extract successive overlapping length-N subsegments from it, say $\{\mathbf{x}_{(i)}\}_{i=1}^{L-N+1}$, and compute[2]:

$$\mathbf{R}_{xx} \approx \frac{1}{L-N+1}\sum_{i=1}^{L-N+1} \mathbf{x}_{(i)}\mathbf{x}_{(i)}^\dagger \tag{7}$$

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ be the eigenvalues of $\mathbf{R}_{xx}$. Since $\mathrm{Rank}(\mathbf{ACA}^\dagger) = K$, it can be shown that $\lambda_{K+1} = \lambda_{K+2} = \ldots = \lambda_N = \sigma^2$. These are commonly referred to as the noise eigenvalues, and their corresponding eigenvectors $\mathbf{U_e} = [\mathbf{u}_{K+1}, \mathbf{u}_{K+2}, \ldots, \mathbf{u}_N]$ as the noise eigenvectors. Using (6), it is easy to see that

$$\mathbf{ACA}^\dagger\mathbf{U_e} = \mathbf{0} \tag{8}$$

[2]Strictly speaking, since $\mathbf{x}_{(i)}$'s in (7) are extracts from the same signal $x(n)$, the statistical independence of the phases of c_k's is not really true. However, most of the prior literature ignores this subtle detail. Fortunately, even when c_k's are completely deterministic, it can be shown that the sample autocorrelation matrix constructed in (7) can be decomposed as (6) with a diagonal $\mathbf{C}$ (with diagonal elements $|c_k|^2$), when $L \gg N$.

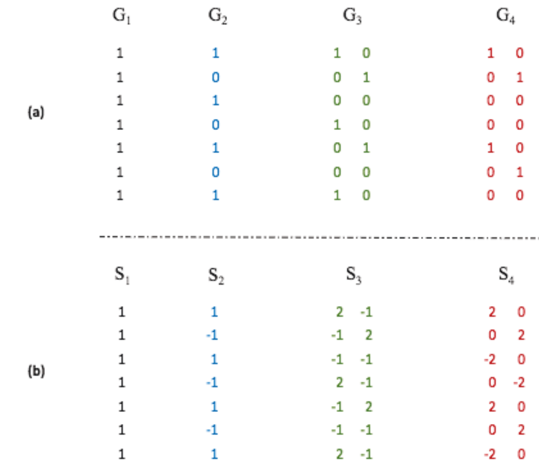

Fig. 2: *Simple Integer Alternatives to Complex-Exponentials*: Bases of (a) The Natural Basis Subspaces, and (b) The Ramanujan Subspaces.

But since $\mathbf{AC}$ has full column rank, this is equivalent to:

$$\mathbf{A}^\dagger\mathbf{U_e} = \mathbf{0} \tag{9}$$

That is, the complex-exponentials in (4) turn out to be orthogonal to the noise eigenspace. So given the noise eigenvectors, one can use the following criterion to find the frequencies in the signal:

$$\min_{\omega \in (-\pi, \pi]} \|\mathbf{a}^\dagger(\omega)\mathbf{U_e}\|_2^2 \tag{10}$$

where $\mathbf{a}(\omega) = [1, e^{j\omega}, e^{2j\omega}, \ldots, e^{(N-1)j\omega}]^T$. It can be proved that as long as $N > K$ the only complex-exponentials that are orthogonal to the noise eigenspace are those in (4) [12], [13].

Now, if a signal has the harmonic structure in (3), [4] proposes to modify the search over all frequencies in (10) in the following way:

$$\min_{\omega \in (-\pi, \pi]} \min_{K} \frac{\|\mathbf{B}^\dagger(\omega)\mathbf{U_e}\|_F^2}{KN(N-K)} \tag{11}$$

where $\mathbf{B}(\omega) = [\mathbf{a}(0), \mathbf{a}(\omega), \mathbf{a}(2\omega), \ldots, \mathbf{a}((K-1)\omega)]$. The factor of $KN(N-K)$ is a normalization term. The resulting algorithm was called the Harmonic MUSIC (HMUSIC) algorithm [4]. The HMUSIC algorithm was further generalized to the case of mixtures of periodic signals in [3] in the following way:

$$\min_{\{K_l\}_{l=0}^{M-1}} \min_{\{\omega_l\}_{l=0}^{M-1}} \sum_{l=0}^{M-1} \frac{\|\mathbf{B}^\dagger(\omega_l)\mathbf{U_e}\|_F^2}{KN(N-K)} \tag{12}$$

where M is the number of component periods in the mixture, and K is the total signal space dimension.

Although these algorithms were shown to offer good estimation performance in the context of pitch estimation [4], [3], for signals that can be approximated well by the integer period model of (1) (such as DNA and Protein repeats), we can have much simpler techniques, with a significantly higher accuracy as well. We shall present such new techniques next.

III. The Proposed Techniques

Our proposed alternatives to conventional MUSIC and its variations are based on the recently developed Nested Periodic Subspaces (NPSs) [15], [16]. NPSs are essentially a set of subspaces that can span all integer periodic sequences satisfying (1). For instance, consider the two examples of NPSs shown in Fig. 2. These are the bases for the Natural Basis and the Ramanujan Subspaces [18]. For every integer P, the period-P subspace has dimension (P), which is equal to the number of positive integers smaller than and co-prime to P (the Euler totient function).

To span signals of period-P, all subspaces that have divisors of P as periods are used. For example, to span period 4 signals, we must use the subspaces with periods 1, 2 and 4 in Fig. 2. Gauss' well known result $_{d|P} \; (d) = P$ ensures that the dimensions of these subspaces do add up to P. Stating this formally, it was shown in [16] that when using NPSs, the LCM of the periods of the subspaces involved in spanning a signal must be exactly equal to the period of the signal[3]. For e.g., if the subspaces with periods 1, 2, 3 and 6 are involved in spanning a signal, its period must be $\mathrm{LCM}(1, 2, 3, 6) = 6$.

This idea easily extends to the case of mixtures of periodic signals. For example, if the input is a mixture of period 4 and period 6 signals, then the subspaces with periods 1, 2, 3, 4 and 6 are involved in spanning it. Since these numbers are essentially divisors of 4 and 6, we can conclude that the input signal is a mixture of periods 4 and 6. Dictionaries based on NPSs were shown to offer several new advantages over traditional period estimation techniques in [15].

Taking inspiration from MUSIC, we can formulate an eigenspace approach using NPSs in a similar fashion. For example, if $x(n)$ is a noisy version of a signal satisfying (1), it can be expressed in matrix notation as:

$$\mathbf{x} = \mathbf{A}\mathbf{c} + \mathbf{e} \tag{13}$$

This equation is similar to (5), but now $\mathbf{A}$ is an $N \times K$ matrix whose columns are the basis functions of the corresponding NPSs, (for example, those shown in Fig. 2). Here K is the total number of NPS basis vectors needed to span the signal. The resulting auto-correlation matrix can again be decomposed as (6), and the eigenvalues of $\mathbf{R}_{xx}$ can be arranged as $_1 \quad _2 \ldots \quad _N$, with $_{K+1} = \; _{K+2} = \ldots = \; _N = \; ^2$. The corresponding noise eigenvectors $\mathbf{U_e} = [\mathbf{u}_{K+1}, \; \mathbf{u}_{K+2}, \; \ldots \; , \mathbf{u}_N]$ will satisfy:

$$\mathbf{A}\mathbf{C}\mathbf{A}^{\dagger}\mathbf{U_e} = \mathbf{0} \tag{14}$$

If we assume that the coefficients of the signal along the NPS basis vectors are statistically independent, then $\mathbf{C} = \mathrm{E}[\mathbf{c}\mathbf{c}^{\dagger}]$ turns out to be full rank[4]. From the linear independence property of NPSs [16], it can be shown that, as long as $N \quad K$, $\mathbf{A}$ has full column rank. So (14) can be rewritten as:

$$\mathbf{A}^{\dagger}\mathbf{U_e} = \mathbf{0} \tag{15}$$

This implies that, the basis vectors of the NPSs involved in spanning the signal will be orthogonal to the noise eigenspace. For instance, if we know a priori that the component periods in $x(n)$ lie in a range 1 to P_{max}, we can check the orthogonality of $\mathbf{U_e}$ with the corresponding NPS basis vectors one-by-one. In the simulations of Sec. IV, for every integer P, we compute:

$$\frac{1}{(P)} \sum_{m=1}^{(P)} \frac{1}{\left\| \mathbf{U_e}^{\dagger}\mathbf{a_P^{(m)}} \right\|_2^2} \tag{16}$$

[3]For arbitrary union-of-subspaces representations of periodic signals, this property does not hold. See [16].

[4]The full rank of $\mathbf{C}$ was verified to be true with probability one in the Monte-Carlo simulations of Sec. IV.

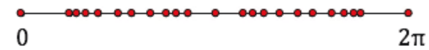

Fig. 3: *The Non-Uniform Farey Grid*: The true frequencies needed to span periods in the range $1 \quad P \quad 8$. See Sec. III-A for details.

where $\{\mathbf{a_P^{(m)}}\}_{m=1}^{(P)}$ are the P^{th} NPS's basis vectors, and the additional (P) term in the denominatior is a normalizing factor. *A plot of* (16)*, with the integer P as the x-axis is the discrete pseudospectrum plot based on Nested Periodic Subspaces*.

Remark: A crucial step in the above formulation was the full column rank of $\mathbf{A}$. While there can be several union-of-subspaces models for periodic signals, the choice of NPSs is important, since they are unique in offering this linear independence property. We refer the reader to [16] for more details on this.

Just like in the case of MUSIC, one can derive precise bounds on how large N must be such that, the only NPS basis vectors that are orthogonal to $\mathbf{U_e}$ will be the ones actually involved in spanning the signal. For example, it can be shown that in the case of a single periodic signal, if the period of $x(n)$ is known a priori to lie in the range 1 to P_{max}, then it is sufficient for N to be $\quad 2P_{max} - 2$. Similar bounds can be derived for mixtures of periodic signals. Deriving these results needs a much broader discussion, so their development is omitted here. They will be presented elsewhere.

A. Comparison with Conventional MUSIC

While the above formulation was inspired by, and follows closely, the steps in the MUSIC framework, there are some important differences:

Integer Basis Vectors: A key feature of the new techniques is that we can evaluate the *psuedospectrum* in (16) using a finite number of simple integer vectors as shown in Fig. 2, instead of complex exponentials with a continuous parameter . One can even have NPSs with randomly chosen integer basis functions (Fig. 4(f)).

Non-Uniform Frequency Grid: The Ramanujan subspaces (Fig. 2(b)) can alternatively be spanned by a basis of complex-exponentials. For instance, the P^{th} Ramanujan subspace can be spanned by: $\{e^{j\frac{2 \; k}{P}(n)} : \gcd(k, P) = 1\}$. So in (16), we can either use the integer valued basis vectors shown in Fig. 2(b), or these complex-exponentials.

If we choose to use the complex-exponential basis, our method still differs from conventional MUSIC and its variants in important ways. Firstly, in prior works, the period of $e^{j\frac{2 \; k}{P}n}$ was interpreted as P/k. However, the strict integer definition of period in (1) results in $P/\gcd(P, k)$. In this work, we follow the latter interpretation, and one can see that the complex-exponentials spanning the P^{th} Ramanujan subspace do have period P. Further, these complex-exponentials lie on a *non-uniform grid known as the Farey grid* [19], as shown in Fig. 3. We evaluate (15) on this Farey grid, unlike the uniform grids in [4], [3].

These differences result in a significantly higher accuracy over prior MUSIC based techniques. We shall demonstrate this using a number of examples in the next section.

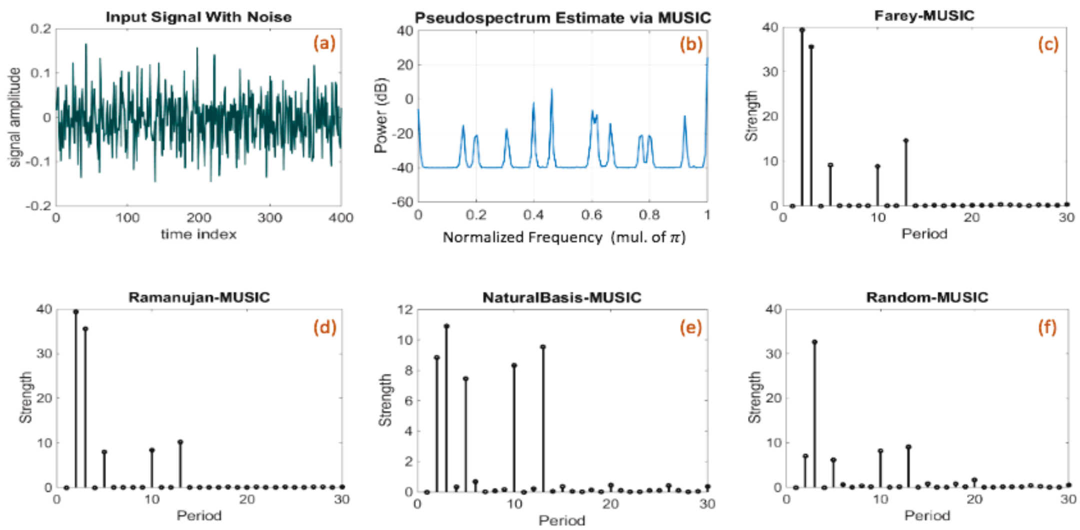

Fig. 4: Demonstrating the proposed techniques on a computer-generated signal. (a) The noisy periodic signal, (b) conventional MUSIC, (c)-(f) The new NPS-MUSIC methods. (d)-(f) are based on integer valued NPS basis vectors. See Sec. IV for details.

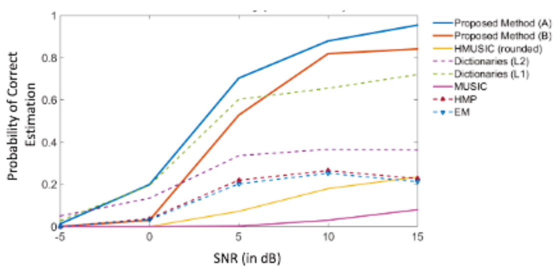

Fig. 5: Probability of Estimating both the component periods exactly. See Sec. IV for details.

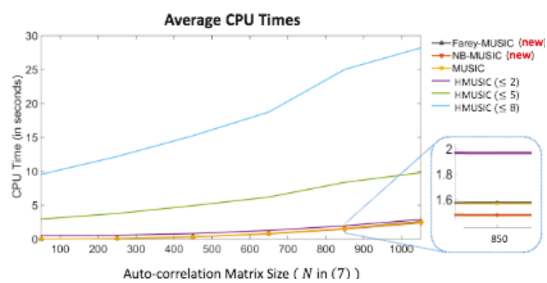

Fig. 6: A comparison of the CPU Times. See Sec. IV for details.

IV. Simulation Results

We will start with a simple demonstration. Fig. 4(a) shows a sum of randomly generated signals with periods 3, 10 and 13 and SNR 5dB. The signal length (L in (7)) was 400. The length of the subsegments $\mathbf{x}_{(i)}$'s in (7) was $N = 101$. K, the dimension of the signal subspace in (5) and (13), turns out to be 24 for this choice of periods. In practice, the true value of K is unknown a priori, and we estimate it using a simple metric: all eigenvalues of $\mathbf{R}_{xx}$ smaller than 5% of the maximum eigenvalue were considered as noise eigenvalues. Fig. 4(b) shows the conventional MUSIC pseudospectrum. The peaks correspond to periods 12.79, 9.85, 6.56, 5.02, 4.34, 3.32, 3.24, 3.01, 2.59, 2.51 and 2.17. Notice that it is quite inconvenient to spot the true periods 3, 10 and 13 from this set. Fig. 4(c) to (f) show the pseudospectra computed using (16) with various NPSs. All these plots have clean peaks at periods 2, 3, 5, 10 and 13. Using the LCM property of NPSs, these correspond to periods 3, 10 and 13.

Fig. 5 compares the accuracy of various techniques. For each SNR, 400 Monte Carlo trials, each with a signal of length $L = 100$, containing periods 3 and 10, were used. N was chosen as 41 in (7). K turns out to be 12 in this case. Since the period can only take discrete values in our periodicity model in (1), Cramer-Rao bounds cannot be defined unlike in [4], where the fundamental frequency can take continuous values. Since our problem is closer to a detection framework, in Fig. 5, instead of plotting Mean Squared Error[5], we have used a much stricter metric: the fraction of times the detected periods were exactly equal to the set $\{3, 10\}$. Proposed method (A) uses (16) with the Ramanujan subspaces (Farey basis), with the true K given as an input. Proposed method (B) is the same, but with K being estimated using the simple 5% criteria as above. For a strong comparison, HMUSIC was given the actual number of periodic signals in the mixture (M in (12)) as an input, and its estimates were rounded to the nearest integers. But it still does not perform well compared to the proposed methods. The MUSIC algorithm, with its estimates rounded to the nearest integers, and after discarding peaks which are harmonics of a fundamental, performs worse than HMUSIC. This is because it does not take into account the harmonic structure in the signal. Both the l_1 and l_2 norm based dictionary techniques of [15], constructed using Ramanujan subspaces, are also compared. Although the l_1 norm based dictionaries perform quite well, they come at a tremendously high computational complexity (discussed below). In addition, we have also compared the Expectation Maximization and the Harmonic Matching Pursuit techniques described in [5]. Both these methods, while performing better than HMUSIC and MUSIC, are far from satisfactory.

Fig. 6 compares the average CPU times using MATLAB

[5]MSE is not an appropriate metric to use in many applications. For example, in Fig. 1, proteins with Ankyrin repeats are known to have periods in the range 30 - 40. So it might be alright to estimate 66 as the period, instead of say 40, since we can readily deduce that 66 might actually indicate period 33 repeats.

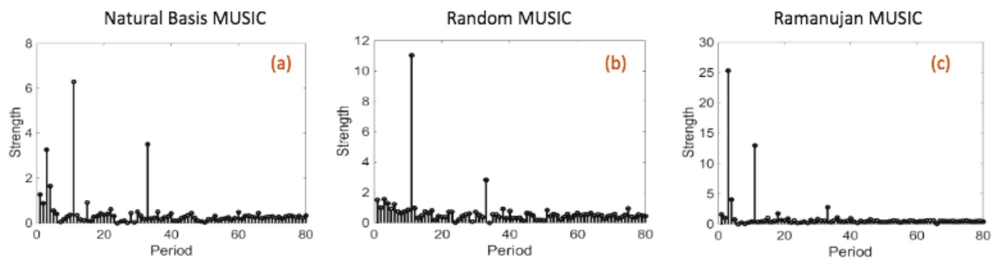

Fig. 7: Pseudospectra of the proposed NPS based techniques for the Ankyrin protein repeats shown in Fig. 1. See Sec. IV for details.

2014b on a 2.4GHz CPU with 8GB RAM, for the various eigenspace approaches, for different sizes of the autocorrelation matrix (N in (7)). The total datalength of the signal, L was chosen as $3N$, and the dimension of the signal subspace K was fixed at 25 for simplicity. MUSIC and HMUSIC were implemented with a uniform frequency grid of the same size as the Farey grid. Notice that, our Natural Basis (NB) MUSIC is the fastest. Farey-MUSIC and MUSIC are similar to each other in terms of CPU time due to identical grid sizes. In Fig. 6, HMUSIC(T) uses (12), with the prior knowledge that the number of fundamental frequencies M T. The complexity of HMUSIC(T) increases exponentially with T. This is because, according to the theory of partitions [6], the number of ways in which K_l's in (12) can be chosen to add up to the signal space dimension K increases exponentially with M. In contrast, since we check the NPS basis vectors one-by-one in (16), the complexity of our proposed techniques does not depend on T. The NPS based l_1 norm dictionary techniques, which showed a good performance in Fig. 5, are much more complex than the eigenspace methods. For example, for a datalength of 250, the average CPU time required by the Ramanujan dictionary was around 51s (implemented using CVX (http://cvxr.com/cvx/)). In contrast, the proposed Farey-MUSIC technique takes 0.1s for an autocorrelation matrix of size 250×250, constructed using $L = 750$ samples. These increase to 150s and 0.35s respectively for $N = 450$. From Fig. 6 and 5, it is evident that our methods offer much better accuracy than the prior variants of MUSIC, keeping the complexity low at the same time.

Lastly, in Fig. 7, we show the results of applying the proposed methods using Ramanujan (integer basis), Natural Basis and Random Integer NPSs to the AnkyrinR protein repeats shown in Fig. 1 (The Farey basis can also be used; it was shown earlier in Fig. 1). The Kyte-Doolittle scale [8] was used to map amino acids to numbers. All four plots have clear peaks at 33 and its divisors. Notice that the Ramanujan (integer basis) plot in Fig. 7(c) has a weak peak at 33. However, the LCM of the peaks at 11 and 3 indicate the presence of the period 33 repeats.

V. Conclusion

A new family of MUSIC-like period estimation techniques have been derived, based on the Nested Periodic Subspaces of [15]. These offer significantly better accuracy and computational simplicity than existing techniques for integer periods. The model in (1) is especially relevant to applications such as repeats in proteins and DNA sequences. Many proteins have concurrent repeats at various scales in the 3D structure, and these usually manifest as mixtures of periods when using the subspace techniques. An initial analysis of our techniques on such repeats reveals promising results, an example of which is shown in Fig. 7. A more extensive study in this direction will follow in our future efforts.

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