INDIAN JOURNAL OF MATHEMATICS
DHARMA PRAKASH GUPTA MEMORIAL VOLUME
Vol. 60, No.1, 2018, 65-84

ANGULAR DECOMPOSITION OF TENSOR PRODUCTS OF A VECTOR

Gregory S. Adkins

(*Received* 24 August 2017; *Revised* 13 November 2017)

The tensor product of L copies of a single vector, such as $p_{i_1} \cdots p_{i_L}$, can be analyzed in terms of angular momentum. When $p_{i_1} \cdots p_{i_L}$ is decomposed into a sum of components $(p_{i_1} \cdots p_{i_L})_\ell^L$, each characterized by angular momentum ℓ, the components are in general complicated functions of the p_i vectors, especially so for large ℓ. We obtain a compact expression for $(p_{i_1} \cdots p_{i_L})_\ell^L$ explicitly in terms of the p_i valid for all L and ℓ. We use this decomposition to perform three-dimensional Fourier transforms of functions like $p^n \hat{p}_{i_1} \cdots \hat{p}_{i_L}$ that are useful in describing particle interactions.

I. Introduction

Three-dimensional Fourier transforms of the general form

$$I_{n;i_1 \cdots i_L}(\vec{r}\,) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}}\, p^n \hat{p}_{i_1} \cdots \hat{p}_{i_L}, \tag{1}$$

(where $\hat{p} = \frac{\vec{p}}{p}$ and $p = |\vec{p}\,|$) have a wide variety of uses. For example,

$$I_{-2}(\vec{r}\,) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}}\, \frac{1}{p^2} = \frac{1}{4\pi r} \tag{2}$$

is the Fourier representation of the Coulomb potential. Two related transforms that occur in the study of fermion-fermion interactions

---

2010 Mathematics Subject Classification: 42B10, 33C55, 46N50, 46F10.

Key words and phrases: harmonic analysis; Fourier transforms; applications in quantum physics; Fourier transforms of distributions.

([8], [4], Sect. 39 [3], Sect. 83) are

$$I_{-1;i}(\vec{r}) = \int \tfrac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} \tfrac{p_i}{p^2} = \tfrac{i\hat{x}_i}{4\pi r^2}, \tag{3a}$$

$$I_{0;ij}(\vec{r}) = \int \tfrac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} \tfrac{p_i p_j}{p^2} = \tfrac{1}{3}\delta_{ij}\delta(\vec{r}) - \tfrac{3}{4\pi r^3}\left(\hat{x}_i\hat{x}_j - \tfrac{1}{3}\delta_{ij}\right). \tag{3b}$$

(In our notation the position vector has components $\vec{r} = (x_1, x_2, x_3) = (x, y, z)$, and $\hat{r} = \frac{\vec{r}}{r}$ with components $\hat{x}_i = \frac{x_i}{r}$ is the associated unit vector. In order to avoid ambiguity, we will always use subscripts to indicate Cartesian components of a vector and superscripts to denote powers of the magnitude of a vector, so $p^n = |\vec{p}|^n$ is a power while p_i is a component.) The structure of three dimensional Fourier transforms such as (1) is organized by angular momentum. One sees that both the original function $\frac{1}{p^2}$ and the transform $\frac{1}{4\pi r}$ of (2) are scalars under rotation. The original function $\frac{p_i}{p^2}$ of (3a) is a vector with $\ell = 1$ because the components $\hat{p}_i$ can be expressed linearly in terms of spherical harmonics $Y_1^m(\hat{p})$ with $\ell = 1$. The transform $\frac{i\hat{x}_i}{4\pi r^2}$ of (3a) also has $\ell = 1$ as $\hat{x}_i$ can be expressed linearly in terms of $Y_1^m(\hat{r})$. The original function in (3b), $\hat{p}_i\hat{p}_j$, is a combination of $\ell = 0$ and $\ell = 2$:

$$\hat{p}_i\hat{p}_j = (\hat{p}_i\hat{p}_j)_0^2 + (\hat{p}_i\hat{p}_j)_2^2 = \left(\frac{1}{3}\delta_{ij}\right) + \left(\hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij}\right), \tag{4}$$

where $(\hat{p}_i\hat{p}_j)_0^2 = \frac{1}{3}\delta_{ij}$ is the $\ell = 0$ component and $(\hat{p}_i\hat{p}_j)_2^2 = \hat{p}_i\hat{p}_j - \frac{1}{3}\delta_{ij}$ is the $\ell = 2$ component. (In general we will write $(p_{i_1}\cdots p_{i_L})_\ell^L$ for the component of $p_{i_1}\cdots p_{i_L}$ of angular momentum ℓ.) We know that $(\hat{p}_i\hat{p}_j)_2^2$ has $\ell = 2$ because it can be expressed linearly in terms of $Y_2^m(\hat{p})$: $(\hat{p}_i\hat{p}_j)_2^2 = \sum_m C_{ij}^{2m} Y_2^m(\hat{p})$. It is apparent that the $\ell = 0$ and $\ell = 2$ components of $\hat{p}_i\hat{p}_j$ behaves differently under the Fourier transform, acquiring different radial factors. It is generally true that in transforms like (1) it is useful to decompose $\hat{p}_{i_1}\cdots\hat{p}_{i_L}$ into components of definite ℓ and deal with each component separately.

The purpose of this work is to show how the decomposition of $p_{i_1} \cdots p_{i_L}$ can be done and to give explicit expressions for the components of $(p_{i_1} \cdots p_{i_L})^L_\ell$ having various values of angular momentum ℓ. This is a generalization of (4) to an arbitrary number of vectors L. Our derivations are presented in terms of unit vectors because the relation

$$(p_{i_1} \cdots p_{i_L})^L_\ell = p^L \, (\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_\ell \tag{5}$$

allows us to immediately obtain the general case.

This work is organized as follows. In II we will review the method for performing three-dimensional Fourier transforms like (1) making use of angular decomposition. In III we obtain the general expression for the component of $\hat{p}_{i_1} \cdots \hat{p}_{i_L}$ of angular momentum ℓ. In IV, we give some examples and applications of our results. Sec. V contains a summary and final comments.

More general studies of the relation between Cartesian and spherical components of tensors have been done, ([9], [10], [17], [22], [23]) but the results of those studies are not in a form useful for our purposes here.

II. Three-Dimensional Fourier Transforms Using Angular Decomposition

A systematic procedure exists for the evaluation of transforms such as (1) based on the decomposition of $\hat{p}_{i_1} \cdots \hat{p}_{i_L}$ into components of definite angular momentum. [1] Our purpose in this section is to review this procedure. We begin by noting that any function of angles, such as $\hat{p}_{i_1} \cdots \hat{p}_{i_L}$, can be written in terms of spherical harmonics:

$$\hat{p}_{i_1} \cdots \hat{p}_{i_L} = \sum_{\ell=0 \,\mathrm{or}\, 1}^{L} \sum_{m=-\ell}^{\ell} C^{\ell m}_{i_1 \cdots i_L} Y^m_\ell(\hat{p}) \tag{6}$$

for some constants $C^{\ell m}_{i_1\cdots i_L}$. The values of ℓ that enter this sum begin with 0 or 1 depending on whether L is even or odd and go up by 2s to L. There are no values of ℓ greater than L because $\hat{p}$ has $\ell = 1$ and the combination of L objects having $\ell = 1$ can lead to angular momentum L at the most. Matching the parity $(-1)^L$ of $\hat{p}_{i_1}\cdots\hat{p}_{i_L}$ to the parity $(-1)^\ell$ of $Y^m_\ell(\hat{p})$ gives the requirement that only odd or only even values of ℓ can contribute. We define $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ to be the component of $\hat{p}_{i_1}\cdots\hat{p}_{i_L}$ of angular momentum ℓ

$$(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell \equiv \sum_{m=-\ell}^{\ell} C^{\ell m}_{i_1\cdots i_L} Y^m_\ell(\hat{p}), \tag{7}$$

so that

$$\hat{p}_{i_1}\cdots\hat{p}_{i_L} = \sum_{\ell=0 \text{ or } 1}^{L} (\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell \,. \tag{8}$$

It follows that any transform of the form given in (1) can be expressed as a linear combination of transforms like

$$I_{n\ell m}(\vec{r}\,) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} p^n Y^m_\ell(\hat{p}). \tag{9}$$

It is convenient to express the exponential in (9) as a Rayleigh expansion: ([25], p. 368), ([16], p. 1466), ([2], p. 770)

$$e^{i\vec{p}\cdot\vec{r}} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(pr) P_\ell(\hat{p}\cdot\hat{r}), \tag{10}$$

where the $j_\ell(x)$ are spherical Bessel functions ([2], Sect. 11.7) and the $P_\ell(x)$ are Legendre polynomials. We substitute (10) into (9) and use the addition theorem of spherical harmonics ([2], Sect. 12.8])

$$P_\ell(\hat{p}\cdot\hat{r}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^m_\ell(\hat{r}) Y^{m*}_\ell(\hat{p}) \tag{11}$$

to factor the angular dependence present in $P_\ell(\hat{p}\cdot\hat{r})$ into parts involving the angles of $\hat{p}$ and $\hat{r}$ separately. We integrate over the angles of $\hat{p}$ using orthogonality

$$\int d\Omega_p Y_\ell^{m*}(\hat{p}) Y_{\ell'}^{m'}(\hat{p}) = \delta_{\ell\ell'}\delta_{mm'}, \tag{12}$$

where $d\Omega_p = d\theta_p \sin\theta_p d\phi_p$ is the element of solid angle for $\hat{p}$, to write the transform as

$$I_{n\ell m}(\vec{r}) = \frac{i^\ell}{2\pi^2} Y_\ell^m(\hat{r}) \int_0^\infty dp\, p^{n+2} j_\ell(pr). \tag{13}$$

The integral

$$R_{n\ell}(r) \equiv \int_0^\infty dp\, p^{n+2} j_\ell(pr) \tag{14}$$

converges for $-(\ell+3) < n < -1$ (when n and ℓ are real, as here), and has the value $R_{n\ell}(r) = \chi_{n\ell}/r^{n+3}$, where ([14], Integral 6.561, p. 684)

$$\chi_{n\ell} = 2^{n+1}\sqrt{\pi}\, \frac{\Gamma\left(\frac{\ell+3+n}{2}\right)}{\Gamma\left(\frac{\ell-n}{2}\right)}. \tag{15}$$

We can extend the useful range of n by generalizing (14) to

$$R_{n\ell}(r) = \lim_{\lambda\to 0^+} \int_0^\infty dp\, e^{-\lambda p} p^{n+2} j_\ell(pr), \tag{16}$$

which is also given by $R_{n\ell}(r) = \frac{\chi_{n\ell}}{r^{n+3}}$ for all n in the larger range $-(\ell+3) < n < \ell$. When $n = \ell$ the integral contains a delta function: [1]

$$R_{\ell\ell}(r) = \lim_{\lambda\to 0^+} \int_0^\infty dp\, e^{-\lambda p} p^{\ell+2} j_\ell(pr) = \frac{2\pi^2(2\ell+1)!!}{r^\ell}\delta(\vec{r}). \tag{17}$$

We always integrate over the spherical angles before doing the radial integration as part of the definition of these possibly singular integrals. (Non-spherical regularization alternatives have been considered by Hnizdo. [15] More general results for Fourier transforms of the form (9) have been obtained by Samko. [21]) All in all, we see that the initial transform (1) can be written as

$$
\begin{aligned}
I_{n;i_1\cdots i_L}(\vec{r}) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}}\, p^n \hat{p}_{i_1} \cdots \hat{p}_{i_L} \\
&= \sum_{\ell=0\,\mathrm{or}\,1}^{L} \frac{i^\ell}{2\pi^2} R_{n\ell}(r)\, (\hat{x}_{i_1} \cdots \hat{x}_{i_L})^L_\ell\,,
\end{aligned} \tag{18}
$$

where $(\hat{x}_{i_1} \cdots \hat{x}_{i_L})^L_\ell$ is defined in terms of $Y^m_\ell(\hat{r})$ just as $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_\ell$ is in terms of $Y^m_\ell(\hat{p})$. It follows that if we can arrive at a useful expression for $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_\ell$, then we will be able to perform Fourier transforms of the form shown in (1) in a systematic way.

III. Angular Decomposition of $\hat{p}_{i_1} \cdots \hat{p}_{i_L}$

Our goal in this section is to obtain an explicit and useful expression for the component $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_\ell$ of angular momentum ℓ. We will discuss both a constructive method most useful for low values of L and a general result valid for all L. Both approaches make use of the explicit solution for the constants $C^{\ell m}_{i_1\cdots i_L}$ in (7):

$$
C^{\ell m}_{i_1\cdots i_L} = \int d\Omega_p\, Y^{m*}_\ell(\hat{p})\, \hat{p}_{i_1} \cdots \hat{p}_{i_L}, \tag{19}
$$

obtained through use of the orthogonality of the spherical harmonics. From this it is easy to see that the constants $C^{\ell m}_{i_1\cdots i_L}$, and thus the components $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_\ell$, are completely symmetric in all indices. The constructive method also uses the tracelessness of the maximum angular momentum component $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})^L_L$, which follows from the tracelessness of $C^{Lm}_{i_1\cdots i_L}$, which is a consequence of the fact that an object composed of $L-2$ parts, each part of unit angular momentum, has no overlap with an object of angular momentum L:

$$
C^{Lm}_{i_1\cdots i_L} \delta_{i_{L-1} i_L} = \int d\Omega_p\, Y^{m*}_L(\hat{p})\, \hat{p}_{i_1} \cdots \hat{p}_{i_{L-2}} = 0. \tag{20}
$$

The most convenient way to obtain $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ for small values of L is by straightforward construction. We illustrate the constructive approach with a number of examples. The procedure starts with the maximum angular momentum component $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_L$, which is completely symmetric and traceless. This maximum angular momentum component can be written as $\hat{p}_{i_1}\cdots\hat{p}_{i_L}$ plus a linear combination of symmetric terms involving fewer momentum factors (but still with the same parity) added in with unknown coefficients. The condition of tracelessness determines the coefficients.

As a first example of the constructive approach, consider the case $L=3$. The maximum angular momentum component is

$$(\hat{p}_i\hat{p}_j\hat{p}_k)^3_3 = \hat{p}_i\hat{p}_j\hat{p}_k - \frac{1}{5}\left(\hat{p}_i\delta_{jk} + \hat{p}_j\delta_{ki} + \hat{p}_k\delta_{ij}\right), \tag{21}$$

where the $-\frac{1}{5}$ coefficient was determined by the tracelessness condition. The other component, $(\hat{p}_i\hat{p}_j\hat{p}_k)^3_1$, is the difference $\hat{p}_i\hat{p}_j\hat{p}_k - (\hat{p}_i\hat{p}_j\hat{p}_k)^3_3$:

$$(\hat{p}_i\hat{p}_j\hat{p}_k)^3_1 = \frac{1}{5}\left(\hat{p}_i\delta_{jk} + \hat{p}_j\delta_{ki} + \hat{p}_k\delta_{ij}\right). \tag{22}$$

It is clear that $(\hat{p}_i\hat{p}_j\hat{p}_k)^3_1$ has $\ell = 1$ because each of its terms is linear in $\hat{p}$.

As a second example of explicit construction, we consider the term with $L=4$. The term with maximal angular momentum is

$$\begin{aligned}(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_4 = \hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4} &- \frac{1}{7}\left(\hat{p}_{i_1}\hat{p}_{i_2}\delta_{i_3i_4} + \text{perms}\right)_{6\ \text{terms}} \\ &+ \frac{1}{35}\left(\delta_{i_1i_2}\delta_{i_3i_4} + \text{perms}\right)_{3\ \text{terms}},\end{aligned} \tag{23}$$

where the coefficients $-\frac{1}{7}$ and $\frac{1}{35}$ were obtained by applying the tracelessness condition. We are only writing one representative permutation of indices–the others are represented by "+ perms" and an indication of how many independent permutations in all there are. We identify the $\ell = 2$ component $(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_2$ by isolating the term in the difference $\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4} - (\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_4$ that is quadratic in $\hat{p}$ and subtracting the appropriate momentum-independent terms so that each part of $(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_2$ has $\ell = 2$:

$$(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_2 = \frac{1}{7}\Big((\hat{p}_{i_1}\hat{p}_{i_2})^2_2\,\delta_{i_3 i_4} + \text{perms}\Big)_{6\text{ terms}}. \tag{24}$$

The $\ell = 0$ component is the remainder:

$$(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4})^4_0 = \frac{1}{15}\,(\delta_{i_1 i_2}\delta_{i_3 i_4} + \text{perms})_{3\text{ terms}}. \tag{25}$$

We have also constructed the $L = 5$ decomposition and just give the results:

$$\begin{aligned}(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4}\hat{p}_{i_5})^5_5 = \hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4}\hat{p}_{i_5} - \frac{1}{9}\,(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\delta_{i_4 i_5} + \text{perms})_{10\text{ terms}}\\ + \frac{1}{63}\,(\hat{p}_{i_1}\delta_{i_2 i_3}\delta_{i_4 i_5} + \text{perms})_{15\text{ terms}},\end{aligned} \tag{26a}$$

$$(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4}\hat{p}_{i_5})^5_3 = \frac{1}{9}\big(\,(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3})^3_3\,\delta_{i_4 i_5} + \text{perms}\big)_{10\text{ terms}}, \tag{26b}$$

$$(\hat{p}_{i_1}\hat{p}_{i_2}\hat{p}_{i_3}\hat{p}_{i_4}\hat{p}_{i_5})^5_1 = \frac{1}{35}\big(\hat{p}_{i_1}\delta_{i_2 i_3}\delta_{i_4 i_5} + \text{perms}\big)_{15\text{ terms}}. \tag{26c}$$

The basis of our general construction of $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ is an inductive argument using a recursion relation giving a component with angular momentum ℓ in terms of components with lower values of ℓ. We will propose a general expression for $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ and show that it satisfies both the recursion relation and the appropriate initial values.

We can obtain a useful expression for $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ by using (19) in (7) along with the addition theorem for spherical harmonics:

$$\begin{aligned}(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell &= \int d\Omega_{p'}\,\hat{p}'_{i_1}\cdots\hat{p}'_{i_L}\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(\hat{p}')Y_\ell^m(\hat{p})\\ &= (2\ell+1)\int\frac{d\Omega_{p'}}{4\pi}\,\hat{p}'_{i_1}\cdots\hat{p}'_{i_L}\,P_\ell(\hat{p}'\cdot\hat{p}).\end{aligned} \tag{27}$$

It can be seen from this expression both that $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L = 0$ for $\ell > L$, and that $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L = 0$ unless $L - \ell$ is even (by use of a parity argument). For any particular value of ℓ we could write out $P_\ell(\hat{p}' \cdot \hat{p})$ as a polynomial of order ℓ and perform the angular integral using ([7], Appendix A)

$$\int \frac{d\Omega}{4\pi}\, \hat{x}_{i_1} \cdots \hat{x}_{i_N} = \frac{\delta_{N,\text{even}}}{(N+1)!!} \left(\delta_{i_1 i_2} \cdots \delta_{i_{N-1} i_N} + \text{perms}\right)_{(N-1)!!\ \text{terms}}. \tag{28}$$

(We use the value $(-1)!! = 1$ where necessary.) It is easy to perform the integral in (27) for $\ell = 0$ and $\ell = 1$ where $P_0(\hat{p}' \cdot \hat{p}) = 1$ and $P_1(\hat{p}' \cdot \hat{p}) = \hat{p}' \cdot \hat{p} = \hat{p}'_j \hat{p}_j$ (with an implied sum over j from 1 to 3). One finds that

$$\begin{aligned}
&(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_0^L \\
&= \int \frac{d\Omega_{p'}}{4\pi}\, \hat{p}'_{i_1} \cdots \hat{p}'_{i_L} \\
&= \frac{\delta_{L,\text{even}}}{(L+1)!!} \left(\delta_{i_1 i_2} \cdots \delta_{i_{L-1} i_L} + \text{perms}\right)_{(L-1)!!\ \text{terms}},
\end{aligned} \tag{29a}$$

$$\begin{aligned}
&(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_1^L \\
&= 3 \int \frac{d\Omega_{p'}}{4\pi}\, \hat{p}'_{i_1} \cdots \hat{p}'_{i_L} \hat{p}'_j\, \hat{p}_j = 3\, (\hat{p}_{i_1} \cdots \hat{p}_{i_L} \hat{p}_j)_0^{L+1}\, \hat{p}_j \\
&= \frac{3\delta_{L,\text{odd}}}{(L+2)!!} \left(\hat{p}_{i_1} \left(\delta_{i_2 i_3} \cdots \delta_{i_{L-1} i_L} + \text{perms}\right)_{(L-2)!!\ \text{terms}} + \text{perms}\right)_{L\ \text{terms}}.
\end{aligned} \tag{29b}$$

These results will serve as initial values for the inductive argument. The recursion relation for $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L$ is

$$(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L = \frac{2\ell+1}{\ell} \left\{ (\hat{p}_{i_1} \cdots \hat{p}_{i_L} \hat{p}_j)_{\ell-1}^{L+1}\, \hat{p}_j - \frac{\ell-1}{2\ell-3}\, (\hat{p}_{i_1} \cdots \hat{p}_{i_L})_{\ell-2}^L \right\}, \tag{30}$$

which follows from (27) and the recursion relation for Legendre polynomials

$$\ell P_\ell(x) = (2\ell-1)xP_{\ell-1}(x) - (\ell-1)P_{\ell-2}(x). \tag{31}$$

In order to write down a general expression for $(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell$ and perform the inductive proof of its correctness it will be useful to introduce a little notation. First, we will use the usual summation symbol to represent a 'sum over permutations' instead of the '+ perms' notation used up until now. Specifically, we will write

$$\sum_{(L-1)!!} \big(\delta_{i_1 i_2}\cdots\delta_{i_{L-1}i_L}\big) \tag{32}$$

for the sum over the $(L-1)!!$ independent permutations of the indices of the quantity in parentheses. This sum over permutations doesn't include ones that trivially equal one another, which is why there are only $(L-1)!!$ permutations in (32), but would be $L!$ permutations in a sum over permutations of $A^1_{i_1}\cdots A^L_{i_L}$, but only one in a sum over permutations of $S_{i_1\cdots i_L}$ if $S_{i_1\cdots i_L}$ is completely symmetric. Also, we define the new symbol

$$X^{L,\ell}_{i_1\cdots i_L} \equiv \sum_{\binom{L}{\ell}} \Big(\hat{p}_{i_1}\cdots\hat{p}_{i_\ell} \sum_{(L-\ell-1)!!} \big(\delta_{i_{\ell+1}i_{\ell+2}}\cdots\delta_{i_{L-1}i_L}\big)\Big) \tag{33}$$

to represent the completely symmetric object with L indices formed of ℓ momentum unit vectors and $\frac{L-\ell}{2}$ Kronecker deltas. The $\binom{L}{\ell}$ notation represents the combinatoric factor $\frac{L!}{\ell!(L-\ell)!}$ for the number of ways to pick ℓ indices out of a collection of L indices. We can represent $X^{L,\ell}_{i_1\cdots i_L}$ slightly more compactly as

$$X^{L,\ell}_{i_1\cdots i_L} = \sum_{\binom{L}{\ell}(L-\ell-1)!!} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_\ell}\delta_{i_{\ell+1}i_{\ell+2}}\cdots\delta_{i_{L-1}i_L}\big). \tag{34}$$

We note that the $X^{L,\ell}_{i_1\cdots i_L}$ symbol only makes sense when L and ℓ are either both even or both odd–we define it to be zero otherwise. We also define $X^{L,\ell}_{i_1\cdots i_L}$ to be zero if L or ℓ is negative or if ℓ is greater than L. With the new notation we can write the results of (29) as

$$(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_0 = \frac{1}{(L+1)!!}X^{L,0}_{i_1\cdots i_L}, \tag{35a}$$

$$(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_1 = \frac{3}{(L+2)!!}X^{L,1}_{i_1\cdots i_L}. \tag{35b}$$

Identities among $X^{L,\ell}_{i_1\cdots i_L}$ quantities can often be found by simple counting. For instance, consider the following identity for the symmetrized product of two Xs:

$$\sum_{\binom{L+N}{L}}\left(X^{L,\ell}_{i_1\cdots i_L}X^{N,n}_{i_{L+1}\cdots i_{L+N}}\right) = \kappa X^{L+N,\ell+n}_{i_1\cdots i_{L+N}}. \tag{36}$$

Both sides of (36) involve the same set of $L+N$ indices, both are symmetric in all indices, and both have exactly $\ell+n$ momentum unit vectors in each term, so the two sides of (36) are proportional. Since all terms enter with the same sign and there are no cancellations, the constant of proportionality κ can be found simply by counting the total number of terms on each side. On the left there are $\binom{L+N}{L}\binom{L}{\ell}(L-\ell-1)!!\binom{N}{n}(N-n-1)!!$ terms, while on the right there are $\binom{L+N}{\ell+n}(L+N-\ell-n-1)!!$ terms. The constant κ is the ratio:

$$\kappa = \binom{L+N-\ell-n}{L-\ell}\binom{\ell+n}{\ell}\frac{(L-\ell-1)!!(N-n-1)!!}{(L+N-\ell-n-1)!!}. \tag{37}$$

Two additional identities that will be useful to us involve the contraction $X^{L,\ell}_{i_1\cdots i_L}\hat{p}_{i_L}$ of an X with $\hat{p}$ and the contraction $X^{L,\ell}_{i_1\cdots i_L}\delta_{i_{L-1}i_L}$

of two indices of an X. For the first identity, we note that

$$X^{L,\ell}_{i_1\cdots i_L} = \sum_{\binom{L-1}{\ell}} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_\ell} \sum_{(L-\ell-1)!!} \big(\delta_{i_{\ell+1}i_{\ell+2}}\cdots\delta_{i_{L-1}i_L}\big)\big)$$
$$+ \sum_{\binom{L-1}{\ell-1}} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_{\ell-1}}\hat{p}_{i_L} \sum_{(L-\ell-1)!!} \big(\delta_{i_\ell i_{\ell+1}}\cdots\delta_{i_{L-2}i_{L-1}}\big)\big), \tag{38}$$

where in the first term it is understood that index i_L is definitely on a δ, while in the second term index i_L is attached to a $\hat{p}$. Contraction with $\hat{p}_{i_L}$ then gives two corresponding terms: $\hat{p}_{i_L}$ times the first term of (38) has $L-1$ indices and $\ell+1$ factors of $\hat{p}$, while $\hat{p}_{i_L}$ times the second term of (38) has $L-1$ indices and only $\ell-1$ powers of $\hat{p}$, so that $X^{L,\ell}_{i_1\cdots i_L}\hat{p}_{i_L} = \alpha X^{L-1,\ell+1}_{i_1\cdots i_{L-1}} + \beta X^{L-1,\ell-1}_{i_1\cdots i_{L-1}}$ for some constants α and β. Counting terms allows us to identity the constants to be $\alpha = \ell+1$ and $\beta = 1$, so that

$$X^{L,\ell}_{i_1\cdots i_L}\hat{p}_{i_L} = (\ell+1)X^{L-1,\ell+1}_{i_1\cdots i_{L-1}} + X^{L-1,\ell-1}_{i_1\cdots i_{L-1}}. \tag{39}$$

For the second identity we write $X^{L,\ell}_{i_1\cdots i_L}$ as

$$X^{L,\ell}_{i_1\cdots i_L} = \sum_{\binom{L-2}{\ell}} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_\ell}\big\{ \sum_{(L-\ell-3)!!} \big(\delta_{i_{\ell+1}i_{\ell+2}}\cdots\delta_{i_{L-1}i_L}\big)$$
$$+ \sum_{(L-\ell-2)(L-\ell-3)!!} \big(\delta_{i_{\ell+1}i_{\ell+2}}\cdots\delta_{i_{L-3}i_{L-1}}\delta_{i_{L-2}i_L}\big)\big\}\big)$$
$$+ \sum_{\binom{L-2}{\ell-1}} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_{\ell-1}}\hat{p}_{i_{L-1}} \sum_{(L-\ell-1)!!} \big(\delta_{i_\ell i_{\ell+1}}\cdots\delta_{i_{L-2}i_L}\big)\big)$$
$$+ \sum_{\binom{L-2}{\ell-1}} \big(\hat{p}_{i_1}\cdots\hat{p}_{i_{\ell-1}}\hat{p}_{i_L} \sum_{(L-\ell-1)!!} \big(\delta_{i_\ell i_{\ell+1}}\cdots\delta_{i_{L-2}i_{L-1}}\big)\big)$$

$$+\sum_{\binom{L-2}{\ell-2}}\bigl(\hat{p}_{i_1}\cdots\hat{p}_{i_{\ell-2}}\hat{p}_{i_{L-1}}\hat{p}_{i_L}\sum_{(L-\ell-1)!!}\bigl(\delta_{i_{\ell-1}i_\ell}\cdots\delta_{i_{L-3}i_{L-2}}\bigr)\bigr),\tag{40}$$

where it is understood that in the first two terms the indices i_{L-1} and i_L are definitely on δs, on the same δ in the first and on different δs in the second; in the third term i_{L-1} is on a $\hat{p}$ while i_L is on a δ; in the fourth i_L is on a $\hat{p}$ and i_{L-1} on a δ; and in the last term both i_{L-1} and i_L are on $\hat{p}$s. Contraction of i_{L-1} with i_L gives rise to two structures: $X^{L,\ell}_{i_1\cdots i_L}\delta_{i_{L-1}i_L} = \rho X^{L-2,\ell}_{i_1\cdots i_{L-2}} + \sigma X^{L-2,\ell-2}_{i_1\cdots i_{L-2}}$, with the first four terms of (40) contributing to ρ and only the last contributing to σ. Again, we use term counting to identify values for ρ and σ, finding $\rho = L+\ell+1$ and $\sigma = 1$. (The four contributions to ρ are, in order, 3, $L-\ell-2$, ℓ and ℓ.) The final form of the contraction identity is

$$X^{L,\ell}_{i_1\cdots i_L}\delta_{i_{L-1}i_L} = (L+\ell+1)X^{L-2,\ell}_{i_1\cdots i_{L-2}} + X^{L-2,\ell-2}_{i_1\cdots i_{L-2}}.\tag{41}$$

We propose the following form for the general decomposition formula

$$(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell = \frac{(2\ell+1)(L-\ell-1)!!}{(L-\ell)!(L+\ell+1)!!}\sum_{n=0\,\mathrm{or}\,1}^{\ell}(-1)^{\frac{\ell-n}{2}}\times \frac{(L-n)!(\ell+n-1)!!(\ell-n-1)!!}{(\ell-n)!(L-n-1)!!}X^{L,n}_{i_1\cdots i_L},\tag{42}$$

where the sum starts with $n=0$ and includes only even values of n if ℓ is even, otherwise it starts with $n=1$ and includes only odd values of n if ℓ is odd. We proceed to verify the correctness of this formula by: (i), showing that it is consistent with the initial values of (35a), (35b); and (ii), verifying that it satisfies the recursion relation (30). Verification of consistency with the initial values is immediate by substituting $\ell=0$ and $\ell=1$ into (42) and noting that in each case there is only one term in the sum and that it agrees with (35a), (35b).

For step (ii), verification that (42) satisfies the recursion relation (30), we substitute (42) into the right hand side of (30) and obtain two terms. The first of these is

$$\frac{2\ell+1}{\ell}\frac{(2\ell-1)(L-\ell+1)!!}{(L-\ell+2)!(L+\ell+1)!!}\sum_{n'=0\text{ or }1}^{\ell-1}(-1)^{\frac{\ell-n'-1}{2}}\times$$
$$\frac{(L-n'+1)!(\ell+n'-2)!!(\ell-n'-2)!!}{(\ell-n'-1)!(L-n')!!}\left((n'+1)\,X^{L,n'+1}_{i_1\cdots i_L}+X^{L,n'-1}_{i_1\cdots i_L}\right) \tag{43}$$

and the second is

$$-\frac{2\ell+1}{\ell}\frac{\ell-1}{2\ell-3}\frac{(2\ell-3)(L-\ell+1)!!}{(L-\ell+2)!(L+\ell-1)!!}\sum_{n=0\text{ or }1}^{\ell-2}(-1)^{\frac{\ell-n-2}{2}}\times$$
$$\frac{(L-n)!(\ell+n-3)!!(\ell-n-3)!!}{(\ell-n-2)!(L-n-1)!!}\,X^{L,n}_{i_1\cdots i_L}. \tag{44}$$

We shift the summation index in the first part of (43) according to $n'\longrightarrow n-1$ and in the second part by $n'\longrightarrow n+1$, so that all terms are proportional to $X^{L,n}_{i_1\cdots i_L}$. We add the two parts of (43) to (44) and, after some algebraic simplification, find that the sum is equal to (42). Thus (42) satisfies the recursion relation and by induction is correct for all values of ℓ. (An expression consistent with (42) is given in ([5], [6]) but is more general because not restricted to three dimensions of space. The consistency of (42) with Theorem 1 of ([5], [6]) is established through use of the identity $\Delta^n p_{i_1}\cdots p_{i_L} = 2^n n! p^{L-2n} X^{L,L-2n}_{i_1\cdots i_L}$, where Δ is the Laplacian.)

It is useful to find an expression for the component of $\hat{p}_{i_1}\cdots\hat{p}_{i_L}$ having maximal angular momentum. With $\ell\longrightarrow L$ in the general decomposition formula (42) we find that

$$(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_L=\sum_{n=0\text{ or }1}^{L}(-1)^{\frac{L-n}{2}}\frac{(L+n-1)!!}{(2L-1)!!}X^{L,n}_{i_1\cdots i_L}. \tag{45}$$

This expression is traceless on all pairs of indices as required by (20) and as can be confirmed by applying the trace identity (41) to (45).

Using (45), the product identity (36), and expression (35a) for $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_0^L$, it is easy to see that $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L$ can be written in an alternative, and illuminating, form:

$$(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L = \frac{(2\ell+1)!!(L-\ell+1)!!}{(L+\ell+1)!!} \sum_{\binom{L}{\ell}} (\hat{p}_{i_1} \cdots \hat{p}_{i_\ell})_\ell^\ell \left(\hat{p}_{i_{\ell+1}} \cdots \hat{p}_{i_L}\right)_0^{L-\ell} . \tag{46}$$

This form displays clearly the fact that every sub-part of $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L$ has angular momentum ℓ among some subset of momentum unit vectors and angular momentum zero among the rest. This behavior is illustrated by the examples shown in (24), (26b), and (26c) (since $(\hat{p}_i)_1^1 = \hat{p}_i$).

IV. Applications

As discussed in [1] and Sec. II, a useful class of three-dimensional Fourier transforms (that of functions like $p^n \hat{p}_{i_1} \cdots \hat{p}_{i_L}$) can be conveniently done after separation of the various angular momenta in the tensor product. With a Fourier transform pair $\Phi(\vec{p})$ and $\Psi(\vec{r})$ defined through

$$\Psi(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} \, e^{i\vec{p}\cdot\vec{r}} \Phi(\vec{p}), \tag{47a}$$

$$\Phi(\vec{p}) = \int d^3r \, e^{-i\vec{p}\cdot\vec{r}} \Psi(\vec{r}), \tag{47b}$$

it is generally true that the angular momenta of $\Phi(\vec{p})$ and $\Psi(\vec{r})$ are the same. It follows that the transform pairs can be represented by

$$\Phi(\vec{p}) = \phi(p) Y_\ell^m(\hat{p}) \iff \Psi(\vec{r}) = \psi(r) Y_\ell^m(\hat{r}) \tag{48}$$

or

$$\Phi(\vec{p}) = \phi(p)\,(\hat{p}_{i_1}\cdots\hat{p}_{i_L})^L_\ell \iff \Psi(\vec{r}) = \psi(r)\,(\hat{x}_{i_1}\cdots\hat{x}_{i_L})^L_\ell\,, \tag{49}$$

where the radial functions $\phi(p)$ and $\psi(r)$ are related by

$$\psi(r) = \frac{i^\ell}{2\pi^2}\int_0^\infty dp\,p^2 j_\ell(pr)\phi(p), \tag{50a}$$

$$\phi(p) = 4\pi(-i)^\ell\int_0^\infty dr\,r^2 j_\ell(pr)\psi(r). \tag{50b}$$

In the case that $\phi(p) = p^n$ the transform is $\psi(r)$ where

$$\psi(r) = \begin{cases} \frac{i^\ell}{2\pi^2}\frac{\chi_{n\ell}}{r^{n+3}} & -(\ell+3) < n < \ell\,, \\ \frac{i^\ell(2\ell+1)!!}{r^\ell}\delta(\vec{r}) & n = \ell\,. \end{cases} \tag{51}$$

Since the transform pair (50a), (50b) is essentially the Hankel transform, ([18], [11], Sect. 15) many additional ϕ, ψ pairs are also available, for example those that relate momentum space and coordinate space versions of the Coulomb wave functions. [19] Results for the examples given in Sec. I are immediate consequences of angular decomposition and the transforms contained in (51).

An interesting use of Fourier transforms of the type considered here is to find unusual differential identities. Consider the Fourier transform of $f(p)\,(p_{i_1}\cdots p_{i_L})^L_\ell$. The momentum vectors can be converted into derivatives when acting on the exponential in the Fourier transform, leading to

$$\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}}f(p)\,(p_{i_1}\cdots p_{i_L})^L_\ell = (-i)^L\,(\partial_{i_1}\cdots\partial_{i_L})^L_\ell\int\frac{d^3p}{(2\pi)^3}\,e^{i\vec{p}\cdot\vec{r}}f(p). \tag{52}$$

On the other hand, the transform can be evaluated explicitly using (51). Comparison leads to a differential identity. For example, comparison of the two approaches for the transform of $\frac{1}{p^2}\,(p_{i_1}\cdots p_{i_k})^k_k$

leads to the identity

$$(\partial_{i_1}\cdots\partial_{i_k})^k_k\,\frac{1}{r}=\frac{(-1)^k(2k-1)!!}{r^{k+1}}\,(\hat{x}_{i_1}\cdots\hat{x}_{i_k})^k_k\,. \tag{53}$$

(The same identity expressed in terms of spherical harmonics has been given by Rowe. [20] General derivatives of inverse powers of r have been worked out by Estrada and Kanwal from a distribution point of view. [12]) For $k=2$, and with use of the Poisson equation $\partial^2\frac{1}{r}=-4\pi\delta(\vec{r}\,)$, one obtains the familiar identity ([13], [26])

$$\partial_i\partial_j\frac{1}{r}=-\frac{4\pi}{3}\delta_{ij}\delta(\vec{r}\,)+\frac{3}{r^3}\big(\hat{x}_i\hat{x}_j-\frac{1}{3}\delta_{ij}\big), \tag{54}$$

as also follows from (3b) of Sec. I. A second interesting identity of this class follows from consideration of the transform of $(p_{i_1}\cdots p_{i_k})^k_k$:

$$(\partial_{i_1}\cdots\partial_{i_k})^k_k\,\delta(\vec{r}\,)=\frac{(-1)^k(2k+1)!!}{r^k}\,(\hat{x}_{i_1}\cdots\hat{x}_{i_k})^k_k\,\delta(\vec{r}\,). \tag{55}$$

Other differential identities can be obtained as easily.

V. Conclusion

Decomposition of tensor products of a vector is useful for many purposes. For example, in quantum mechanics, if one knows the angular momentum structure of an operator, one can bring the powerful and efficient methods of general angular momentum theory to bear on a calculation of matrix elements or expectation values. [24] As one simple example, we define the angular integral

$$\begin{aligned}K&\equiv\int d\Omega_p\,Y^{m*}_\ell(\hat{p})\,(\hat{p}_{i_1}\hat{p}_{i_3}\hat{p}_{i_3})^3_3\,(\hat{p}_{i_4})^1_1\\&=\int d\Omega_p\,Y^{m*}_\ell(\hat{p})\Big\{p_{i_1}p_{i_2}p_{i_3}-\frac{1}{5}\big(p_{i_1}\delta_{i_2i_3}+p_{i_2}\delta_{i_3i_1}+p_{i_3}\delta_{i_1i_2}\big)\Big\}p_{i_4}.\end{aligned} \tag{56}$$

The general rules for the combination of angular momentum assure us that the integral K vanishes unless $\ell = 4$ or $\ell = 2$. We know this because the combination of two objects with angular momenta 3 and 1 gives $(3-1) \leq \ell_{\rm tot} \leq (3+1)$ where the combined angular momentum $\ell_{\rm tot}$ is an integer, and is even since parity is preserved. As another important use of angular decomposition, we have seen that Fourier transforms involving tensor products of vectors are organized according to angular momentum. For example, the transform

$$I_{0;ijk}(\vec{r}\,) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}}\, \hat{p}_i\hat{p}_j\hat{p}_k \tag{57}$$

is evaluated by decomposing the product $\hat{p}_i\hat{p}_j\hat{p}_k$ into its angular momentum components $(\hat{p}_i\hat{p}_j\hat{p}_k)_3^3$ and $(\hat{p}_i\hat{p}_j\hat{p}_k)_1^3$ as in (21) and (22) and using the results of the previous section:

$$\begin{aligned} I_{0;ijk}(\vec{r}\,) & \\ = \sum_{\ell=1,\,\mathrm{odd}}^{3} \frac{i^\ell}{2\pi^2}\frac{\chi_{0\ell}}{r^3}\,(\hat{x}_i\hat{x}_j\hat{x}_k)_\ell^3 &= \frac{i}{2\pi^2}\frac{2}{r^3}\,(\hat{x}_i\hat{x}_j\hat{x}_k)_1^3 - \frac{i}{2\pi^2}\frac{8}{r^3}\,(\hat{x}_i\hat{x}_j\hat{x}_k)_3^3 \\ = \frac{-i}{\pi^2 r^3}\big\{4\hat{x}_i\hat{x}_j\hat{x}_k - (\hat{x}_i\delta_{jk} + \hat{x}_j\delta_{ki} + \hat{x}_k\delta_{ij})\big\}. & \end{aligned} \tag{58}$$

In this work we have given explicit and convenient expressions for the various angular momentum components of a tensor product of vectors. Specifically, (42) expresses the angular momentum ℓ component of the tensor product $\hat{p}_{i_1}\hat{p}_{i_2}\cdots\hat{p}_{i_L}$ in terms of the elementary $X^{L,n}_{i_1\cdots i_L}$ objects, while (45) and (46) together give an especially compact form for the same thing. These expressions are the main results of this paper.

In closing, we make note of a curious and nontrivial summation identity that holds for non-negative integers L and n with $L \geq n$ and

L and n either both even or both odd:

$$\sum_{\ell=n}^{L} \frac{(-1)^{\frac{\ell-n}{2}}(2\ell+1)(L-\ell-1)!!(\ell+n-1)!!(\ell-n-1)!!}{(L-\ell)!(L+\ell+1)!!(\ell-n)!} = \delta_{n,L}, \tag{59}$$

where the sum is over even or odd ℓ depending on whether L and n are even or odd. This summation identity is derived from (8) expressing $\hat{p}_{i_1} \cdots \hat{p}_{i_L}$ as a sum of angular momentum components $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L$ when (42) is used to give an explicit formula for $(\hat{p}_{i_1} \cdots \hat{p}_{i_L})_\ell^L$ followed by interchange of the order of summation.

Acknowledgement

This material is based upon work supported by the National Science Foundation through Grants PHY-1404268 and PHY-1707489.

References

[1] G. S. Adkins, Three-dimensional Fourier transforms, integrals of spherical Bessel functions, and novel delta function identities, Bulletin Alld. Math. Soc., **31**(2016), 215-246.

[2] G. B. Arfken and H. J. Weber, Mathematical Methods for Physicists, Academic Press, San Diego, 5th edition, 2001.

[3] V. B. Berestetskii, E. M. Lifshitz and L.P. Pitaevskii, Quantum Electrodynamics, Pergamon Press, Oxford, 1982.

[4] H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms, Springer-Verlag, Berlin, 1957.

[5] A. Bezubik, A. Dabrowska and A. Strasburger, A new derivation of the plane wave expansion into spherical harmonics and related Fourier transforms, J. Nonlinear Mathematical Physics, 11 Supplement (2004), 167-173.

[6] A. Bezubik and A. Strasburger, A new form of the spherical expansion of zonal functions and Fourier transforms of SO(d)-finite functions, Symmetry, Integrability and Geometry: Methods and Applications, **2**(2006), 033.

[7] J. M. Bowen, Delta function terms arising from classical point-source fields, Am. J. Phys., **62**(1994), 511-515.

[8] G. Breit, The effect of retardation on the interaction of two electrons, Phys. Rev., **34**(1929), 553-573.

[9] J. A. R. Coope, R. F. Snider and F. R. McCourt, Irreducible cartesian tensors, J. Chem. Phys., **43**(1965), 2269-2275.

[10] J. A. R. COOPE AND R. F. SNIDER, Irreducible Cartesian tensors. II. General formulation, J. Math. Phys., **11**(1970), 1003-1017.
[11] B. DAVIES, Integral Transforms and their Applications, Springer-Verlag, New York, 1978.
[12] R. ESTRADA AND R. P. KANWAL, Regularization and distributional derivatives of $(x_1^2 + x_2^2 + \cdots + x_p^2)^{-\frac{n}{2}}$ in $\mathbf{R}^p$, Proc. R. Soc. London A, **401**(1985), 281-297.
[13] C. P. FRAHM, Some novel delta-function identities, Am. J. Phys., **51**(1983), 826-829.
[14] I. S. GRADSHTEYN AND I. M. RYZHIK, Table of Integrals, Series, and Products, Academic Press, New York, 1980.
[15] V. HNIZDO, Generalized second-order partial derivatives of $\frac{1}{r}$, Eur. J. Phys., **32**(2011), 287-297.
[16] P. M. MORSE AND H. FESHBACH, Methods of Theoretical Physics, McGraw Hill, New York, 1953.
[17] J.-M. NORMAND AND J. RAYNAL, Relations between Cartesian and spherical components of irreducible Cartesian tensors, J. Phys. A: Math. Gen., 15 (1982), 1437-1461.
[18] F. OBERHETTINGER, Tables of Bessel Transforms, Springer-Verlag, Berlin, 1972.
[19] B. PODOLSKY AND L. PAULING, The momentum distribution in hydrogen-like atoms, Phys. Rev., **34**(1929), 109-116.
[20] E. G. P. ROWE, Spherical delta functions and multipole expansions, J. Math. Phys., **19**(1978), 1962-1968.
[21] S. G. SAMKO, On the Fourier transforms of the functions $Y_m(x/|x|)/|x|^{n+\alpha}$, Iz. VUZ. Mat., **22**(1978), 73-78.
[22] A. J. STONE, Transformation between cartesian and spherical tensors, Mol. Phys., **29**(1975), 1461-1471.
[23] A. J. STONE, Properties of Cartesian-spherical transformation coefficients, J. Phys. A: Math. Gen., **9**(1976), 485-497.
[24] D. A. VARSHALOVICH, A. N. MOSKALEV AND V. K. KHERSONSKII, Quantum Theory of Angular Momentum, World Scientific, Singapore, 1988.
[25] G. N. WATSON, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 2nd edition, 1922.
[26] Y. YANG AND R. ESTRADA, Regularization using different surfaces and the second-order derivatives of $\frac{1}{r}$, Applicable Analysis, **92**(2013), 246-258.

DEPARTMENT OF PHYSICS AND ASTRONOMY
FRANKLIN & MARSHALL COLLEGE
LANCASTER
PENNSYLVANIA 17604
U.S.A.

E-mail: gadkins@fandm.edu