L_p DUAL CURVATURE MEASURES

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ABSTRACT. A new family of geometric Borel measures on the unit sphere is introduced. Special cases include the L_p surface area measures (which extend the classical surface area measure of Aleksandrov and Fenchel & Jessen) and L_p-integral curvature (which extends Alkesandrov's integral curvature) in the L_p Brunn-Minkowski theory. It also includes the dual curvature measures (which are the duals of Federer's curvature measures) in the dual Brunn-Minkowski theory. This partially unifies the classical theory of mixed volumes and the newer theory of dual mixed volumes.

1. INTRODUCTION

Surface area measure and integral curvature are two important geometric measures of convex bodies in the Euclidean n-space, $\mathbb{R}^n$. Integral curvature measures the images of the Gauss map of a convex body, while surface area measure measures the inverse images of the Gauss map of a convex body. Both measures are fundamental concepts in the classical Brunn-Minkowski theory of convex bodies. The Minkowski problem characterizing surface area measure and the Aleksandrov problem characterizing integral curvature are two well-known problems. In modern convex geometry, the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory generalize and dualize the classical Brunn-Minkowski theory. The L_p surface area measures were introduced in [38], and L_p integral curvatures were recently defined in [26]. Equally fundamental geometric measures in the dual Brunn-Minkowski theory were only constructed very recently in [25]. They are called dual curvature measures (and are dual to Federer's curvature measures). Minkowski problems associated with these geometric measures are major problems in convex geometric analysis, which are far from being completely solved.

The purpose of this paper is to continue the study begun in [25] and to construct L_p dual curvature measures. It turns out that the L_p surface area measure, L_p integral curvatures, and dual curvature measures are all special cases of the now-to-be introduced L_p dual curvature measures. These lead to a unified concept of mixed volume that includes Minkowski's classical first mixed volume, L_p mixed volumes, L_p entropy, as well as dual mixed volumes as special cases. We shall demonstrate a surprising connection between the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory by establishing geometric inequalities and variational integral formulas for the unified mixed volumes and for the new L_p dual curvature measures. We pose the L_p dual Minkowski problem for L_p dual curvature measure which opens a new direction of study in convex geometric analysis. Detailed explanations are provided below.

The L_p Brunn-Minkowski theory as a generalization of the classical Brunn-Minkowski theory has attracted increasing interest in recent years partly due to its wide range of connections with other subjects such as affine geometry, Banach space theory, harmonic analysis, and partial differential

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equations. The core concept in the L_p Brunn-Minkowski theory (introduced in [38]) is the notion of the L_p *surface area measure* which is a Borel measure (defined on the unit sphere) for each convex body in $\mathbb{R}^n$ that contains the origin in its interior. The L_p-cosine transform (a spherical variant of the Fourier transform) of the L_p surface area measure turns out to yield a finite dimensional Banach norm. The associated affine isoperimetric inequality for the volume of the unit ball of this Banach norm is known as the L_p *Petty projection inequality* which was established in [39] and is a profound strengthening of the classical isoperimetric inequality. The Radon-Nikodym derivative of the L_p surface area measure with respect to the spherical Lebesgue measure is called the L_p *curvature* function. The integral of the L_p curvature function (raised to an appropriate power) over the unit sphere is L_p *affine surface area* which has been a focus of study in affine geometry and valuation theory, see e.g. [20, 35, 42, 50, 53, 54]. Finding the necessary and sufficient conditions for a given measure to guarantee that it is the L_p surface area measure is the existence problem called the L_p *Minkowski problem* posed in [38]. Solving the L_p Minkowski problem requires solving a degenerate and singular Monge-Ampère type equation on the unit sphere. The problem has been solved for $p \geq 1$, see [10, 27, 38], but critical cases for $p < 1$ remain open, see e.g. [7, 10, 51, 52, 63, 64]. The solution of the L_p Minkowski problem and the L_p Petty projection inequality are key tools used for establishing the L_p *affine Sobolev inequality* and its relatives, see [11, 23, 40, 41, 59].

Let $\mathcal{K}_o^n$ denote the class of convex bodies (compact convex subsets) in Euclidean n-space, $\mathbb{R}^n$, that contain the origin in their interiors. The *support function*, $h_Q : S^{n-1} \to (0, \infty)$ of the convex body $Q \in \mathcal{K}_o^n$, determines Q uniquely and is defined, for $v \in S^{n-1}$, by $h_Q(v) = \max\{v \cdot x : x \in Q\}$, where $v \cdot x$ is the standard inner product of v and x in $\mathbb{R}^n$. The basic operation between convex bodies is the Minkowski combination (vector sum). For $K, L \in \mathcal{K}_o^n$ and $s, t \geq 0$, the Minkowski combination $sK + tL$ is defined by $sK + tL = \{sx + ty : x \in K, y \in L\}$, or equivalently,

$$h_{sK+tL} = sh_K + th_L.$$

The fundamentally important *surface area measure* $S(K, \cdot)$ of a convex body K can be defined by the variational formula,

$$\frac{d}{dt} V(K + tL)\Big|_{t=0^+} = \int_{S^{n-1}} h_L(v)\, dS(K, v),$$

which holds for each $L \in \mathcal{K}_o^n$. The integral above times $\frac{1}{n}$ is Minkowski's first mixed volume of K and L,

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(v)\, dS(K, v).$$

The mixed volume V_1 is an extension of three functionals: volume, surface area and mean width. It is the most studied mixed volume within the classical Brunn-Minkowski theory.

An extension of Minkowski combinations studied by Firey in the early 1960's, defines the L_p Minkowski combination (also known as the Minkowski-Firey combination), $s \cdot K +_p t \cdot L$, for each $p \geq 1$, each pair $K, L \in \mathcal{K}_o^n$, and $s, t \geq 0$, by

$$h^p_{s\cdot K +_p t\cdot L} = sh^p_K + th^p_L.$$

The concept of L_p mixed volume was defined in the 1990's after introducing the fundamental concept of L_p *surface area measure* (see [38]). The L_p surface area measure $S_p(K, \cdot)$ of a convex body $K \in \mathcal{K}_o^n$ can be defined by the variational formula,

$$\frac{d}{dt} V(K +_p t \cdot L)\Big|_{t=0^+} = \frac{1}{p} \int_{S^{n-1}} h^p_L(v)\, dS_p(K, v), \tag{1.1}$$

which holds for each $L \in \mathcal{K}_o^n$. The integral above (times p/n) is called the L_p mixed volume of K and L:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v)\, dS_p(K, v). \tag{1.2}$$

It generalizes Minkowski's first mixed volume $V_1(K, L)$ of K and L. The important L_p *Minkowski inequality* for L_p mixed volume states that for $p \geq 1$,

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p,$$

with equality if and only if K, L are dilates when $p > 1$ and if and only if K, L are homothets in the case where $p = 1$.

The dual Brunn-Minkowski theory was developed in the 1970s (in [36]) as the dual theory to the classical Brunn-Minkowski theory based on a conceptual duality in convex geometry, see Schneider [48], Chapter 9. The dual Brunn-Minkowski theory studies dual mixed volumes of star bodies, see [17, 36, 37, 55, 56]. A star body $Q \subset \mathbb{R}^n$ is a compact star-shaped set about the origin whose radial function $\rho_Q : S^{n-1} \to (0, \infty)$, defined for $u \in S^{n-1}$ by $\rho_Q(u) = \max\{\lambda > 0 : \lambda u \in Q\}$, is continuous. Denote the set of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_o^n$. Obviously, $\mathcal{K}_o^n \subset \mathcal{S}_o^n$. In the late 1980's, the important concept of *intersection body* in the dual Brunn-Minkowski theory was introduced (in [37]). This brought remarkable applications of Radon transforms and Fourier transforms, tools from harmonic analysis, to convex geometry, see e.g., [14, 16, 17, 29–31, 37, 57, 58].

The dual Brunn-Minkowski theory is a theory of dual mixed volumes of star bodies. For $q \in \mathbb{R}$, the q-th dual mixed volume of $K, Q \in \mathcal{S}_o^n$, is defined by

$$\widetilde{V}_q(K, Q) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) \rho_Q^{n-q}(u)\, du, \tag{1.3}$$

where the integration is with respect to spherical Lebesgue measure. The basic geometric invariants associated with a star body are the dual quermassintegrals, also called dual volumes: For $q \neq 0$, the q-th dual volume $\widetilde{V}_q(K)$ of $K \in \mathcal{S}_o^n$ is defined by $\widetilde{V}_q(K) = \widetilde{V}_q(K, B)$, where B is the origin-centered unit ball in $\mathbb{R}^n$. One of the reasons that the q-th dual volume is important in geometric tomography is that for integer values $q = 1, \ldots, n-1$, and each $K \in \mathcal{S}_o^n$,

$$\widetilde{V}_q(K) = c_{n,q} \int_{G(n,q)} \mathrm{vol}_q(K \cap \xi)\, d\xi,$$

where vol_q denotes volume in $\mathbb{R}^q$, and the integration is with respect to the rotation invariant probability measure on $G(n, q)$, the Grassmannian of q dimensional subspaces of $\mathbb{R}^n$. The constant $c_{n,q}$ is trivially determined by taking K to be B.

Very recently, a new family of geometric measures were discovered in [25]. These measures are the long-sought duals (in the dual Brunn-Minkowski theory) of Federer's curvature measures (which are fundamental in the classical Brunn-Minkowski theory). The new measures are called *dual curvature measures*. For real $q \neq 0$, the q-th dual curvature measure $\widetilde{C}_q(K, \cdot)$ of convex body $K \in \mathcal{K}_o^n$ is a Borel measure on the unit sphere that may be defined via the variational formula,

$$\frac{d}{dt} \widetilde{V}_q(K + tL)\Big|_{t=0^+} = q \int_{S^{n-1}} h_L(v) h_K(v)^{-1} d\widetilde{C}_q(K, v), \tag{1.4}$$

which holds for every $L \in \mathcal{K}_o^n$. In the same way that L_p surface area measures play a critical role in the L_p Brunn-Minkowski theory, dual curvature measures can be seen to be a central concept within the dual Brunn-Minkowski theory.

The Minkowski problem (including both existence and uniqueness) associated with the dual curvature measures, is called the *dual Minkowski problem*, and is a major open problem in the dual Brunn-Minkowski theory. It requires solving a degenerate and singular Monge-Ampère type equation on the unit sphere. Existence of solutions for the dual Minkowski problem for even data within the class of origin-symmetric convex bodies was established in [25]. Very recently existence for the critical cases of the even dual Minkowski problem were established in [62] and [5], and a complete solution to the dual Minkowski problem with negative indices was given in [61].

Aleksandrov's *integral Gauss curvature*, or simply *integral curvature*, generalizes *total Gauss curvature* of a smooth convex body to become a geometric measure for all convex bodies (without any smoothness restrictions). It measures the Gauss image of points on the boundary of a convex body parameterized by the radial directions of the points. The *Aleksandrov problem* asks for necessary and sufficient conditions so that a given measure on the unit sphere is the integral curvature of a convex body. Aleksandrov solved his problem by using his topological mapping lemma and polytope approximation. (See Oliker [45] for an alternate approach.) It was recently discovered in [26] that the integral curvature and the Aleksandrov problem have natural extensions in the L_p Brunn-Minkowski theory. The concept of L_p *integral curvature* was introduced in [26], and the associated L_p Aleksandrov problem was posed (in [26]) as well. The paper [26] develops a radically new approach to studying the Aleksandrov problem and its L_p extension.

The singular case $q = 0$ of dual volume leads to the notion of dual entropy of a star body. For $K \in \mathcal{S}_o^n$, the *dual entropy* $\widetilde{E}(K)$ may be defined by

$$\widetilde{E}(K) = \frac{1}{n}\int_{S^{n-1}} \log \rho_K(u)\, du.$$

The L_p *integral curvature*, $J_p(K,\cdot)$, of a convex body $K \in \mathcal{K}_o^n$, introduced in [26], may be defined via the variational formula,

$$\frac{d}{dt}\widetilde{E}(K +_p t\cdot L)\Big|_{t=0^+} = \frac{1}{np}\int_{S^{n-1}} h_L^p(v)\, dJ_p(K^*, v), \tag{1.5}$$

which holds for all $L \in \mathcal{K}_o^n$ and where K^* is the polar body of K, defined by

$$K^* = \{x \in \mathbb{R}^n : x\cdot y \le 1 \text{ for all } y \in K\}.$$

In view of the fact that volume is a special dual volume and dual entropy is the singular case of dual volumes, it is a natural question to ask if the three families of geometric measures — the L_p surface area measure $S_p(K,\cdot)$, the q-th dual curvature measure $\widetilde{C}_q(K,\cdot)$, and the L_p integral curvature $J_p(K,\cdot)$ might all belong to one large family of geometric measures associated with a convex body $K \in \mathcal{K}_o^n$. It is the purpose of this paper to show that such a family of geometric measures of convex bodies does exist. They will be called L_p *dual curvature measures* and they unify L_p surface area measures, dual curvature measures, and L_p integral curvatures. The three variational formulas (1.1), (1.4), and (1.5) are special cases of one general variational formula for L_p dual curvature measures. Using the new L_p dual curvature measures, we introduce the notion of L_p *dual mixed volumes* which unifies L_p mixed volumes of convex bodies in the L_p Brunn-Minkowski theory and dual mixed volumes of star bodies in the dual Brunn-Minkowski theory. Thus, parts of the L_p Brunn-Minkowski theory and the dual Brunn-Minkowski theory can finally be unified. The L_p dual curvature measures are the core concept of this unification.

The L_p dual curvature measures, $\widetilde{C}_{p,q}$, are a two-parameter family of Borel measures on the unit sphere. Specifically, for $p, q \in \mathbb{R}$, a convex body $K \in \mathcal{K}_o^n$, and a star body $Q \in \mathcal{S}_o^n$, we define the

Borel measure $\widetilde{C}_{p,q}(K,Q,\cdot)$ on S^{n-1} by

$$\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u)) h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_Q(u)^{n-q}\, du, \tag{1.6}$$

for each continuous $g : S^{n-1} \to \mathbb{R}$, where α_K is the radial Gauss map that associates to almost (with respect to spherical Lebesgue measure) each $u \in S^{n-1}$ the unique outer unit normal at the point $\rho_K(u)u \in \partial K$.

The L_p surface area measures, the dual curvature measures and the L_p integral curvatures are special cases, of the L_p dual curvature measures in the sense that for $p, q \in \mathbb{R}$, and $K \in \mathcal{K}_o^n$,

$$\widetilde{C}_{p,n}(K,B,\cdot) = \frac{1}{n} S_p(K,\cdot), \tag{1.7}$$

$$\widetilde{C}_{0,q}(K,B,\cdot) = \widetilde{C}_q(K,\cdot), \tag{1.8}$$

$$\widetilde{C}_{p,0}(K,B,\cdot) = \frac{1}{n} J_p(K^*,\cdot), \tag{1.9}$$

where B is the unit ball in $\mathbb{R}^n$.

Using L_p dual curvature measures, we can define L_p dual mixed volumes. For $p, q \in \mathbb{R}$, and convex bodies $K, L \in \mathcal{K}_o^n$, and a star body $Q \in \mathcal{S}_o^n$, the L_p dual mixed volume $\widetilde{V}_{p,q}(K,L,Q)$ is defined by

$$\widetilde{V}_{p,q}(K,L,Q) = \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(K,Q,v). \tag{1.10}$$

The L_p mixed volume and the dual mixed volume will be shown to be the special cases,

$$\widetilde{V}_{p,q}(K,L,K) = V_p(K,L), \tag{1.11}$$

$$\widetilde{V}_{p,q}(K,K,Q) = \widetilde{V}_q(K,Q). \tag{1.12}$$

The L_p dual mixed volume has the following integral formula in terms of support functions, radial functions, and the radial Gauss map,

$$\widetilde{V}_{p,q}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} h_L(\alpha_K(u))^p h_K(\alpha_K(u))^{-p}\rho_K(u)^q\rho_Q(u)^{n-q}\, du. \tag{1.13}$$

The following inequality for L_p dual mixed volume is a generalization of the L_p Minkowski inequality for L_p mixed volume. Suppose $1 \le \frac{q}{n} \le p$. If $K, L \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{p,q}(K,L,Q)^n \ge V(K)^{q-p}V(L)^pV(Q)^{n-q}, \tag{1.14}$$

with equality, when $q > n$, if and only if K, L, and Q are dilates, while when $q = n$ and $p > 1$, with equality if and only if K and L are dilates, while when $q = n$ and $p = 1$, with equality if and only if K and L are homothets.

To simplify stating the general variational formula that defines the L_p dual curvature measures, we introduce the *normalized power function*. For $q \in \mathbb{R}$, and $t \in (0,\infty)$, define $t^{\bar{q}}$, by

$$t^{\bar{q}} = \begin{cases} \frac{1}{q}t^q & q \neq 0, \\ \log t & q = 0. \end{cases} \tag{1.15}$$

For $q \in \mathbb{R}$ and star bodies $K, Q \in \mathcal{S}_o^n$, the normalized dual mixed volume $\widetilde{V}_{\bar{q}}(K, Q)$ is defined by

$$\widetilde{V}_{\bar{q}}(K, Q) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{\rho_K}{\rho_Q}\right)^{\bar{q}}(u) \rho_Q^n(u)\, du. \tag{1.16}$$

Note that for $q \neq 0$, we have $q\widetilde{V}_{\bar{q}}(K, Q) = \widetilde{V}_q(K, Q)$, while for $q = 0$ the normalized dual mixed volume $\widetilde{V}_{\bar{q}}(K, Q)$ differs considerably from the standard definition of $\widetilde{V}_q(K, Q)$ — not just by a multiplicative constant.

Recall that L_p Minkowski combinations have recently been extended so that they are defined for all $p \in \mathbb{R}$ (see §2 for details). One of the goals of this work is to demonstrate that for $p, q \in \mathbb{R}$, a convex body $K \in \mathcal{K}_o^n$ and a star body $Q \in \mathcal{S}_o^n$, there is a variational formula that can define the L_p dual curvature measure $\widetilde{C}_{p,q}(K, Q, \cdot)$:

$$\frac{d}{dt} \widetilde{V}_{\bar{q}}(K +_p t \cdot L, Q)\Big|_{t=0^+} = \int_{S^{n-1}} h_L^{\bar{p}}(v)\, d\widetilde{C}_{p,q}(K, Q, v), \tag{1.17}$$

which holds for every $L \in \mathcal{K}_o^n$. This unifies (1.1), (1.4), and (1.5). This is also the key fact needed to make it possible to solve the Minkowski problems associated with L_p dual curvature measures by using a variational method.

The obvious major problem of study regarding the new L_p dual curvature measures is the L_p *dual Minkowski problem* — a general Minkowski problem that unifies the L_p Minkowski problem, the dual Minkowski problem, and the L_p Aleksandrov problem. The L_p dual Minkowski problem concerns both the existence and the uniqueness questions. The existence problem is to find necessary and sufficient conditions so that a given measure on the unit sphere is the L_p dual curvature measure of a convex body in $\mathbb{R}^n$. The uniqueness question asks to what extent a convex body is uniquely determined by its L_p dual curvature measure. Recall that the L_p Minkowski problem, the dual Minkowski problem, and the L_p Aleksandrov problem are only partially solved. Important special cases are largely open, for example, the *centro-affine Minkowski problem* and the *logarithmic Minkowski problem*. Solving the new L_p dual Minkowski problem requires solving a degenerate and singular Monge-Ampère equation on the unit sphere S^{n-1} of the following type: For fixed $p, q \in \mathbb{R}$,

$$h^{1-p} \|\bar{\nabla} h + h\iota\|^{q-n} \det(\bar{\nabla}^2 h + hI) = f,$$

where $\|\cdot\|$ is a given n-dimensional Banach norm, $f : S^{n-1} \to [0, \infty)$ is the given "data" function, $h : S^{n-1} \to (0, \infty)$ is the function to be found, and $\iota : S^{n-1} \to S^{n-1}$ is the identity map. Here, $\bar{\nabla} h$ and $\bar{\nabla}^2 h$ denote the gradient vector and the Hessian matrix of h, respectively, with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix.

2. Preliminaries

Schneider's book [48] is our standard reference for the basics regarding convex bodies. The books [13, 18] are also good references.

Throughout $\mathbb{R}^n$ denotes n-dimensional Euclidean space. For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of x. For $x \in \mathbb{R}^n \setminus \{0\}$, we will use both $\bar{x}$ and $\langle x \rangle$ to abbreviate $x/|x|$, and for $E \subset \mathbb{R}^n \setminus \{0\}$, we write $\bar{E}$ for $\{\bar{x} : x \in E\}$. The origin-centered unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ is always denoted by B, and its boundary by S^{n-1}.

For the set of continuous functions defined on the unit sphere S^{n-1} write $C(S^{n-1})$, and for $f \in C(S^{n-1})$ write $\|f\|_\infty = \max_{v \in S^{n-1}} |f(v)|$. We shall view $C(S^{n-1})$ as endowed with the topology induced by this *max-norm*.

If $K \subset \mathbb{R}^n$ is compact and convex, then $h(K,\cdot) = h_K : \mathbb{R}^n \to \mathbb{R}$, the support function of K, is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

The support function is convex and homogeneous of degree 1. A compact convex subset of $\mathbb{R}^n$ is uniquely determined by its support function. From the definition of the support function, we see immediately that for $\phi \in \mathrm{SL}(n)$, for the support function of ϕK, the image of K under ϕ, we have for all $x \in \mathbb{R}^n$,

$$h(\phi K, x) = h(K, \phi^t x), \tag{2.1}$$

where ϕ^t denotes the transpose of ϕ.

The gradient of h_K in $\mathbb{R}^n$ is denoted by ∇h_K. When h_K is viewed as restricted to the unit sphere S^{n-1}, the gradient of h_K on S^{n-1} with respect to the standard metric of S^{n-1} is denoted by $\bar{\nabla} h_K$. Since $h_K : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous of degree 1, at each point $v \in S^{n-1}$ where h_K is differentiable,

$$\nabla h_K(v) = \bar{\nabla} h_K(v) + h_K(v)v. \tag{2.2}$$

Denote by $\mathcal{K}^n$ the space of compact convex sets in $\mathbb{R}^n$ endowed with the *Hausdorff metric*; i.e. the distance between $K, L \in \mathcal{K}^n$ is $\|h_K - h_L\|_\infty$. By a *convex body* in $\mathbb{R}^n$ we will always mean a compact convex set with nonempty interior. Denote by $\mathcal{K}_o^n$ the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors.

Let $K \subset \mathbb{R}^n$ be compact and star-shaped with respect to the origin; i.e., the line segment joining each point of K to the origin is completely contained in K. The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\},$$

for $x \neq 0$. From the definition of the radial function, we see immediately that for $\phi \in \mathrm{SL}(n)$, we have

$$\rho_{\phi K}(x) = \rho_K(\phi^{-1}x), \tag{2.3}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

A compact star-shaped (about the origin) set is uniquely determined by its radial function on S^{n-1}. Denote by $\mathcal{S}^n$ the set of compact star-shaped sets. A star body is a compact star-shaped set with respect to the origin whose radial function is continuous and positive. If K is a star body, then obviously

$$\partial K = \{\rho_K(u)u : u \in S^{n-1}\} = \{\rho_K(x)x : x \in \mathbb{R}^n \setminus \{0\}\} = \{x \in \mathbb{R}^n : \rho_K(x) = 1\}.$$

Denote by $\mathcal{S}_o^n$ the space of star bodies in $\mathbb{R}^n$ endowed with the *radial metric*; i.e., the distance between $K, L \in \mathcal{S}_o^n$, is $\|\rho_K - \rho_L\|_\infty$. Note that $\mathcal{K}_o^n \subset \mathcal{S}_o^n$ and that on the space $\mathcal{K}_o^n$ the Hausdorff metric and radial metric are equivalent, and thus $\mathcal{K}_o^n$ is a subspace of $\mathcal{S}_o^n$.

For a convex body $K \in \mathcal{K}_o^n$, the *polar body* K^* of K is the convex body in $\mathbb{R}^n$ defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } y \in K\}.$$

From this definition, we easily see that on $\mathbb{R}^n \setminus \{0\}$,

$$\rho_K = 1/h_{K^*} \quad \text{and} \quad h_K = 1/\rho_{K^*}. \tag{2.4}$$

It follows that

$$K^{**} = K, \tag{2.5}$$

a fact we shall frequently make use of.

Throughout, $\Omega \subset S^{n-1}$ will denote a set that is both closed and that cannot be contained in any closed hemisphere of S^{n-1}. The *Wulff shape* $[h] \in \mathcal{K}_o^n$, of a continuous function $h : \Omega \to (0, \infty)$, also known as the *Aleksandrov body* of h, is the convex body defined by

$$[h] = \bigcap_{v \in \Omega} \{x \in \mathbb{R}^n : x \cdot v \le h(v)\}. \tag{2.6}$$

Because of the restrictions placed on Ω, we see that $[h] \in \mathcal{K}_o^n$. If $K \in \mathcal{K}_o^n$, then it is easily seen that

$$[h_K] = K.$$

Let $\rho : \Omega \to (0, \infty)$ be continuous. Since $\Omega \subset S^{n-1}$ is assumed to be closed, and ρ is continuous, $\{\rho(u)u : u \in \Omega\}$ is a compact set in $\mathbb{R}^n$. Hence, the convex hull $\langle\rho\rangle$ generated by ρ,

$$\langle\rho\rangle = \operatorname{conv}\{\rho(u)u : u \in \Omega\},$$

is compact as well (see Schneider [48], Theorem 1.1.11). Since Ω is not contained in any closed hemisphere of S^{n-1}, the compact convex set $\langle\rho\rangle$ contains the origin in its interior; i.e., $\langle\rho\rangle \in \mathcal{K}_o^n$. Obviously, if $K \in \mathcal{K}_o^n$,

$$\langle\rho_K\rangle = K. \tag{2.7}$$

We shall make frequent use of the easily-established fact that the support function of $\langle\rho\rangle$ is given by:

$$h_{\langle\rho\rangle}(v) = \max_{u \in \Omega}(v \cdot u)\rho(u), \tag{2.8}$$

for all $v \in S^{n-1}$.

The Wulff shape $[h]$ determined by h and the convex hull $\langle 1/h\rangle$ generated by the function $1/h$ are easily shown (see [25]) to be polar reciprocals of each other; i.e.,

$$[h]^* = \langle 1/h\rangle.$$

For $K, L \subset \mathbb{R}^n$ that are compact and convex, and real $s, t \ge 0$, the *Minkowski combination*, $sK + tL \subset \mathbb{R}^n$, is the compact, convex set defined by

$$sK + tL = \{sx + ty : x \in K \text{ and } y \in L\},$$

and its support function is given by

$$h_{sK+tL} = sh_K + th_L. \tag{2.9}$$

If K and L contain the origin, then for $p \ge 1$, the L_p Minkowski combination, also known as the Minkowski-Firey combination, $s \cdot K +_p t \cdot L \subset \mathbb{R}^n$, is the compact, convex set whose support function is given by,

$$h(s \cdot K +_p t \cdot L, \cdot)^p = sh(K, \cdot)^p + th(L, \cdot)^p. \tag{2.10}$$

Note that "$\cdot$" is used with the subscript p implied.

Using the concept of Wulff shape, the definition of an L_p Minkowski combination can be extended so as to be defined for $p < 1$ and even negative s or t: Fix a real $p \ne 0$. For $K, L \in \mathcal{K}_o^n$, and $s, t \in \mathbb{R}$ such that $sh_K^p + th_L^p$ is a strictly positive function S^{n-1}, define the L_p *Minkowski combination*, $s \cdot K +_p t \cdot L \in \mathcal{K}_o^n$, by

$$s \cdot K +_p t \cdot L = [(sh_K^p + th_L^p)^{1/p}]. \tag{2.11}$$

When $p = 0$, define $s \cdot K +_0 t \cdot L$ by

$$s \cdot K +_0 t \cdot L = [h_K^s h_L^t]. \tag{2.12}$$

Since, h_K, h_L are strictly positive functions on S^{n-1}, it follows that $s \cdot K +_0 t \cdot L$ is defined for all $s, t \in \mathbb{R}$.

For $\phi \in \mathrm{SL}(n)$ and $p \neq 0$,

$$s \cdot \phi K +_p t \cdot \phi L = \phi(s \cdot K +_p t \cdot L). \tag{2.13}$$

If also $s + t = 1$, then (2.13) hold for $p = 0$ as well. That (2.13) holds follows from the observation that since the support function is homogeneous of degree 1 we can re-write (2.11), by using (2.6), as an intersection over $\mathbb{R}^n \backslash \{0\}$, rather than S^{n-1},

$$s \cdot \phi K +_p t \cdot \phi L = \bigcap_{y \neq 0} \{x \in \mathbb{R}^n : x \cdot y \leq (sh_{\phi K}(y)^p + th_{\phi L}(y)^p)^{1/p}\}$$

and then use (2.1). If $s + t = 1$, then we can re-write (2.12), by using (2.6), as an intersection over $\mathbb{R}^n \backslash \{0\}$,

$$s \cdot \phi K +_0 t \cdot \phi L = \bigcap_{y \neq 0} \{x \in \mathbb{R}^n : x \cdot y \leq h_{\phi K}(y)^s h_{\phi L}(y)^t\}$$

and then use (2.1).

For $K, L \subset \mathbb{R}^n$ that are compact and star-shaped (with respect to the origin), and real $s, t \geq 0$, the *radial combination*, $sK \,\widetilde{+}\, tL \subset \mathbb{R}^n$, is the compact star-shaped set defined by

$$sK \,\widetilde{+}\, tL = \{sx + ty : x \in K \text{ and } y \in L, \text{ whenever } x \cdot y = |x||y|\}.$$

Obviously, $x \cdot y = |x||y|$ means that either $y = \alpha x$ or $x = \alpha y$ for some $\alpha \geq 0$. The radial function of the radial combination of two star-shaped sets is the combination of their radial functions; i.e.,

$$\rho(sK \,\widetilde{+}\, tL, \cdot) = s\rho(K, \cdot) + t\rho(L, \cdot).$$

For fixed real q, the *radial q-combination* $s \cdot K \,\widetilde{+}_q\, t \cdot L$ is defined by

$$\rho(s \cdot K \,\widetilde{+}_q\, t \cdot L, \cdot)^q = s\rho(K, \cdot)^q + t\rho(L, \cdot)^q, \quad q \neq 0, \tag{2.14}$$

$$\rho(s \cdot K \,\widetilde{+}_0\, t \cdot L, \cdot) = \rho(K, \cdot)^s \rho(L, \cdot)^t. \tag{2.15}$$

Note that in order to have a natural definition of $s \cdot K \,\widetilde{+}_0\, t \cdot L$ whose radial function is homogeneous of degree -1 it is necessary in (2.15) that $s + t = 1$. Of course, one could use (2.15) to define $s \cdot K \,\widetilde{+}_0\, t \cdot L$ as above on S^{n-1}, exclusively, and then extend the function obtained to $\mathbb{R}^n \setminus \{0\}$ by declaring it to be homogeneous of degree -1.

For $\phi \in \mathrm{SL}(n)$ and $q \neq 0$,

$$s \cdot \phi K \,\widetilde{+}_q\, t \cdot \phi L = \phi(s \cdot K \,\widetilde{+}_q\, t \cdot L). \tag{2.16}$$

If also $s + t = 1$, then (2.16) hold for $q = 0$ as well. All this follows from (2.3).

For $q \in \mathbb{R} \backslash \{0\}$, and star bodies $K, L \in \mathcal{S}_o^n$, the q-th dual mixed volume $\widetilde{V}_q(K, L)$ is defined by

$$\widetilde{V}_q(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^q(u) \rho_L^{n-q}(u)\, du = \frac{q}{n} \lim_{t \to 0} [V(L \,\widetilde{+}_q\, t \cdot K) - V(L)]/t, \tag{2.17}$$

where the second equality follows from (2.14) and the polar coordinate formula for volume. ¿From (2.17) and (2.16) we get the well-known fact that for $\phi \in \mathrm{SL}(n)$,

$$\widetilde{V}_q(\phi K, \phi L) = \widetilde{V}_q(K, L). \tag{2.18}$$

The *q-th dual volume* $\widetilde{V}_q(K)$ of a star body $K \in \mathcal{S}_o^n$ is defined by

$$\widetilde{V}_q(K) = \widetilde{V}_q(K, B),$$

where B is the unit ball.

The *dual mixed entropy* $\widetilde{E}(K, L)$ of star bodies $K, L \in \mathcal{S}_o^n$ is defined by

$$\widetilde{E}(K, L) = \frac{1}{n} \int_{S^{n-1}} \log\Big(\frac{\rho_K(u)}{\rho_L(u)}\Big) \rho_L^n(u)\, du. \tag{2.19}$$

Note that $\widetilde{E}(K, L) = \widetilde{V}_{\bar{0}}(K, L)$. As was the case in (2.18), for the dual mixed entropy we have that for $\phi \in \mathrm{SL}(n)$,

$$\widetilde{E}(\phi K, \phi L) = \widetilde{E}(K, L). \tag{2.20}$$

To establish this, use the polar-coordinate formula for volume together with definition (2.15), and then (2.19) to see that

$$\lim_{t\to 0} \frac{V((1-t)\cdot L \mathbin{\widetilde{+}_0} t\cdot K) - V(L)}{t} = n\widetilde{E}(K, L).$$

This together with the fact that $\mathrm{SL}(n)$-transformations leave volume V unaltered, definition (2.15), and (2.16), give (2.20).

We shall make use of the fact that a function $h \in C^2(S^{n-1})$ is the support function of a convex body provided that the matrix $\bar{\nabla}^2 h + hI$ is positive definite, where $\bar{\nabla}^2 h$ is the Hessian matrix of h on S^{n-1}, with respect to an orthonormal frame. This fact follows from Theorem 1.5.13 of Schneider [48] if we switch from Euclidean derivatives to spherical derivatives.

The case $p = 1$ of the following lemma is well known (see Schneider [48]). For real $p \neq 0$ it was established by Kiderlen [28]. Thus, only the case $p = 0$ will require proof.

Lemma 2.1. *For each* $p \in \mathbb{R}$, *the set*

$$\big\{ch_K^{\bar{p}} - ch_B^{\bar{p}} : K \in \mathcal{K}_o^n,\ c > 0\big\}$$

is dense in $C(S^{n-1})$.

Proof. It is sufficient to show that given a $g \in C^2(S^{n-1})$ there exist $K \in \mathcal{K}_o^n$ and $c > 0$ so that

$$g = ch_K^{\bar{p}} - ch_B^{\bar{p}},$$

for $p = 0$. This will be the case provided

$$h = e^{tg}$$

is the support function of a convex body in $\mathcal{K}_o^n$ for some sufficiently small $t > 0$.

The function h is obviously in $C^2(S^{n-1})$. An easy calculation gives

$$\bar{\nabla}^2 h = ht^2 \bar{\nabla} g \otimes \bar{\nabla} g + ht\bar{\nabla}^2 g.$$

It follows that $\bar{\nabla}^2 h + hI \to I$ uniformly as $t \to 0$, and thus $\bar{\nabla}^2 h + hI$ is positive definite when t is sufficiently small. □

We shall use Lemma 2.1 to show that two Borel measures are equal provided they agree when integrated against the $\bar{p}$-th powers of support functions of bodies in $\mathcal{K}_o^n$.

We shall require the following definition.

Definition 2.2. *Suppose $p \in \mathbb{R}$. If μ is a Borel measure on S^{n-1} and $\phi \in \mathrm{SL}(n)$ then, $\phi_p^{\lrcorner}\mu$, the L_p image of μ under ϕ, is a Borel measure such that,*

$$\int_{S^{n-1}} f(u)\, d\phi_p^{\lrcorner}\mu(u) = \int_{S^{n-1}} |\phi^{-1}u|^p f(\langle \phi^{-1}u\rangle)\, d\mu(u),$$

for each Borel $f : S^{n-1} \to \mathbb{R}$.

3. The radial Gauss map

Suppose K is a convex body in $\mathbb{R}^n$. For each $v \in \mathbb{R}^n \setminus \{0\}$, the hyperplane

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}$$

is called the *supporting hyperplane to K with outer normal v*.

It will be convenient to call vector $u \in \mathbb{R}^n \setminus \{0\}$ a *regular radial vector* for K provided the boundary point $\rho_K(u)u$ is *smooth*; i.e., $\rho_K(u)u \in H_K(v_1) \cap H_K(v_2)$, for $v_1, v_2 \in S^{n-1}$ is only possible when $v_1 = v_2$. We recall that a convex body is called *smooth* provided each boundary point is smooth; i.e., provided each boundary point of the convex body has a unique supporting hyperplane passing through it. A vector $v \in \mathbb{R}^n \setminus \{0\}$ is called a *regular normal vector* for the body K provided $H_K(v) \cap \partial K$ consists of a single point.

The *spherical image* of $\sigma \subset \partial K$ is defined by

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

The *reverse spherical image* of $\eta \subset S^{n-1}$ is defined by

$$\boldsymbol{x}_K(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K.$$

Let $\sigma_K \subset \partial K$ be the set consisting of all $x \in \partial K$, for which the set $\boldsymbol{\nu}_K(\{x\})$, which we frequently abbreviate as $\boldsymbol{\nu}_K(x)$, contains more than a single element. It is well known that $\mathcal{H}^{n-1}(\sigma_K) = 0$ (see p. 84 of Schneider [48]). On precisely the set of regular radial vectors of ∂K is defined the function

$$\nu_K : \partial K \setminus \sigma_K \to S^{n-1}, \tag{3.1}$$

by letting $\nu_K(x)$ be the unique element in $\boldsymbol{\nu}_K(x)$, for each $x \in \partial K \setminus \sigma_K$. The functions ν_K is called the *spherical image map* of K and is known to be continuous (see Lemma 2.2.12 of Schneider [48]). It will occasionally be convenient to abbreviate $\partial K \setminus \sigma_K$ by $\partial' K$. Since $\mathcal{H}^{n-1}(\sigma_K) = 0$, when the integration is with respect to $\mathcal{H}^{n-1}$, it will be immaterial if the domain is over subsets of $\partial' K$ or ∂K.

The set $\eta_K \subset S^{n-1}$ consisting of all $v \in S^{n-1}$, for which the set $\boldsymbol{x}_K(v)$ contains more than a single element, is of $\mathcal{H}^{n-1}$-measure 0 (see Theorem 2.2.11 of Schneider [48]). On precisely the set of regular unit normal vectors for K is defined the function

$$x_K : S^{n-1} \setminus \eta_K \to \partial K, \tag{3.2}$$

by letting $x_K(v)$ be the unique element in $\boldsymbol{x}_K(v)$, for each $v \in S^{n-1} \setminus \eta_K$. The function x_K is called the *reverse spherical image map* and is well known to be continuous (see Lemma 2.2.12 of Schneider [48]). Note that by extending x_K to $\mathbb{R}^n \setminus \{0\}$, by making it a function homogeneous of degree 0, we obtain a natural definition of x_K on the set of regular normal vectors of K.

Lemma 3.1. *Suppose $K \in \mathcal{K}_o^n$. The vector $u \in \mathbb{R}^n \setminus \{0\}$ is a regular radial vector for K if and only if u is a regular normal vector for K^*.*

Proof. To see this note that $u \neq 0$ is a regular radial vector for K if and only if for $v_1, v_2 \in S^{n-1}$,

$$\rho_K(u)u \in H_K(v_1) \cap H_K(v_2) \quad \Longrightarrow \quad v_1 = v_2,$$

equivalently if and only if,

$$\rho_K(u)(u \cdot v_1) = h_K(v_1) \text{ and } \rho_K(u)(u \cdot v_2) = h_K(v_2) \quad \Longrightarrow \quad v_1 = v_2,$$

equivalently, using (2.4), if and only if,

$$\rho_{K^*}(v_1)(v_1 \cdot u) = h_{K^*}(u) \text{ and } \rho_{K^*}(v_2)(v_2 \cdot u) = h_{K^*}(u) \quad \Longrightarrow \quad v_1 = v_2,$$

equivalently, if and only if,

$$\rho_{K^*}(v_1)v_1 \in H_{K^*}(u) \text{ and } \rho_{K^*}(v_2)v_2 \in H_{K^*}(u) \quad \Longrightarrow \quad v_1 = v_2,$$

equivalently, if and only if u is a regular normal vector for K^*. □

It is well known (see Corollary 1.73 of Schneider [48]) that for $K \in \mathcal{K}_o^n$ the support function $h_K : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$ is differentiable precisely on the set of regular normal vectors for K, and

$$x_K = \nabla h_K, \tag{3.3}$$

on the set of regular normal vectors for K.

We will need to make use of the well-know fact that if $K \in \mathcal{K}_o^n$,

$$u \text{ is an outer normal at } x \in \partial K \Longrightarrow x \text{ is an outer normal at } u\rho_{K^*}(u) \in \partial K^*. \tag{3.4}$$

To establish this we need show is that $h_K(u) = x \cdot u$ implies $h_{K^*}(x) = \rho_{K^*}(u)(u \cdot x)$, or by (2.4), that $h_K(u) = \rho_K(x)(x \cdot u)$. But this follows immediately from the observation that $x \in \partial K$ means $\rho_K(x) = 1$.

Lemma 3.2. *If $K \in \mathcal{K}_o^n$, then for $x \in \partial K \setminus \sigma_K$,*

$$\nu_K(x) = -\frac{\nabla \rho_K(x)}{|\nabla \rho_K(x)|} = -\frac{\nabla \rho_K(\bar{x})}{|\nabla \rho_K(\bar{x})|}, \tag{3.5}$$

and

$$\nu_K(x) = \frac{\nabla h_{K^*}(\bar{x})}{|\nabla h_{K^*}(\bar{x})|}. \tag{3.6}$$

Proof. First, observe that a point $y \in \partial K$ will have a unique outer unit normal precisely if y is a regular radial vector for K, by definition of a regular radial vector for K. By Lemma 3.1 this is the case precisely when y is a regular normal vector of K^*, and from (3.3), we know that this will be the case precisely when $h_{K^*} = 1/\rho_K$ is differentiable at y and that in this case,

$$x_{K^*}(y) = \nabla h_{K^*}(y) = \nabla(1/\rho_K)(y) = -\rho_K(y)^{-2} \nabla \rho_K(y). \tag{3.7}$$

Since $x_{K^*}(y)$ is a point of ∂K^* with normal y, it follows from (3.4) and (2.5) that $x_{K^*}(y)$ is a normal at $y\rho_K(y) = y \in \partial K$, and from (3.7) we see that

$$\nu_K(y) = -\frac{\nabla \rho_K(y)}{|\nabla \rho_K(y)|} = \frac{\nabla h_{K^*}(y)}{|\nabla h_{K^*}(y)|} = \frac{\nabla h_{K^*}(\bar{y})}{|\nabla h_{K^*}(\bar{y})|},$$

for each regular vector $y \in \partial K$. Note that to obtain the results involving $\bar{x}$ or $\bar{y}$ (as opposed to x or y) we are making use of the positive homogeneity of the support and radial functions. □

For $K \in \mathcal{K}_o^n$, define the *radial map* of K,

$$r_K : S^{n-1} \to \partial K \qquad \text{by} \qquad r_K(u) = \rho_K(u)u \in \partial K,$$

for $u \in S^{n-1}$. Note that $r_K^{-1} : \partial K \to S^{n-1}$ is just the restriction to ∂K of the map $\bar{\cdot} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$.

For $\omega \subset S^{n-1}$, define the *radial Gauss image of* ω by

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)) \subset S^{n-1},$$

or equivalently,

$$\boldsymbol{\alpha}_K(\omega) = \{v \in S^{n-1} : r_K(u) \in H_K(v) \text{ for some } u \in \omega\}, \tag{3.8}$$

and thus, for $u \in S^{n-1}$,

$$\boldsymbol{\alpha}_K(u) = \{v \in S^{n-1} : r_K(u) \in H_K(v)\}. \tag{3.9}$$

Define the *radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$

$$\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1} \qquad \text{by} \qquad \alpha_K = \nu_K \circ r_K,$$

where $\omega_K = \overline{\sigma_K} = r_K^{-1}(\sigma_K)$. Since $r_K^{-1} = \bar{\cdot}$ is a bi-Lipschitz map between the spaces ∂K and S^{n-1} it follows that ω_K has spherical Lebesgue measure 0. Observe that if $u \in S^{n-1} \setminus \omega_K$, then $\boldsymbol{\alpha}_K(u)$ contains only the element $\alpha_K(u)$. Since both ν_K and r_K are continuous, α_K is continuous. Note that for $x \in \partial' K$,

$$\alpha_K(\bar{x}) = \nu_K(x), \tag{3.10}$$

and hence, for $x \in \partial' K$,

$$h_K(\alpha_K(\bar{x})) = h_K(\nu_K(x)) = x \cdot \nu_K(x). \tag{3.11}$$

If $u \in S^{n-1} \setminus \omega_K$, then from the definition of ω_K, we see that $x = u\rho_K(u) \in \partial K \setminus \sigma_K$, with $\bar{x} = u$. Hence from (3.10) we have $\alpha_K(u) = \alpha_K(\bar{x}) = \nu_K(x)$ and by appealing to (3.5) and (3.6), we get:

$$\alpha_K(u) = -\frac{\nabla \rho_K(u)}{|\nabla \rho_K(u)|} = \frac{\nabla h_{K^*}(u)}{|\nabla h_{K^*}(u)|}, \qquad u \in S^{n-1} \setminus \omega_K. \tag{3.12}$$

For $\eta \subset S^{n-1}$, define the *reverse radial Gauss image* of η by

$$\boldsymbol{\alpha}_K^*(\eta) = r_K^{-1}(\boldsymbol{x}_K(\eta)) = \langle \boldsymbol{x}_K(\eta) \rangle. \tag{3.13}$$

Thus,

$$\boldsymbol{\alpha}_K^*(\eta) = \{\bar{x} : \ x \in \partial K \text{ where } x \in H_K(v) \text{ for some } v \in \eta\}.$$

In particular, if η contains only the single vector $v \in S^{n-1}$, we see that

$$\boldsymbol{\alpha}_K^*(v) = \{\bar{x} : \ x \in \partial K \text{ where } x \in H_K(v)\} \tag{3.14}$$

Define the *reverse radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$,

$$\alpha_K^* : S^{n-1} \setminus \eta_K \to S^{n-1}, \qquad \text{by} \qquad \alpha_K^* = r_K^{-1} \circ x_K. \tag{3.15}$$

Note that since both r_K^{-1} and x_K are continuous, α_K^* is continuous.

Note for a subset $\eta \subset S^{n-1}$,

$$\boldsymbol{\alpha}_K^*(\eta) = \{u \in S^{n-1} : r_K(u) \in H_K(v) \text{ for some } v \in \eta\}, \tag{3.16}$$

and hence for $u \in S^{n-1}$ and $\eta \subset S^{n-1}$, we see from (3.9) that

$$u \in \boldsymbol{\alpha}_K^*(\eta) \quad \Longleftrightarrow \quad \boldsymbol{\alpha}_K(u) \cap \eta \neq \varnothing. \tag{3.17}$$

Thus, for $\eta_1, \eta_2 \subseteq S^{n-1}$,

$$\eta_1 \subseteq \eta_2 \quad \Longrightarrow \quad \boldsymbol{\alpha}_K^*(\eta_1) \subseteq \boldsymbol{\alpha}_K^*(\eta_1).$$

If η is the singleton containing only $v \in S^{n-1}$, then (3.17) reduces to

$$u \in \boldsymbol{\alpha}^*_K(v) \iff v \in \boldsymbol{\alpha}_K(u). \tag{3.18}$$

If $u \notin \omega_K$, then $\boldsymbol{\alpha}_K(u) = \{\alpha_K(u)\}$, and (3.17) becomes

$$u \in \boldsymbol{\alpha}^*_K(\eta) \iff \alpha_K(u) \in \eta, \tag{3.19}$$

and hence (3.19) holds for almost all $u \in S^{n-1}$, with respect to spherical Lebesgue measure.

We shall need to make use of the fact that for $u, v \in S^{n-1}$,

$$u \in \boldsymbol{\alpha}_{K^*}(v) \iff v \in \boldsymbol{\alpha}_K(u). \tag{3.20}$$

To see this note that from (3.9), it follows that,

$$u \in \boldsymbol{\alpha}_{K^*}(v) \quad \iff \quad H_{K^*}(u) \quad \text{is a support hyperplane at } \rho_{K^*}(v)v,$$

that is,

$$u \in \boldsymbol{\alpha}_{K^*}(v) \quad \iff \quad h_{K^*}(u) = (u \cdot v)\, \rho_{K^*}(v),$$

and, by (2.4), this is the case if and only if

$$h_K(v) = (v \cdot u)\, \rho_K(u) = v \cdot r_K(u),$$

or equivalently, using (3.9) again, if and only if,

$$v \in \boldsymbol{\alpha}_K(u).$$

For $u \in S^{n-1}$, define $(u, \boldsymbol{\alpha}_K(u))$ by

$$(u, \boldsymbol{\alpha}_K(u)) = \{(u, v) \in S^{n-1} \times S^{n-1} : v \in \boldsymbol{\alpha}_K(u)\}.$$

The following will be a basic fact needed.

Lemma 3.3. *Suppose $K \in \mathcal{K}^n_o$ and that $\theta \subset S^{n-1} \times S^{n-1}$ is a Borel set. Then the subset of S^{n-1},*

$$\omega = \{u \in S^{n-1} : (u, \boldsymbol{\alpha}_K(u)) \cap \theta \neq \varnothing\},$$

is spherical Lebesgue measurable.

Proof. Let $\iota : S^{n-1} \to S^{n-1}$ be the identity map, $\iota(u) = u$, for $u \in S^{n-1}$. Since $\alpha_K : S^{n-1}\backslash\omega_K \to S^{n-1}$ is continuous, the map $(\iota, \alpha_K) : S^{n-1}\backslash\omega_K \to S^{n-1} \times S^{n-1}$ is continuous, where $S^{n-1}\backslash\omega_K$ is viewed as the topological space with the relative topology inherited from S^{n-1}. Thus, since $\theta \subset S^{n-1} \times S^{n-1}$ is a Borel set, the set

$$\omega_0 = \{u \in S^{n-1}\backslash\omega_K : (u, \alpha_K(u)) \in \theta\} = \omega \cap (S^{n-1}\backslash\omega_K)$$

is a Borel set in $S^{n-1}\backslash\omega_K$. Note that $\omega\backslash\omega_0 \subseteq \omega_K$ and hence $\omega\backslash\omega_0$ has Lebesgue measure 0.

Since the Borel structure of $S^{n-1} \setminus \omega_K$ coincides with the relative Borel structure inherited from S^{n-1}, there is a Borel set ω_1 in S^{n-1} so that

$$\omega_0 = \omega_1 \cap (S^{n-1} \setminus \omega_K) = \omega_1 \setminus (\omega_1 \cap \omega_K).$$

Since ω_1 is Borel measurable and $\omega_1 \cap \omega_K$ has Lebesgue measure 0 on S^{n-1}, it follows that ω_0 is Lebesgue measurable on S^{n-1}. Since the set $\omega \setminus \omega_0$ has Lebesgue measure 0 on S^{n-1}, we conclude that the set ω is Lebesgue measurable on S^{n-1}. □

Suppose $\eta \subseteq S^{n-1}$. Note that (3.17) tells us that $u \in \boldsymbol{\alpha}^*_K(\eta)$ if and only if $\boldsymbol{\alpha}_K(u) \cap \eta \neq \varnothing$, which can happen if and only if $(u, \boldsymbol{\alpha}_K(u)) \cap (S^{n-1} \times \eta) \neq \varnothing$. Thus,

$$\boldsymbol{\alpha}^*_K(\eta) = \{u \in S^{n-1} : (u, \boldsymbol{\alpha}_K(u)) \cap (S^{n-1} \times \eta) \neq \varnothing\}. \tag{3.21}$$

The following corollary of our Lemma 3.3 is Lemma 2.2.14 of Schneider [48].

Corollary 3.4. *If $K \in \mathcal{K}_o^n$, and $\eta \subseteq S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}_K^*(\eta) = \langle \boldsymbol{x}_K(\eta) \rangle \subset S^{n-1}$ is spherical Lebesgue measurable.*

If $g : S^{n-1} \to \mathbb{R}$ is a Borel function, then $g \circ \alpha_K$ is spherical Lebesgue measurable because it is just the composition of a Borel function g and a continuous function α_K in $S^{n-1} \setminus \omega_K$ with ω_K having Lebesgue measure 0. Moreover, if g is a bounded Borel function, then $g \circ \alpha_K$ is spherical Lebesgue integrable. In particular, $g \circ \alpha_K$ is spherical Lebesgue integrable, for each continuous function $g : S^{n-1} \to \mathbb{R}$.

From [25] we need:

Lemma 3.5. *Suppose $K_i \in \mathcal{K}_o^n$ with $\lim_{i\to\infty} K_i = K_0 \in \mathcal{K}_o^n$. Let $\omega = \cup_{i=0}^{\infty} \omega_{K_i}$ be the set (of $\mathcal{H}^{n-1}$-measure 0) off of which all of the α_{K_i} are defined. Then if $u_i \in S^{n-1} \setminus \omega$ are such that $\lim_{i\to\infty} u_i = u_0 \in S^{n-1} \setminus \omega$, then $\lim_{i\to\infty} \alpha_{K_i}(u_i) = \alpha_{K_0}(u_0)$.*

Recall that
$$(u, \boldsymbol{\alpha}_K(u)) = \{(u, v) \in S^{n-1} \times S^{n-1} : v \in \boldsymbol{\alpha}_K(u)\},$$
and that $\iota : S^{n-1} \to S^{n-1}$ is the identity map, $\iota(u) = u$, for $u \in S^{n-1}$. For $\omega \subset S^{n-1}$, define
$$(\iota, \boldsymbol{\alpha}_K)(\omega) = \cup_{u\in\omega}(u, \boldsymbol{\alpha}_K(u)) \subseteq S^{n-1} \times S^{n-1}.$$
For $\theta \subseteq S^{n-1} \times S^{n-1}$, define
$$(\iota, \boldsymbol{\alpha}_K)^*(\theta) = \{u \in S^{n-1} : (u, \boldsymbol{\alpha}_K(u)) \cap \theta \neq \varnothing\}. \tag{3.22}$$
As a trivial observation, note that for $\omega \subseteq S^{n-1}$,
$$(\iota, \boldsymbol{\alpha}_K)^*(\omega \times S^{n-1}) = \omega. \tag{3.23}$$

Obviously, for $\theta_1, \theta_2 \subseteq S^{n-1} \times S^{n-1}$,
$$\theta_1 \subseteq \theta_2 \quad \Longrightarrow \quad (\iota, \boldsymbol{\alpha}_K)^*(\theta_1) \subseteq (\iota, \boldsymbol{\alpha}_K)^*(\theta_2). \tag{3.24}$$

Lemma 3.6. *Suppose $K \in \mathcal{K}_o^n$. If $\{\theta_j\}$ is a sequence of subsets of $S^{n-1} \times S^{n-1}$, then*
$$(\iota, \boldsymbol{\alpha}_K)^* \left(\cup_j \theta_j\right) = \cup_j (\iota, \boldsymbol{\alpha}_K)^*(\theta_j).$$

Proof. If $v \in \cup_j \theta_j$, then $v \in \theta_{j_1}$ for some j_1, and $(\iota, \boldsymbol{\alpha}_K)^*(v) \subseteq (\iota, \boldsymbol{\alpha}_K)^*(\theta_{j_1}) \subseteq \cup_j (\iota, \boldsymbol{\alpha}_K)^*(\theta_j)$. Thus, $(\iota, \boldsymbol{\alpha}_K)^* \left(\cup_j \theta_j\right) \subseteq \cup_j (\iota, \boldsymbol{\alpha}_K)^*(\theta_j)$. If $u \in \cup_j (\iota, \boldsymbol{\alpha}_K)^*(\theta_j)$, then for some j_2, we have $u \in (\iota, \boldsymbol{\alpha}_K)^*(\theta_{j_2}) \subseteq (\iota, \boldsymbol{\alpha}_K)^* \left(\cup_j \theta_j\right)$, by (3.24). Thus, $(\iota, \boldsymbol{\alpha}_K)^* \left(\cup_j \theta_j\right) \supseteq \cup_j (\iota, \boldsymbol{\alpha}_K)^*(\theta_j)$. □

Lemma 3.7. *Suppose $K \in \mathcal{K}_o^n$. If $\{\theta_j\}$ is a sequence of pairwise disjoint sets in $S^{n-1} \times S^{n-1}$, then the sequence $\{(\iota, \boldsymbol{\alpha}_K)^*(\theta_j) \setminus \omega_K\}$ is pairwise disjoint as well.*

Proof. Suppose there exists a $u \in S^{n-1}$ such that for $j_1 \neq j_2$,
$$u \in (\iota, \boldsymbol{\alpha}_K)^*(\theta_{j_1}) \setminus \omega_K \quad \text{and} \quad u \in (\iota, \boldsymbol{\alpha}_K)^*(\theta_{j_2}) \setminus \omega_K.$$
Definition (3.22) tells us this means that
$$(u, \boldsymbol{\alpha}_K(u)) \cap \theta_{j_1} \neq \varnothing \quad \text{and} \quad (u, \boldsymbol{\alpha}_K(u)) \cap \theta_{j_2} \neq \varnothing.$$
But since $u \notin \omega_K$ we know that $\boldsymbol{\alpha}_K(u)$ is a singleton, which contradicts the assumption that $\theta_{j_1} \cap \theta_{j_2} \neq \varnothing$. □

Lemma 3.8. *For $K \in \mathcal{K}_o^n$, and Borel sets $\omega, \eta \subseteq S^{n-1}$,*
$$(\iota, \boldsymbol{\alpha}_K)^*(\omega \times \eta) = \omega \cap \boldsymbol{\alpha}_{K^*}(\eta).$$

Proof. Fix $u \in S^{n-1}$. From definition (3.22) and using (3.20), we have

$$\begin{aligned} u \in (\iota, \boldsymbol{\alpha}_K)^*(\omega \times \eta) &\iff (u, \boldsymbol{\alpha}_K(u)) \cap (\omega \times \eta) \neq \varnothing \\ &\iff u \in \omega, \exists v \in \eta : v \in \boldsymbol{\alpha}_K(u) \\ &\iff \exists v \in \eta : (u, v) \in \omega \times \eta \text{ and } v \in \boldsymbol{\alpha}_K(u) \\ &\iff \exists v \in \eta : (u, v) \in \omega \times \eta \text{ and } u \in \boldsymbol{\alpha}_{K^*}(v) \\ &\iff u \in \omega \cap \boldsymbol{\alpha}_{K^*}(\eta). \end{aligned}$$

□

The reverse radial Gauss image of a convex body and the radial Gauss image of its polar body are related. Combine (3.21) and definition (3.22) to get

$$(\iota, \boldsymbol{\alpha}_K)^*(S^{n-1} \times \eta) = \boldsymbol{\alpha}_K^*(\eta), \tag{3.25}$$

which when combined with Lemma 3.8, immediately yields:

Corollary 3.9. *If $K \in \mathcal{K}_o^n$, then*

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta),$$

for each $\eta \subseteq S^{n-1}$.

Since $\boldsymbol{\alpha}_K^*(v) = \{\alpha_K^*(v)\}$, for almost all $v \in S^{n-1}$, with respect to spherical Lebesgue measure, and $\boldsymbol{\alpha}_{K^*}(v) = \{\alpha_{K^*}(v)\}$, for almost all $v \in S^{n-1}$, with respect to spherical Lebesgue measure, Corollary 3.9 gives:

Lemma 3.10. *If $K \in \mathcal{K}_o^n$, then*

$$\alpha_K^* = \alpha_{K^*},$$

almost everywhere with respect to spherical Lebesgue measure.

When Lemma 3.10 is combined with (3.12), we get

$$\alpha_K^*(v) = \frac{\nabla h_K(v)}{|\nabla h_K(v)|}, \tag{3.26}$$

for almost all v, with respect to spherical Lebesgue measure.

By using the spherical image and the reverse spherical image, one can define the integral curvature, surface area measures, and their L_p extensions.

The surface area measure $S(K, \cdot)$ can be defined, for Borel $\eta \subseteq S^{n-1}$, by

$$S(K, \eta) = \mathcal{H}^{n-1}(\boldsymbol{x}_K(\eta)). \tag{3.27}$$

If ∂K is smooth with positive Gauss curvature, the surface area measure of K is absolutely continuous with respect to Lebesgue measure, S, on S^{n-1}, and the density is the reciprocal Gauss curvature, when viewed as a function of the outer unit normals of ∂K. The density has an explicit description in terms of the support function and its Hessian matrix on S^{n-1},

$$\frac{dS(K, \cdot)}{dS} = \det(\bar{\nabla}^2 h_K + h_K I), \tag{3.28}$$

where $\bar{\nabla}^2 h_K$ denotes the Hessian matrix of h_K and I is the identity matrix with respect to an orthonormal frame on S^{n-1}. For all this see Schneider [48].

For convex bodies K, L in $\mathbb{R}^n$, the classical mixed volume, $V_1(K, L)$, has the integral representation:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(v)\, dS(K, v), \tag{3.29}$$

which holds for each convex body L. The celebrated Minkowski mixed-volume inequality states that

$$V_1(K,L)^n \geq V(K)^{n-1}V(L), \tag{3.30}$$

with equality if and only if K, L are homothets.

For $p \in \mathbb{R}$, the L_p surface area measure $S_p(K,\cdot)$ of $K \in \mathcal{K}_o^n$, introduced in [38], may be defined by,

$$dS_p(K,\cdot) = h_K^{1-p}\, dS(K,\cdot), \tag{3.31}$$

or equivalently, by

$$S_p(K,\eta) = \int_{\boldsymbol{x}_K(\eta)} (x\cdot\nu(x))^{1-p}\, d\mathcal{H}^{n-1}(x), \tag{3.32}$$

for each Borel $\eta \subseteq S^{n-1}$.

For $p \in \mathbb{R}$, and $K, L \in \mathcal{K}_o^n$, the L_p mixed volume $V_p(K,L)$ is defined by

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L^p(v)\, dS_p(K,v). \tag{3.33}$$

For $p \geq 1$, the L_p-Minkowski inequality for the L_p-mixed volume is

$$V_p(K,L)^n \geq V(K)^{n-p}V(L)^p, \tag{3.34}$$

with equality if and only if either K, L are dilates when $p > 1$, or K, L are homothets when $p = 1$.

The p-th dual curvature measure $\widetilde{C}_p(K,\cdot)$, introduced in [25], is a Borel measure on S^{n-1} defined for each $K \in \mathcal{K}_o^n$ and each $p \in \mathbb{R}$. As shown in [25], one convenient way of defining $\widetilde{C}_p(K,\cdot)$ is via the integral representation

$$\int_{S^{n-1}} g(v)\, d\widetilde{C}_p(K,v) = \frac{1}{n}\int_{\partial' K} x\cdot\nu_K(x)\, g(\nu_K(x))|x|^{p-n}\, d\mathcal{H}^{n-1}(x), \tag{3.35}$$

which holds for each bounded Borel function $g : S^{n-1} \to \mathbb{R}$.

The integral curvature measure $J(K,\cdot)$ on S^{n-1} can be defined, for Borel $\omega \subseteq S^{n-1}$, by

$$J(K,\omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)); \tag{3.36}$$

that is, $J(K,\omega)$ is the spherical Lebesgue measure of $\boldsymbol{\alpha}_K(\omega)$. The *integral curvature measure of* K was introduced by Aleskandrov.

For each $p \in \mathbb{R}$, the L_p integral curvature measure $J_p(K,\cdot)$ was introduced in [26] and is the Borel measure on S^{n-1} that may be defined by

$$dJ_p(K,\cdot) = \rho_K^p dJ(K,\cdot), \tag{3.37}$$

or, equivalently (as shown in [26]) by

$$J_p(K,\omega) = \int_\omega \rho_K^p(u)\, dJ(K,u) = \int_{\boldsymbol{\alpha}_K(\omega)} \rho_K^p(\alpha_K^*(v))\, dv, \tag{3.38}$$

for each Borel $\omega \subseteq S^{n-1}$.

The following integral identity was established in [25]: If $q \in \mathbb{R}$, and $K \in \mathcal{K}_o^n$, while $f : S^{n-1} \to \mathbb{R}$ is bounded and Lebesgue integrable, then

$$\int_{S^{n-1}} f(u)\rho_K(u)^q\, du = \int_{\partial K} f(\bar{x})|x|^{q-n}(x\cdot\nu_K(x))\, d\mathcal{H}^{n-1}(x). \tag{3.39}$$

Also established in [25] was the following integral identity: If $K \in \mathcal{K}_o^n$ is strictly convex, and $f : S^{n-1} \to \mathbb{R}$ and $F : \partial K \to \mathbb{R}$ are both continuous, then

$$\int_{S^{n-1}} f(v)F(\nabla h_K(v))h_K(v)\,dS(K,v) = \int_{\partial K} (x \cdot \nu_K(x))\, f(\nu_K(x))F(x)\,d\mathcal{H}^{n-1}(x), \tag{3.40}$$

where ∇h_K is the gradient of h_K in $\mathbb{R}^n$. Recall that while ν_K is defined only on $\partial K \setminus \sigma_K$, the set σ_K has $\mathcal{H}^{n-1}$-measure 0. We shall require a slight extension of (3.40). Specifically, if $p \in \mathbb{R}$, while $K \in \mathcal{K}_o^n$ is strictly convex, and $f : S^{n-1} \to \mathbb{R}$ and $F : \partial K \to \mathbb{R}$ are both continuous, then

$$\int_{S^{n-1}} f(v)F(\nabla h_K(v))\,dS_p(K,v) = \int_{\partial K} (x \cdot \nu_K(x))^{1-p}\, f(\nu_K(x))F(x)\,d\mathcal{H}^{n-1}(x). \tag{3.41}$$

To derive (3.41) from (3.40) merely replace f by $h_K^{-p} f$ in (3.40), and observe that for $\mathcal{H}^{n-1}$-almost all $x \in \partial K$, specifically, for all $x \in \partial K \setminus \sigma_K$, from (3.11) we have $h_K(\nu_K(x)) = x \cdot \nu_K(x)$, and then use (3.31).

4. L_p DUAL CURVATURE MEASURES

For a star body $Q \in \mathcal{S}_o^n$, define $\|\cdot\|_Q : \mathbb{R}^n \to [0,\infty)$ by letting

$$\|x\|_Q = \begin{cases} 1/\rho_Q(x) & x \neq 0 \\ 0 & x = 0. \end{cases} \tag{4.1}$$

Note that $\|\cdot\|_Q$ is a continuous, and a positively homogeneous function of degree 1. When Q is an origin-symmetric convex body in $\mathbb{R}^n$, then $\|\cdot\|_Q$ is an ordinary norm in $\mathbb{R}^n$, and $(\mathbb{R}^n, \|\cdot\|_Q)$ is the n-dimensional Banach space whose unit ball is Q.

We first define the dual notions of the metric projection map p_K and the distance function $d(K,\cdot)$. Suppose $K \in \mathcal{K}_o^n$. Define the *radial projection map* $\widetilde{p}_K : \mathbb{R}^n \setminus K \to \partial K$ by

$$\widetilde{p}_K(x) = \rho_K(x)x = r_K(\bar{x}), \tag{4.2}$$

for $x \in \mathbb{R}^n \setminus K$.

For $x \in \mathbb{R}^n$, the *radial distance* $\widetilde{d}_Q(K,x)$ of x to K, is defined by

$$\widetilde{d}_Q(K,x) = \begin{cases} \|x - \widetilde{p}_K(x)\|_Q & x \notin K \\ 0 & x \in K. \end{cases} \tag{4.3}$$

Let

$$\widetilde{u}_K(x) = \bar{x}. \tag{4.4}$$

Suppose $Q \in \mathcal{S}_o^n$ and $K \in \mathcal{K}_o^n$. For $t \geq 0$, a Borel $\theta \subseteq S^{n-1} \times S^{n-1}$, a Lebesgue measurable $\omega \subseteq S^{n-1}$, and a Borel $\eta \subseteq S^{n-1}$, define the *local dual parallel bodies*

$$\widetilde{A}_t(K,Q,\eta) = \{x \in \mathbb{R}^n : \ 0 \leq \widetilde{d}_Q(K,x) \leq t \text{ with } \widetilde{p}_K(x) \in \boldsymbol{x}_K(\eta)\}, \tag{4.5}$$

$$\widetilde{B}_t(K,Q,\omega) = \{x \in \mathbb{R}^n : 0 \leq \widetilde{d}_Q(K,x) \leq t \text{ with } \widetilde{u}_K(x) \in \omega\}, \tag{4.6}$$

$$\widetilde{E}_t(K,Q,\theta) = \{x \in \mathbb{R}^n : \ 0 \leq \widetilde{d}_Q(K,x) \leq t \text{ with } (\widetilde{u}_K(x), \boldsymbol{\alpha}_K(\widetilde{u}_K(x))) \cap \theta \neq \varnothing\}. \tag{4.7}$$

These local dual parallel bodies have Steiner type formulas as shown in the following theorem.

Theorem 4.1. *Suppose $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$. For $t \ge 0$, a Borel $\theta \subseteq S^{n-1} \times S^{n-1}$, a Lebesgue measurable $\omega \subseteq S^{n-1}$, and a Borel $\eta \subseteq S^{n-1}$,*

$$V(\widetilde{A}_t(K,Q,\eta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \widetilde{C}_i(K,Q,\eta), \tag{4.8}$$

$$V(\widetilde{B}_t(K,Q,\omega)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \widetilde{S}_i(K,Q,\omega), \tag{4.9}$$

$$V(\widetilde{E}_t(K,Q,\theta)) = \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \widetilde{\Theta}_i(K,Q,\theta), \tag{4.10}$$

where $\widetilde{C}_i(K,Q,\cdot)$ and $\widetilde{S}_i(K,Q,\cdot)$ are Borel measures on S^{n-1} given by

$$\widetilde{C}_i(K,Q,\eta) = \frac{1}{n} \int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^i(u) \rho_Q^{n-i}(u)\, du, \tag{4.11}$$

$$\widetilde{S}_i(K,Q,\omega) = \frac{1}{n} \int_{\omega} \rho_K^i(u) \rho_Q^{n-i}(u)\, du, \tag{4.12}$$

and $\widetilde{\Theta}_i(K,Q,\cdot)$ is a Borel measure on $S^{n-1} \times S^{n-1}$ given by

$$\widetilde{\Theta}_i(K,Q,\theta) = \frac{1}{n} \int_{(\iota,\boldsymbol{\alpha}_K)^*(\theta)} \rho_K^i(u) \rho_Q^{n-i}(u)\, du, \tag{4.13}$$

Proof. Using (4.1), (4.2), (4.3), and (4.4), we can re-write (4.6) as

$$\widetilde{B}_t(K,Q,\omega) = \{x \in \mathbb{R}^n : 0 \le |x| \le \rho_K(\bar{x}) + t\rho_Q(\bar{x}) \text{ with } \bar{x} \in \omega\ \}. \tag{4.14}$$

Write $x = \rho u$, with $\rho \ge 0$ and $u \in S^{n-1}$, and for the volume of $\widetilde{B}_t(K,Q,\omega)$ we get:

$$\begin{aligned} V(\widetilde{B}_t(K,Q,\omega)) &= \int_{u\in\omega} \left(\int_0^{\rho_K(u)+t\rho_Q(u)} \rho^{n-1}\, d\rho \right) du \\ &= \frac{1}{n} \int_{u\in\omega} (\rho_K(u) + t\rho_Q(u))^n\, du \\ &= \frac{1}{n} \sum_{i=0}^{n} \binom{n}{i} t^{n-i} \int_{u\in\omega} \rho_K^i(u)\rho_Q^{n-i}(u)\, du, \end{aligned}$$

where the integration is with respect to Lebesgue measure in S^{n-1}. This gives (4.9) and (4.12).

In (4.5), the condition that $\widetilde{p}_K(x) \in \boldsymbol{x}_K(\eta)$, or equivalently by (4.2) that $r_K(\bar{x}) \in \boldsymbol{x}_K(\eta)$, is by (3.13), the same as $\bar{x} \in r_K^{-1}(\boldsymbol{x}_K(\eta)) = \langle \boldsymbol{x}_K(\eta) \rangle = \boldsymbol{\alpha}_K^*(\eta)$. Thus, (4.5) can be written as

$$\widetilde{A}_t(K,Q,\eta) = \{x \in \mathbb{R}^n : 0 \le |x| \le \rho_K(\bar{x}) + t\rho_Q(\bar{x}) \text{ with } \bar{x} \in \boldsymbol{\alpha}_K^*(\eta)\ \}. \tag{4.15}$$

Since $\eta \subset S^{n-1}$ is a Borel set, we know from Corollary 3.4 that $\boldsymbol{\alpha}_K^*(\eta)$ is a Lebesgue measurable subset of S^{n-1}. A quick glance at (4.14) and (4.15) allows us to see that

$$\widetilde{A}_t(K,Q,\eta) = \widetilde{B}_t(K,Q,\boldsymbol{\alpha}_K^*(\eta)).$$

Now (4.9) yields,

$$V(\widetilde{A}_t(K,Q,\eta)) = V(\widetilde{B}_t(K,Q,\boldsymbol{\alpha}_K^*(\eta))) = \sum_{i=0}^{n}\binom{n}{i} t^{n-i}\widetilde{S}_i(K,Q,\boldsymbol{\alpha}_K^*(\eta)),$$

and by defining

$$\widetilde{C}_i(K,Q,\eta) = \widetilde{S}_i(K,Q,\boldsymbol{\alpha}_K^*(\eta)). \tag{4.16}$$

we get both (4.8) and (4.11).

From its integral representation (4.12), we see that $\widetilde{S}_i(K,Q,\cdot)$ is a Borel measure that is obviously absolutely continuous with respect to spherical Lebesgue measure. In particular, $\widetilde{S}_i(K,Q,\cdot)$ will assume the same value on sets that differ by sets of spherical Lebesgue measure 0.

Write (4.7) as

$$\widetilde{E}_t(K,Q,\theta) = \{x\in\mathbb{R}^n : 0 \le |x| \le \rho_K(\bar{x}) + t\rho_Q(\bar{x}) \text{ with } (\bar{x},\boldsymbol{\alpha}_K(\bar{x}))\cap\theta \ne \varnothing\ \}. \tag{4.17}$$

Write $x=\rho u$, with $\rho\ge 0$ and $u\in S^{n-1}$, and for the volume of $\widetilde{E}_t(K,Q,\theta)$, we get

$$\begin{aligned} V(\widetilde{E}_t(K,Q,\theta)) &= \int_{(u,\boldsymbol{\alpha}_K(u))\cap\theta\ne\varnothing}\left(\int_0^{\rho_K(u)+t\rho_Q(u)}\rho^{n-1}\,d\rho\right)du \\ &= \frac{1}{n}\int_{(u,\boldsymbol{\alpha}_K(u))\cap\theta\ne\varnothing}(\rho_K(u)+t\rho_Q(u))^n\,du \\ &= \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\int_{(u,\boldsymbol{\alpha}_K(u))\cap\theta\ne\varnothing}\rho_K^i(u)\rho_Q^{n-i}(u)\,du \\ &= \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\int_{(\iota,\boldsymbol{\alpha}_K)^*(\theta)}\rho_K^i(u)\rho_Q^{n-i}(u)\,du, \end{aligned}$$

where the last identity comes directly from definition (3.22). This gives (4.10) and (4.13), and the critical fact that

$$\widetilde{\Theta}_i(K,Q,\theta) = \widetilde{S}_i(K,Q,(\iota,\boldsymbol{\alpha}_K)^*(\theta)). \tag{4.18}$$

We now show that $\widetilde{\Theta}_i(K,Q,\cdot)$ is a Borel measure on $S^{n-1}\times S^{n-1}$. For the empty set $\varnothing$, we have from (4.18) and definition (3.22) that,

$$\widetilde{\Theta}_i(K,Q,\varnothing) = \widetilde{S}_i(K,Q,(\iota,\boldsymbol{\alpha}_K)^*(\varnothing)) = \widetilde{S}_i(K,Q,\varnothing) = 0.$$

Let $\{\theta_j\}$ be a sequence of pairwise disjoint Borel sets in $S^{n-1}\times S^{n-1}$. From Lemma 3.3 and definition (3.22), together with the fact that ω_K has spherical Lebesgue measure 0, we know from Lemma 3.7 that $\{(\iota,\boldsymbol{\alpha}_K)^*(\theta_j)\setminus\omega_K\}$ is a sequence of pairwise disjoint Lebesgue measurable sets. From (4.18), Lemma 3.6, the fact that ω_K has measure 0, and the fact from Lemma 3.7 that the $\{(\iota,\boldsymbol{\alpha}_K)^*(\theta_j)\setminus\omega_K\}$ are pairwise disjoint together with the fact that $\widetilde{S}_i(K,Q,\cdot)$ is a measure, the fact that ω_K has spherical Lebesgue measure 0, and (4.18) again, we have

$$\begin{aligned}\widetilde{\Theta}_i(K,Q,\textstyle\bigcup_j \theta_j) &= \widetilde{S}_i(K,Q,\ (\iota,\boldsymbol{\alpha}_K)^*(\textstyle\bigcup_j \theta_j))\\ &= \widetilde{S}_i(K,Q,\ \textstyle\bigcup_j (\iota,\boldsymbol{\alpha}_K)^*(\theta_j))\\ &= \widetilde{S}_i(K,Q,\ (\textstyle\bigcup_j (\iota,\boldsymbol{\alpha}_K)^*(\theta_j))\setminus\omega_K)\\ &= \widetilde{S}_i(K,Q,\ \textstyle\bigcup_j ((\iota,\boldsymbol{\alpha}_K)^*(\theta_j)\setminus\omega_K))\\ &= \textstyle\sum_j \widetilde{S}_i(K,Q,(\iota,\boldsymbol{\alpha}_K)^*(\theta_j)\setminus\omega_K)\\ &= \textstyle\sum_j \widetilde{S}_i(K,Q,(\iota,\boldsymbol{\alpha}_K)^*(\theta_j))\\ &= \textstyle\sum_j \widetilde{\Theta}_i(K,Q,\theta_j).\end{aligned}$$

This shows that $\widetilde{\Theta}_i(K,Q,\cdot)$ is a Borel measure on $S^{n-1}\times S^{n-1}$. □

We call the measure $\widetilde{\Theta}_i(K,Q,\cdot)$ the *generalized i-th dual curvature measure of K relative to Q* or the *i-th dual support measure of K relative to Q*. Call the measure $\widetilde{S}_i(K,Q,\cdot)$ the *i-th dual area measure of K relative to Q* and the measure $\widetilde{C}_i(K,Q,\cdot)$ the *i-th dual curvature measure of K relative to Q*. From (2.17), (4.11) together with definition (3.13), and (4.12), we see that the total measures of the i-th dual area measure and the i-th dual curvature measure are the i-th dual mixed volume $\widetilde{V}_i(K,Q)$; i.e.,

$$\widetilde{V}_i(K,Q) = \widetilde{S}_i(K,Q,S^{n-1}) = \widetilde{C}_i(K,Q,S^{n-1}). \tag{4.19}$$

The dual curvature measures and the dual area measures are the marginal measures of the dual support measures; i.e., for Borel $\eta,\omega\subseteq S^{n-1}$,

$$\widetilde{C}_i(K,Q,\eta) = \widetilde{\Theta}_i(K,Q,S^{n-1}\times\eta), \tag{4.20}$$

$$\widetilde{S}_i(K,Q,\omega) = \widetilde{\Theta}_i(K,Q,\omega\times S^{n-1}). \tag{4.21}$$

To see this, observe that (4.20) follows from (4.18), (3.25), and (4.16) while (4.21) follows from (4.18) and (3.23).

The integral representations (4.11) and (4.12) show that the dual curvature and dual area measures can be extended.

Definition 4.2. *Suppose $q\in\mathbb{R}$. For $K\in\mathcal{K}_o^n$ and $Q\in\mathcal{S}_o^n$, define the q-th dual area measure $\widetilde{S}_q(K,Q,\cdot)$ by letting*

$$\widetilde{S}_q(K,Q,\omega) = \frac{1}{n}\int_{\omega}\rho_K^q(u)\rho_Q^{n-q}(u)\,du, \tag{4.22}$$

for each Lebesgue measurable $\omega\subseteq S^{n-1}$, and the q-th dual curvature measure $\widetilde{C}_q(K,Q,\cdot)$ by letting

$$\widetilde{C}_q(K,Q,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)}\rho_K^q(u)\rho_Q^{n-q}(u)\,du, \tag{4.23}$$

for each Borel $\eta \subseteq S^{n-1}$. *Moreover, for each* $p \in \mathbb{R}$, *define the* (p,q)*-th dual curvature measure* $\widetilde{C}_{p,q}(K,Q,\cdot)$ *by*

$$d\widetilde{C}_{p,q}(K,Q,\cdot) = h_K^{-p} d\widetilde{C}_q(K,Q,\cdot). \tag{4.24}$$

A trivial, but important observation is that

$$\widetilde{C}_{0,q}(K,Q,\cdot) = \widetilde{C}_q(K,Q,\cdot). \tag{4.25}$$

Note that from definition (4.23) and the fact that (3.19) holds off of the set ω_K of spherical Lebesgue measure 0, we have for each Borel $\eta \subseteq S^{n-1}$,

$$\begin{aligned} \int_{S^{n-1}} \mathbb{1}_\eta(u)\, d\widetilde{C}_q(K,Q,u) &= \widetilde{C}_q(K,Q,\eta) \\ &= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \rho_K^q(u)\rho_Q^{n-q}(u)\, du \\ &= \frac{1}{n}\int_{S^{n-1}} \mathbb{1}_{\boldsymbol{\alpha}_K^*(\eta)}(u)\rho_K^q(u)\rho_Q^{n-q}(u)\, du \\ &= \frac{1}{n}\int_{S^{n-1}} \mathbb{1}_\eta(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)\, du. \end{aligned} \tag{4.26}$$

Observe that $\widetilde{C}_q(K,Q,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure. From (4.26) we will deduce that for each bounded Borel $f : S^{n-1} \to \mathbb{R}$, we have

$$\int_{S^{n-1}} f(u)\, d\widetilde{C}_q(K,Q,u) = \frac{1}{n}\int_{S^{n-1}} f(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)\, du. \tag{4.27}$$

Since (4.27) is shown to hold for indicator functions of Borel sets by (4.26), we know that (4.27) holds for linear combinations of indicator functions of Borel sets; i.e. simple functions $\phi : S^{n-1} \to \mathbb{R}$, given by

$$\phi = \sum_{i=1}^m c_i \mathbb{1}_{\eta_i},$$

with $c_i \in \mathbb{R}$ and Borel $\eta_i \subset S^{n-1}$. Now choose a sequence of simple functions $\phi_k : S^{n-1} \to \mathbb{R}$ converging to the bounded Borel function $f : S^{n-1} \to \mathbb{R}$. Note that since f is bounded the ϕ_k may be chosen to be uniformly bounded. Then $\phi_k \circ \alpha_K$ converges pointwise to $f \circ \alpha_K$ on $S^{n-1} \setminus \omega_K$. Since $f : S^{n-1} \to \mathbb{R}$ is a Borel function and the radial Gauss map $\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1}$ is continuous, $f \circ \alpha_K$ is a Borel function on $S^{n-1} \setminus \omega_K$. Since f is bounded and ω_K has spherical Lebesgue measure 0, we conclude that f is $\widetilde{C}_q(K,Q,\cdot)$-integrable and $f \circ \alpha_K$ is spherical Lebesgue integrable on S^{n-1}. Since $\widetilde{C}_q(K,Q,\cdot)$ is a finite measure, by taking the limit $k \to \infty$ we get (4.27).

Proposition 4.3. *Suppose* $p, q \in \mathbb{R}$. *If* $K \in \mathcal{K}_o^n$ *while* $Q \in \mathcal{S}_o^n$, *then*

$$\widetilde{C}_{p,q}(K,Q,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} h_K(\alpha_K(u))^{-p}\rho_K^q(u)\rho_Q^{n-q}(u)\, du, \tag{4.28}$$

for each Borel set $\eta \subseteq S^{n-1}$.

Proof. To establish (4.28), note that from (4.24), (4.27), and (3.19), we have

$$
\begin{aligned}
\widetilde{C}_{p,q}(K,Q,\eta) &= \int_{S^{n-1}} \mathbb{1}_\eta(u)\, d\widetilde{C}_{p,q}(K,Q,u)\\
&= \int_{S^{n-1}} \mathbb{1}_\eta(u) h_K(u)^{-p} d\widetilde{C}_q(K,Q,u)\\
&= \frac{1}{n}\int_{S^{n-1}} \mathbb{1}_\eta(\alpha_K(u)) h_K(\alpha_K(u))^{-p} \rho_K^q(u)\rho_Q^{n-q}(u)\, du\\
&= \frac{1}{n}\int_{S^{n-1}} \mathbb{1}_{\boldsymbol{\alpha}_K^*(\eta)}(u) h_K(\alpha_K(u))^{-p} \rho_K^q(u)\rho_Q^{n-q}(u)\, du\\
&= \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} h_K(\alpha_K(u))^{-p} \rho_K^q(u)\rho_Q^{n-q}(u)\, du,
\end{aligned}
$$

for each Borel $\eta \subseteq S^{n-1}$.

Verification that $\widetilde{C}_q(K,Q,\cdot)$ is a Borel measure for each $q\in\mathbb{R}$, is the same as for the cases treated above where $q=1,\dots,n$.

Obviously, the total measures of the q-th dual curvature measure and the q-th dual area measure are the q-th dual mixed volume; i.e.,

$$
\widetilde{V}_q(K,Q) = \widetilde{S}_q(K,Q,S^{n-1}) = \widetilde{C}_q(K,Q,S^{n-1}). \tag{4.29}
$$

It follows immediately from glancing at definitions (4.22) and (4.23) that,

$$
\widetilde{C}_q(K,Q,\eta) = \widetilde{S}_q(K,Q,\boldsymbol{\alpha}_K^*(\eta)). \tag{4.30}
$$

□

5. PROPERTIES OF THE L_p DUAL CURVATURE MEASURES

Lemma 5.1. *Suppose $p,q\in\mathbb{R}$. If $K\in\mathcal{K}_o^n$ while $Q\in\mathcal{S}_o^n$, then for each Borel set $\eta\subseteq S^{n-1}$, and each bounded, Borel function $g: S^{n-1}\to\mathbb{R}$,*

$$
\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u)) h_K^{-p}(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)\, du, \tag{5.1}
$$

$$
\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K,Q,v) = \frac{1}{n}\int_{\partial' K} g(\nu_K(x))(x\cdot\nu_K(x))^{1-p}\, \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x), \tag{5.2}
$$

$$
\widetilde{C}_{p,q}(K,Q,\eta) = \frac{1}{n}\int_{x\in\boldsymbol{x}_K(\eta)} (x\cdot\nu_K(x))^{1-p}\, \|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x). \tag{5.3}
$$

Proof. Since $h_K^{-p}: S^{n-1}\to\mathbb{R}$ is a bounded Borel function, from (4.27) with $f=gh_K^{-p}$, we have

$$
\int_{S^{n-1}} g(v) h_K^{-p}(v)\, d\widetilde{C}_q(K,Q,v) = \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u)) h_K^{-p}(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)\, du,
$$

which, in light of (4.24), is the desired result (5.1).

By using (5.1) and letting $f = (g \circ \alpha_K)(h_K^{-p} \circ \alpha_K)\rho_Q^{n-q}$ in (3.39), the homogeneity of ρ_Q, (3.10), (3.11), and finally the fact that $\|\cdot\|_Q = 1/\rho_Q$, we have

$$\begin{aligned}\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K,Q,v) &= \frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))h_K^{-p}(\alpha_K(u))\rho_K^q(u)\rho_Q^{n-q}(u)\, du \\ &= \frac{1}{n}\int_{\partial' K} g(\alpha_K(\bar{x}))h_K^{-p}(\alpha_K(\bar{x}))|x|^{q-n}\rho_Q^{n-q}(\bar{x})(x\cdot\nu_K(x))\, d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{n}\int_{\partial' K} g(\nu_K(x))(x\cdot\nu_K(x))^{1-p}\|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x).\end{aligned}$$

This establishes (5.2).

Take $g = \mathbb{1}_\eta$ in (5.2). Recall that $\nu_K(x) \in \eta \;\Leftrightarrow\; x \in \boldsymbol{x}_K(\eta)$, for almost all x with respect to spherical Lebesgue measure. And we immediately obtain (5.3). □

Example 5.2 [L_p dual curvature measures of polytopes] Let $P \in \mathcal{K}_o^n$ be a polytope with outer unit normals $v_1, \dots, v_m$. Let Δ_i be the cone that consists of the set of all rays emanating from the origin and passing through the facet of P whose outer normal is v_i. Recalling that we abbreviate $\boldsymbol{\alpha}_P^*(\{v_i\})$ by $\boldsymbol{\alpha}_P^*(v_i)$, from (3.14) we have

$$\boldsymbol{\alpha}_P^*(v_i) = S^{n-1}\cap\Delta_i, \quad \text{and} \quad \alpha_P(u) = v_i, \text{ for almost all } u \in \Delta_i \cap S^{n-1}. \tag{5.4}$$

If $\eta \subset S^{n-1}$ is a Borel set such that $\{v_1,\dots,v_m\}\cap\eta = \varnothing$, then $\boldsymbol{\alpha}_P^*(\eta)$ has spherical Lebesgue measure 0. Therefore, the (p,q)-dual curvature measure $\widetilde{C}_{p,q}(P,Q,\cdot)$ is discrete and is concentrated on $\{v_1,\dots,v_m\}$. From Proposition 4.3, and (5.4), we see that

$$\widetilde{C}_{p,q}(P,Q,\cdot) = \sum_{i=1}^m c_i\delta_{v_i}, \tag{5.5}$$

where, δ_{v_i} denotes the delta measure concentrated at v_i, and

$$c_i = \frac{1}{n}h_P^{-p}(v_i)\int_{S^{n-1}\cap\Delta_i}\rho_P^q(u)\rho_Q^{n-q}(u)\, du. \tag{5.6}$$

Example 5.3 [L_p dual curvature measures of strictly convex bodies] Suppose $K \in \mathcal{K}_o^n$ is strictly convex. If $g : S^{n-1} \to \mathbb{R}$ is continuous, then (5.2), and (3.41) together with the fact that $\partial K \setminus \partial' K$ has measure 0, give

$$\begin{aligned}\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K,Q,v) &= \frac{1}{n}\int_{\partial' K}(x\cdot\nu_K(x))^{1-p}g(\nu_K(x))\|x\|_Q^{q-n}\, d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{n}\int_{S^{n-1}} g(v)\|\nabla h_K(v)\|_Q^{q-n}\, dS_p(K,v).\end{aligned}$$

Using (3.31), this shows that

$$d\widetilde{C}_{p,q}(K,Q,\cdot) = \frac{1}{n}\|\nabla h_K\|_Q^{q-n}\, dS_p(K,\cdot) = \frac{1}{n}h_K^{1-p}\|\nabla h_K\|_Q^{q-n}\, dS(K,\cdot). \tag{5.7}$$

Example 5.4 [L_p dual curvature measures of smooth convex bodies] Suppose $K \in \mathcal{K}_o^n$ has a C^2 boundary with everywhere positive Gauss curvature. Since in this case $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, it follows that $\widetilde{C}_{p,q}(K,Q,\cdot)$ is absolutely continuous with respect to the spherical Lebesgue measure, and from (5.7), (3.28), and (2.2), we have

$$\frac{d\widetilde{C}_{p,q}(K,Q,v)}{dv} = \frac{1}{n} h_K^{1-p}(v) \|\bar{\nabla} h_K(v) + h_K(v)v\|_Q^{q-n} \det(\bar{\nabla}^2 h_K(v) + h_K(v)I), \tag{5.8}$$

where $\bar{\nabla} h_K(v)$ denotes the gradient of h_K on S^{n-1} at v and $\bar{\nabla}^2 h_K$ denotes the Hessian matrix of h_K with respect to an orthonormal frame on S^{n-1}.

The weak convergence of L_p dual curvature measures is an important property and is contained in the following proposition.

Proposition 5.2. *Suppose $p, q \in \mathbb{R}$ and $Q \in \mathcal{S}_o^n$. If $K_i \in \mathcal{K}_o^n$ with $K_i \to K_0 \in \mathcal{K}_o^n$, then $\widetilde{C}_{p,q}(K_i, Q, \cdot) \to \widetilde{C}_{p,q}(K_0, Q, \cdot)$, weakly.*

Proof. Suppose $g : S^{n-1} \to \mathbb{R}$ is continuous. From (5.1) we know that

$$\int_{S^{n-1}} g(v)\, d\widetilde{C}_{p,q}(K_i, Q, v) = \frac{1}{n} \int_{S^{n-1}} g(\alpha_{K_i}(u)) h_{K_i}^{1-p}(\alpha_{K_i}(u)) \rho_{K_i}^q(u) \rho_Q^{n-q}(u)\, du,$$

for all i. Since $K_i \to K_0$, with respect to the Hausdorff metric, we know that both $h_{K_i} \to h_{K_0}$ and $\rho_{K_i} \to \rho_{K_0}$, uniformly on S^{n-1}, and using Lemma 3.5 that $\alpha_{K_i} \to \alpha_{K_0}$, almost everywhere on S^{n-1}. Thus,

$$\int_{S^{n-1}} g(\alpha_{K_i}(u)) h_{K_i}^{1-p}(\alpha_{K_i}(u)) \rho_{K_i}^q(u) \rho_Q^{n-q}(u)\, du \longrightarrow \int_{S^{n-1}} g(\alpha_{K_0}(u)) h_{K_0}^{1-p}(\alpha_{K_0}(u)) \rho_{K_0}^q(u) \rho_Q^{n-q}(u)\, du.$$

It follows that $\widetilde{C}_{p,q}(K_i, Q, \cdot) \to \widetilde{C}_{p,q}(K_0, Q, \cdot)$, weakly. □

The absolute continuity of the L_p dual curvature measure with respect to the surface area measure is contained in the following proposition.

Proposition 5.3. *Suppose $p, q \in \mathbb{R}$. If $K \in \mathcal{K}_o^n$ and $Q \in \mathcal{S}_o^n$, then the dual curvature measure $\widetilde{C}_{p,q}(K, Q, \cdot)$ is absolutely continuous with respect to the surface area measure $S(K, \cdot)$.*

Proof. Suppose $\eta \subset S^{n-1}$ is such that $S(K, \eta) = 0$, or equivalently by definition (3.27), $\mathcal{H}^{n-1}(\boldsymbol{x}_K(\eta)) = 0$. Now (5.3) states that,

$$\widetilde{C}_{p,q}(K, Q, \eta) = \frac{1}{n} \int_{\boldsymbol{x}_K(\eta)} \|x\|_Q^{q-n} (x \cdot \nu_K(x))^{1-p}\, d\mathcal{H}^{n-1}(x) = 0,$$

since the integration is over a set of measure 0. □

Cone-volume measure has become an increasing important concept; see e.g., [3,4,7,24,43,44,46,63]. The following proposition tells us that the L_p surface area measure including the classical surface area measure, the dual curvature measure including the cone-volume measure, the L_p integral curvature including Aleksandrov's integral curvature, are all special cases of the L_p dual curvature measure.

Proposition 5.4. *If $K \in \mathcal{K}_o^n$ and $p, q \in \mathbb{R}$, then*

$$\widetilde{C}_{p,q}(K, K, \cdot) = \frac{1}{n} S_p(K, \cdot), \tag{5.9}$$

$$\widetilde{C}_{p,n}(K, B, \cdot) = \frac{1}{n} S_p(K, \cdot), \tag{5.10}$$

$$\widetilde{C}_{0,q}(K, B, \cdot) = \widetilde{C}_q(K, \cdot), \tag{5.11}$$

$$\widetilde{C}_{p,0}(K, B, \cdot) = \frac{1}{n} J_p(K^*, \cdot). \tag{5.12}$$

Proof. Let $\eta \subset S^{n-1}$ be a Borel set. From (5.3), we have

$$\widetilde{C}_{p,n}(K, B, \eta) = \frac{1}{n} \int_{\boldsymbol{x}_K(\eta)} (x \cdot \nu_K(x))^{1-p} \, d\mathcal{H}^{n-1}(x) = \widetilde{C}_{p,q}(K, K, \eta),$$

where the right equality follows from the simple observation that for all $x \in \partial K$ we have $\|x\|_K = 1/\rho_K(x) = 1$, from the very definition of the radial function. But, (3.32) states that the integral above is just $\frac{1}{n} S_p(K, \eta)$, which establishes both (5.9) and (5.10) simultaneously.

Identity (5.11) follows from the definition of dual curvature measure (5.2) and the definition of the the q-th dual curvature measure (3.35).

From the definition of $(p, 0)$-th dual curvature measure, Proposition 4.3, with $q = 0$ and $Q = B$, (2.4), (2.5) together with Lemma 3.10, and finally (3.38), we have

$$\begin{aligned} \widetilde{C}_{p,0}(K, \eta) &= \frac{1}{n} \int_{\boldsymbol{\alpha}^*_K(\eta)} h_K(\alpha_K(u))^{-p} du \\ &= \frac{1}{n} \int_{\boldsymbol{\alpha}_{K^*}(\eta)} \rho_{K^*}(\alpha^*_{K^*}(u))^p du \\ &= \frac{1}{n} J_p(K^*, \eta), \end{aligned}$$

which gives (5.12). □

Let $\mathcal{M}(S^{n-1})$ denote the set of Borel measures on S^{n-1}. We shall now show that, for fixed indices $p, q \in \mathbb{R}$, and a fixed star body $Q \in \mathcal{S}_o^n$, the functional $\mathcal{K}_o^n \to \mathcal{M}(S^{n-1})$, defined by $K \mapsto \widetilde{C}_{p,q}(K, Q, \cdot)$ is a valuation; i.e., if $K, L \in \mathcal{K}_o^n$, are such that $K \cup L \in \mathcal{K}_o^n$ then

$$\widetilde{C}_{p,q}(K, Q, \cdot) + \widetilde{C}_{p,q}(L, Q, \cdot) = \widetilde{C}_{p,q}(K \cap L, Q, \cdot) + \widetilde{C}_{p,q}(K \cup L, Q, \cdot).$$

Towards that end, we shall employ Weil's Approximation Lemma: If $K, L \in \mathcal{K}_o^n$ are such that $K \cup L$ is convex, then K and L may be approximated by sequences of bodies $K_i, L_i \in \mathcal{K}_o^n$ that are both strictly convex and smooth and such that $K_i \cup L_i \in \mathcal{K}_o^n$. The simple and elegant proof (below) is due to Wolfgang Weil: Simply let

$$K_i = ((K + \tfrac{1}{i} B)^* + \tfrac{1}{i} B)^* \quad \text{and} \quad L_i = ((L + \tfrac{1}{i} B)^* + \tfrac{1}{i} B)^*.$$

Other than the fact that polarity $^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is both continuous and an involution, there are three key observations required to establish Weil's result: (1) If $K \cup L \in \mathcal{K}_o^n$ then $K^* \cup L^* = (K \cap L)^* \in \mathcal{K}_o^n$. See Schneider [48], Thm 1.6.3. (2) If $K \cup L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, then the smooth bodies $K + \varepsilon B$ and $L + \varepsilon B$ are such that

$$(K + \varepsilon B) \cup (L + \varepsilon B) = K \cup L + \varepsilon B \in \mathcal{K}_o^n.$$

(3) Polars of smooth bodies are strictly convex and visa versa, which is Lemma 3.1.

We appeal to Proposition 5.2 together with Weils Approximation Lemma in order to complete our proof.

Lemma 5.5. *For $p, q \in \mathbb{R}$ and a fixed $Q \in \mathcal{S}_o^n$, the functional*

$$\widetilde{C}_{p,q}(\cdot, Q, \cdot) : \mathcal{K}_o^n \longrightarrow \mathcal{M}(S^{n-1}),$$

defined by $K \mapsto \widetilde{C}_{p,q}(K, Q, \cdot)$, is a valuation.

Proof. We shall make use of the fact that when $K, L \in \mathcal{K}_o^n$, are such that $K \cup L \in \mathcal{K}_o^n$, then $h_{K\cup L} = \max\{h_K, h_L\}$ and $h_{K\cap L} = \min\{h_K, h_L\}$. We shall also make use of the fact that ν_K and ν_L are defined $\mathcal{H}^{n-1}$ almost everywhere on the boundaries of K and L respectively.

First, we assume that K and L are both strictly convex. For a fixed $\theta \subset S^{n-1}$, write θ as the union of three disjoint pieces $\theta = \theta_0 \cup \theta_K \cup \theta_L$, where

$$\theta_K = \{u \in \theta : h_K(u) > h_L(u)\}, \quad \theta_L = \{u \in \theta : h_K(u) < h_L(u)\}.$$

while

$$\theta_0 = \{u \in \theta : h_K(u) = h_L(u)\}.$$

Now,

$$\int_{x\in\boldsymbol{x}_{K\cup L}(\theta_K)} (x \cdot \nu_{K\cup L}(x))^{1-p} \, \|x\|_Q^{q-n} \, d\mathcal{H}^{n-1}(x) = \int_{x\in\boldsymbol{x}_K(\theta_K)} (x \cdot \nu_K(x))^{1-p} \, \|x\|_Q^{q-n} \, d\mathcal{H}^{n-1}(x),$$

while

$$\int_{x\in\boldsymbol{x}_{K\cap L}(\theta_K)} (x \cdot \nu_{K\cap L}(x))^{1-p} \, \|x\|_Q^{q-n} \, d\mathcal{H}^{n-1}(x) = \int_{x\in\boldsymbol{x}_L(\theta_K)} (x \cdot \nu_L(x))^{1-p} \, \|x\|_Q^{q-n} \, d\mathcal{H}^{n-1}(x).$$

Alternatively, using (5.3), we can write this as

$$C_{p,q}(K \cup L, Q, \theta_K) = C_{p,q}(K, Q, \theta_K) \quad \text{and} \quad C_{p,q}(K \cap L, Q, \theta_K) = C_{p,q}(L, Q, \theta_K). \tag{5.13}$$

Similarly

$$C_{p,q}(K \cup L, Q, \theta_L) = C_{p,q}(L, Q, \theta_L) \quad \text{and} \quad C_{p,q}(K \cap L, Q, \theta_L) = C_{p,q}(K, Q, \theta_L). \tag{5.14}$$

It is also the case that,

$$C_{p,q}(K \cup L, Q, \theta_0) = C_{p,q}(K, Q, \theta_0), \quad \text{and} \quad C_{p,q}(K \cap L, Q, \theta_0) = C_{p,q}(L, Q, \theta_0). \tag{5.15}$$

To see this last fact, observe that the strict convexity of K and L forces $\boldsymbol{x}_{K\cup L}(\theta_0) = \boldsymbol{x}_{K\cap L}(\theta_0)$.

Now, using the fact that $C_{p,q}(\,\cdot\,, Q, \cdot)$ is a measure in the third argument on S^{n-1}, together with the fact that the union $\theta = \theta_0 \cup \theta_K \cup \theta_L$ is disjoint, by adding (5.13), (5.14), and (5.15) we obtain the desired result that

$$\widetilde{C}_{p,q}(K \cap L, Q, \theta) + \widetilde{C}_{p,q}(K \cup L, Q, \theta) = \widetilde{C}_{p,q}(K, Q, \theta) + \widetilde{C}_{p,q}(L, Q, \theta),$$

which is the desired result.

For arbitrary $K, L \in \mathcal{K}_o^n$, we appeal to Proposition 5.2 in order to use the weak continuity of $\widetilde{C}_{p,q}(\cdot, Q, \cdot)$ in the first argument. □

6. Variational formulas for dual mixed volumes and dual mixed entropy

Let Ω be a closed subset of S^{n-1} that is not contained in any closed hemisphere. Let $f : \Omega \to \mathbb{R}$ be continuous, and $\delta > 0$. Let $h_t : \Omega \to (0, \infty)$ be a positive continuous function defined for each $t \in (-\delta, \delta)$ by

$$\log h_t(v) = \log h_0(v) + tf(v) + o(t, v), \tag{6.1}$$

where $o(t, \cdot) : \Omega \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o(t, \cdot)/t = 0$, uniformly on Ω. Denote by

$$[h_t] = \{x \in \mathbb{R}^n : x \cdot v \le h_t(v) \text{ for all } v \in \Omega\},$$

the Wulff shape determined by h_t. We shall call $[h_t]$ a *logarithmic family of Wulff shapes generated by* (h_0, f). If h_0 is the support function h_K of a convex body K, we also write $[h_t]$ as $[K, f, t]$.

Let $g : \Omega \to \mathbb{R}$ be continuous, and $\delta > 0$. Let $\rho_t : \Omega \to (0, \infty)$ be a positive continuous function defined for each $t \in (-\delta, \delta)$ by

$$\log \rho_t(u) = \log \rho_0(u) + tg(u) + o(t, u), \tag{6.2}$$

where again $o(t, \cdot) : \Omega \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o(t, \cdot)/t = 0$, uniformly on Ω. Denote by

$$\langle \rho_t \rangle = \operatorname{conv}\{\rho_t(u)u : u \in S^{n-1}\}$$

the convex hull generated by ρ_t. We will call $\langle \rho_t \rangle$ a *logarithmic family of convex hulls generated by* (ρ_0, g). If ρ_0 is the radial function ρ_K of a convex body K, we also write $\langle \rho_t \rangle$ as $\langle K, g, t \rangle$.

The following theorem gives the variational formulas for dual mixed volumes and dual mixed entropy of a logarithmic family of convex hulls.

Theorem 6.1. *Suppose $\Omega \subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}, and $\rho_0 : \Omega \to (0, \infty)$ and $g : \Omega \to \mathbb{R}$ are continuous. If $\langle \rho_t \rangle$ is a logarithmic family of convex hulls generated by (ρ_0, g), then for $Q \in \mathcal{S}_o^n$ and $q \neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(\langle \rho_t \rangle^*, Q) - \widetilde{V}_q(\langle \rho_0 \rangle^*, Q)}{t} = -q \int_\Omega g(u)\, d\widetilde{C}_q(\langle \rho_0 \rangle^*, Q, u),$$

and

$$\lim_{t\to 0} \frac{\widetilde{E}(\langle \rho_t \rangle^*, Q) - \widetilde{E}(\langle \rho_0 \rangle^*, Q)}{t} = -\int_\Omega g(u)\, d\widetilde{C}_0(\langle \rho_0 \rangle^*, Q, u).$$

For $Q = B$, the unit ball in $\mathbb{R}^n$, Theorem 6.1 was proved in [25]. When Q is an arbitrary star body in $\mathcal{S}_o^n$, the proof of Theorem 6.1 is very similar and thus omitted.

The following theorem gives the variational formulas for dual mixed volumes and dual mixed entropy of a logarithmic family of Wulff shapes.

Theorem 6.2. *Suppose $\Omega \subset S^{n-1}$ is a closed set not contained in any closed hemisphere of S^{n-1}. If $h_0 : \Omega \to (0, \infty)$ and $f : \Omega \to \mathbb{R}$ are continuous, and $[h_t]$ is a logarithmic family of Wulff shapes generated by (h_0, f), then, for $Q \in \mathcal{S}_o^n$ and $q \neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q([h_t], Q) - \widetilde{V}_q([h_0], Q)}{t} = q \int_\Omega f(v)\, d\widetilde{C}_q([h_0], Q, v),$$

and

$$\lim_{t\to 0} \frac{\widetilde{E}([h_t], Q) - \widetilde{E}([h_0], Q)}{t} = \int_\Omega f(v)\, d\widetilde{C}_0([h_0], Q, v).$$

Again, for $Q = B$, Theorem 6.2 was proved in [25]. When Q is a star body in $\mathcal{S}_o^n$, the proof of Theorem 6.2 is similar and thus will be omitted.

We state the special cases of Theorems 6.1 and 6.2 for logarithmic families of convex hulls and Wulff shapes generated by convex bodies.

Theorem 6.3. *Suppose $K \in \mathcal{K}_o^n$ and $g : S^{n-1} \to \mathbb{R}$ is continuous. Then, for $Q \in \mathcal{S}_o^n$ and $q \neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(\langle K^*, g, t\rangle^*, Q) - \widetilde{V}_q(K, Q)}{t} = -q \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K, Q, v),$$

and

$$\lim_{t\to 0} \frac{\widetilde{E}(\langle K^*, g, t\rangle^*, Q) - \widetilde{E}(K, Q)}{t} = -\int_{S^{n-1}} g(v)\, d\widetilde{C}_0(K, Q, v).$$

Theorem 6.4. *Suppose $K \in \mathcal{K}_o^n$, and $f : S^{n-1} \to \mathbb{R}$ is continuous. Then, for $Q \in \mathcal{S}_o^n$ and $q \neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q([K, f, t], Q) - \widetilde{V}_q(K, Q)}{t} = q \int_{S^{n-1}} f(v)\, d\widetilde{C}_q(K, Q, v),$$

and

$$\lim_{t\to 0} \frac{\widetilde{E}([K, f, t], Q) - \widetilde{E}(K, Q)}{t} = \int_{S^{n-1}} f(v)\, d\widetilde{C}_0(K, Q, v).$$

The following theorem gives the variational formulas of dual mixed volumes and dual mixed entropy with respect to L_p Minkowski combinations.

Theorem 6.5. *Suppose $p \neq 0$ and $q \neq 0$. If $Q \in \mathcal{S}_o^n$ and $K, L \in \mathcal{K}_o^n$, then,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(K +_p t\cdot L, Q) - \widetilde{V}_q(K, Q)}{t} = \frac{q}{p} \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(K, Q, v), \tag{6.3}$$

$$\lim_{t\to 0} \frac{\widetilde{V}_q(K +_0 t\cdot L, Q) - \widetilde{V}_q(K, Q)}{t} = q \int_{S^{n-1}} \log h_L(v)\, d\widetilde{C}_q(K, Q, v), \tag{6.4}$$

$$\lim_{t\to 0} \frac{\widetilde{E}(K +_p t\cdot L, Q) - \widetilde{E}(K, Q)}{t} = \frac{1}{p} \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,0}(K, Q, v), \tag{6.5}$$

$$\lim_{t\to 0} \frac{\widetilde{E}(K +_0 t\cdot L, Q) - \widetilde{E}(K, Q)}{t} = \int_{S^{n-1}} \log h_L(v)\, d\widetilde{C}_0(K, Q, v). \tag{6.6}$$

Proof. For small t, define h_t by

$$\begin{aligned} h_t^p &= h_K^p + t h_L^p, \quad p \neq 0,\\ h_t &= h_K h_L^t, \qquad p = 0. \end{aligned} \tag{6.7}$$

From (2.11) and (2.12), the Wulff shape $[h_t] = K +_p t\cdot L$.

From (6.7), it follows immediately that, for sufficiently small t,

$$\begin{aligned} \log h_t &= \log h_K + \frac{t}{p}\frac{h_L^p}{h_K^p} + o(t, \cdot), && p \neq 0,\\ \log h_t &= \log h_K + t \log h_L, && p = 0. \end{aligned}$$

Let $f = \frac{1}{p}\frac{h_L^p}{h_K^p}$ when $p \neq 0$, and let $f = \log h_L$ when $p = 0$. The desired formulas now follow directly from Theorem 6.4 and (4.24). $\square$

By using the normalized power function, we can write the formulas in Theorem 6.5 as one single formula.

Theorem 6.6. *Suppose* $p, q \in \mathbb{R}$*. For* $Q \in \mathcal{S}_o^n$*, and* $K, L \in \mathcal{K}_o^n$*,*

$$\frac{d}{dt}\widetilde{V}_{\bar{q}}(K+_p t\cdot L, Q)\Big|_{t=0} = \int_{S^{n-1}} h_L^{\bar{p}}(v)\, d\widetilde{C}_{p,q}(K,Q,v).$$

For 0-Minkowski-Firey linear combinations it will be helpful to have an affine version of Theorem 6.5. This is contained in:

Theorem 6.7. *Suppose* $q \neq 0$*. If* $Q \in \mathcal{S}_o^n$ *and* $K, L \in \mathcal{K}_o^n$*, then,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q((1-t)\cdot K+_0 t\cdot L, Q) - \widetilde{V}_q(K,Q)}{t} = q\int_{S^{n-1}} \log\frac{h_L(v)}{h_K(v)}\, d\widetilde{C}_q(K,Q,v), \tag{6.8}$$

$$\lim_{t\to 0} \frac{\widetilde{E}((1-t)\cdot K+_0 t\cdot L, Q) - \widetilde{E}(K,Q)}{t} = \int_{S^{n-1}} \log\frac{h_L(v)}{h_K(v)}\, d\widetilde{C}_0(K,Q,v). \tag{6.9}$$

Proof. Let

$$h_t = h_K^{1-t} h_L^t.$$

From (2.12), the Wulff shape $[h_t] = (1-t)\cdot K +_0 t\cdot L$. From the above definition of h_t, it follows immediately that, for sufficiently small t,

$$\log h_t = \log h_K + t\log\frac{h_L}{h_K}.$$

Let $f = \log\frac{h_L}{h_K}$. The desired formulas now follow directly from Theorem 6.4. □

In stating our next theorem we recall Definition 2.2.

Theorem 6.8. *Suppose* $p \neq 0$ *and* $q \neq 0$*. Then for all* $Q \in \mathcal{S}_o^n$ *and* $K, L \in \mathcal{K}_o^n$*, and* $\phi \in \mathrm{SL}(n)$*,*

$$\widetilde{C}_{p,q}(\phi K, \phi Q, \cdot) = \phi^t {}_p\!\lrcorner\, \widetilde{C}_{p,q}(K,Q,\cdot) \tag{6.10}$$

$$\widetilde{C}_{p,0}(\phi K, \phi Q, \cdot) = \phi^t {}_p\!\lrcorner\, \widetilde{C}_{p,0}(K,Q,\cdot) \tag{6.11}$$

$$\widetilde{C}_{q}(\phi K, \phi Q, \cdot) = \phi^t {}_0\!\lrcorner\, \widetilde{C}_{q}(K,Q,\cdot) \tag{6.12}$$

$$\widetilde{C}_{0}(\phi K, \phi Q, \cdot) = \phi^t {}_0\!\lrcorner\, \widetilde{C}_{0}(K,Q,\cdot) \tag{6.13}$$

Observe that the case $p \neq 0$ and $q = 0$ is handled by (6.11). The case $p = 0$ and $q \neq 0$ is handled by (6.12), while the case $p = 0$ and $q = 0$ is handled by (6.13).

Recall that Haberl & Parapatits [21] classified measure-valued operators on $\mathcal{K}_o^n$ that are $\mathrm{SL}(n)$-contravariant of degree p, which corresponds to the transformation behavior in Theorem 6.8, but our measures depend on an additional star body.

Proof. From (2.16), (2.18), and (6.3), we see that for all $K, L \in \mathcal{K}_o^n$ and all $Q \in \mathcal{S}_o^n$,

$$\int_{S^{n-1}} h_{\phi L}^p(v)\, d\widetilde{C}_{p,q}(\phi K, \phi Q, v) = \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(K,Q,v), \quad \text{for all } \phi \in \mathrm{SL}(n),$$

or equivalently, that for all $K, L \in \mathcal{K}_o^n$ and all $Q \in \mathcal{S}_o^n$,

$$\int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(\phi K, \phi Q, v) = \int_{S^{n-1}} h_{\phi^{-1}L}^p(v)\, d\widetilde{C}_{p,q}(K,Q,v), \quad \text{for all } \phi \in \mathrm{SL}(n). \tag{6.14}$$

From Definition 2.2, the fact that support functions are positively homogeneous of degree 1, followed by (2.1), and (6.14), we have

$$\begin{aligned}\int_{S^{n-1}} h_L^p(v)\, d\phi^t_{\,p}\lrcorner\, \widetilde{C}_{p,q}(K,Q,v) &= \int_{S^{n-1}} h_L^p(\phi^{-t}v)\, d\widetilde{C}_{p,q}(K,Q,v)\\ &= \int_{S^{n-1}} h_{\phi^{-1}L}^p(v)\, d\widetilde{C}_{p,q}(K,Q,v)\\ &= \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(\phi K,\phi Q,v).\end{aligned}$$

But this shows that the measures $\phi^t_{\,p}\lrcorner\, \widetilde{C}_{p,q}(K,Q,\cdot)$ and $\widetilde{C}_{p,q}(\phi K,\phi Q,\cdot)$ when integrated against the p-th power of support functions of bodies in $\mathcal{K}_o^n$ are identical and thus Lemma 2.1 now shows that indeed

$$\widetilde{C}_{p,q}(\phi K,\phi Q,\cdot) = \phi^t_{\,p}\lrcorner\, \widetilde{C}_{p,q}(K,Q,\cdot),$$

which establishes (6.10).

The proof of (6.11) is identical to the proof of (6.10): As long as $p \neq 0$, it will be the case that (6.14) continues to hold even if $q = 0$ provided we appeal to (6.5) and (2.20) when previously we had turned to (6.3) and (2.18).

From (2.16), (2.18), and (6.8), we see that for all $K, L \in \mathcal{K}_o^n$ and all $Q \in \mathcal{S}_o^n$,

$$\int_{S^{n-1}} \log \frac{h_{\phi^{-1}L}(v)}{h_K(v)}\, d\widetilde{C}_q(K,Q,v) = \int_{S^{n-1}} \log \frac{h_L(v)}{h_{\phi K}(v)}\, d\widetilde{C}_q(\phi K,\phi Q,v), \tag{6.15}$$

for all $\phi \in \mathrm{SL}(n)$.

In (6.15) choose $L = B$. Then by (2.1) we see that $h_{\phi^{-1}L}(v) = h_L(\phi^{-t}v) = |\phi^{-t}v|$, and (6.15) becomes:

$$\int_{S^{n-1}} \log h_K(v)\, d\widetilde{C}_q(K,Q,v) = \int_{S^{n-1}} \log|\phi^{-t}v|\, d\widetilde{C}_q(K,Q,v) + \int_{S^{n-1}} \log h_{\phi K}(v)\, d\widetilde{C}_q(\phi K,\phi Q,v), \tag{6.16}$$

which holds for all $\phi \in \mathrm{SL}(n)$, all $K \in \mathcal{K}_o^n$, and all $Q \in \mathcal{S}_o^n$. Combining (6.15) and (6.16) gives

$$\int_{S^{n-1}} \log \frac{h_{\phi^{-1}L}(v)}{|\phi^{-t}v|}\, d\widetilde{C}_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)\, d\widetilde{C}_q(\phi K,\phi Q,v),$$

or using (2.1),

$$\int_{S^{n-1}} \log h_L(\langle \phi^{-t}v\rangle)\, d\widetilde{C}_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)\, d\widetilde{C}_q(\phi K,\phi Q,v), \tag{6.17}$$

which holds for all $\phi \in \mathrm{SL}(n)$, all $K, L \in \mathcal{K}_o^n$, and all $Q \in \mathcal{S}_o^n$. Equivalently,

$$\int_{S^{n-1}} \log h_L(v)\, d\phi^t_{\,0}\lrcorner\, \widetilde{C}_q(K,Q,v) = \int_{S^{n-1}} \log h_L(v)\, d\widetilde{C}_q(\phi K,\phi Q,v), \tag{6.18}$$

which holds for all $\phi \in \mathrm{SL}(n)$, all $K, L \in \mathcal{K}_o^n$, and all $Q \in \mathcal{S}_o^n$. Using Lemma 2.1, we see that (6.18) yields

$$\widetilde{C}_q(\phi K,\phi Q,\cdot) = \phi^t_{\,0}\lrcorner\, \widetilde{C}_q(K,Q,\cdot),$$

for all $\phi \in \mathrm{SL}(n)$, all $K \in \mathcal{K}_o^n$, and all $Q \in \mathcal{S}_o^n$. This establishes (6.12).

The proof of (6.13) is identical to the proof of (6.12) except that instead of appealing to (6.8) and (2.18) we appeal to (6.9) and (2.20). $\square$

7. L_p DUAL MIXED VOLUMES

For convex bodies $K, L \in \mathcal{K}_o^n$, recall that the L_p mixed volume, $V_p(K, L)$, has the integral representation,

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(v)^p \, dS_p(K,v).$$

From (5.1), with $q = n$, $Q = B$, and $g = h_L^p$, we see that

$$\int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,n}(K,B,v) = \frac{1}{n}\int_{S^{n-1}} h_L^p(\alpha_K(u)) h_K^{-p}(\alpha_K(u))\rho_K^n(u)\, du.$$

But (5.10), tells us that $\widetilde{C}_{p,n}(K,B,\cdot) = \frac{1}{n}S_p(K,\cdot)$. This shows that the L_p mixed volume, $V_p(K,L)$, has a dual integral formulation: If $K, L \in \mathcal{K}_o^n$, then

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \Big(\frac{h_L}{h_K}\Big)^p (\alpha_K(u))\rho_K^n(u)\, du. \tag{7.1}$$

This leads us to define L_p *dual mixed volumes*, a unification which includes L_p mixed volumes and dual mixed volumes.

Definition 7.1. *Suppose $p, q \in \mathbb{R}$. If $K, L \in \mathcal{K}_o^n$, and $Q \in \mathcal{S}_o^n$, define the L_p dual mixed volume, or (p,q)-mixed volume, $\widetilde{V}_{p,q}(K,L,Q)$, by*

$$\widetilde{V}_{p,q}(K,L,Q) = \int_{S^{n-1}} h_L^p(v)\, d\widetilde{C}_{p,q}(K,Q,v). \tag{7.2}$$

By using (5.1) with $g = h_L^p$, definition (7.2) can be written as a dual formula,

$$\widetilde{V}_{p,q}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}} \Big(\frac{h_L}{h_K}\Big)^p (\alpha_K(u))\Big(\frac{\rho_K}{\rho_Q}\Big)^q(u)\rho_Q^n(u)\, du. \tag{7.3}$$

Proposition 7.2. *Suppose $p, q \in \mathbb{R}$. If $K, L \in \mathcal{K}_o^n$, and $Q \in \mathcal{S}_o^n$, then*

$$\widetilde{V}_{p,q}(K,K,K) = V(K), \tag{7.4}$$

$$\widetilde{V}_{p,q}(K,K,Q) = \widetilde{V}_q(K,Q), \tag{7.5}$$

$$\widetilde{V}_{p,q}(K,L,K) = V_p(K,L), \tag{7.6}$$

$$\widetilde{V}_{0,q}(K,L,Q) = \widetilde{V}_q(K,Q), \tag{7.7}$$

$$\widetilde{V}_{p,n}(K,L,Q) = V_p(K,L). \tag{7.8}$$

Proof. Identity (7.4) follow from (7.3) and the polar coordinate formula for volume. Identity (7.5) follow from (7.3) and the definition of dual mixed volumes (1.3). Identity (7.6) follow from (7.3) and (7.1). Identity (7.7) follow from (7.3) and the definition of dual mixed volumes (1.3). Identity (7.8) follow from (7.3) and (7.1). □

Proposition 7.3. *The L_p dual mixed volume is $\mathrm{SL}(n)$-invariant, in that for $p, q \in \mathbb{R}$, and $K, L \in \mathcal{K}_o^n$, with $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_{p,q}(\phi K, \phi L, \phi Q) = \widetilde{V}_{p,q}(K,L,Q),$$

for each $\phi \in SL(n)$.

Proof. For $p = 0$ the result follows from (7.7) and the $\mathrm{SL}(n)$-invariance of dual mixed volumes (2.18). We assume $p \neq 0$. By definition (7.2), from (6.10) and (6.11), the fact that support functions are positively homogeneous of degree 1, (2.1), and finally definition (7.2) again

$$\begin{aligned}
\widetilde{V}_{p,q}(\phi K, \phi L, \phi Q) &= \int_{S^{n-1}} h^p_{\phi L}(v)\, d\widetilde{C}_{p,q}(\phi K, \phi Q, v) \\
&= \int_{S^{n-1}} h^p_{\phi L}(v)\, d\phi^t_{\frac{}{p}\lrcorner} \widetilde{C}_{p,q}(K, Q, v) \\
&= \int_{S^{n-1}} h^p_{\phi L}(\phi^{-t} v)\, d\widetilde{C}_{p,q}(K, Q, v) \\
&= \int_{S^{n-1}} h^p_{L}(v)\, d\widetilde{C}_{p,q}(K, Q, v) \\
&= \widetilde{V}_{p,q}(K, L, Q)
\end{aligned}$$

□

The following inequality for L_p dual mixed volume is a generalization of the L_p Minkowski inequality for mixed volume.

Theorem 7.4. *Suppose p, q are such that $1 \leq \frac{q}{n} \leq p$. If $K, L \in \mathcal{K}^n_o$ and $Q \in \mathcal{S}^n_o$, then*

$$\widetilde{V}_{p,q}(K, L, Q)^n \geq V(K)^{q-p} V(L)^p V(Q)^{n-q}, \tag{7.9}$$

with equality if and only if K, L, Q are dilates when $1 < \frac{q}{n} < p$, while only K and L need be dilates when $q = n$ and $p > 1$, and K and L are homothets when $q = n$ and $p = 1$.

Proof. From (7.3), the Hölder inequality, (7.1), and the L_p Minkowski inequality (3.34), we have

$$\begin{aligned}
\widetilde{V}_{p,q}(K, L, Q) &= \frac{1}{n} \int_{S^{n-1}} \Big(\frac{h_L}{h_K}\Big)^p (\alpha_K(u)) \Big(\frac{\rho_K}{\rho_Q}\Big)^q (u) \rho^n_Q(u)\, du \\
&= \frac{1}{n} \int_{S^{n-1}} \Big[\Big(\frac{h_L}{h_K}\Big)^{\frac{np}{q}} (\alpha_K(u)) \rho^n_K(u)\Big]^{\frac{q}{n}} [\rho^n_Q(u)]^{\frac{n-q}{n}}\, du \\
&\geq \Big(\frac{1}{n} \int_{S^{n-1}} \Big(\frac{h_L}{h_K}\Big)^{\frac{np}{q}} (\alpha_K(u)) \rho^n_K(u)\, du\Big)^{\frac{q}{n}} \Big(\frac{1}{n} \int_{S^{n-1}} \rho^n_Q(u)\, du\Big)^{\frac{n-q}{n}} \\
&= V_{\frac{np}{q}}(K, L)^{\frac{q}{n}} V(Q)^{\frac{n-q}{n}} \\
&\geq V(K)^{\frac{q-p}{n}} V(L)^{\frac{p}{n}} V(Q)^{\frac{n-q}{n}}.
\end{aligned}$$

The equality conditions follow from the equality conditions of the Hölder inequality and the L_p Minkowski inequality (3.34) for L_p mixed volumes. □

Over the past two decades valuation theory has become an ever more important part of convex geometric analysis. See e.g. [6], [19], [32], [33], [34], [35], [49], and [60]. The L_p dual mixed volume is a valuation with respect to each entry.

Proposition 7.5. *The L_p dual mixed volume $\widetilde{V}_{p,q}(K, L, Q)$ is a valuation over $\mathcal{K}^n_o$ with respect to both K and L, and is a valuation over $\mathcal{S}^n_o$ with respect to Q.*

Proof. That the L_p dual mixed volume $\widetilde{V}_{p,q}(K, L, Q)$ is a valuation on $\mathcal{S}_o^n$ with respect to the third argument can be seen easily by writing (7.3) as

$$\widetilde{V}_{p,q}(K, L, Q) = \frac{1}{n} \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p (\alpha_K(u)) \rho_K^q(u) \rho_Q^{n-q}(u)\, du,$$

and observing that for $Q_1, Q_2 \in \mathcal{S}_o^n$ we have

$$\rho_{Q_1 \cup Q_2}^{n-q} + \rho_{Q_1 \cap Q_2}^{n-q} = \rho_{Q_1}^{n-q} + \rho_{Q_2}^{n-q}, \qquad \text{on } S^{n-1}.$$

That the L_p dual mixed volume $\widetilde{V}_{p,q}(K, L, Q)$ is a valuation on $\mathcal{K}_o^n$ with respect to the second argument can be seen easily by looking at definition (7.2) and using the fact that if $L_1, L_2 \in \mathcal{K}_o^n$, are such that $L_1 \cup L_2 \in \mathcal{K}_o^n$, we have

$$h_{L_1 \cup L_2}^p + h_{L_1 \cap L_2}^p = h_{L_1}^p + h_{L_2}^p, \qquad \text{on } S^{n-1},$$

since $h_{L_1 \cup L_2} = \max\{h_{L_1}, h_{L_2}\}$ and $h_{L_1 \cap L_2} = \min\{h_{L_1}, h_{L_2}\}$. From (7.2) and Lemma 5.5, we see that $\widetilde{V}_{p,q}(K, L, Q)$ is a valuation in the first argument. □

8. The L_p dual Minkowski problem

The existence and uniqueness problems for L_p dual curvature measures are the central problems to be studied here. The L_p dual Minkowski existence problem for L_p dual curvature measure may be stated as follows:

Problem 8.1. *Suppose $p, q \in \mathbb{R}$, and $Q \in \mathcal{S}_o^n$ are fixed. Given a Borel measure $\mu \in \mathcal{M}(S^{n-1})$, what are necessary and sufficient conditions on μ so that there exists a convex body $K \in \mathcal{K}_o^n$ whose dual curvature measure $\widetilde{C}_{p,q}(K, Q, \cdot)$ is the given measure μ?*

The case where $q = n$ is the L_p Minkowski problem. The case where $p = 0$ and $Q = B$ is the dual Minkowski problem. The case where $q = 0$ and $Q = B$ is the L_p Aleksandrov problem.

When the given data measure μ has a density f, from (5.8) we see that, the L_p dual Minkowski problem is equivalent to solving the following Monge-Ampère type equation on S^{n-1}:

$$h^{1-p} \|\nabla h\|_Q^{q-n} \det(\bar{\nabla}^2 h + hI) = f, \tag{8.1}$$

where h is the unknown function on S^{n-1}, and ∇h is the gradient in $\mathbb{R}^n$ of the extension of h to $\mathbb{R}^n$ as a function that is positively homogeneous of degree 1, and where $\bar{\nabla}^2$ is the Hessian matrix of h, with respect to an orthonormal frame on S^{n-1}.

The uniqueness problem for L_p dual curvature measures is:

Problem 8.2. *For fixed $p, q \in \mathbb{R}$ and $Q \in \mathcal{S}_o^n$, if $K, L \in \mathcal{K}_o^n$, are such that*

$$\widetilde{C}_{p,q}(K, Q, \cdot) = \widetilde{C}_{p,q}(L, Q, \cdot),$$

then how is K related to L?

We now establish uniqueness of the solution to the problem for the case of polytopes when $q \le p$.

Theorem 8.3. *Let $P, P' \in \mathcal{K}_o^n$ be polytopes and let $Q \in \mathcal{S}_o^n$. Suppose*

$$\widetilde{C}_{p,q}(P, Q, \cdot) = \widetilde{C}_{p,q}(P', Q, \cdot).$$

Then $P = P'$ when $q < p$ and P' is a dilate of P when $q = p$.

Proof. From (5.5), we know that the curvature measures of polytopes are discrete, and that $\widetilde{C}_{p,q}(P,Q,\cdot) = \widetilde{C}_{p,q}(P',Q,\cdot)$ means that P and P' must have the same outer unit normals; say $v_1,\dots,v_m$ and that

$$\widetilde{C}_{p,q}(P,Q,\cdot) = \widetilde{C}_{p,q}(P',Q,\cdot) = \sum_{i=1}^{m} c_i\delta_{v_i},$$

where

$$c_i = \frac{1}{n}h_P(v_i)^{-p}\int_{S^{n-1}\cap\Delta_i}\rho_P^q(u)\rho_Q^{n-q}(u)\,du = \frac{1}{n}h_{P'}(v_i)^{-p}\int_{S^{n-1}\cap\Delta_i'}\rho_{P'}^q(u)\rho_Q^{n-q}(u)\,du \tag{8.2}$$

and Δ_i and Δ_i' are the cones formed by the origin and the facets of P and P' with normal v_i, respectively.

Suppose $P \neq P'$. Clearly $P \subsetneq P'$ is not possible. Let λ be the maximal number so that $\lambda P \subseteq P'$. Then $\lambda < 1$. Since λP and P' have the same outer unit normals, there is a facet of λP that is contained in a facet of P'. Denote the outer unit normal of those facets by v_{i_1}. We have

$$\begin{aligned} h_{\lambda P}(v_{i_1}) &= h_{P'}(v_{i_1}),\\ \Delta_{i_1} &\subseteq \Delta_{i_1}',\\ \rho_{\lambda P}(u) &= \rho_{P'}(u), \qquad \text{for all } u\in\Delta_{i_1}. \end{aligned}$$

Therefore,

$$h_{\lambda P}(v_{i_1})^{-p}\int_{S^{n-1}\cap\Delta_{i_1}}\rho_{\lambda P}^q(u)\rho_Q^{n-q}(u)\,du \le h_{P'}(v_{i_1})^{-p}\int_{S^{n-1}\cap\Delta_{i_1}'}\rho_{P'}^q(u)\rho_Q^{n-q}(u)\,du, \tag{8.3}$$

with equality if and only if $\Delta_{i_1} = \Delta_{i_1}'$. This and (8.2) give that

$$\lambda^{q-p} \le 1. \tag{8.4}$$

But $\lambda < 1$ implies that $\lambda^{q-p} > 1$ if $q < p$; a contradiction.

If $q = p$, then (8.2) forces equality in (8.3). Thus, $\Delta_{i_1} = \Delta_{i_1}'$, and the facets of λP and P' with outer unit normal v_{i_1} are the same. Let v_{i_2} be the outer unit normal to a facet adjacent to the facet whose outer unit normal is v_{i_1}. Then the facet of λP with outer unit normal v_{i_2} is contained in the facet of P' with outer unit normal v_{i_2}. The same argument yields that these two facets are also the same. Continuing in this manner allows us to conclude that $\lambda P = P'$. □

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