A CALCULUS PROOF OF THE CRAMÉR–WOLD THEOREM

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(Communicated by Mark M. Meerschaert)

ABSTRACT. We present a short, elementary proof not involving Fourier transforms of the theorem of Cramér and Wold that a Borel probability measure is determined by its values on half-spaces.

1. INTRODUCTION

In this note, we give a brief and elementary proof, not involving Fourier transforms, of a theorem of Cramér and Wold.

The fundamental theorem of [Cramér and Wold (1936)] states that a Borel probability measure on Euclidean space is determined by the values it assigns to all half-spaces (equivalently, by its projections to lines through the origin). This theorem is proved easily with the aid of Fourier analysis. However, generations of probabilists have learned from the editions of the textbook of [Billingsley (1995)] that despite the elementary statement of the theorem, no proof was known that did not use Fourier transforms. That changed with the publication of [Walther (1997)], who used Gaussians. Walther's proof depends on a nice idea, but its implementation uses $1\frac{1}{2}$ pages of calculations. See Section 8.7 of [Pollard (2002)] for another presentation of Walther's proof. By contrast, our proof uses only natural constructions and avoids calculations.

A brief and somewhat inaccurate outline of our proof is the following. Using Crofton's measure on half-spaces, we show that knowledge of $\mu(S)$ for all half-spaces S determines the μ-average distance $f_\mu(x)$ to every point x. We then show that a suitable power of the Laplacian applied to f_μ yields a constant times μ. Thus, integral geometry combined with differentiation recovers μ.

2. PROOF

Let $\mathcal{S}$ be the set of closed half-spaces $S \subset \mathbb{R}^n$.

The Cramér–Wold Theorem. *Let μ and ν be Borel probability measures on $\mathbb{R}^n$ such that $\mu(S) = \nu(S)$ for all $S \in \mathcal{S}$. Then $\mu = \nu$.*

Proof. Let σ be the (infinite) Borel measure on $\mathcal{S}$ that is invariant under isometries, normalized so that

$$(2.1) \qquad \sigma\big(\{0 \in S, x \notin S\}\big) = \|x\|/2$$

for $\|x\| = 1$. The measure σ goes back to [Crofton (1868)] (in two dimensions); it can be constructed as follows. (See Theorem 5.1.1 of [Schneider and Weil (2008)] for a generalization.) Let Ω_{n-1} denote hypersurface area measure on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and let λ denote Lebesgue measure on $\mathbb{R}$. Write $\varphi \colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathcal{S}$ for the map

$$\varphi(\omega, p) := \{ x \in \mathbb{R}^n ; \ \langle \omega, x \rangle \geq p \}.$$

Then $\sigma := \alpha_n \cdot \varphi_*(\Omega_{n-1} \times \lambda)$ for some constant α_n whose value does not concern us. It is clear that σ is invariant under rotations about the origin and under reflections in hyperplanes that pass through the origin. Translation invariance amounts to the property that for $y \in \mathbb{R}^n$, the pushforward by $\varphi_y(\omega, p) := \varphi(\omega, p) - y$ is the same measure. But since

$$\varphi(\omega, p) - y = \{ x - y \in \mathbb{R}^n ; \ \langle \omega, x \rangle \geq p \} = \{ x \in \mathbb{R}^n ; \ \langle \omega, x + y \rangle \geq p \}$$
$$= \{ x \in \mathbb{R}^n ; \ \langle \omega, x \rangle \geq p - \langle \omega, y \rangle \} = \varphi(\omega, p - \langle \omega, y \rangle),$$

isometry invariance of λ gives this property. The isometry invariance of σ implies that $\sigma(\{ 0 \in S, \ x \notin S \})$ is a function of $\|x\|$ alone; additivity for collinear segments shows that it is a linear function. Thus, we may choose α_n so that (2.1) holds.

From (2.1) and isometry invariance, we have

$$\|x\| = \int_{\mathcal{S}} |1_S(0) - 1_S(x)|^2 \, d\sigma(S).$$

Integrating with respect to a signed measure μ on $\mathbb{R}^n$ with compact support, we obtain

$$\int_{\mathbb{R}^n} \|x\| \, d\mu(x) = \int_{\mathcal{S}} \int_{\mathbb{R}^n} |1_S(0) - 1_S(x)|^2 \, d\mu(x) \, d\sigma(S)$$
$$= \int_{\mathcal{S}} \Big[1_S(0)\big(1 - 2\mu(S)\big) + \mu(S) \Big] \, d\sigma(S).$$

The choice of 0 was arbitrary, so making another choice and subtracting, we get

$$\int_{\mathbb{R}^n} \big(\|y - x\| - \|x\| \big) \, d\mu(x) = \int_{\mathcal{S}} \Big[\big(1_S(y) - 1_S(0) \big)\big(1 - 2\mu(S) \big) \Big] \, d\sigma(S).$$

By taking a limit, we see that this equation holds for every finite signed measure μ.

Define

$$f_\mu(y) := \int_{\mathbb{R}^n} \big(\|y - x\| - \|x\| \big) \, d\mu(x).$$

We have shown that the function $S \mapsto \mu(S)$ determines f_μ. It remains to show that f_μ determines μ.

The idea is that if $n = 2m - 1$ is odd, then $\Delta^m f_\mu = c_m \mu$ for some constant c_m, using the fundamental solution of the Laplacian, Δ. This then establishes the Cramér–Wold theorem in odd dimensions. But since an even dimension embeds in the next higher dimension, the Cramér–Wold theorem follows in even dimensions as well. That is, we may identify a measure μ on $\mathbb{R}^{2m}$ with a measure μ' on $\mathbb{R}^{2m} \times \{0\} \subset \mathbb{R}^{2m+1}$. The function $S \mapsto \mu(S)$ on half-spaces $S \subset \mathbb{R}^{2m}$ determines the values $\mu'(S')$ for half-spaces $S' \subset \mathbb{R}^{2m+1}$. Since this determines μ', the theorem follows for μ.

We now show that $\Delta^m f_\mu = c_m \mu$ in an appropriate sense for μ on $\mathbb{R}^{2m-1}$. Recall Green's second identity, which says that for a bounded domain $D \subset \mathbb{R}^n$ with C^1

boundary ∂D having outward unit normal $\mathbf{n}$ and two functions $\phi, \psi \in C^2(\overline{D})$, we have

$$\int_D (\phi \Delta \psi - \psi \Delta \phi) = \int_{\partial D} (\phi \nabla_{\mathbf{n}} \psi - \psi \nabla_{\mathbf{n}} \phi).$$

Recall also that if $F \colon \mathbb{R}^n \to \mathbb{R}$ is such that $F(x) = G(\|x\|)$ depends only on $r := \|x\|$, then

$$(\Delta F)(x) = G''(r) + \frac{n-1}{r} G'(r).$$

In particular, $\Delta r^k = k(k+n-2)r^{k-2}$. If the support of ψ lies in the interior of a ball $B(\mathbf{0}, R)$ and $\phi(x) = r^k$ with $k > -n+2$, then letting D be $B(\mathbf{0}, R) \setminus B(\mathbf{0}, \epsilon)$ with $\epsilon \to 0$ shows that $\int_{\mathbb{R}^n} \phi \Delta \psi = \int_{\mathbb{R}^n} \psi \Delta \phi$. Similarly, if $k = -n+2$, then $\int_{\mathbb{R}^n} \phi \Delta \psi = \beta_{n-1} \psi(\mathbf{0})$, where β_{n-1} is the surface area of $\mathbb{S}^{n-1}$.

To show that f_μ determines $\int g \, d\mu$ for all $g \in C_c^\infty(\mathbb{R}^{2m-1})$, we now prove that with $c_m := 2(-2\pi)^{m-1}(2m-2)!!$, where !! denotes the double factorial, we have

$$\int g \, d\mu = c_m^{-1} \int_{\mathbb{R}^{2m-1}} f_\mu(y)(\Delta^m g)(y) \, d\lambda(y),$$

where now λ denotes Lebesgue measure on $\mathbb{R}^{2m-1}$. Fubini's theorem yields

$$\int_{\mathbb{R}^{2m-1}} f_\mu(y)(\Delta^m g)(y) \, d\lambda(y) = \int_{\mathbb{R}^{2m-1}} \int_{\mathbb{R}^{2m-1}} (\|y-x\| - \|x\|)(\Delta^m g)(y) \, d\lambda(y) \, d\mu(x).$$

Applying the preceding Green formulas (translated to x) repeatedly to the inner integral, we obtain

$$\int_{\mathbb{R}^{2m-1}} (\|y-x\| - \|x\|)(\Delta^m g)(y) \, d\lambda(y) = \int_{\mathbb{R}^{2m-1}} \Delta_y^{m-1}(\|y-x\| - \|x\|) \Delta g(y) \, d\lambda(y)$$
$$= c_m g(x),$$

as desired. $\qquad \square$

Our inversion formula $\mu = c_m^{-1} \Delta^m f_\mu$ in $\mathbb{R}^{2m-1}$ is similar to a well-known inversion formula for the Radon transform due to [Radon (1917)] and [John (1955), p. 13]: If $f \in C_c^1(\mathbb{R}^n)$, then writing $J(\omega, p) := \int_{\langle \omega, x \rangle = p} f(x) \, dx$ for the integral of f on a hyperplane perpendicular to $\omega \in \mathbb{S}^{n-1}$, we have

$$f(x) = \begin{cases} \frac{1}{2}(2\pi)^{1-n}(-\Delta_x)^{(n-1)/2} \int_{\mathbb{S}^{n-1}} J(\omega, \langle \omega, x \rangle) \, d\Omega_{n-1}(\omega) & \text{if } n \text{ is odd,} \\ -(2\pi)^{-n}(-\Delta_x)^{(n-2)/2} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \frac{dJ(\omega, p)}{p - \langle \omega, x \rangle} \, d\Omega_{n-1}(\omega) & \text{if } n \text{ is even.} \end{cases}$$

Apparently it was not realized until pointed out by [Rényi (1952)] that the theorem of [Cramér and Wold (1936)] generalized the injectivity results of Radon, John, and others.

The injectivity of the map $\mu \mapsto \int_{\mathbb{R}^n} \|x\| \, d\mu(x)$ for probability measures μ with finite first moment holds in other spaces as well. On metric spaces of negative type, it is equivalent to strong negative type. See Remark 3.4 of [Lyons (2013)] for details and references. That paper also shows its relevance to statistics.

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