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Branching Random Walks with Immigration

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Abstract. The note contains several results on the existence of limits for the first two moments of the popular model in the population dynamics: branching random walk on multidimensional lattices with immigration and infinite number of initial particles. Additional result concerns the Lyapunov stability of the moments with respect to small perturbations of the parameteres of the model such as mortality rate, the rate of the birth of $(k-1)$ offsprings and, finally, the immigration rate.

Keywords: random walks on multidimensional lattices, branching processes with continious time, contact model, immigration, correlation functions.

1. Introduction

The models we will study below give good description of the demographic situations in such European countries as Germany, Sweden, Denmark etc. However we will use the neutral terminology: particle, reactions, transformations.

2. Main section

The subject of our study is the particle field $n(t,x)$, $t \geq 0$, $x \in \mathbb{Z}^d$. We assume that $n(0,x)$ are independent identically distributed random variables with the finite exponential moment (say, Poissonian distribution with the parameter $\lambda > 0$). The evolution of the field $n(t,x)$, $t > 0$ includes several independent ingredients.

Migration (random walk). Each particle at the moment $t > 0$ in the point $x \in \mathbb{Z}^d$ spends in this point the random time τ up to the first transformation, at the moment $(t+\tau+0)$ there are several options:

1. First it can be the jump $x \to x+z$ with probability $a(z)$. We assume that $a(z) = a(-z)$ and the intencity of the jumps (diffusivity) equals $\kappa > 0$. The generator of the corresponding (underlying) random walk

has the form

$$(\mathcal{L}\psi)(x) = \kappa \sum_{z\neq 0} [\psi(x+z) - \psi(x)]a(z), \quad a(z) = a(-z), \quad \sum_{z\neq 0} a(z) = 1$$

2. Secondly, each particle can die, the **mortality rate** we denote $\mu > 0$ (i.e during time $(t, t+dt)$ particle annihilate with the probability μdt).
3. Each particle (independent on others) can produce n new particles (i.e, if you wish, the parental particle produces $n-1$ new particles and still stays in the point $x \in \mathbb{Z}^d$). Let β_n, $n \geq 0$ is the intencity of the transformation for the single parental particle into n particles. Let's introduce the corresponding **infinitesimal generating function**

$$F(z) = \mu - (\mu + \sum_{n\geq 2} \beta_n)z + \sum_{n\geq 2} \beta_n z^n$$

   We will assume that $F(z)$ is an analytic function in the circle $|z| < 1 + \delta$, $\delta > 0$ (i.e the intencities β_n as the functions of n are exponentially decreasing).

   We assume that new particles (offsprings) start their evolution from the same birth place independenlty on others (like in classical Kolmogorov-Petrovski-Piskunov paper [1])
4. The new moment in our model is the **immigration**. For any $x \in \mathbb{Z}^d$ and time interval $(t, t+dt)$ the new particle (independently on the $n(t,x)$, $x \in \mathbb{Z}^d$) can appear in the site x with probability kdt (k is the rate of immigration).

In the usual case of branching random walk (the contact model in the terminology of [2]) there is no immigration ($k \equiv 0$) and $\beta_2 = \mu$ (critical case). Then under condition of transience of the random walk with generator $\mathcal{L}$ (see [4]) there is the limiting state (steady state) $n(\infty, x)$ for $t \to \infty$. The study of this state can be based on the direct Kolmogorov equations like in [2] or on the backward equations (which are much simpler), see [4].

But in the presence of the immigration we have to use forward Kolmogorov equations. Their derivation is based on the representations:

$$n(t+dt, x) = n(t, x) + \xi(dt, x)$$

where $\xi(dt,x)$ is the random variable

$$\xi(dt,x)=\begin{cases} n-1, & \text{with probability } \beta_n n(t,x)dt,\ n\geq 3\\ 1, & \text{with probability } \beta_2 n(t,x)dt+kdt+\kappa\sum_{z\neq 0}a(-z)\times\\ & \times n(t,x+z)dt\\ -1, & \text{with probability } \mu n(t,x)dt+\kappa n(t,x)dt\\ 0, & \text{with probability } 1-\sum_{n\geq 3}\beta_n n(t,x)dt-\\ & -(\beta_2+\mu+\kappa)n(t,x)dt-kdt-\\ & -\sum_{z\neq 0}a(-z)n(t,x+z)dt \end{cases}$$

Applying the method of the conditional expectations we can derive now the equations for the first two moments $m_1(t,x)=\mathsf{E}n(t,x)$, $m_2(t,x,y)=\mathsf{E}n(t,x)n(t,y)$.

Equation for $m_1(t,x)$ has the form

$$\frac{\partial m_1}{\partial t}=\mathcal{L}m_1+(\beta-\mu)m_1+k,\quad m_1(0,x)=\mathsf{E}n(0,x)\equiv a>0$$

Here $\beta=\sum_{n\geq 2}(n-1)\beta_n$. Exactly the same equations cover the case when $\beta=\beta(x)$, $\mu=\mu(x)$, $k=k(x)$ are the bounded functions on the lattice $\mathbb{Z}^d$.

In the case of constant coefficients β, μ, k the equation for the first moment can be solved:

$$m_1(t,x)=\frac{k}{\beta-\mu}(e^{(\beta-\mu)t}-1)+e^{(\beta-\mu)t}a.$$

Thus $m_1(t,x)\to\infty$, $t\to\infty$ for $\beta\geq\mu$, $k>0$. And for $\mu>\beta$

$$m_1(t,x)\to\frac{k}{\mu-\beta},\quad t\to\infty$$

independently on the initial conditions.

The next result presents the Lyapunov stability of the first moment.

Theorem 1 *Let coefficients $\beta_n(x)$, $n\geq 2$, $\mu(x)$, $k(x)$, $x\in\mathbb{Z}^d$ are bounded and $\mu(x)-\beta(x)\geq\delta_1>0$, $k(x)\geq\delta_2>0$. Then for the bounded initial condition there exists*

$$m_1(\infty,x)=\lim_{t\to\infty}m_1(t,x)$$

Let's stress that in the co-called contact model (see [1], [4]) the limiting state exists only in the critical case $\mu(x)=\beta(x)$ and this state is unstable with respect of any sufficiently small in L^∞-norm perturbations (including random perturbations) of the parameteres of the model.

The equation for the second correlation function

$$m_2(t,x,y) = \mathsf{E}n(t,x)n(t,y)$$

is more complex:

$$\frac{\partial m_2}{\partial t} = \kappa\mathcal{L}_x m_2 + \kappa\mathcal{L}_y m_2 + 2(\beta-\mu)m_2 + 2\kappa a(x-y)\Phi(m_1) + \delta(x-y)\Psi(m_1)$$

The functions $\Phi(x)$ and $\Psi(x)$ depend linearly on the first moment, i.e can be considered as known ones.

The equation for $m_2(t,x,y)$ and the initial data (due to translation invariance of the problem) can be simplified since $m_2(t,x,y) = f(t,x-y)$ and for $z = x - y$

$$\frac{\partial f}{\partial t} = 2\mathcal{L}_z f + 2(\beta-\mu)f + s(t,z)$$

$s(t,z)$ is the known function (related to the first moment $m_1(t,z)$). If $\mu > \beta$ and $k > 0$ the second moment has a limit if $t \to \infty$ and

$$f(\infty,z) = \left(\frac{k}{\mu-\beta}\right)^2 + \tilde{f}(z)$$

and the limiting correlation function $\tilde{f}(z)$ is exponentially decreasing.

In the case of the binary branching $\beta = \beta_2$ (i.e $\beta_n = 0$, $n \geq 3$) equations for the second moment studied in [3].

We already have expression for the limit of the third correlation function $m_3(t,x_1,x_2,x_3) = \mathsf{E}n(t,x_1)n(t,x_2)n(t,x_3)$ and can prove the convergence together with asymptotic formulas for

$$m_3(\infty,x_1,x_2,x_3) = \tilde{m}(x_1,x_2,x_3).$$

3. Conclusions

The proof of all results, formulated above and their developments (including the study of the higher correlation functions and the existence uniqueness theorem for the steady state) will be published in one of the applied probability journals.

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