Degradable states and one-way entanglement distillation

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***Abstract*—We derive an upper bound on the one-way distillable entanglement of bipartite quantum states. To this end, we revisit the notion of degradable, conjugate degradable, and antidegradable bipartite quantum states [1]. We prove that for degradable and conjugate degradable states the one-way distillable entanglement is equal to the coherent information, and thus given by a single-letter formula. Furthermore, it is well-known that the one-way distillable entanglement of antidegradable states is zero. We use these results to derive an upper bound for arbitrary bipartite quantum states, which is based on a convex decomposition of a bipartite state into degradable and antidegradable states. This upper bound is always at least as good an upper bound as the entanglement of formation. Applying our bound to the qubit depolarizing channel, we obtain an upper bound on its quantum capacity that is strictly better than previously known bounds in the high noise regime. We also transfer the concept of approximate degradability [2] to quantum states and show that this yields another easily computable upper bound on the one-way distillable entanglement. Moreover, both methods of obtaining upper bounds on the one-way distillable entanglement can be combined into a generalized one.**

I. INTRODUCTION

One-way entanglement distillation is the task in which two parties (Alice and Bob) aim to convert n copies of a shared bipartite quantum state into m_n ebits using local operations and forward (or one-way) classical communication (LOCC). More precisely, given a mixed state $\rho_{AB}^{\otimes n}$, Alice and Bob's goal is to obtain, via one-way LOCC, a state that is close to $\Phi_+^{\otimes m_n}$ (with respect to a suitable distance measure), where $|\Phi_+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ denotes an ebit, i.e., a maximally entangled state of Schmidt rank 2. If there is a one-way entanglement distillation protocol such that the distance between final and target state vanishes asymptotically, then the normalized number of ebits, $\lim_{n\to\infty} m_n/n$, is called an achievable rate for one-way entanglement distillation. The *one-way distillable entanglement* $D_\to(\rho_{AB})$ is defined as the supremum over all achievable rates.

Devetak and Winter [3] proved the *hashing bound*, establishing that the *coherent information* $I(A\rangle B)_\rho$ is an achievable rate for one-way entanglement distillation:

$$D_\to(\rho_{AB}) \geq I(A\rangle B)_\rho := S(B)_\rho - S(AB)_\rho, \qquad (1)$$

where $S(A)_\rho := -\operatorname{Tr}(\rho_A \log \rho_A)$ is the von Neumann entropy. Furthermore, they derived the following regularized formula:

$$D_\to(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} D_\to^{(1)}(\rho_{AB}^{\otimes n}), \qquad (2)$$

where $D_\to^{(1)}(\rho_{AB})$ is defined as

$$D_\to^{(1)}(\rho_{AB}) := \max_T \sum_m \lambda_m I(A\rangle B)_{\rho_m},$$

and the maximization is over *quantum instruments* $T\colon A \to A'M$ with $T(\cdot) := \sum_m T_m(\cdot) \otimes |m\rangle\langle m|_M$, where $\{|m\rangle\}_m$ is an orthonormal basis for the classical register M, for each m the map $T_m\colon A \to A'$ is completely positive, and $\sum_m T_m$ is trace-preserving. We set $\rho_m := \frac{1}{\lambda_m}(T_m \otimes \mathrm{id}_B)(\rho_{AB})$, with $\lambda_m := \operatorname{Tr}(T_m(\rho_{AB}))$ denoting the probability of obtaining the outcome m of T.

As in the case of the quantum capacity, the regularization in (2) renders the distillable entanglement intractable to compute in most cases. Hence, it is desirable to identify classes of bipartite states for which (2) reduces to a single-letter formula that can be computed efficiently. Moreover, we are interested in useful upper bounds on $D_\to(\rho_{AB})$ for arbitrary bipartite states. In the present work, we address both problems. We prove that for a degradable bipartite quantum state [1], defined in Section II, the one-way distillable entanglement is equal to its coherent information, and this result can be extended to conjugate degradable states [4]. Moreover, it is well-known that the one-way distillable entanglement of antidegradable states is equal to zero.

We use these results to derive a generic upper bound on the one-way distillable entanglement of arbitrary bipartite states, starting from a decomposition of the latter into degradable and antidegradable states. Our upper bound is always less than or equal to the entanglement of formation $E_F(\cdot)$, a known upper bound on $D_\to(\cdot)$ [5]. We apply our result to the Choi state τ of the qubit depolarizing channel $\mathcal{D}_p$ for noise parameter $p \in [0, 0.25]$, obtaining an upper bound on its quantum capacity $Q(\mathcal{D}_p) = D_\to(\tau)$ that is tighter than the best known upper bound obtained by Sutter et al. [2] in the high-noise regime. We also introduce and discuss the notion of approximately (anti)degradable states, which is inspired by and analogous to the notion of approximately degradable

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quantum channels in [2]. Approximate degradability leads to an alternative upper bound on the one-way distillable entanglement. Moreover, both approaches (decomposition into (anti)degradable parts and approximate degradability) can be combined into a generalized method. For a more detailed discussion including proofs of the following results, we refer to the full version [6] available on the arXiv.

II. Degradable and antidegradable states

The central objects in our discussion are degradable and antidegradable bipartite quantum states. In analogy to degradable quantum channels [7], we call a bipartite state ρ_{AB} with purification $|\phi\rangle_{ABE}$ and 'complementary' state $\rho_{AE} = \operatorname{Tr}_B \phi_{ABE}$ *degradable*, if there is a quantum operation $\mathcal{D}\colon B \to E$ such that

$$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB}). \tag{3}$$

Equivalently, ρ_{AB} is degradable, if there is an isometry $U\colon \mathcal{H}_B \to \mathcal{H}_{E'} \otimes \mathcal{H}_G$ with $\mathcal{H}_{E'} \cong \mathcal{H}_E$ such that for the state $|\varphi\rangle_{AE'GE} = U|\phi\rangle_{ABE}$ we have $\varphi_{AE} = \varphi_{AE'} = \phi_{AE}$. A state is called *conjugate degradable*, if the degradability condition holds up to complex conjugation, that is, $\varphi_{AE} = \phi_{AE} = \mathcal{C}(\varphi_{AE'})$, where $\mathcal{C}$ denotes entry-wise complex conjugation with respect to a fixed basis of $E' \cong E$. Finally, ρ_{AB} is called *antidegradable*, if there is an isometry $V\colon \mathcal{H}_E \to \mathcal{H}_{B'} \otimes \mathcal{H}_F$ with $\mathcal{H}_{B'} \cong \mathcal{H}_B$ such that for the state $|\psi\rangle_{ABB'F} = V|\phi\rangle_{ABE}$ we have $\psi_{AB'} = \psi_{AB} = \phi_{AB}$.

III. Main result

As mentioned in Section I, for (conjugate) degradable states ρ_{AB} the one-way distillable entanglement $D_{\to}(\cdot)$ is equal to the coherent information: $D_{\to}(\rho_{AB}) = I(A\rangle B)_\rho$. Antidegradable states σ_{AB} are useless for one-way entanglement distillation: $D_{\to}(\sigma_{AB}) = 0$. Furthermore, an adaption of the "additivity implies convexity"-argument by Wolf and Pérez-García [8] shows that $D_{\to}(\cdot)$ is convex on mixtures of degradable and antidegradable states.

We can use these results to derive a general upper bound on $D_{\to}(\rho_{AB})$ for bipartite states ρ_{AB} with a decomposition of the form

$$\rho_{AB} = \sum_{i=1}^{k} p_i \rho_i + \sum_{i=k+1}^{l} p_i \sigma_i, \tag{4}$$

where the states ρ_i are degradable, and σ_i are antidegradable. The following constitutes our main result:

Theorem 1. *Let ρ_{AB} be a bipartite state. Then*

$$D_{\to}(\rho_{AB}) \leq \min \sum_{i=1}^{k} p_i I(A\rangle B)_{\rho_i} \leq E_F(\rho_{AB}),$$

where the minimization is over all decompositions of ρ_{AB} of the form in (4).

In Theorem 1, $E_F(\cdot)$ denotes the entanglement of formation [5], [9], defined as

$$E_F(\rho_{AB}) := \min_{\{p_i, \psi^i_{AB}\}_i} \sum\nolimits_i p_i S(\psi^i_A),$$

where the minimization is over all pure-state ensembles $\{p_i, \psi^i_{AB}\}_i$ satisfying

$$\rho_{AB} = \sum\nolimits_i p_i |\psi^i\rangle\langle\psi^i|_{AB}.$$

IV. Upper bound on the quantum capacity of the depolarizing channel

Our main result, Theorem 1, can be used to obtain an upper bound on the quantum capacity of the *qubit depolarizing channel* $\mathcal{D}_p$, defined for $p \in (0,1)$ and a state ρ of a qubit as

$$\mathcal{D}_p(\rho) := (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z),$$

where X, Y, and Z are the Pauli operators.

We set $\mathcal{H}_A = \mathcal{H}_{A'} = \mathbb{C}^2$, and denote by $\tau_{A'A} := (\mathrm{id}_{A'} \otimes \mathcal{D}_p)(\Phi_{A'A})$ the Choi state of the depolarizing channel. Bennett et al. [5] proved that the quantum capacity $Q(\mathcal{D}_p)$ of the depolarizing channel is equal to the one-way distillable entanglement $D_{\to}(\tau_{A'A})$ of its Choi state:

$$Q(\mathcal{D}_p) = D_{\to}(\tau_{A'A}).$$

Hence, upper bounds on $Q(\mathcal{D}_p)$ can be obtained from upper bounds on $D_{\to}(\tau_{A'A})$ via Theorem 1, for which the starting point is a decomposition of a bipartite state into degradable and antidegradable states as in (4). To obtain such a decomposition of $\tau_{A'A}$, we make use of a result by Wolf and Pérez-García [8], which states that all qubit-qubit quantum channels with a qubit environment are either degradable or antidegradable. This is easily extended to quantum states: every qubit-qubit quantum state of rank 2 is either degradable or antidegradable. Decompositions of $\tau_{A'A}$ into degradable and antidegradable states can therefore be obtained from the following procedure: We first decompose $\tau_{A'A}$ into $2k$ pure states for some $k \in \mathbb{N}$,

$$\tau_{A'A} = \sum_{i=1}^{2k} p_i \psi_i. \tag{5}$$

Note that every $2k \times 2k$ unitary matrix gives rise to such a pure-state decomposition [10]. We then group two of them at a time together into k states ω_i of rank 2:

$$\tau_{A'A} = \overbrace{(p_1 + p_2)}^{=:q_1} \underbrace{\left(\frac{p_1}{p_1+p_2}\psi_1 + \frac{p_2}{p_1+p_2}\psi_2\right)}_{=:\omega_1} + \dots,$$

which leads to a decomposition of $\tau_{A'A}$ of the form

$$\tau_{A'A} = \sum_{j=1}^{k} q_j \omega_j, \tag{6}$$

where for $j = 1, \dots, k$ we define $q_j := p_{2j-1} + p_{2j}$ and

$$\omega_j := \frac{p_{2j-1}}{q_j}\psi_{2j-1} + \frac{p_{2j}}{q_j}\psi_{2j}.$$

Hence, Theorem 1 yields the following upper bound on $D_{\to}(\tau_{A'A}) = Q(\mathcal{D}_p)$:

$$Q(\mathcal{D}_p) = D_{\to}(\tau_{A'A}) \leq \min_U \sum_{j\colon \omega_j \text{ deg.}} q_j I(A'\rangle A)_{\omega_j}, \tag{7}$$

where the minimization is over all $2k \times 2k$ unitary matrices U determining the pure-state decomposition (5), and the sum is over all j such that ω_j is degradable.

We applied the method outlined above to $\mathcal{D}_p$ with $p \in [0, 1/4]$. To obtain (6), we chose $k = 4$ and generated an 8×8 unitary at random in MATLAB. We then used the MATLAB built-in function `fmincon` to minimize (7) over all 8×8 unitaries. This process was repeated 500 times to avoid local minima. The resulting upper bound on $Q(\mathcal{D}_p)$ is shown as the red line in Figure 1 below. We also compare it to the best known upper bound on $Q(\mathcal{D}_p)$ (blue) obtained by Sutter et al. [2] (see also [11]), demonstrating that our bound obtained from Theorem 1 is strictly better in the high noise regime ($p \gtrsim 0.069$). The channel coherent information

$$Q^{(1)}(\mathcal{D}_p) := \max_{|\phi\rangle_{A'A}} I(A'\rangle A)_{(\mathrm{id}_{A'} \otimes \mathcal{D}_p)(\phi_{A'A})}, \tag{8}$$

which is a lower bound to $Q(\mathcal{D}_p)$ by the hashing inequality (1), is plotted as a dashed line.

The improvement of the upper bound to $Q(\mathcal{D}_p)$ stems from the fact that our method can be understood as a generalization of the additive extensions method of [11] (the main contribution for the bound derived in [2] for high values of p), which is based on a decomposition of the channel into degradable channels. The corresponding decomposition of the Choi state is a decomposition into degradable states as in (4), and hence a candidate for the minimization in Theorem 1. Our method yields a tighter bound, since we minimize over all decompositions into degradable states not necessarily corresponding to Choi states of quantum channels. For example, the eigenvectors of the Choi state τ_{AB} provide a degradable decomposition of τ_{AB}, whereas there seems to be no corresponding degradable decomposition for channels.

V. Approximate degradability and a generalized method

In [2], the authors introduced the concept of an approximate degradable quantum channel and used it to derive computable upper bounds on the quantum capacity of a given quantum channel. In analogy to [2], we can define the corresponding notion of an approximate (anti)degradable state. A state is called ε-degradable if the degradability condition (3) is only satisfied up to a parameter ε:

$$\min_{\mathcal{D}^{B\to E}} \frac{1}{2} \left\|\rho_{AE} - \mathcal{D}^{B\to E}(\rho_{AB})\right\|_1 \leq \varepsilon. \tag{9}$$

We define the *degradability parameter* $\mathrm{dg}(\rho_{AB})$ as the left-hand side of (9). Similarly, the *antidegradability parameter* $\mathrm{adg}(\rho_{AB})$ of a state ρ_{AB} is defined as

$$\mathrm{adg}(\rho_{AB}) := \min_{\mathcal{A}^{E\to B}} \frac{1}{2} \left\|\rho_{AB} - \mathcal{A}^{E\to B}(\rho_{AE})\right\|_1, \tag{10}$$

and we call a state ε-antidegradable, if $\mathrm{adg}(\rho_{AB}) \leq \varepsilon$. Both parameters can be efficiently computed, since they can be obtained as the solutions of semidefinite programs (SDPs):

Lemma 2. *$\mathrm{dg}(\rho_{AB})$ is the solution of the SDP*

$$\begin{aligned} \text{min.:} & \frac{1}{4}(\operatorname{Tr} X_{AE} + \operatorname{Tr} Y_{AE}) \\ \text{subj. to:} & \begin{pmatrix} X_{AE} & Z_{AE} - \rho_{AE} \\ Z_{AE} - \rho_{AE} & Y_{AE} \end{pmatrix} \geq 0 \\ & \tau_{B'E} \geq 0 \\ & \tau_{B'} = I_B \\ & X_{AE}, Y_{AE} \geq 0, \end{aligned}$$

where $Z_{AE} = \operatorname{Tr}_{B'}\left[\left(\rho_{AB'}^{T_B} \otimes I_E\right)(I_A \otimes \tau_{B'E})\right]$ with $B' \cong B$ and $\rho_{AB'} = \rho_{AB}$, and where $\tau_{B'E}$ is the Choi state of the CPTP map $\mathcal{D}^{B\to E}$ over which we optimize in (9).

Similarly, $\mathrm{adg}(\rho_{AB})$ *is the solution of the SDP*

$$\begin{aligned} \text{min.:} & \frac{1}{4}(\operatorname{Tr} X_{AB} + \operatorname{Tr} Y_{AB}) \\ \text{subj. to:} & \begin{pmatrix} X_{AB} & W_{AB} - \rho_{AB} \\ W_{AB} - \rho_{AB} & Y_{AB} \end{pmatrix} \geq 0 \\ & \tau_{E'B} \geq 0 \\ & \tau_{E'} = I_{E'} \\ & X_{AB}, Y_{AB} \geq 0, \end{aligned}$$

where $W_{AB} = \operatorname{Tr}_{E'}\left[\left(I_B \otimes \rho_{AE}^{T_E}\right)(I_A \otimes \tau_{E'B})\right]$ with $E' \cong E$ and $\rho_{AE'} = \rho_{AE}$, and where $\tau_{E'B}$ is the Choi state of the CPTP map $\mathcal{A}^{E\to B}$ over which we optimize in (10).

The notion of approximate degradability leads to an efficiently computable upper bound on the one-way distillable entanglement, which mirrors the corresponding result for quantum channels in [2] and can be proven along similar lines:

Theorem 3. *Let ρ_{AB} be a bipartite state with purification $|\phi\rangle_{ABE}$, and $\delta > 0$ be such that $\mathrm{dg}(\rho_{AB}) \leq \delta$. Then,*

$$\begin{aligned} I(A\rangle B)_\rho &\leq D_\to(\rho_{AB}) \\ &\leq I(A\rangle B)_\rho + 4\delta \log |E| + 2(1+\delta)\, h\!\left(\frac{\delta}{1+\delta}\right), \end{aligned}$$

where $h(\cdot)$ denotes the binary entropy.

There is a generalized method of finding upper bounds on the one-way distillable entanglement (resp. the quantum capacity of a teleportation-simulable channel) that encompasses both approximate degradability (AD) bounds and additive extension (AE) bounds. The former provides the best upper bounds for very low noise, while the latter does best for higher noise levels (cf. Figure 1). By searching for approximately degradable extensions of quantum states (or channels, for that matter) we can do no worse than either of these two known methods. In detail, the two methods can be combined as follows.

For a given bipartite state ρ_{AB}, fix $k \in \mathbb{N}$ and consider an extension $\tilde{\rho}_{ABC}$ with $|C| = k$, such that $\operatorname{Tr}_C \tilde{\rho}_{ABC} = \rho_{AB}$. We assume C to be in Bob's possession, and consider entanglement distillation with respect to the $A|BC$ bipartition in the following. Computing the degradability parameter $\varepsilon = \mathrm{dg}(\tilde{\rho}_{ABC})$ of this extension and using Theorem 3, we obtain an upper bound on the one-way distillable entanglement

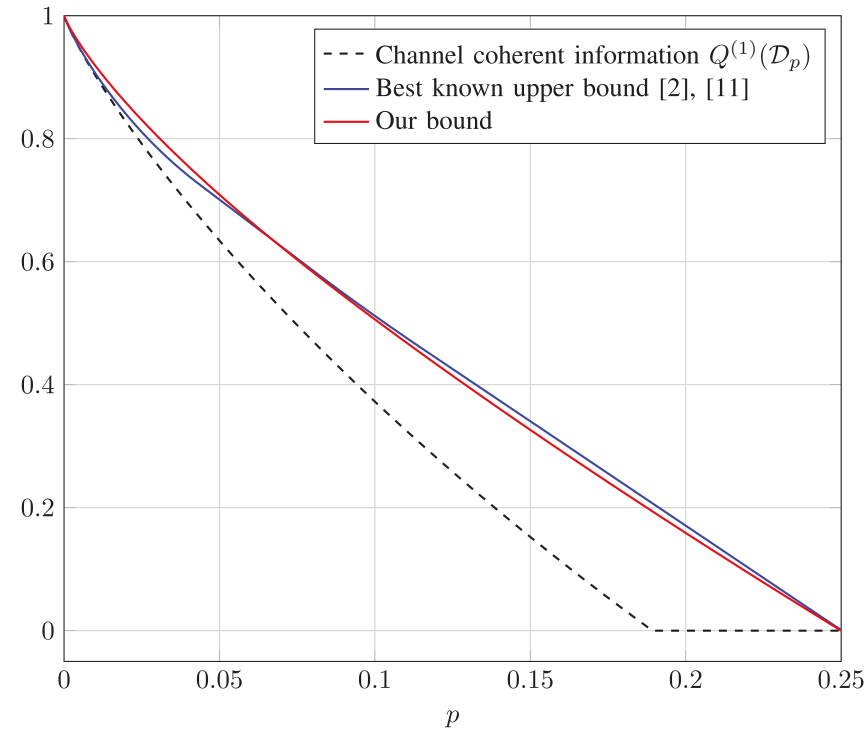

Fig. 1. Upper and lower bounds on the quantum capacity $Q(\mathcal{D}_p)$ of the depolarizing channel for the interval $p \in [0, 0.25]$. The hashing bound [3] yields the channel coherent information $Q^{(1)}(\mathcal{D}_p)$ defined in (8) as a lower bound on $Q(\mathcal{D}_p)$ (dashed). Our upper bound (red) obtained by the method outlined in Section IV is compared to the upper bound obtained in [2], [11] (blue).

$D_{\to}(\tilde{\rho}_{ABC})$ of $\tilde{\rho}_{ABC}$, which in turn is an upper bound on $D_{\to}(\rho_{AB})$. We can then optimize this bound over all extensions $\tilde{\rho}_{ABC}$ with $|C| = k$. Restricting to trivial extensions of ρ_{AB}, this bound reduces to the AD bound (Theorem 3). Restricting to 'flagged' (anti)degradable extensions of the form

$$\tilde{\rho}_{ABC} = \sum_{c=1}^{k} \tilde{\rho}_{AB}^{c} \otimes |c\rangle\langle c|_C,$$

where the states $\tilde{\rho}_{AB}^c$ are either degradable or antidegradable, the bound reduces to the AE bound (Theorem 1). In this case, we have $\rho_{AB} = \sum_c \tilde{\rho}_{AB}^c$. We leave a thorough numerical investigation of this generalized method to future work.

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