Nonsmooth Barrier Functions with Applications to Multi-Robot Systems

Paul Glotfelter, Jorge Cortés, and Magnus Egerstedt

Abstract—**As multi-agent systems become more wide-spread and versatile, the ability to satisfy multiple system-level constraints grows increasingly important. In applications ranging from automated cruise control to safety in robot swarms, barrier functions have emerged as a tool to provably meet such constraints by guaranteeing forward invariance of desirable sets. However, satisfying multiple constraints typically implies formulating multiple barrier functions, which would be ameliorated if the barrier functions could be composed together as Boolean logic formulae. The use of $\max$ and $\min$ operators, which yields nonsmooth functions, represents one path to accomplish Boolean compositions of barrier functions, and this work extends previously established concepts for barrier functions to a class of nonsmooth barrier functions that operate on systems described by differential inclusions. We validate our results by deploying Boolean compositions of nonsmooth barrier functions onto a team of mobile robots.**

Index Terms—**Robotics, Lyapunov methods, autonomous systems**

I. INTRODUCTION

NUMEROUS applications utilize multi-agent systems to achieve objectives in a robust and decentralized manner, including rendezvous, where agents must meet in a decentralized fashion; coverage control, in which agents must cover an area of importance; and flocking, which mimics biological systems (e.g., [1], [2], [3]). As the number of agents increases, accomplishing objectives while satisfying multiple system-level constraints becomes a concern. For example, collision avoidance and connectivity maintenance typically must be ensured throughout the maneuver (e.g., [4]), which translates into the constraints that agents do not collide and do not lose connectivity. As such, the ability to provably guarantee the satisfaction of multiple constraints grows increasingly relevant.

Recently, [5] utilized barrier functions for constraint satisfaction by ensuring forward invariance of a set that encodes such safety requirements, and, subsequently, barrier functions have been used to encode a variety of system constraints across different domains, such as adaptive cruise control [5], [6], collision avoidance for ground vehicles [7], unmanned aerial vehicles [8], and remote-access robotics testbeds [9].

The above-referenced literature on barrier functions addresses a single, sufficiently smooth barrier function that operates on a continuous dynamical system. Recently, [10] achieves a form of Boolean composition through products and sums of barrier functions. However, the construction in [10] forgoes the robustness qualities of the zeroing barrier functions in [6] and restricts the system to lie strictly in the interior of the invariant set. In this paper, we retain the robustness properties associated with zeroing barrier functions (see [6]) while supporting Boolean composition of barrier functions by utilizing $\max$ and $\min$ operators of multiple component barrier functions. However, the use of $\max$ and $\min$ operators introduces points of nondifferentiablity into the composite barrier functions, preventing the existing results from applying. Though not considered with regard to barrier functions, nonsmooth Lyapunov functions have been extensively studied (e.g., [11], [12], [13], [14]). The tools developed for nonsmooth Lyapunov functions will also prove highly useful for Nonsmooth Barrier Functions (NBFs), and in this paper, we show how to extend the previously established concepts within the smooth barrier function literature to a rich class of NBFs.

It should be noted that NBFs are not the only possible tools for composition of system-level constraints in multi-agent systems. For example, potential functions and Lyapunov-like barrier functions represent an approach that also permits some degree of composition [15], [16], [17]. The major difference between this work and these other approaches lies in the fact that the work in this paper explicitly allows for guaranteed Boolean composition of these objects (i.e., composition with Boolean $\wedge$, $\vee$, $\neg$ operators).

Additionally, the above-mentioned prior methods are often formulated with respect to a particular task (e.g., obstacle avoidance) or a particular dynamical system (e.g., differential drive robots). Another strength of this work is that the NBF framework is mathematically agnostic to the particular task under consideration.

This work provides three main results with experimental validation. First, this work presents a framework that permits the application of NBFs to a class of systems described by differential inclusions. Second, this work addresses some computational requirements imposed by the nonsmooth nature of the NBF framework, demonstrating that validation of NBFs can be feasibly performed under certain assumptions. Third, Boolean compositional NBFs are achieved via $\max$ and $\min$ operators and are formulated as Quadratic Programs (QPs).

This article unfolds as follows. Sec. II covers background material regarding differential inclusions and discusses some tools from nonsmooth analysis. Sec. III applies these concepts to NBFs for dynamical systems that are described by differential inclusions and introduces convenient computational

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methods to check whether a candidate function is a valid NBF. Sec. IV considers a special case of the results in Sec. III to compose a number of barrier functions with Boolean logic via $\min$ and $\max$ operations. Finally, Sec. V shows the successful deployment of a Boolean compositional NBF onto a team of mobile robots.

II. Background Material

This section introduces notation and background material, including generalized gradients, differential inclusions, and set-valued Lie derivatives. These tools are necessary to deal properly with the nondifferentiable points of NBFs.

A. Notation

We denote by $\mathbb{R}_{\geq 0}$ the set of nonnegative real numbers. For an integer $k > 0$, we use the shorthand notation $[k]$ to denote the set $\{1, \ldots, k\}$. The symbol $\circ$ denotes function composition. The abbreviation $a.e.$ stands for almost everywhere in the sense of Lebesgue measure. The expression $\langle \cdot, \cdot \rangle$ represents the inner product of two vectors. The abbreviation co stands for the convex hull of a set. A function $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ belongs to extended class-$\mathcal{K}$ if $\overline{\alpha}$ is continuous, strictly increasing, and $\overline{\alpha}(0) = 0$. The function $\overline{\alpha}$ is a class-$\mathcal{K}$ function when restricted to $\mathbb{R}_{\geq 0}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class-$\mathcal{KL}$ if it is class-$\mathcal{K}$ in its first argument and, for each fixed r, $\beta(r, \cdot)$ is continuous, strictly decreasing, and $\lim_{s\to\infty} \beta(r, s) = 0$.

B. Differential Inclusions

Differential inclusions have emerged as a tool to analyze certain types of dynamical systems. For example, differential equations with discontinuous right-hand sides have been extensively studied (e.g., in [18]) by transforming the discontinuous differential equation into a differential inclusion.

When formulating NBFs, we allow for applications to differential inclusions, potentially facilitating forward-set-invariance analysis of such systems; though, these results also apply to systems modeled by continuous differential equations. Given a set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, consider the differential inclusion represented by

$$\dot{x}(t) \in F(x(t)). \tag{1}$$

We assume that F is locally bounded; upper semi-continuous (see [19, Sidebar 7]); and takes nonempty, compact, convex values. These properties ensure the existence (but not uniqueness) of solutions to (1) (see [19, Prop. S1]). A Carathéodory solution to (1) is an absolutely continuous trajectory $x : [0, t_1] \to \mathcal{D} \subset \mathbb{R}^n$ such that $\dot{x}(t) \in F(x(t)),\ a.e.\ t \in [0, t_1]$, $x(0) = x_0$, with $\mathcal{D}$ an open, connected set and $0 < t_1$. Later references to solutions to (1) always assume this definition.

In general, this article focuses on guaranteeing that a set is forward invariant with respect to a differential inclusion, meaning that every solution that starts in the set stays in the set. This notion of forward invariance has been called strong forward invariance in other work (cf. [19]). This article simply refers to this property as forward invariance.

Definition 1. *A set $\mathcal{C}$ is forward invariant, with respect to* (1)*, if $x(0) \in \mathcal{C}$ implies that $x(t) \in \mathcal{C}$, for every $t \in [0, t_1]$ and for every Carathéodory solution of* (1) *starting from $x(0)$.*

C. Nonsmooth Analysis

Here, we review some basic notions on nonsmooth analysis that are necessary to analyze the nonsmooth functions that result from applying $\max$ and $\min$ operators to smooth functions (e.g., $|x| = \max\{-x, x\}$). The generalized gradient of locally Lipschitz functions is a tool that deals with the nondifferentiable points of nonsmooth functions [20]. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz near x is there exist $\delta, L > 0$ such that $\|f(x_1) - f(x_2)\| \leq L\|x_1 - x_2\|$, for every $x_1, x_2 \in B(x, \delta)$. If a function is Lipschitz near every point in its domain of definition, we refer to the function as locally Lipschitz. Next, we define the generalized gradient.

Theorem 1 ([20, Theorem 2.5.1])**.** *Let f be Lipschitz near x, and suppose S is any set of Lebesgue measure zero in $\mathbb{R}^n$. Then, the generalized gradient of a function $\partial f(x)$ is*

$$\partial f(x) = co\{\lim_{i\to\infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \ \cup \ \Omega_f\},$$

where Ω_f represents the zero-measure set where f is nondifferentiable.

Often, some regularity is assumed to imbue the generalized gradient with some desirable properties.

Definition 2 ([20, Definition 2.3.4])**.** *A function f is regular at x provided that for all $v \in \mathbb{R}^n$, the one-sided directional derivative $f'(x;\ v) = \lim_{h\downarrow 0} h^{-1}(f(x+hv) - f(x))$ exists and that $f'(x; v) = f^{\circ}(x; v)$, where the generalized directional derivative $f^{\circ}(x; v)$ is given by*

$$f^{\circ}(x; v) = \limsup_{\substack{y\to x\\ h\downarrow 0}} \frac{f(y + hv) - f(y)}{h}.$$

If the component functions are regular, the generalized gradient of their point-wise $\max$ or $\min$ can be easily computed, as the next result shows.

Proposition 2 ([20, Proposition 2.3.12])**.** *Let $\{f_i\}$ be a finite collection of functions ($i = 1, 2, \ldots, k$) Lipschitz near x. Then, the function f defined by*

$$f(x') = \max_{i\in[k]}\{f_i(x')\}$$

is Lipschitz near x as well. Let $I(x')$ denote the set of indices i for which $f_i(x') = f(x')$. Then,

$$\partial f(x) \subset co\{\partial f_i(x) \mid i \in I(x)\},$$

and if f_i is regular at x for each $i \in I(x)$, then equality holds; and f is regular at x.

This property becomes of particular interest when considering Boolean compositions of NBFs in Sec. IV. In particular, Prop. 2 implies that the behavior of the generalized gradients of the component functions encapsulates the behavior of the generalized gradient of the $\max$ (or $\min$).

D. Set-Valued Lie Derivatives

Following [19], this section formulates set-valued Lie derivatives for nonsmooth functions with respect to systems described by differential inclusions. Set-valued Lie derivatives encapsulate the behavior of these nonsmooth functions by combining possible directions between the generalized gradient and the differential inclusion. In [11], these objects are used to analyze nonsmooth Lyapunov functions; however, the same tool may be applied to NBFs. The authors of [11] introduce the following strong version of a set-valued Lie derivative.

Lemma 2.1 ([11, Lemma 1])**.** *Let $x : [0, t_1] \to \mathcal{D} \subset \mathbb{R}^n$ be a Carathéodory solution to* (1)*, and let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz, regular function. Then, $[0, t_1] \ni t \mapsto \dot{h}(x(t))$ is absolutely continuous, and*

$$\dot{h}(x(t)) \in \mathcal{L}_F^S h(x(t)), \ a.e. \ t \in [0, t_1], \tag{2}$$

where, for each $x' \in \mathcal{D}$,

$$\mathcal{L}_F^S h(x') = \{a \in \mathbb{R} \mid \exists \ v \in F(x') \ s.t. \ \langle \xi , v \rangle = a, \ \forall \ \xi \in \partial h(x')\}.$$

Interestingly, the work [12] extends the strong set-valued Lie derivative of Lem. 2.1 to the larger class of so-called non-pathological functions, a class that contains regular functions as a subset.

Remark 2.1. *If the regularity assumption on h is removed from the hypothesis of Lemma 2.1, then* (2) *still holds with the weaker set-valued Lie derivative defined by*

$$\mathcal{L}_F^W h(x') = \{a \in \mathbb{R} \mid \exists \ v \in F(x'), \exists \ \xi \in \partial h(x') \ s.t. \ \langle \xi , v \rangle = a\}, \tag{3}$$

for each $x' \in \mathcal{D}$. This statement follows from [20, Prop. 2.2.2]. •

Regarding Rem. 2.1, the weak set-valued Lie derivative generates substantially more values than the strong set-valued Lie derivative but only requires a locally Lipschitz assumption. As such, the weak set-valued Lie derivative lends itself to the Boolean composition of barrier functions (see Sec. IV), as the regularity property is not necessarily preserved through nested compositions of $\max$ and $\min$ operators (e.g., a point-wise minimum of point-wise maximums). This condition occurs because regularity of some function f does not imply that $-f$ is regular.

III. Nonsmooth Barrier Functions

This section contains the main results of the paper. Initially, the section introduces the definitions of candidate and valid NBFs and then provides sufficient conditions to guarantee the forward-set-invariance properties of NBFs. Finally, this segment discusses useful computational methods to check these conditions.

A. Candidate and Valid Nonsmooth Barrier Functions

Here, we define the concepts of candidate and valid NBFs. Note that, in Def. 3, the function h is not necessarily differentiable. Importantly, if a candidate NBF is a valid NBF, then the set $\mathcal{C}$, as in Def. 3, is forward invariant. Valid and candidate NBFs are defined as follows.

Definition 3. *A continuous function $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, where $\mathcal{D}$ is an open, connected set, is a candidate NBF if the set $\mathcal{C} = \{x' \in \mathcal{D} \mid h(x') \geq 0\}$ is nonempty.*

Definition 4. *A continuous candidate NBF $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is a valid NBF for* (1) *if $x(0) \in \mathcal{C}$ implies that there exists a class-$\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$h(x(t)) \geq \beta(h(x(0)), t), \ \forall \ t \in [0, t_1],$$

for all Carathéodory solutions $x : [0, t_1] \to \mathbb{R}^n$ of (1) *starting from $x(0)$.*

B. Sufficient Conditions for Valid NBFs

This section provides sufficient conditions that allow us to determine whether a candidate NBF is in fact a valid NBF. Toward this end, the following result will be useful.

Lemma 2.2. *Let $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, extended class-$\mathcal{K}$ function and $h : [0, t_1] \to \mathbb{R}$ be an absolutely continuous function. If $\dot{h}(t) \geq -\overline{\alpha}(h(t))$, for almost every $t \in [0, t_1]$, and $h(0) \geq 0$, then there exists a class-$\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $h(t) \geq \beta(h(0), t)$, and $h(t) \geq 0, \ \forall \ t \in [0, t_1]$.*

Proof. To prove this result, we utilize a differential inequality. Toward this end, let

$$\dot{z}(t) = -\overline{\alpha}(z(t)), \ z(0) = h(0).$$

Because $\overline{\alpha}$ is locally Lipschitz, solutions $z(t)$ exist and are unique, and since $z(0) \geq 0$ and the restriction of an extended class-$\mathcal{K}$ function to $\mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, the solution $z(t)$ is a class-$\mathcal{KL}$ function β such that

$$z(t) = \beta(z(0), t).$$

Therefore, the solution $z(t)$ is valid over $[0, t_1]$. Then, because

$$\dot{h}(t) \geq -\overline{\alpha}(h(t)), \ a.e. \ t \in [0, t_1],$$

$h(t) \geq z(t)$, $\forall t \in [0, t_1]$, by [21, Thm. 1.10.2]. Thus,

$$h(t) \geq \beta(h(0), t), \ \forall \ t \in [0, t_1],$$

proving the first claim. Because β is a class-$\mathcal{KL}$ function, $\beta(h(0), t) \geq 0, \ \forall \ t \in [0, t_1]$; thus, $h(t) \geq 0, \ \forall t \in [0, t_1]$. □

The following result states a sufficient condition for a candidate NBF to be valid in terms of its strong set-valued Lie derivative when evaluated along solutions to (1).

Theorem 3. *Let $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz, regular function that is a candidate NBF. If there exists a locally Lipschitz extended class-$\mathcal{K}$ function $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ such that the strong set-valued Lie derivative satisfies*

$$\min \mathcal{L}_F^S h(x') \geq -\overline{\alpha}(h(x')), \ \forall \ x' \in \mathcal{D}, \tag{4}$$

then h is a valid NBF for (1)*.*

Proof. Let $x(0) \in \mathcal{C}$. By Lem. 2.1, each solution of (1) satisfies

$$\dot{h}(x(t)) \in \mathcal{L}_F^S h(x(t)), \ a.e. \ t \in [0, t_1].$$

Thus, at $a.e.$ $t \in [0, t_1]$

$$\dot{h}(x(t)) \geq \min \mathcal{L}_F^S h(x(t)) \geq -\overline{\alpha}(h(x(t))).$$

This condition implies that at $a.e.$ $t \in [0, t_1]$

$$\frac{d}{dt}(h \circ x)(t) \geq -\overline{\alpha}((h \circ x)(t)),$$

when $h \circ x$ is viewed as a function of t. Since $x(0) \in \mathcal{C}$, $(h \circ x)(0) \geq 0$. Directly applying Lem. 2.2 yields that h is a valid NBF, as defined in Def. 4. □

Remark 3.1. *The same result holds if we remove the assumption that* h *is regular and instead the inequality* (4) *holds with the weak set-valued Lie derivative* $\mathcal{L}_F^W h$ *defined in* (3). •

Remark 3.2. *By a similar argument, if* $x(0) \in \mathcal{D} - \mathcal{C}$ *(i.e.,* $h(0) < 0$*) and the solution exists for all* $t \in [0, \infty)$*, then we may show that* $-h(x(t)) \leq \beta(-h(x(0)), t)$ *(i.e., that* $x(t)$ *asymptotically returns to* $\mathcal{C}$*).* •

As the eventual goal of this work is to apply NBFs to a group of mobile robots, the computational requirements of verifying the NBF inequality conditions become a concern. Toward this end, the following property of the usual inner product on two convex hulls becomes of use. In the interest of space efficiency, we omit this proof and note that it follows from Caratheódory's theorem for convex hulls.

Lemma 3.1. *Let* $\bar{A} \subset co\, A \subset \mathbb{R}^n$*,* $\bar{B} \subset co\, B \subset \mathbb{R}^n$*. If for every* $a \in A$*,* $b \in B$*,* $\langle a\,, b \rangle \geq c,\ c \in \mathbb{R}$*, then for every* $\bar{a} \in \bar{A}$*,* $\bar{b} \in \bar{B}$*,* $\langle \bar{a}\,, \bar{b} \rangle \geq c$*.*

Next, we present the second of this article's main results. We omit the proof and note that it follows from Lem. 3.1 and the version of Thm. 3 described in Rem. 3.1.

Theorem 4. *Let* $h : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ *be locally Lipschitz function which is a candidate NBF. Let* $\mathcal{E}_f,\ \mathcal{E}_h : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *be set-valued maps such that*

$$F(x') \subset co\, \mathcal{E}_f(x'), \quad \partial h(x') \subset co\, \mathcal{E}_h(x'),$$

for all $x' \in \mathcal{D}$*. If there exists a locally Lipschitz extended class-*$\mathcal{K}$ *function* $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ *such that for every* $x' \in \mathcal{D}$*,* $\xi \in \mathcal{E}_h(x')$*, and* $v \in \mathcal{E}_f(x')$*,*

$$\langle \xi\,, v \rangle \geq -\overline{\alpha}(h(x')),$$

then h *is a valid NBF for* (1)*.*

In Sec. IV, Thm. 4 facilitates the validation of candidate NBFs that are defined by $\max$ or $\min$ operations of smooth functions by expressing these sufficient conditions in terms of the component functions.

IV. Boolean Logic via Max/Min

This sections covers applications of $\max$ and $\min$ functions to the Boolean composition of barrier functions. In particular, this section demonstrates that these operators encode a system of Boolean logic falling into the NBF framework in Sec. III. We also cover a QP-based formulation of these Boolean compositional NBFs with respect to a class of control-affine systems.

A. Composition by Boolean Logic

Throughout this section, we assume that a finite set of functions $h_i : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$, $i \in [k]$, are candidate NBFs. Within this framework, $\max$ represents a Boolean $\vee$ operation: that is, if $h_{[k]}^{\max} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_{[k]}^{\max}(x') = \max_{i \in [k]}\{h_i(x')\}, \tag{5}$$

for $x' \in \mathcal{D}$, is a candidate and valid NBF for (1), then at each $t \in [0, t_1]$, there exists at least one $j \in [k]$ such that $h_j(x(t)) \geq 0$. Similarly, we note that $\min$ represents a Boolean $\wedge$ operation: that is, if $h_{[k]}^{\min} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_{[k]}^{\min}(x') = \min_{i \in [k]}\{h_i(x')\}, \tag{6}$$

for $x' \in \mathcal{D}$, is a candidate and valid NBF for (1), then at each $t \in [0, t_1]$, $h_j(x(t)) \geq 0,\ \forall\ j \in [k]$. Furthermore, $-h$ represents $\neg h$. These expressions allow for the application of De Morgan's laws in that $h_1 \vee h_2 = \neg(\neg h_1 \wedge \neg h_2)$, permitting full Boolean composition.

B. Min/Max Barrier Functions

Having noted the utility of $\min$ and $\max$ as Boolean operators, we focus on the criteria that these Boolean compositional NBFs must satisfy to be covered under the results of Sec. III. In the interest of space efficiency, we omit the proof of this result and note that it follows from Prop. 2 and Thm. 4. Prop. 5 holds for the $\min$ operator as well.

Proposition 5. *Let* $h_i : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$*,* $i \in [k]$*, be a finite set of locally Lipschitz functions which are candidate NBFs, and let* $h_{[k]}^{\max} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ *be defined as in* (5)*. For each* $x' \in \mathcal{D}$*, let*

$$J(x') = \{j \in [k] \mid h_j(x') = \max_{i \in [k]}\{h_i(x')\}\},$$

and consider the set-valued map $\mathcal{E}_h : \mathcal{D} \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *defined by*

$$\mathcal{E}_h(x') = \bigcup_{j \in J(x')} \partial h_j(x').$$

If $h_{[k]}^{\max}$ *is a candidate NBF and there exists a locally Lipschitz extended class-*$\mathcal{K}$ *function* $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ *such that for every* $x' \in \mathcal{D}$*,* $\xi \in \mathcal{E}_h(x')$*, and* $v \in F(x')$*,*

$$\langle \xi\,, v \rangle \geq -\overline{\alpha}(h_{[k]}^{\max}(x')),$$

then $h_{[k]}^{\max}$ *is a valid NBF for* (1)*.*

C. Quadratic-Program-Based Controllers

The formulation of a smooth barrier function with respect to control-affine systems produces an affine constraint on the system, and coupling this affine constraint with the minimization of a quadratic cost, at each point in time, results in a quadratic program (e.g., [5], [10]). This section provides similar results for NBFs with respect to a class of control-affine systems. In the nonsmooth case, the component functions generate a series of constraints, rather than a single constraint, that must be enforced point-wise in time. In the interest of space, we

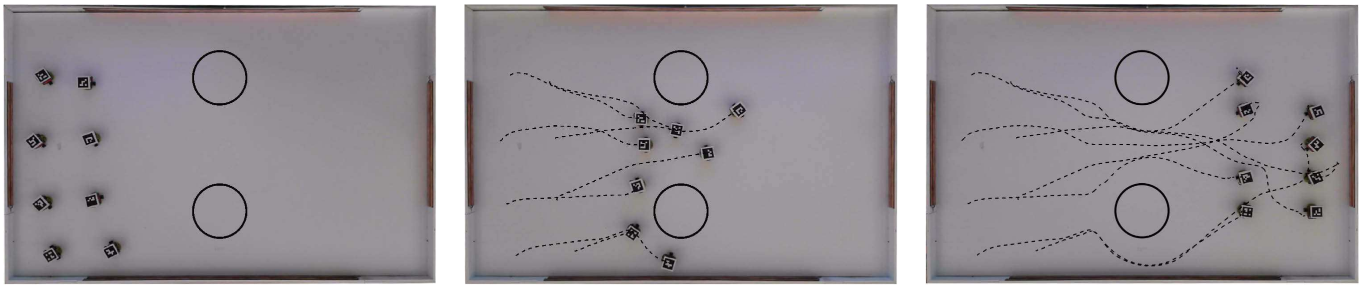

Fig. 1. A group of 8 differential-drive robots in the Robotarium successfully navigate through a pair of obstacles (circles) to their desired destination (crosses) and avoid inter-robot collisions. This task is accomplished by solving online for a QP-based controller with respect to the NBF in (7) that encodes and enforces safety requirements.

omit the proof and note that it follows from [22, Thm. 1] and Prop. 5.

Proposition 6. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $u : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, and consider the control-affine system $\dot{x}(t) = f(x(t)) + G(x(t))u(x(t))$. Let $h^{\min}_{[k]} : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be defined as in (6), where each $h_i : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable candidate NBF with a locally Lipschitz derivative. Consider the functions $w^* : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ and $u^* : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$\begin{aligned} &w^*(x') = \min_{(u,w)\in\mathbb{R}^{m+1}} w \\ &\textit{s.t. } \nabla h_i(x')^T(f(x')+G(x')u)+\overline{\alpha}(h_i(x'))-w \geq 0, \forall\, i \in [k] \end{aligned}$$

and

$$\begin{aligned} &u^*(x') = \arg\min_{u\in\mathbb{R}^m} u^T H(x')u + b(x')^T u \\ &\textit{s.t. } \nabla h_i(x')^T(f(x') + G(x')u)+\overline{\alpha}(h_i(x')) \geq 0,\ \forall\, i \in [k], \end{aligned}$$

where $H : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is locally Lipschitz, symmetric, positive definite and $b : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz. If $h^{\min}_{[k]}$ is a candidate NBF and $w^*(x') > 0$, for all $x' \in \mathcal{D}$, then u^* is locally Lipschitz; and $h^{\min}_{[k]}$ is a valid NBF for the closed-loop system under the controller u^*.

Intuitively, w^* in the above result gives some notion of the width of the feasible set of solutions. If the feasible set has non-zero width at all points, then a locally Lipschitz solution may be selected from the feasible set. In general, the computational complexity of a QP depends on the decision variables, the constraints, and the utilized solver. For an excellent survey of these methods for multi-agent systems, we refer the reader to [8].

V. Experimental Results

This section features a group of robots in the Robotarium (see [9]), which is a remote-access, multi-agent robotics test bed. The agents attempt to achieve a navigation objective by utilizing a given controller that accomplishes the desired goal but disregards safety measures: inter-agent collisions and static obstacles. In this experiment, a QP wraps the existing controller in an NBF framework such that it simultaneously satisfies multiple safety requirements and fulfills the intent behind the original controller.

Consider a team of 8 planar, single-integrator agents each with state $x_i(t) \in \mathbb{R}^2,\ i \in [8]$, and dynamics $\dot{x}_i(t) = u_i(x(t))$. To solve the ensemble problem via QP, we stack the states and inputs into vectors $x(t) = \left[x_1(t)^T \ \ldots \ x_8(t)^T\right]^T$, where $x(t) \in \mathbb{R}^{16}$ and $u(x(t))$ is defined in the same fashion. The agents' objective is to drive from their initial condition to some pre-specified goal points $x^g \in \mathbb{R}^{16}$, which is accomplished by use of a locally Lipschitz proportional controller

$$u^{obj}(x(t)) = x^g - x(t).$$

To avoid collisions with other agents, the following compositional candidate NBF applies to each pair of agents

$$h^c(x(t)) = \bigwedge_{i=1}^{8} \bigwedge_{j=i+1}^{8} \|x_i(t) - x_j(t)\|^2 - (\delta^c)^2,$$

where $\delta^c > 0$. Similarly, each agent avoids collisions with two circular obstacles in the plane via the NBF

$$h^o(x(t)) = \bigwedge_{i=1}^{8} \bigwedge_{j=1}^{2} \|x_i(t) - o_j\|^2 - (\delta^o)^2,$$

where $o_j \in \mathbb{R}^2$ indicates the static position of an obstacle and $\delta^o > 0$. The final Boolean compositional barrier function is given by

$$h^{\min}(x(t)) = h^c(x(t)) \wedge h^o(x(t)). \tag{7}$$

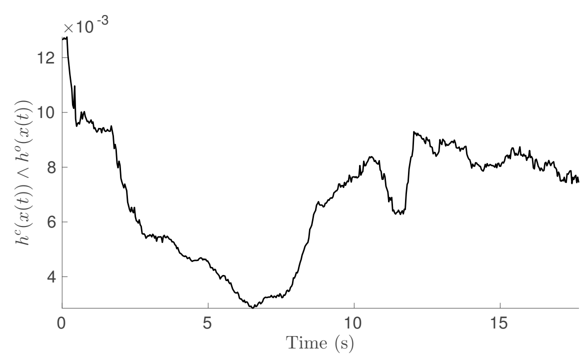

Fig. 2. Value of Boolean compositional NBF in (7) over the course of the experiment. Because the NBF remains positive over time, all safety objectives are simultaneously satisfied.

Now, we examine the derivatives of the component barrier functions of h^c and h^o. Taking a component barrier function in h^c with agents i and j yields

$$\frac{d}{dt}\left(\|x_i(t)-x_j(t)\|^2-(\delta^c)^2\right)=A^{ij}(x(t))u.$$

Here, the superscript A^{ij} indicates that this vector describes the derivative for agents i and j. A^{ij} maps to a row vector whose indices satisfy

$$A_i^{ij}(x')=2(x_i'-x_j')^T,\ A_j^{ij}(x')=-A_i^{ij}(x'),\ A_k^{ij}(x')=0,$$

where $k \neq i, j$ and the subscript indicates a particular two-dimensional element of $A^{ij}(x')$. Importantly, A^{ij} is locally Lipschitz.

Similarly, each component function of h^o will have a derivative for agent i and obstacle j

$$\frac{d}{dt}\left(\|x_i(t)-o_j\|^2-(\delta^o)^2\right)=B^{ij}(x(t))u,$$

where the superscript B^{ij} indicates that this function is between agent i and obstacle j. B^{ij} maps to a row vector whose indices satisfy

$$B_i^{ij}(x')=2(x_i'-o_j)^T,\ B_k^{ij}(x')=0,\ k\neq i,$$

where the subscript indicates a particular two-dimensional element in $B^{ij}(x')$. In this case, B^{ij} is also locally Lipschitz.

Now, we utilize the QP formulation noted in Prop. 6 with the objective function $u^Tu-2u^{obj}(x(t))^Tu$, which is equivalent to minimizing the squared norm $\|u-u^{obj}(x(t))\|^2$. This cost attempts, at each point in time, to minimally modify the existing controller $u^{obj}(x(t))$ such that the modified controller achieves the safety objectives. In this experiment, we assume that the selection $\overline{\alpha}(h^{\min}(x(t)))=\gamma h^{\min}(x(t))^3$, $\gamma>0$ makes w^*, as defined in Prop. 6, satisfy the condition $w^*(x')>0$ for all $x\in\mathbb{R}^{16}$.

The QP is formulated as in Prop. 6 with the parameters $\gamma=1000,\ \delta^c=0.04,\ \delta^o=0.1$; and we deploy the resulting controller onto the Robotarium's team of unicycle-modeled robots using the method in [23, Sec. 5].

Fig. 1 displays the mobile robots during this experiment, and Fig. 2 shows the NBF of (7) during the course of the experiment. The Boolean compositional NBF in (7) starts positive and remains positive over the course of the experiment; thus, all component barrier functions are simultaneously satisfied. Furthermore, as a result of the minimally invasive modification, the robots also arrive at the desired goal position, satisfying their original navigation objective and the NBF. Additionally, we note that the width of the feasible set remains strictly greater than zero, validating the application of Prop. 6.

VI. Conclusions

We have introduced a class of Nonsmooth Barrier Functions (NBFs), showing that existing results for smooth barrier functions apply to NBFs and allowing formulation of Boolean compositional NBFs via $\max$ and $\min$ operators. Furthermore, we have provided results that illustrate some computational methods for these conditions, allowing one to validate a class of NBFs with quadratic programs. To validate these results, a Boolean compositional NBF was deployed onto a team of mobile robots in the Robotarium. Future work on this topic could include temporal logic specifications for NBFs.

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