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Partial differential equation modeling of malware propagation in social networks with mixed delays

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ABSTRACT

With the wide applications of social networks, government and individuals increasingly emphasize information networks security. This paper is devoted to investigating a reaction–diffusion malware propagation model with mixed delays to describe the process of social networks. Applying matrix theory for characteristic values, we establish the local stability conditions of a positive equilibrium point. Based on the linear approximation method of nonlinear systems, the Hopf bifurcation at the positive equilibrium point is considered. Additionally, we identify some sensitive parameters in the process of malware propagation that are significant for control theory. Finally, numerical simulations are performed to illustrate the theoretical results.

1. Introduction

With the rapid development of the Internet, social networking has become the main platform for information spreading and diffusion. Establishing various social relations through social networking is the most important channel for information dissemination in our everyday life. More and more researches focus on social networking. Zhao, Wu and Xu [1] revealed complex dynamics of both the structure and the diffusion of large-scale online social networks and found that selecting proper weak ties can make the information diffuse quickly. Zheng, Lu and Zhao [2] considered an unknown–known–approved–exhausted four-status model. In view of effects of social reinforcement, they obtained that redundant signals can improve the probability of approval. In [3], according to frequent social activities of users, Liu and Zhang introduced a new link rewiring strategy based on the Fermi function, and obtained an interesting result: informed individuals tend to break old links and reconnect to their second-order friends who have more uninformed neighbors. With the widespread use of social networks, speed and scope of information transmission have been greatly promoted. Hence, social networks play a significant role in economic and social activities. However, the convenience of social networks, enhances the spread of various types of malware, which threatens the stability growth and rapid development of economies and simultaneously destroys social harmony and stability. Afer, Du, and Yin [4] carefully discussed relevant features to Android malware behavior captured at API level and gave different methods for controlling the spreading of Android malwares in social networks. Faghani and Nguyen [5] discussed recent advances on the topic of malware spreading through use of online social networking. Three malware propagation techniques, cross site scripting, Trojan and clickjacking types, and their characteristics were addressed by relevant models and numerical simulations in [5]. All in all, great deal of malware has

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appeared widely in social networks, and some highly destructive malwares continue to emerge. Given this grim situation, we need to devote substantial attention to the spread of malware.

During the past decade, considerable attention has been paid to application and control problems of malware spreading models. Particularly, communication protocols, hardware design, resource efficiency, battlefield surveillance and home security have been extensively studied, see for example, [6–10]. Cheng et al. [6] studied the ripple-based spread of hybrid malware in generalized social networks including personal and spatial social relations. Nature, dynamics, and defense implications for malware propagation in online social networks were considered in [7]. Peng, Wang and Yu [8] studied a special social network (smart phone social network) and discussed the dynamic properties of malware programs in smart phone social networks. Chain exploitation of social networks malware was addressed by Sood and Enbody [9,10]. Where malware wireless sensor networks are concerned, the most important feature may be mobility, because of the wide applications in our everyday life [11]. For example, in an intelligent factory, nodes may be attached to equipment to collect information, such as running condition, efficiency of equipment and maintenance of equipment [12]. The purpose is to ensure, by monitoring equipment conditions, that the equipment is always running with high efficiency. For more applications to malware spreading models; see, for example, [13–15]. In fact, as Khan et al. [15] point out, the advantages of malware spreading models over static wireless sensor networks include enhanced target tracking, improved coverage, energy efficiency, and superior channel capacity. Owing to their wide application, malware wireless sensor networks are becoming malware targets [14]. Injecting malware often happens on the Internet, which causes damage to some nodes, especially mobile nodes, such as network paralysis, loss of data, and loss of online account.

To reduce or eliminate the damage caused by malware, we must first understand the dynamic characteristics of malware propagation. Malware propagation systems have been studied by some authors, and some interesting results have been obtained. Since the malware propagation model is based on the epidemic model, we shall review the related works for the epidemic model. Newman [16] studied a large class of standard epidemiological models that can be solved exactly on a wide variety of networks, and proposed a percolation theory based upon evaluation of the spread of an epidemic on graphs with given degree distributions. However, the temporal dynamics of epidemic spread were not considered by Newman. Liu et al. [17] considered the spreading behavior of malware across mobile. Based on the theory of complex networks, the spreading threshold that monitors the dynamics of the model was calculated, and the properties of malware epidemics were investigated in [17]. Wang [18,19] also obtained the threshold for a kind of malware to propagate in social networks, where all the nodes were supposed to be stationary.

We note that the above malware spreading models have been constructed by ordinary differential equations. As is well known, partial differential equations (PDEs) can depict the real world more accurately. Many complex models can be established only by PDEs. However, the malware spreading models by PDEs are rarely small. We find Wang et al. [20–22] constructed an intuitive cyber-distance among online users to study both temporal and spatial patterns of information diffusion process on social networks by using PDEs. Using real data coming from Digg (an online social network), Wang verified the reliability of the PDE model. Zhu, Zhao and Wang [23–25] studied several reaction–diffusion malware propagation models and obtained some results for stability and bifurcation of positive equilibrium points. Dai et al. [26] studied a partial differential equation with a Robin boundary condition in online social networks and discussed temporal and spatial properties of social networks. In general, matrix theory cannot be used easily in PDE models, because PDEs have complex natures and no standard characteristic equations. Hence, the research for malware propagation models with PDEs is just getting started, there are still many problems to be solved. In the present paper, we will try to construct a reaction–diffusion malware propagation model (PDE model) for developing the above research.

On the other hand, delays exist extensively in certain dynamic systems, such as various engineering, biological, and economic systems (see, for example, [27–31]). It is well known that delay falls into many categories, such as constant delay (discrete delay), time-varying delay and distribution delay. For the dynamical behavior analysis of delayed networks system, different types of time delays, have been taken into account, using a variety of techniques that include Lyapunov functional method, M-matrix theory, topological degree theory, and techniques of inequality analysis; see, for example, [32–39]. Recently, Zhu, Zhao and Wang [23,24] studied two kinds of reaction–diffusion malware propagation model with discrete delay, as follows:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d\nabla^2 S + rS(1 - S/k) - \beta SI(t-\tau) - \varepsilon_1 S - \eta S \\ \dfrac{\partial I}{\partial t} = d\nabla^2 I + \beta SI(t-\tau) - \varepsilon_2 I - \eta I \\ \dfrac{\partial R}{\partial t} = d\nabla^2 S + \varepsilon_1 S + \varepsilon_2 I - \eta R \\ 0 < x\,, y < L, \ \ t > 0, \\ \dfrac{\partial S}{\partial \phi} = \dfrac{\partial I}{\partial \phi} = \dfrac{\partial R}{\partial \phi} = 0, \ \ x, y = 0, L, \ \ t \geq 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} \frac{\partial S}{\partial t} = d\nabla^2 S + A - \beta SI(t) - \eta S + \delta R(t-\tau) \\ \frac{\partial I}{\partial t} = d\nabla^2 I + \beta SI(t) - \varepsilon I - \eta I \\ \frac{\partial R}{\partial t} = d\nabla^2 S + \varepsilon I - \eta R - \delta R(t-\tau) \\ 0 < x\,, y < L, \ \ t > 0, \\ \frac{\partial S}{\partial \phi} = \frac{\partial I}{\partial \phi} = \frac{\partial R}{\partial \phi} = 0, \ \ x, y = 0, L, \ \ t \geq 0, \end{cases} \tag{1.2}$$

where $S(x, y, t)$, $I(x, y, t)$ and $R(x, y, t)$ denote the densities of susceptible, infected and removed individuals with a position of (x, y) at time t, respectively. After that, they [25] studied complex dynamic behavior of a rumor propagation model:

$$\begin{cases} \frac{\partial I}{\partial t} = d\frac{\partial I^2}{\partial x^2} + rI(1 - I/K) - k\beta I \int_{\Omega}\int_{-\infty}^{t} G(t-\xi, x, y)f(t-\xi)S(\xi, y)d\xi dy \\ \frac{\partial S}{\partial t} = d\frac{\partial S^2}{\partial x^2} + k\beta I \int_{\Omega}\int_{-\infty}^{t} G(t-\xi, x, y)f(t-\xi)S(\xi, y)d\xi dy, \ x \in \Omega, \ \ t > 0, \\ \frac{\partial I}{\partial \nu} = \frac{\partial S}{\partial \nu} = 0, \ \ x, y \in \partial\Omega, \ t \geq 0. \end{cases} \tag{1.3}$$

The results in [23–25] showed that the delay was also admitted as one of the main reasons for resulting in bifurcations and instability. Hence, this paper is devoted to investigating a malware spreading model with mixed delays (including discrete delay and distribution delay) by using PDEs. We use the delay to accomplish the control of the Hopf bifurcation in malware models. Our main contributions are summarized as follows.

(1) We develop a new malware propagation model with mixed delays by using PDEs and the SIR model in the epidemic theory. To the best of our knowledge, few results have been obtained for the latter model. Our new model is more accurate about the actual situation; hence, it will be more important for the applications.

(2) We study the sensitivity of some parameters in the malware model, guaranteeing the controllability of the model. Thus, we can control the stability interval, and critical value occurs Hopf bifurcation by sensitive parameters which have important implications for practical applications.

(3) It is non-trivial to establish a unified framework to handle the reaction–diffusion terms, the sensitive parameters, and mixed delays influence. We develop some mathematical techniques (including matrix theory, Hopf analysis, and the like) for overcoming these difficulties.

The remaining structure of this paper is arranged as follows. In Section 2, we consider the model formulation. Section 3 deals with local stability and Hopf bifurcation of the equilibrium points for the present model. In Section 4, some numerical simulations are presented to illustrate our theoretical results. Finally, conclusions are drawn in Section 5.

2. Modeling malware propagation model with mixed delays

A malware system is a 2-dimensional space $\Omega = [0, L] \times [0, L]$, where L is positive constant. It consists of many mobile nodes. Denote (x, y) as the geographic position of a node. At any time t, a node is classified as either external or internal according to whether it is connected to the networks or not at that time. For malware propagation models, all nodes are divided into three classes depending on their states: susceptible (healthy), infected, and removed (immunized). To examine the heterogeneity induced by the presence of nodes with different geographic positions, denote $S(x, y, t)$, $I(x, y, t)$, $R(x, y, t)$ the densities of susceptible, infected and removed individuals with a position of the node at time t, respectively. In view of the SIR epidemic model and the characteristics of malware propagation models, we give the following three facts:

(1) When nodes lie in state I, users can immunize their nodes with countermeasures.

(2) Some part of all recovered nodes go through a temporary immunity with probability δ; others with probability γ.

(3) Dead nodes cannot be infected by any malware residing in other nodes. In addition, if a malware resided in the dead nodes, it immediately disappeared from the dead nodes.

In malware propagation model (1.1), the densities of infected individuals $I(x, y, t)$ are influenced by discrete delay, so the term $\beta I(t-\tau)$ is added in (1.1). In the malware propagation model (1.2), densities of removed individuals $R(x, y, t)$ are influenced by discrete delay, so the term $\delta\, R(t-\tau)$ is added in (1.2). In a real malware propagation model, I and R may be influenced by different classes of delay at the same time, including discrete and distribution delays. As the above analysis, our model can be represented by the following reaction–diffusion equations with mixed delays:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d\nabla^2 S + A - \beta SI(t-\tau) - \eta S + \delta R(t-\tau) + \gamma \displaystyle\int_{t-\tau}^{t} R(s)ds \\ \dfrac{\partial I}{\partial t} = d\nabla^2 I + \beta SI(t-\tau) - \varepsilon I - \eta I \\ \dfrac{\partial R}{\partial t} = d\nabla^2 S + \varepsilon I - \eta R - \delta R(t-\tau) - \gamma \displaystyle\int_{t-\tau}^{t} R(s)ds \\ 0 < x, y < L, \quad t > 0, \\ \dfrac{\partial S}{\partial \phi} = \dfrac{\partial I}{\partial \phi} = \dfrac{\partial R}{\partial \phi} = 0, \quad x, y = 0, L, \ t \geq 0, \end{cases} \tag{2.1}$$

with initial condition

$$\begin{cases} S(x, y, t) = \phi_1(x, y, t), \\ I(x, y, t) = \phi_2(x, y, t), \\ R(x, y, t) = \phi_3(x, y, t), \\ (t, x, y) \in [-\tau, 0] \times [0, L] \times [0, L], \end{cases} \tag{2.2}$$

where τ is non-negative constant, $\phi = (\phi_1, \phi_2, \phi_3)^\top$ is the outside normal vector of $\partial\Omega$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the Laplace operator, initial condition (2.2) satisfies

$$\phi_i \in C([-\tau, 0], X), \ i = 1, 2, 3, X = \{u \in W^{2,2}, \frac{\partial u}{\partial \phi} = 0 \text{ on } \partial\Omega\}.$$

The other parameters' meanings can be seen in Table 1.

3. Local stability and Hopf bifurcation

In this section, we will study the local stability and Hopf bifurcation of system (2.1) and prove that the delay τ is the bifurcation parameter.

From homogeneous Neumann boundary conditions (2.2), $E^1 = (\frac{A}{\eta}, 0, 0)^\top$ is an equilibrium point for any feasible parameters in (2.1), and (2.1) always has a unique positive equilibrium point $E^* = (S^*, I^*, R^*)^\top$ provided that the condition

$(H_1)\, A\beta > (\eta + \delta + \gamma\tau)(\eta + \varepsilon)$,

where

$$S^* = \frac{\varepsilon + \eta}{\beta}, \ I^* = \frac{A\beta(\eta + \delta + \gamma\tau) - \eta(\eta + \varepsilon)(\eta + \delta + \gamma\tau)}{\beta(\varepsilon + \eta)(\eta + \delta + \gamma\tau) - \beta\varepsilon(\delta + \gamma\tau)}, R^* = \frac{A\beta\varepsilon - \eta\varepsilon(\eta + \varepsilon)}{\beta(\varepsilon + \eta)(\eta + \delta + \gamma\tau) - \beta\varepsilon(\delta + \gamma\tau)}.$$

Let $\tilde{S} = S - S^*, \tilde{I} = I - I^*, \tilde{R} = R - R^*$, then system (2.1) can be transformed into the following form

$$\begin{cases} \dfrac{\partial S}{\partial t} = d\nabla^2 S - (\beta I^* + \eta)S - \beta S^* I(t-\tau) + \delta R(t-\tau) + \gamma \displaystyle\int_{t-\tau}^{t} R(s)ds - \beta SI(t-\tau) \\ \dfrac{\partial I}{\partial t} = d\nabla^2 I + \beta I^* S + (-\eta - \varepsilon)I + \beta S^* I(t-\tau) + \beta SI(t-\tau) \\ \dfrac{\partial R}{\partial t} = d\nabla^2 S + \varepsilon I - \eta R - \delta R(t-\tau) - \gamma \displaystyle\int_{t-\tau}^{t} R(s)ds \\ 0 < x, y < L, \quad t > 0, \\ \dfrac{\partial S}{\partial \phi} = \dfrac{\partial I}{\partial \phi} = \dfrac{\partial R}{\partial \phi} = 0, \quad x, y = 0, L, \ t \geq 0. \end{cases} \tag{3.1}$$

Thus, the positive equilibrium point E^* of (2.1) is changed into the zero equilibrium point $E^0 = (0, 0, 0)^\top$ of (3.1). Let

$$U(t) = (u_1(t), u_2(t), u_3(t))^\top = (S(t, x, y), I(t, x, y), R(t, x, y))^\top,$$

then (3.1) can be rewritten as an abstract differential equation in the phase space $C([-\tau, 0], X)$ as follows:

$$\dot{U} = D\triangle u(t) + L(u_t) + f(u_t), \tag{3.2}$$

where

$$D = \operatorname{diag}\{d, d, d\}, \ \triangle = \operatorname{diag}\{\partial^2/\partial x^2 + \partial^2/\partial y^2, \partial^2/\partial x^2 + \partial^2/\partial y^2, \partial^2/\partial x^2 + \partial^2/\partial y^2\},$$

$$U_t(s) = U(t + s), \ s \in [-\tau, 0],$$

Table 1
Symbols and their meanings of system (2.1).

Parameters	Notes
A	The number of susceptible nodes
d	The diffusion coefficient of users, being used to describe the mobility of users
β	The constant contact rate for $S(t)$ and $I(t-\tau)$
η	The death rate of nodes
δ	The rate constant for nodes becoming susceptible again after recovery
γ	The number of new recovered devices from the recovered $R(t)$
ε	The rate constant for nodes leaving infective class $I(t)$ for recovered class $R(t)$

$L : C \to X$ is a linear operator, which is defined by

$$L(\phi) = \begin{pmatrix} -(\beta I^* + \eta)\phi_1(0) - \beta S^*\phi_2(-\tau) + \delta\phi_3(-\tau) \\ \beta I^*\phi_1(0) + (-\eta - \varepsilon)\phi_2(0) + \beta S^*\phi_2(-\tau) \\ \varepsilon\phi_2(0) - \eta\phi_3(0) - \delta\phi_3(-\tau) \end{pmatrix},$$

$f : C \to X$ is a nonlinear operator, which is defined by

$$f(\phi) = \begin{pmatrix} -\beta\phi_1(0)\phi_2(0) + \gamma\displaystyle\int_{-\tau}^{0}\phi_3(s)ds \\ \beta\phi_1(0)\phi_2(-\tau) \\ -\gamma\displaystyle\int_{-\tau}^{0}\phi_3(s)ds \end{pmatrix},$$

where $\phi(\theta) = U_t(\theta),\ \theta \in [-\tau, 0],\ \phi = (\phi_1, \phi_2, \phi_3)^\top$. The linearized system of (3.2) at E^0 is

$$\dot{U} = D\triangle u(t) + L(u_t)$$

and its characteristic equation is

$$\lambda\omega - D\triangle\omega - L(e^{\lambda\cdot}\omega) = 0, \tag{3.3}$$

where $\omega \neq 0,\ \omega \in \mathrm{Dom}(\triangle) \subset X$. From the properties of the Laplacian operator, the operator D on X has the eigenvalues $-(m^2 + n^2),\ m, n \in N_0 = \{1, 2, \ldots\}$ with the relative eigenfunctions on X are

$$\xi^1_{mn} = (\alpha_{mn}, 0, 0)^\top, \xi^2_{mn} = (0, \alpha_{mn}, 0)^\top, \xi^3_{mn} = (0, 0, \alpha_{mn})^\top,$$

where $\alpha_{mn} = \cos(mx)\cos(ny)$. Obviously, $(\xi^1_{mn}, \xi^2_{mn}, \xi^3_{mn})_0^\infty$ construct a basis of the phase space X. Therefore, any element ω in X can be expanded as Fourier series in the following form

$$\omega = \sum_{m,n=0}^{\infty} W_{mn}\begin{pmatrix}\xi^1_{mn}\\ \xi^2_{mn}\\ \xi^3_{mn}\end{pmatrix}, W_{mn} = (\langle\omega, \xi^1_{mn}\rangle, \langle\omega, \xi^2_{mn}\rangle)\langle\omega, \xi^3_{mn}\rangle. \tag{3.4}$$

Obviously,

$$L(\phi^\top(\beta^1_{mn}, beta^2_{mn}, beta^3_{mn})^\top) = L(\phi)^\top(\beta^1_{mn}, beta^2_{mn}, beta^3_{mn}),\ \ m, n \in N_0. \tag{3.5}$$

From (3.4) and (3.5), (3.3) is equivalent to

$$\sum_{m,n=0}^{\infty} W_{mn}\left[\lambda I_3 + D(m^2+n^2) - \begin{pmatrix} -\beta I^* - \eta & -\beta S^* e^{-\lambda\tau} & \delta e^{-\lambda\tau} \\ \beta I^* & -\eta - \varepsilon + \beta S^* e^{-\lambda\tau} & 0 \\ 0 & \varepsilon & -\eta - \delta e^{-\lambda\tau} \end{pmatrix}\right]\begin{pmatrix}\xi^1_{mn}\\ \xi^2_{mn}\\ \xi^3_{mn}\end{pmatrix} = 0. \tag{3.6}$$

Thus, by (3.6) we have

$$\begin{aligned}&\lambda^3 + (3dh + a_1)\lambda^2 + (3d^2h^2 + a_2dh + a_3)\lambda + d^3h^3 + a_4d^2h^2 + a_5dh + a_6\\ &\quad + [a_7\lambda^2 + (a_8dh + a_9)\lambda + \delta d^2h^2 + a_{10}dh + a_{11}]e^{-\lambda\tau} = 0,\end{aligned} \tag{3.7}$$

where

$$h = m^2 + n^2, a_1 = 3\eta + \beta I^* + \varepsilon,$$

$$a_2 = 5\eta + 2\beta I^* + 2\varepsilon,\ a_3 = \eta(2\eta + \beta I^* + \varepsilon) + (\beta I^* + \eta)(\eta + \varepsilon),$$

$$a_4 = 3\eta + \varepsilon + \beta I^*,$$

$$a_5 = \eta(2\eta + \varepsilon + \beta I^*) + (\eta + \varepsilon)(\beta I^* + \eta),$$

$$a_6 = (\beta I^* + \eta)(\eta + \varepsilon) + 2\delta\beta^2 I^* S^* + \delta\beta\eta S^*,$$

$$a_7 = \delta + \beta S^*,\quad a_8 = 2\delta + \beta S^*,$$

$$a_9 = 2\eta\delta + \beta I^*\delta + \epsilon\delta + \beta^2 S^* I^* + 2\beta\eta S^*,$$

$$a_{10} = \delta(\beta I^* + 2\eta + \varepsilon) + 2\beta^2 S^* I^* + \beta\eta S^*,$$

$$a_{11} = \delta\eta(\beta\varepsilon I^*\delta + \eta) + \varepsilon\delta\eta + \eta(2\beta^2 S^* I^* + \beta\eta S^*).$$

When $h = 0$, (3.7) becomes

$$\lambda^3 + a_1\lambda^2 + a_3\lambda + a_6 + (a_7\lambda^2 + a_9\lambda + a_{11})e^{-\lambda\tau} = 0. \tag{3.8}$$

For $\forall h \in N_0$, since $a_6 + a_{11} > 0$, we obtain that $\lambda = 0$ is not a root of (3.8). When $\tau = 0$, (3.7) is equivalent to the following cubic equation

$$\begin{aligned}&\lambda^3 + (3dh + a_1 + a_7)\lambda^2 + [3d^2h^2 + (a_2 + a_8)dh + a_3 + a_9]\lambda + d^3h^3 + (a_4 + \delta)d^2h^2\\ &\quad + (a_5 + a_{10})dh + a_6 + a_{11} = 0.\end{aligned} \tag{3.9}$$

Obviously,

$$a_1 + a_7 > 0, a_2 + a_8 > 0,\ a_3 + a_9 > 0,$$

$$a_4 + \delta > 0,\ a_5 + a_{10} > 0,\ a_6 + a_{11} > 0.$$

If the following condition holds:
(H_2) $\varepsilon \geq \delta^2 > 1$,
then we have

$$\begin{aligned}&(3dh + a_1 + a_7)[3d^2h^2 + (a_2 + a_8)dh + a_3 + a_9] - [d^3h^3 + (a_4 + \delta)d^2h^2 + (a_5 + a_{10})dh + a_6 + a_{11}]\\
&= 8d^3h^3 + [11\eta + 5\beta I^* + 5\varepsilon + 6\delta + 3\beta S^* + 3(a_1 + a_7)]d^2h^2\\
&+ [21\eta^2 + 10\varepsilon\eta + 10\beta\eta I^* + 2\beta I^*\varepsilon + \delta\beta I^*\\
&+ 3\delta\eta + \varepsilon\delta + 2\beta\eta S^* + 5\beta S^* + 2\varepsilon\beta S^*\\
&+ 3\delta\beta S^* + \beta^2(S^*)^2 + 2\varepsilon\delta + 2\delta^2 + (\beta I^* + \varepsilon)(a_2 + a_8)]dh\\
&+ (2\varepsilon - 1)\beta\eta I^* + (\delta - 1)\beta\varepsilon I^* + (3\delta - 1)\eta^2\\
&+ (2\delta - 1)\varepsilon\eta + (\varepsilon - 2\delta)\beta^2 I^* S^*\\
&+ (\varepsilon - \delta^2)\beta\eta I^* + \kappa > 0,\end{aligned}$$

where

$$\kappa = 3\eta^2\beta I^* + 2\beta^2 I^*\eta + \varepsilon\beta^2(I^*)^2 + 3\varepsilon\eta^2 + 2\varepsilon^2\eta + \delta\beta I^*\varepsilon + 2\beta^2\varepsilon\eta S^*.$$

By the Routh–Hurwitz criteria, all the roots of (3.9) have negative real parts. Therefore, we have the following theorem.

Theorem 3.1. *Assume that the assumptions (H_1) and (H_2) hold, then the zero equilibrium point E^0 of system (3.1) with $\tau = 0$ is locally asymptotically stable.*

Now we discuss the effect of the delay τ on the stability of E^0. Assume that $i\omega$ is a root of (3.7). Then, for $h \in N_0$, we have

$$\begin{aligned}&-i\omega^3 - (3dh + a_1)\omega^2 + (3d^2h^2 + a_2dh + a_3)i\omega + d^3h^3 + a_4d^2h^2 + a_5dh + a_6\\ &\quad + [-a_7\omega^2 + (a_8dh + a_9)i\omega + \delta d^2h^2 + a_{10}dh + a_{11}][\cos\omega\tau - i\sin\omega\tau] = 0.\end{aligned} \tag{3.10}$$

By (3.10), we have

$$\begin{cases}(a_8dh + a_9)\omega\cos\omega\tau + (a_7 - \delta d^2h^2 - a_{10}dh - a_{11})\sin\omega\tau = \omega^3 - (3d^2h^2 + a_2dh + a_3)\omega\\ (-a_7 + \delta d^2h^2 + a_{10}dh + a_{11})\cos\omega\tau + (a_8dh + a_9)\omega\sin\omega\tau = (3dh + a_1)\omega^2 - d^3h^3 - a_4d^2h^2 - a_5dh - a_6.\end{cases} \tag{3.11}$$

Taking square on both sides of the equations of (3.11) and summing them up, we obtain

$$\begin{aligned}&\omega^6+(3dh+a_1-6d^2h^2-2a_2dh-2a_3)\omega^4\\&+[(3d^2h^2+a_2dh+a_3)^2-2(d^3h^3+a_4d^2h^2+a_5dh+a_6)(3dh+a_1)-(a_8dh+a_9)^2]\omega^2\\&+(d^3h^3+a_4d^2h^2+a_5dh+a_6)^2-(a_7-\delta d^2h^2-a_{10}dh+a_{11})=0.\end{aligned}\tag{3.12}$$

Set $z=\omega^2$, (3.12) is transformed into the following equation

$$\begin{aligned}&z^3+(3dh+a_1-6d^2h^2-2a_2dh-2a_3)z^2\\&+[(3d^2h^2+a_2dh+a_3)^2-2(d^3h^3+a_4d^2h^2+a_5dh+a_6)(3dh+a_1)-(a_8dh+a_9)^2]z\\&+(d^3h^3+a_4d^2h^2+a_5dh+a_6)^2-(a_7-\delta d^2h^2-a_{10}dh+a_{11})=0.\end{aligned}\tag{3.13}$$

When $h=0$, (3.13) becomes

$$z^3+A_1z^2+A_2z+A_3=0,\tag{3.14}$$

where $A_1=a_1-2a_3,\ A_2=a_3^2-2a_1a_6-a_9^2,\ A_3=a_6^2-a_7-a_{11}$. From the results of [40], we give the following theorem for the distribution of roots of (3.14).

Theorem 3.2. *(i) If $a_6^2<a_7+a_{11}$, then (3.14) has at least one positive root.*
(ii) If $a_6^2\geq a_7+a_{11}$, then (3.14) has positive roots if and only if $\tilde z>0,\ r(\tilde z)\leq 0$, where

$$r(z)=z^3+A_1z^2+A_2z+A_3,\ \tilde z=\frac{-A_1+\sqrt{A_1^2-3A_2}}{3}.\tag{3.15}$$

If (3.14) has positive real roots Z_1,Z_2 and Z_3, then we have

$$\omega_1=\sqrt{Z_1},\ \omega_2=\sqrt{Z_2},\ \omega_3=\sqrt{Z_3}.$$

By (3.11), we have

$$\begin{aligned}\cos\omega_k\tau_k=&\frac{(a_8dh+a_9)\omega_k^2[\omega_k^2-(3d^2h^2+a_2dh+a_3)]}{(a_8dh+a_9)^2\omega_k^2+(a_7-\delta d^2h^2-a_{10}dh-a_{11})^2}\\&-\frac{(a_7-\delta d^2h^2-a_{10}dh-a_{11})[(3dh+a_1)\omega_k^2-d^3h^3-a_4d^2h^2-a_5dh-a_6]}{(a_8dh+a_9)^2\omega_k^2+(a_7-\delta d^2h^2-a_{10}dh-a_{11})^2}\end{aligned}$$

and

$$\begin{aligned}\tau_k^j=&\frac{1}{\omega_k}\arccos\Bigg(\frac{(a_8dh+a_9)\omega_k^2[\omega_k^2-(3d^2h^2+a_2dh+a_3)]}{(a_8dh+a_9)^2\omega_k^2+(a_7-\delta d^2h^2-a_{10}dh-a_{11})^2}\\&-\frac{(a_7-\delta d^2h^2-a_{10}dh-a_{11})[(3dh+a_1)\omega_k^2-d^3h^3-a_4d^2h^2-a_5dh-a_6]}{(a_8dh+a_9)^2\omega_k^2+(a_7-\delta d^2h^2-a_{10}dh-a_{11})^2}+2j\pi\Bigg),\ k=1,2,3,\ j\in N_0.\end{aligned}$$

Thus, $\pm i\omega$ is a pair of purely imaginary roots of (3.7). Denote

$$\tau_0=\tau_{k_0}^0=\min_{k=1,2,3}\{\tau_k^0\},\quad \omega_0=\omega_{k_0}.\tag{3.16}$$

We also need the following assumptions:
(H_3) $3-2a_2-6dh>0,\ \ a_1-2a_3>0.$
(H_4) $d>\max\{2a_8a_9+2a_1a_5+6a_6-2a_2a_3,\ 2a_1+6a_4-6a_2,\ 6a_5+2a_1a_4+a_8^2-a_2^2-6a_3\}.$
(H_5) $a_3^2-a_9^2-2a_1a_6>0.$
(H_6) $\delta d^2h^2+a_{10}dh-a_7-a_{11}>0.$

Theorem 3.3. *If (H_3)–(H_6) hold. Then, for all $h,d\geq 1$, (3.12) has no positive real roots.*

Proof. By (H_3), we have

$$3dh+a_1-6d^2h^2-2a_2dh-2a_3=(3-2a_2-6dh)dh+(a_1-2a_3)>0.$$

By (H_4) and (H_5), it is easy to show that

$$\begin{aligned}&(3d^2h^2+a_2dh+a_3)^2-2(d^3h^3+a_4d^2h^2+a_5dh+a_6)(3dh+a_1)-(a_8dh+a_9)^2\\&=3d^4h^4+(6a_2-6a_4-2a_1)d^3h^3+(a_2^2+6a_3-6a_5-2a_1a_4-a_8^2)d^2h^2\\&+(2a_2a_3-2a_8a_9-2a_1a_5-6a_6)dh+a_3^2-a_9^2-2a_1a_6>0.\end{aligned}$$

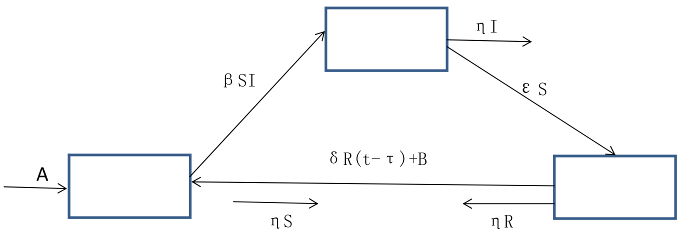

Fig. 1. Flow diagram for system (2.1), where $B = \gamma \int_{t-\tau}^{t} R(s)ds$.

By (H_6), we can obtain

$$\begin{aligned}&(d^3h^3 + a_4d^2h^2 + a_5dh + a_6)^2 - (a_7 - \delta d^2h^2 - a_{10}dh + a_{11})\\&= (d^3h^3 + a_4d^2h^2 + a_5dh + a_6)^2 - a_7 + \delta d^2h^2 + a_{10}dh - a_{11} > 0.\end{aligned}$$

According to Descartes's rule of sign, (3.12) has no positive real roots for $h, d \geq 1$. □

From [11,24], we have the following theorem.

Theorem 3.4. *Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the root of (3.8) near $\tau = \tau_0$ satisfying $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. Assume that $r'(\omega_0^2) > 0$, where $r(z)$ is defined by (3.15). Then $\pm i\omega_0$ is a pair of simple purely imaginary roots of (3.8) and satisfies*

$$\left.\frac{d(Re(\lambda(\tau)))}{d\tau}\right|_{\tau=\tau_0,\lambda=\pm i\omega_0} > 0.$$

Using Theorems 3.1–3.4, we have the following theorem.

Theorem 3.5. *Suppose that (H_1)–(H_6) hold.*
(i) If the conditions of Theorem 3.2, then the zero equilibrium point E^0 of system (3.1) is locally asymptotically stable as $\tau \in [0, \tau_0)$.
(ii) If the condition of (i) is satisfied, and $r'(\omega_0^2) > 0$, then system (3.1) undergoes a Hopf bifurcation at E^0 when $\tau = \tau_0$.
Here τ_0 and ω_0 are defined by (3.16).

4. Numerical examples and control strategies

In this section, we will simulate and analyze the dynamic characteristics of the above model through Matlab, including the trend in the quantity and mixed delays of the users on social networks.

4.1. Numerical examples

In what follows, we will verify the results of Theorems 3.1 and 3.5 by choosing the following parameter values in system (3.1) (see Fig. 1).

(i) For the parameters $A = 1.2,\ \beta = 5,\ \eta = \gamma = 0.2,\ \varepsilon = 2,\ \delta = 1,\ \tau = 0$ and all the conditions of Theorem 3.1 hold. By simple calculation, we have positive equilibrium points of system 2.1 $E^* = (0.4400, 2.0850, 3.4750)^{\top}$. The corresponding waveform is shown in Fig. 2. From Fig. 2, we notice that the number of all nodes converges to the positive equilibrium point E^* of (2.1). The conclusion agrees with Theorem 3.1.

(ii) For the parameters $A = 10,\ \beta = 2,\ \eta = \gamma = \delta = 0.2,\ \varepsilon = 2,\ \delta = 1,\ \tau = 1$, the positive equilibrium point of (2.1) is $E^* = (1.100, 11.2846, 37.6154)^{\top}$. All the conditions of Theorem 3.5 hold. According to Theorem 3.5, the critical value is $\tau_0 = 2.4702$. When $\tau = 1 < \tau_0$, from Fig. 3, we notice that the number of all nodes converges to the positive equilibrium point E^* of (2.1). Let $\tau = 3.2 > \tau_0$. Based on Theorem 3.5, the solutions of (2.1) emerge from the positive equilibrium point E^*, as shown in Fig. 4. From Numerical simulations, we notice that the malware propagation goes into periodic oscillation when $\tau > \tau_0$, while $\tau \in [0, \tau_0]$, the malware propagation is stable. Hence, we can extend range of stability by using the critical value τ_0.

4.2. Control strategies by sensitive parameters

In this subsection, we will discuss the impact of sensitive parameters in (2.1). Our aim is to obtain the proper control strategies according to sensitive parameters of (2.1).

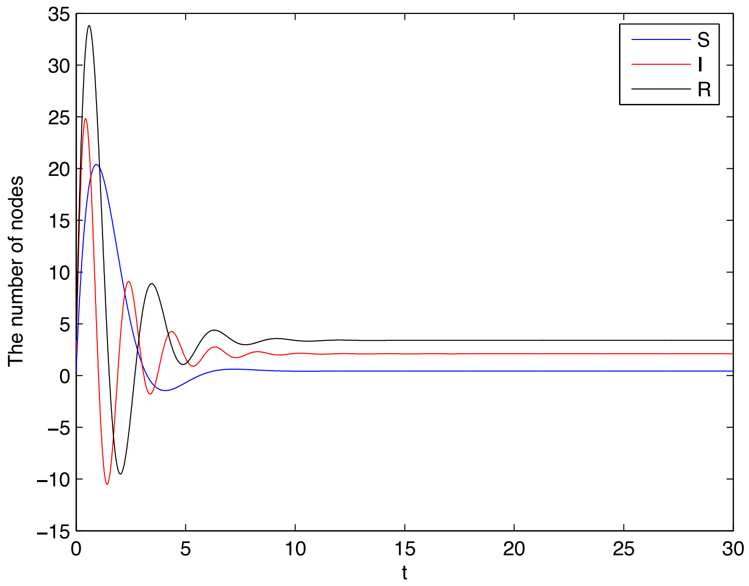

Fig. 2. When the delay $\tau = 0$, the positive equilibrium point E^* is stable.

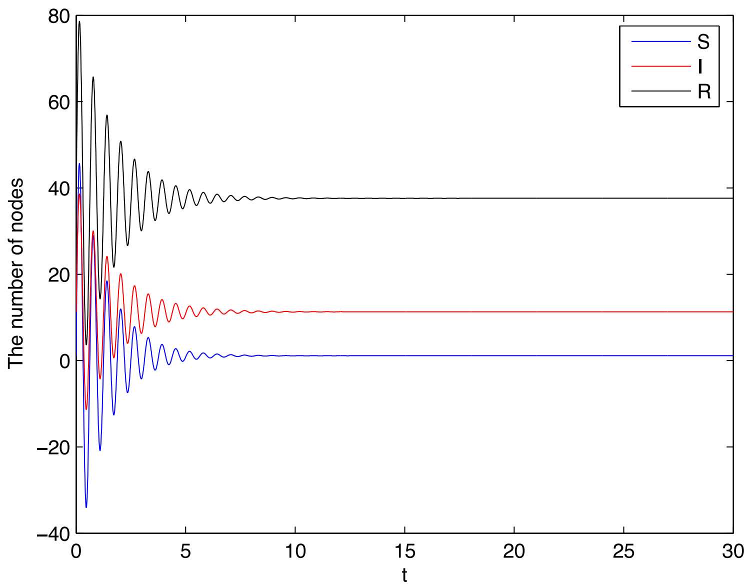

Fig. 3. When $\tau < \tau_0$, the equilibrium point E^* is stable.

(i) Impact on control strategies by infection rate β.
We consider influences of infection rate β on the dynamics of malware propagation. To obtain effective malware control strategies, we assign 5, 6, 7 to β in system (2.1), furthermore, let

$$A = 1,\ \eta = \gamma = 0.2,\ \delta = \tau = 1, \varepsilon = 2.$$

Then all the conditions of Theorem 3.5 hold. The different positive equilibrium point can be seen in Table 1(see Table 2).

From simulation results (see Fig. 5), we find that infection rate β increases which results in the increase of number of infected nodes. Hence, we should decrease the infection rate β for controlling malware propagation.

(ii) Impact on control strategies by transfer rate ε.
To obtain the impact of the transfer rate ε in (2.1), let

$$A = 1,\ \eta = \gamma = 0.2,\ \delta = \tau = 1, \beta = 2.$$

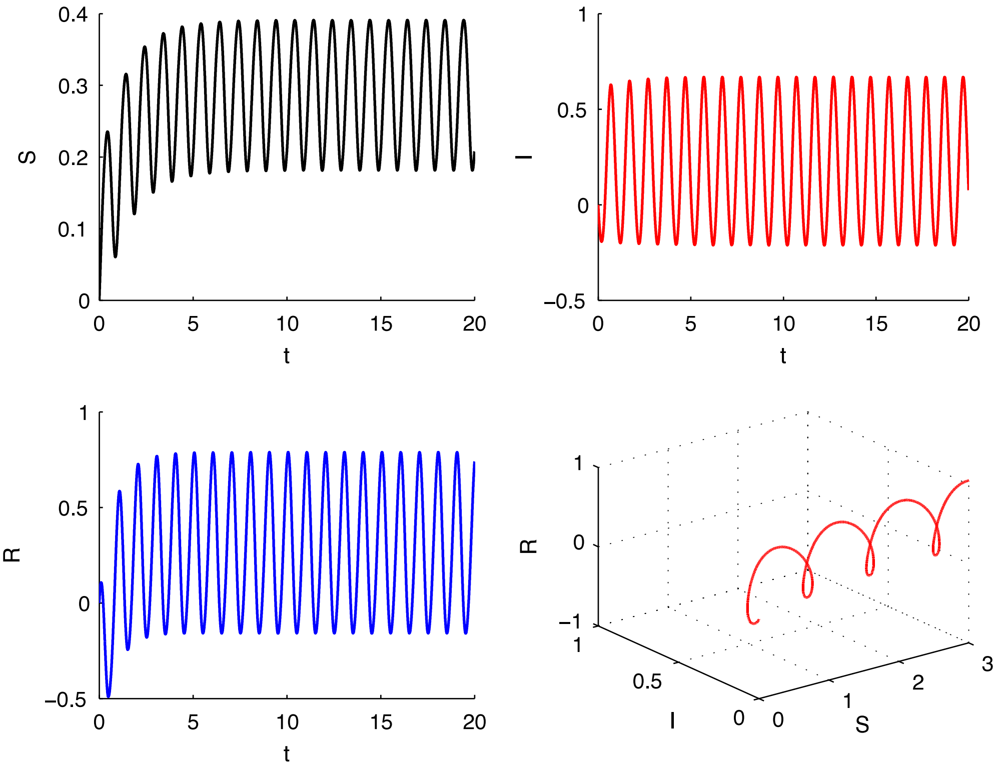

Fig. 4. The bifurcating periodic solutions from positive equilibrium E^* occurs when $\tau > \tau_0$.

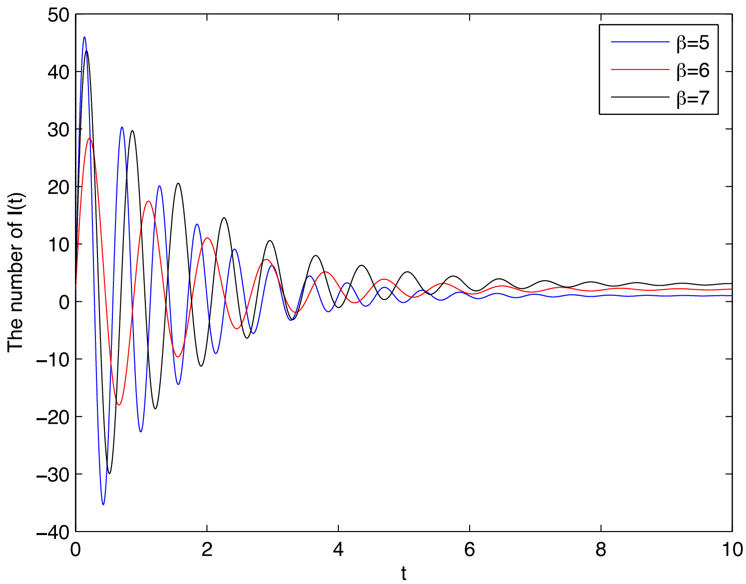

Fig. 5. Impact of different infected rate β on the number of infected nodes.

Assign 0.2, 0.3, 0.4 to ε, respectively. Then all the conditions of Theorem 3.5 hold. For different ε, we can easily obtain the corresponding positive equilibrium points of (2.1); see Table 3.

In view of Table 3, we find that the number of infected nodes decreases when the parameter ε increases. Hence the transfer rate ε is sensitive for the number of infected nodes. To eliminate the impact of malwares, we can reduce the number

Table 2
Equilibrium points of system (2.1) for different infection rate β.

β	Positive equilibrium points
5	$(0.4400, 1.8776, 2.6824)^\top$
6	$(0.3667, 1.9076, 2.7255)^\top$
7	$(0.3143, 1.9294, 2.7563)^\top$

Table 3
Equilibrium points of system (2.1) for different transfer rate ε.

ε	Positive equilibrium points
0.1	$(0.1500, 4.5267, 0.3233)^\top$
0.2	$(0.2000, 4.2000, 0.6000)^\top$
0.3	$(0.2500, 3.9180, 0.8382)^\top$

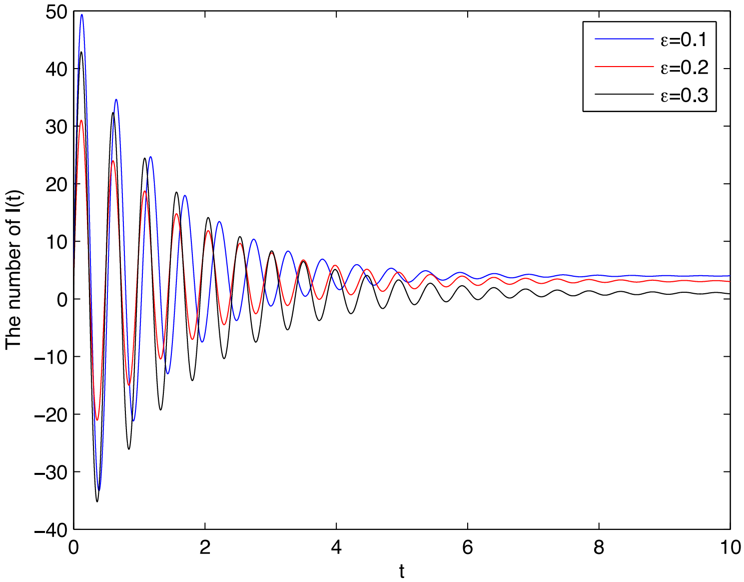

Fig. 6. Impact of different transfer rate ε on the number of infected nodes.

of infected nodes by using the increase of parameter ε. From Fig. 6, we notice that the number of infected nodes decreases when transfer rate ε increases.

Remark 4.1. From numerical results, the value of delay τ is key to the local stability of the malware model (2.1). There exists a critical value τ_0 for different parameters, when the delay $\tau > \tau_0$, Hopf bifurcation occurs. In addition, sensitive parameters β and ε in (2.1) have a large effect on the number of infected nodes. In the present paper, we obtain only local stability of system (2.1). Regarding global stability of (2.1), because of the existence of the delay in (2.1), so far we cannot obtain global stability results for (2.1). We hope that someone will solve the problem in the future.

5. Conclusions

In this article, using SIR epidemic model and PDEs, we study a new malware spreading model with mixed delays. It is noted that we construct a new PDE model which is different from the past ODE model.

Both by theoretical analysis and numerical simulations, we show how time delay affects malware propagation. Also, using delay as a bifurcating parameter, we obtain some conditions for occurrence of Hopf bifurcation. Furthermore, numerical simulations verify the correctness of theoretical analyses. More importantly, we investigate the effects of sensitive parameters β and ε on the scale of a malware spreading model. Simulation results show a great impact on the number of infected nodes. However, many important questions about malware spreading remain to be studied, among them optimal control, clustering for complex networks, global stability, and the direction of bifurcation.

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