UNIQUENESS OF SOLUTIONS
OF MEAN FIELD EQUATIONS IN R^2

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ABSTRACT. In this paper, we prove uniqueness of solutions of mean field equations with general boundary conditions for the critical and subcritical total mass regime, extending the earlier results for null Dirichlet boundary condition. The proof is based on new Bol's inequalities for weak radial solutions obtained from rearrangement of the solutions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain and consider the mean field equation

$$(1) \qquad \begin{cases} \Delta u + \rho \frac{e^u}{\int_\Omega e^u} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suzuki [23] proved that if Ω is simply-connected, then for $0 < \rho < 8\pi$ the equation (1) has a unique solution. Later in [8] the authors extended this result to the case $\rho = 8\pi$. Recently in [5] Bartolucci and Lin extended the result to multiply-connected domains. Indeed they proved the following.

Theorem A (Theorem 2 in [5]). *Let Ω be an open, bounded, and multiply-connected domain of class C^1. Then equation (1) admits at most one solution for $0 < \rho \leq 8\pi$.*

The proof relies on a generalization of the classical Bol's inequality for multiply-connected domains (see Theorem C below). A necessary and sufficient condition for the existence of a solution at the critical parameter $\rho = 8\pi$ is also provided in [5].

In this paper, among other results, we study uniqueness of solutions of the general mean field equation

$$(2) \qquad \begin{cases} \Delta u + \rho \frac{K(x)e^u}{\int_\Omega K(x)e^u} = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

on simply-connected domains, where K is a prescribed positive C^2 function. We shall prove the following uniqueness results for $\rho \leq 8\pi$.

Theorem 1.1. *Let Ω be an open, bounded and simply-connected domain, and let $K \in C^2(\Omega) \cap C(\overline{\Omega})$ be positive. Assume that $v_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$, satisfy*

$$\Delta v_i + K e^{v_i} = f_i(x), \tag{3}$$

where $f_2 \geq f_1 \geq -\Delta \ln(K)$ in Ω. If $v_2 - v_1 > c$ on Ω, $v_2 - v_1 = c$ on $\partial\Omega$ for some $c \in \mathbb{R}$, and

$$\int_\Omega K e^{v_1} dx = \int_\Omega K e^{v_2} dx = \rho,$$

then $\rho > 8\pi$.

The above theorem is equivalent to the next uniqueness result. Indeed Theorem 1.1 follows from Theorem 1.2 by letting $w_i = \ln K + v_i$, i=1,2, and Theorem 1.2 follows from Theorem 1.1 by letting $K \equiv 1$.

Theorem 1.2. *Let Ω be an open, bounded, and simply-connected domain. Assume that $w_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$, satisfy*

$$\Delta w_i + e^{w_i} = f_i(x), \tag{4}$$

where $f_2 \geq f_1 \geq 0$ in Ω. If $w_2 - w_1 > c$ on Ω, $w_2 - w_1 = c$ on $\partial\Omega$ for some $c \in \mathbb{R}$, and

$$\int_\Omega e^{w_1} dx = \int_\Omega e^{w_2} dx = \rho,$$

then $\rho > 8\pi$.

Corollary 1.3. *Let Ω be an open, bounded and simply-connected domain, and let $K \in C^2(\Omega) \cap C(\overline{\Omega})$ be positive. Assume that $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$ satisfy*

$$\Delta u_i + \rho \frac{K e^{u_i}}{\int_\Omega K e^{u_i}} = f_i(x), \tag{5}$$

where $f_2 \geq f_1 \geq -\Delta \ln(K)$ in Ω, and $u_2 - u_1 > c$ on Ω. If $u_2 - u_1 = c$ on $\partial\Omega$ for some $c \in \mathbb{R}$, then $\rho > 8\pi$.

We also present the following uniqueness results on multiply-connected domains.

Theorem 1.4. *Let Ω be an open, bounded and multiply-connected domain, and let $K \in C^2(\Omega) \cap C(\overline{\Omega})$ be positive. Assume that $v_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$, satisfy*

$$\Delta v_i + K e^{v_i} = f_i(x), \tag{6}$$

where $f_2 \geq f_1 \geq -\Delta \ln(K)$. If $v_2 - v_1 > c_2 - c_1$ on Ω, $v_1 + \ln(K) = c_1$ and $v_2 + \ln(K) = c_2$ on $\partial\Omega$ for some $c_1, c_2 \in \mathbb{R}$, and

$$\int_\Omega K e^{v_1} dx = \int_\Omega K e^{v_2} dx = \rho,$$

then $\rho > 8\pi$.

Corollary 1.5. *Let Ω be an open, bounded and multiply-connected domain, and let $K \in C^2(\Omega) \cap C(\overline{\Omega})$ be positive. Assume that $u_i \in C^2(\Omega) \cap C(\overline{\Omega})$, $i = 1, 2$ satisfy*

$$\Delta u_i + \rho \frac{K e^{u_i}}{\int_\Omega K e^{u_i}} = f_i(x), \tag{7}$$

where $f_2 \geq f_1 \geq -\Delta \ln(K)$. If $u_2 - u_1 > c_2 - c_1$ on Ω, $u_1 + \ln(K) = c_1$ and $u_2 + \ln(K) = c_2$ on $\partial\Omega$ for some $c_1, c_2 \in \mathbb{R}$, then $\rho > 8\pi$.

We should mention that Theorems 1.1 and 1.2, and Corollary 1.3 are known in some special cases and also when the wight K is singular (see [4]).

We shall use a new idea from [14] regarding the rearrangement of the solutions according to the standard metric on a sphere (projected to $\mathbb{R}^2$) and compare the total masses of the solutions. In particular, we shall show a reversed Bol's inequality in exterior domain for weak radial solutions (Proposition 3.1).

2. Preliminaries

Bol's isoperimetric inequality plays a crucial role in the proof of our main results. In this section we gather some results on Bol's inequality that will be used in subsequent sections. Let us first recall the classical Bol's isoperimetric inequality; see [2, 4, 6, 8, 21, 23], and [7] for a detailed history of the Bol's inequality.

Theorem B. *Let $\Omega \subseteq \mathbb{R}^2$ be a simply-connected and assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$(8) \qquad \Delta u + e^u \geq 0, \quad \int_\Omega e^u dx \leq 8\pi.$$

Then for every $\omega \Subset \Omega$ of class C^1 the following inequality holds:

$$(9) \qquad \left(\int_{\partial\omega} e^{\frac{u}{2}} \right)^2 \geq \frac{1}{2} \left(\int_\omega e^u \right) \left(8\pi - \int_\omega e^u \right).$$

Moreover the inequality in (9) is strict if $\Delta u + e^u > 0$ somewhere in ω or ω is not simply-connected.

For $\lambda > 0$ the function U_λ defined by

$$(10) \qquad U_\lambda := -2\ln(1 + \frac{\lambda^2|y|^2}{8}) + 2\ln(\lambda)$$

satisfies

$$\Delta U_\lambda + e^{U_\lambda} = 0,$$

and

$$\left(\int_{\partial B_r} e^{\frac{U_\lambda}{2}} \right)^2 = \frac{1}{2} \left(\int_{B_r} e^{U_\lambda} \right) \left(8\pi - \int_{B_r} e^{U_\lambda} \right),$$

for all $r > 0$ and $\lambda > 0$, where B_r denotes the ball of radius r centered at the origin in $\mathbb{R}^2$.

Note that Theorem B requires Ω to be simply-connected but ω can be multiply-connected. Recently in [5] Theorem B is extended to the case where $\Omega \subset \mathbb{R}^2$ is multiply-connected and u is constant on $\partial\Omega$.

Theorem C (Theorem 3 in [5]). *Let Ω be an open and bounded domain of class C^1 in $\mathbb{R}^2$ and assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (8) and $u = c$ on $\partial\Omega$, for some constant $c \in \mathbb{R}$. Then (9) holds for every $\omega \Subset \Omega$. Moreover the inequality is strict if $\Delta u + e^u > 0$ somewhere in ω or ω is not simply-connected.*

Let Ω be an open, bounded, and multiply-connected domain of class C^1 in $\mathbb{R}^2$, and $\overline{\Omega^*}$ be the closure of the union of the bounded components of $\mathbb{R}^2 \setminus \partial\Omega$ and $\Omega^* = \overline{\Omega^*} \setminus \partial\overline{\Omega^*}$. It is easy to see that $\Omega \subset \Omega^*$. Suppose $g \in C(\partial\Omega)$ satisfies

$$(11) \qquad g = G|_{\partial\Omega},$$

where G is Lipschitz continuous in $\overline{\Omega^*}$, G is subharmonic in $\overline{\Omega^*}$ and harmonic in Ω. The following more general result is also proved in [5].

Theorem D (Theorem 4 in [5])**.** *Let Ω be an open, bounded, and multiply-connected domain of class C^1 in $\mathbb{R}^2$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (8) with $u = g$ on $\partial\Omega$, and $g \in C(\partial\Omega)$ satisfies (11). Then (9) holds for every $\omega \Subset \Omega$. Moreover the inequality is strict if $\Delta u + e^u > 0$ somewhere in ω or ω is not simply-connected.*

Next we shall recall some facts about rearrangements according to the metric on $\mathbb{R}^2$ which is the stereographic projection of the standard metric on the unit sphere. Such rearrangments are discussed in detail in [14], but we also include it here for the sake of the readers. Let $\Omega \subset \mathbb{R}^2$ and $\lambda > 0$, and suppose that $u \in C^2(\overline{\Omega})$ satisfies

$$\Delta u + e^u \geq 0.$$

Then any function $\phi \in C^2(\overline{\Omega})$ which is constant on $\partial\Omega$ can be equimeasurably rearranged with respect to the measures $e^u dy$ and $e^{U_\lambda} dy$ (see [2], [4], [8], [21], [23]), where U_λ is defined in (10). More precisely, for $t > \min_{y \in \overline{\Omega}} \phi$ define

$$\Omega_t := \{\phi > t\} \subset\subset \Omega,$$

and let Ω_t^* be the ball centered at the origin in $\mathbb{R}^2$ such that

$$\int_{\Omega_t^*} e^{U_\lambda} dy = \int_{\Omega_t} e^u dy := a(t).$$

Then $a(t)$ is a right-continuous function, and $\phi^* : \Omega^* \to \mathbb{R}$ defined by $\phi^*(y) := \sup\{t \in \mathbb{R} : y \in \Omega_t^*\}$ provides an equimeasurable rearrangement of ϕ with respect to the measure $e^u dy$ and $e^{U_\lambda} dy$, i.e.

$$(12) \qquad \int_{\{\phi^* > t\}} e^{U_\lambda} dy = \int_{\{\phi > t\}} e^u dy, \quad \forall t > \min_{y \in \overline{\Omega}} \phi.$$

We shall need the following lemma.

Proposition 2.1. *Let $u, \varphi \in C^1(\overline{\Omega})$ and assume that ϕ is constant on $\partial\Omega$. Let $\phi^*(r)$ be the equimeasurable rearrangement of ϕ with respect to the measure $e^u dy$ and $e^{U_\lambda} dy$. Then ϕ^* is Lipschitz continuous on $(\epsilon, R - \epsilon)$, for every $\epsilon > 0$, where R is the radius of Ω^*.*

Proof. First note that the function ϕ^* is decreasing and the set

$$\mathcal{T} := \{t \geq \min_{\overline{\Omega}} \phi : \ (\phi^*)^{-1}(t) \ \text{is not a singleton}\}$$

has Lebesgue measure zero. Indeed $(\phi^*)^{-1}(t)$ is a connected closed interval for all $t \in \mathcal{T}$. Let $0 < r_1 < r_2 < R$ and

$$a(t) = \int_{\{\phi^* > t\}} e^{U_\lambda} dy = \int_{\{\phi > t\}} e^u dy, \quad \forall t > \min_{y \in \overline{\Omega}} \phi.$$

For $\phi^*(r_1), \phi^*(r_2) \notin \mathcal{T}$, we have

$$
\begin{aligned}
a(\phi^*(r_2)) - a(\phi^*(r_1)) &= \int_{\{\phi^*(|y|) > \phi^*(r_2)\}} e^{U_\lambda} dy - \int_{\{\phi^*(|y|) > \phi^*(r_2)\}} e^{U_\lambda} dy \\
&= \int_{\{\phi(y) > \phi^*(r_2)\}} e^u dy - \int_{\{\phi(y) > \phi^*(r_2)\}} e^u dy \\
&= \int_{\{\phi^*(r_2) \leq \phi(y) \leq \phi^*(r_1)\}} e^u dy \\
&= \int_{\{\phi^*(r_2) \leq \phi^*(|y|) \leq \phi^*(r_1)\}} e^{U_\lambda} dy.
\end{aligned}
$$

Now let $m := \min\limits_{\overline{\Omega}} e^{u(y)}$, $M_1 := \max\limits_{\overline{\Omega^*}} e^{U_\lambda(y)}$, and $M_2 := \max\limits_{\overline{\Omega}} |\nabla\phi|$. Then it follows from the above equality that

$$
\begin{aligned}
a(\phi^*(r_2)) - a(\phi^*(r_1)) &\leq M_1\mu(\{\phi^*(r_2) \leq \phi^*(|y|) \leq \phi^*(r_1)\}) \\
&= M_1\mu(r_1 \leq |y| \leq r_2) = M_1\pi(r_2^2 - r_1^2) \\
&\leq 2\pi R M_1(r_2 - r_1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
a(\phi^*(r_2)) - a(\phi^*(r_1)) &\geq m\mu(\{\phi^*(r_2) \leq \phi(y) \leq \phi^*(r_1)\}) \\
&\geq \frac{m}{M_2} \int_{\{\phi^*(r_2) \leq \phi(y) \leq \phi^*(r_1)\}} |\nabla\phi| dy \\
&\geq \frac{m}{M_2} \int_{\phi^*(r_2)}^{\phi^*(r_1)} \int_{\{\phi^{-1}(t)\}} ds\, dt \\
&\geq \frac{m}{M_2}(\phi^*(r_1) - \phi^*(r_2)) K(r_1, r_2),
\end{aligned}
$$

where

$$
K(r_1, r_2) = \min_{\{\phi^*(r_2) \leq t \leq \phi^*(r_1)\}} \mathcal{H}^{n-1}(\phi^{-1}(t)) > 0, \quad 0 < r_1 < r_2 < R.
$$

Since $\{\phi^{-1}(t)\} = \partial\{x : \phi(x) > t\}$, it follows from the isoperimetric inequality that if $\phi^*(r_1) < \max\limits_{y\in\overline{\Omega}} \phi - \delta$ and $\phi^*(r_2) > \min\limits_{y\in\overline{\Omega}} \phi + \delta$, for some $\alpha \in (0,1)$, then

$$
K(r_1, r_2) > C > 0, \quad \forall r_2 \text{ with } r_1 < r_2 < R,
$$

for some $C > 0$ independent of ϕ. Hence we have

$$
(13) \qquad 0 \leq \frac{a(\phi^*(r_2)) - a(\phi^*(r_1))}{r_2 - r_1} \leq \frac{2\pi R M_1 M_2}{m K(r_1, r_2)} \leq \frac{2\pi R M_1 M_2}{mC}.
$$

By approximation the above also holds for $\epsilon < r_1 < r_2 < R - \epsilon$. Thus ϕ^* is Lipschitz continuous on $(\epsilon, R - \epsilon)$ for every $\epsilon > 0$. $\qquad\square$

Now let

$$
j(t) := \int_{\{\phi > t\}} |\nabla\phi|^2 dy, \quad j^*(t) := \int_{\{\phi > t\}} |\nabla\phi^*|^2 dy, \quad \forall t > \min_{y\in\overline{\Omega}} \phi;
$$

$$
J(t) := \int_{\{\phi > t\}} |\nabla\phi| dy, \quad J^*(t) := \int_{\{\phi^* > t\}} |\nabla\phi^*| dy, \quad \forall t > \min_{y\in\overline{\Omega}} \phi.
$$

It is easy to see that both $j(t)$ and $J(t)$ are absolutely continuous and decreasing in $t > \min_{y\in\overline{\Omega}} \phi$. If $\phi \equiv C$ on $\partial\Omega$, it can be shown that

$$
(14) \qquad \int_{\{\phi = t\}} |\nabla\phi| ds \geq \int_{\{\phi^* = t\}} |\nabla\phi^*| ds, \quad \text{for } a.e. \ t > \min_{y\in\overline{\Omega}} \phi.
$$

Indeed it follows from Cauchy-Schwarz and Bol's inequalities that

$$
\begin{aligned}
\int_{\{\phi=t\}} |\nabla\phi| ds &\geq \left(\int_{\{\phi=t\}} e^{\frac{u}{2}}\right)^2 \left(\int_{\{\phi=t\}} \frac{e^u}{|\nabla\phi|}\right)^{-1} \\
&= \left(\int_{\{\phi=t\}} e^{\frac{u}{2}}\right)^2 \left(-\frac{d}{dt}\int_{\Omega_t} e^u\right)^{-1} \\
&\geq \frac{1}{2}(\int_{\Omega_t} e^u)(8\pi - \int_{\Omega_t} e^u)(-\frac{d}{dt}\int_{\Omega_t} e^u)^{-1} \\
&= \frac{1}{2}(\int_{\Omega_t^*} e^{U_\lambda})(8\pi - \int_{\Omega_t^*} e^{U_\lambda})(-\frac{d}{dt}\int_{\Omega_t^*} e^{U_\lambda})^{-1} \\
&= \int_{\{\phi^*=t\}} |\nabla\phi^*| ds, \quad \text{for } a.e. \ t > \min_{y\in\overline{\Omega}} \phi.
\end{aligned}
$$

It also follows that $j^*(t), J^*(t)$ are absolutely continuous and decreasing in $t > \min_{y\in\overline{\Omega}} \phi$, since both functions are right-continuous by definition and

$$
0 \leq j^*(t-0) - j^*(t) \leq j(t-0) - j(t) = \int_{\{\phi=t\}} |\nabla\phi|^2 dy = 0, \quad t > \min_{y\in\overline{\Omega}} \phi.
$$

$$
0 \leq J^*(t-0) - J^*(t) \leq J(t-0) - J(t) = \int_{\{\phi=t\}} |\nabla\phi| dy = 0, \quad t > \min_{y\in\overline{\Omega}} \phi.
$$

Therefore we have the following proposition.

Proposition 2.2. *Let $u \in C^2(\overline{\Omega})$ satisfy*

$$
\Delta u + e^u \geq 0 \quad \text{in} \ \Omega,
$$

and let U_λ be given by (10). Suppose $\phi \in C^1(\overline{\Omega})$ and $\phi \equiv C$ on $\partial\Omega$. Define the equimeasurable symmetric rearrangement ϕ^* of ϕ, with respect to the measures $e^u dy$ and $e^{U_\lambda} dy$, by (12). Then ϕ^* is Lipschitz continuous on $(\epsilon, R - \epsilon)$ for every $\epsilon > 0$, and $j^*(t), J^*(t)$ are absolutely continuous and decreasing in $t > \min_{y\in\overline{\Omega}} \phi$ and (14) holds.

3. Bol's type inequalities

We first prove the following lemma.

Lemma 3.1. *Let $\psi \in C(\mathbb{R}^2 \setminus B_R)$ be a decreasing radial function and*

$$
\int_{(\mathbb{R}^2 \setminus B_R)} e^\psi dx < \infty,
$$

for some $R > 0$. Then

$$
\lim_{s\to-\infty} e^s \int_{\{\psi>s\}} dx = 0.
$$

Proof. Since ψ is decreasing,

$$
\frac{3\pi}{4} r^2 e^{\psi(r)} \leq \int_{(B_r \setminus B_{r/2})} e^\psi dx,
$$

for $r > 2R$. Letting $r \to \infty$ we obtain,

$$
\lim_{r\to\infty} r^2 e^{\psi(r)} = 0.
$$

Define
$$r(s) := \sup\{r \geq R : \psi(r) > s\}, \quad s \in \mathbb{R}.$$
Then $r(s)$ is well defined for $s < \psi(R)$ and $\lim_{s \to -\infty} r(s) = \infty$. Since
$$e^s \int_{\{\psi > s\}} dx \leq \pi(r(s)^2 - R^2)e^{\psi(r(s))},$$
we obtain
$$\lim_{s \to -\infty} e^s \int_{\{\psi > s\}} dx = 0.$$
The proof is complete. $\qquad\square$

For the proof of our main results, we shall need the following reversed Bol's inequality.

Proposition 3.1. *Let B_R be the ball of radius R in $\mathbb{R}^2$ $\psi \in C^{0,1}(\mathbb{R}^2 \setminus B_R)$ be a strictly decreasing radial function satisfying*

$$(15) \qquad \int_{\partial B_r} |\nabla \psi| ds \leq 8\pi - \int_{\mathbb{R}^2 \setminus B_r} e^\psi \quad \text{for a.e. } r \in (R, \infty), \quad \text{and} \quad \int_{\mathbb{R}^2 \setminus B_R} e^\psi < 8\pi.$$

Then the following inequality holds:

$$(16) \qquad \left(\int_{\partial B_R} e^{\frac{\psi}{2}}\right)^2 \leq \frac{1}{2}\left(\int_{\mathbb{R}^2 \setminus B_R} e^\psi\right)\left(8\pi - \int_{\mathbb{R}^2 \setminus B_R} e^\psi\right).$$

Moreover if $\int_{\partial B_r} |\nabla \psi| ds \not\equiv 8\pi - \int_{\mathbb{R}^2 \setminus B_r} e^\psi$ on (R, ∞), then the inequality in (16) is strict.

Proof. Let $\beta := \psi(R)$ and define
$$k(s) = 8\pi - \int_{\{\psi < s\}} e^\psi dx, \quad \text{and} \quad \mu(s) = \int_{\{\psi > s\}} dx + \pi R^2,$$
for $s < \beta$. Then
$$-k'(s) = \int_{\{\psi = s\}} \frac{e^\psi}{|\nabla \psi|} = -e^s \mu'(s).$$
Hence
$$(17) \qquad \begin{aligned} -k(s)k'(s) &\geq \int_{\{\psi = s\}} |\nabla \psi| \cdot \int_{\{\psi = s\}} \frac{e^\psi}{|\nabla \psi|} \\ &= \left(\int_{\{\psi = s\}} e^{\psi/2}\right)^2 = e^s\left(\int_{\{\psi = s\}} ds\right)^2 \\ &= e^s \cdot 4\pi\left(\int_{\{\psi > s\}} dx + \pi R^2\right) = 4\pi e^s \mu(s), \end{aligned}$$
for a.e. $s < \beta$. Therefore
$$\frac{d}{ds}\left[e^s \mu(s) - k(s) + \frac{1}{8\pi}k^2(s)\right] = \mu(s) + \frac{1}{4\pi}k'(s)k(s) \leq 0,$$
for a.e. $s < \beta$. Integrating on $(-\infty, \beta)$ and using Lemma 3.1 we get
$$(18) \qquad \left[e^s \mu(s) - k(s) + \frac{1}{8\pi}k^2(s)\right]_{-\infty}^{\beta} = e^\beta \mu(\beta) - k(\beta) + \frac{1}{8\pi}k^2(\beta) \leq 0.$$

Now notice that

$$k(\beta) = 8\pi - \int_{\mathbb{R}^2 \setminus B_R} e^\psi \, dx$$

and

$$e^\beta \mu(\beta) = e^\beta \int_{B_R} dx = \frac{1}{4\pi} e^\beta \left(\int_{\partial B_R} ds \right)^2 = \frac{1}{4\pi} \left(\int_{\partial B_R} e^{\frac{\psi}{2}} \, ds \right)^2.$$

Thus (16) follows from the inequality (18). Finally if $\int_{\partial B_r} |\nabla \psi| ds \not\equiv 8\pi - \int_{\mathbb{R}^2 \setminus B_r} e^\psi$ on (R, ∞), then the inequality (17) will be strict, and consequently (16) will also be strict. $\qquad \square$

Similarly one can prove the following proposition (see Proposition 2.2 in [14]).

Proposition 3.2. *Let B_R be the ball of radius R in $\mathbb{R}^2$ and $u \in C^{0,1}(\overline{B_R})$ be a strictly decreasing radial function satisfying*

$$(19) \qquad \int_{\partial B_r} |\nabla u| ds \leq \int_{B_r} e^u \quad \text{for a.e.} \quad r \in (0, R), \quad \text{and} \quad \int_{B_R} e^u < 8\pi.$$

Then the following inequality holds:

$$(20) \qquad \left(\int_{\partial B_R} e^{\frac{u}{2}} \right)^2 \geq \frac{1}{2} \left(\int_{B_R} e^u \right) \left(8\pi - \int_{B_R} e^u \right).$$

Moreover if $\int_{\partial B_r} |\nabla \psi| ds \not\equiv \int_{B_r} e^\psi$ on $(0, R)$, then the inequality in (20) is strict.

4. Proof of the main results

Lemma 4.1. *Let $R > 0$ and assume that $\psi \in C^{0,1}(\mathbb{R}^2 \setminus B_R)$ is a strictly decreasing radial function that satisfies*

$$(21) \qquad \int_{\partial B_r} |\nabla \psi| \leq 8\pi - \int_{\mathbb{R}^2 \setminus B_r} e^\psi$$

for a.e. $r \in (R, \infty)$ and $\psi = U_{\lambda_1} = U_{\lambda_2}$ on ∂B_R for some $\lambda_2 > \lambda_1$. Then

$$(22) \qquad \int_{\mathbb{R}^2 \setminus B_R} e^{U_{\lambda_2}} \leq \int_{\mathbb{R}^2 \setminus B_R} e^\psi \leq \int_{\mathbb{R}^2 \setminus B_R} e^{U_{\lambda_1}}.$$

Moreover if $\int_{\partial B_r} |\nabla \psi| \not\equiv \int_{\mathbb{R}^2 \setminus B_r} e^\psi$ on $r \in (R, \infty)$, then the inequalities in (22) are also strict.

Proof. Let $m_1 := \int_{\mathbb{R}^2 \setminus B_R} e^{U_{\lambda_1}}$, $m_2 := \int_{\mathbb{R}^2 \setminus B_R} e^{U_{\lambda_2}}$, and $m := \int_{\mathbb{R}^2 \setminus B_R} e^\psi$. Define

$$\beta := \left(\int_{\partial B_R} e^{\frac{\psi}{2}} \right)^2 = \left(\int_{\partial B_R} e^{\frac{U_{\lambda_1}}{2}} \right)^2 = \left(\int_{\partial B_R} e^{\frac{U_{\lambda_2}}{2}} \right)^2.$$

It follows from Proposition 3.1 that

$$\beta \leq \frac{1}{2} m(8\pi - m).$$

On the other hand

$$\beta = \frac{1}{2} m_1(8\pi - m_1) = \frac{1}{2} m_2(8\pi - m_2),$$

i.e. m_1 and m_2 are roots of the quadratic equation

$$x^2 - 8\pi x + 2\beta = 0.$$

Since m satisfies

$$m^2 - 8\pi m + 2\beta \leq 0,$$

we have

$$m_2 \leq m \leq m_1.$$

$\square$

Similarly the following lemma holds (see Lemma 3.3 in [14]).

Lemma 4.2. *Assume that $\psi \in C^{0,1}(\overline{B_R})$ is a strictly decreasing, radial, Lipschitz function, and satisfies*

$$(23) \qquad \int_{\partial B_r} |\nabla \psi| \leq \int_{B_r} e^{\psi}$$

a.e. $r \in (0, R)$ and $\psi = U_{\lambda_1} = U_{\lambda_2}$ for some $\lambda_2 > \lambda_1$ on ∂B_R, and $R > 0$. Then there holds

$$(24) \qquad either \ \int_{B_R} e^{\psi} \leq \int_{B_R} e^{U_{\lambda_1}} \quad or \quad \int_{B_R} e^{\psi} \geq \int_{B_R} e^{U_{\lambda_2}}.$$

Moreover if the inequality in (23) is strict in a set with positive measure in $(0, R)$, then the inequalities in (24) are also strict.

We shall also need the following lemma.

Lemma 4.3. *Assume that $\psi \in C^{0,1}(\overline{B_R})$ is a strictly decreasing radial function satisfying (23) for a.e. $r \in (0, R)$. If*

$$\rho = \int_{B_R} e^{\psi} dx = \int_{B_R} e^{U_{\lambda}} < 8\pi,$$

then $U_{\lambda}(R) \leq \psi(R)$.

Proof. By Proposition 3.2 we have

$$\begin{aligned}
\left(\int_{\partial B_R} e^{\frac{U_{\lambda}}{2}} \right)^2 &= \frac{1}{2} \left(\int_{B_R} e^{U_{\lambda}} \right) \left(8\pi - \int_{B_R} e^{U_{\lambda}} \right) \\
&= \frac{1}{2} \left(\int_{B_R} e^{\psi} \right) \left(8\pi - \int_{B_R} e^{\psi} \right) \\
&\leq \left(\int_{\partial B_R} e^{\frac{\psi}{2}} \right)^2,
\end{aligned}$$

and hence $U_{\lambda}(R) \leq \psi(R)$. $\square$

Now we are ready to prove the main result of this paper, Theorem 1.2.

Proof of Theorem 1.2. First we prove that $\rho \geq 8\pi$. Suppose w_1 and w_2 satisfy the assumptions of Theorem 1.2. Then

$$\Delta(w_2 - w_1) + e^{w_2} - e^{w_1} = f_2 - f_1 \geq 0.$$

Now choose $\lambda > 0$ and $R \in (0, \infty)$ such that

$$(25) \qquad \int_{\Omega} e^{w_1} = \int_{B_R} e^{U_{\lambda}},$$

and let ϕ be the symmetrization of $w_2 - w_1$ with respect to the measures $e^{w_1}dy$ and $e^{U_\lambda}dy$. Then it follows from Proposition 2.2 and Fubini's theorem that

$$
\begin{aligned}
\int_{\{\phi=t\}} |\nabla \phi| &\leq \int_{\{w_2-w_1=t\}} |\nabla(w_2 - w_1)| \\
&\leq \int_{\Omega_t} e^{w_2} - e^{w_1} dx \\
&= \int_{\{\phi>t\}} e^{U_\lambda+\phi} - \int_{\{\phi>t\}} e^{U_\lambda} \\
&= \int_{\{\phi>t\}} e^{U_\lambda+\phi} - \int_{\{\phi=t\}} |\nabla U_\lambda|,
\end{aligned}
$$

for a.e. $t > \inf_\Omega(w_2 - w_1)$. Hence

$$
\tag{26} \int_{\{\phi=t\}} |\nabla(U_\lambda + \phi)| \leq \int_{\{\phi>t\}} e^{(U_\lambda+\phi)} d
$$

for all $t > \inf_\Omega(w_2 - w_1)$. Since ϕ is decreasing in r, $\psi := U_\lambda + \phi$ is a strictly decreasing function, and

$$
\tag{27} \int_{\partial B_r} |\nabla \psi| \leq \int_{B_r} e^\psi dy, \quad a.e. \quad r \in (0, R),
$$

by Proposition 2.2 and the above inequality we see that $\psi \in W^{1,\infty}(B_R)$ and thus by Morrey's inequality $\psi \in C^{0,1}(B_R)$.

Since $w_1 \not\equiv w_2$ and $\int_\Omega e^{w_1} = \int_\Omega e^{w_2}$, then $w_2 < w_1$ on a subset of Ω with positive measure. Hence $\phi(R) < 0$ and consequently $\psi(R) = U_\lambda(R) + \phi(R) < U_\lambda(R)$. This is a contradiction in view of Lemma 4.3, and therefore we must have $\rho \geq 8\pi$.

Next we prove that $\rho > 8\pi$. Suppose $\rho = 8\pi$ and let $\lambda_1 > 0$. With an argument similar to the one above we may show that there exists $\psi = U_{\lambda_1} + \phi \in C^{0,1}(\mathbb{R}^2)$ such that

$$
\int_\Omega e^{w_1} dx = \int_{\mathbb{R}^2} e^{U_{\lambda_1}} = 8\pi = \int_\Omega e^{w_2} dx = \int_{\mathbb{R}^2} e^\psi dx,
$$

and

$$
\tag{28} \int_{\partial B_r} |\nabla \psi| \leq \int_{B_r} e^\psi dx
$$

for a.e. $r \in (0, \infty)$. Since $\int_{\mathbb{R}^2} e^\psi = \int_{\mathbb{R}^2} e^{U_{\lambda_1}}$, there exists $r_0 \in (0, \infty)$ such that $\psi(r_0) = U_{\lambda_1}(r_0)$. There exists a positive constant $\lambda_2 \neq \lambda_1$ such that $U_{\lambda_2}(r_0) = U_{\lambda_1}(r_0) = \psi(r_0)$. Since $\psi > U_{\lambda_1}$ in B_{r_0}, it follows from Proposition 4.2 that $\lambda_1 < \lambda_2$ and

$$
\int_{B_{r_0}} e^\psi \geq \int_{B_{r_0}} e^{U_{\lambda_2}}.
$$

On the other hand $\psi < U_{\lambda_1}$ in $\mathbb{R}^2 \setminus B_{r_0}$ and consequently it follows from Proposition 4.1 that

$$
\int_{\mathbb{R}^2 \setminus B_{r_0}} e^\psi \geq \int_{\mathbb{R}^2 \setminus B_{r_0}} e^{U_{\lambda_2}}.
$$

Hence

$$
\tag{29} 8\pi = \rho = \int_{\mathbb{R}^2} e^\psi \geq \int_{\mathbb{R}^2} e^{U_{\lambda_2}} = 8\pi.
$$

Note that if $f_1 \not\equiv 0$ or $f_2 \not\equiv f_1$, then the inequality in (29) will be strict, which is a contradiciton . Suppose $f_1 \equiv f_2 \equiv 0$. We may assume without loss of generality that $c = w_2 - w_1 \geq 0$ on $\partial\Omega$, since otherwise we can switch w_1 and w_2. By (29) we conclude that the equality in (28) holds for a.e. $r \in (0, \infty)$ and $\psi = U_{\lambda_2}$. It also yields that the equality in (14) must be true for $\phi = w_2 - w_1$ and $t \geq \inf_\Omega \phi$. By the proof of Proposition 2.2, we also know that Bol's inequality (9) on $\omega = \{\phi > t\}$ must be equality, and therefore $\{\phi > t\}$ must be simply-connected for $t \geq \inf_\Omega \phi$ by Theorem D. This is a contradiction since $\{\phi > t\}$ is not simply-connected when $\inf_\Omega \phi < t < 0$. The contradiction implies $\rho > 8\pi$. $\qquad\square$

Proof of Theorem 1.4. The proof follows from Theorem D and the same argument used in the proof of Theorem 1.2. $\qquad\square$

Acknowledgements

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