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The trace on projective representations of quantum groups

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Abstract For certain roots of unity, we consider the categories of weight modules over three quantum groups: small, unrestricted and unrolled. The first main theorem of this paper is to show that there is a modified trace on the projective modules of the first two categories. The second main theorem is to show that category over the unrolled quantum group is ribbon. Partial results related to these theorems were known previously.

Keywords Quantum groups · Modified traces · Pivotal categories

Mathematics Subject Classification 16T20 · 17B37 · 81R50

Introduction

For an odd ordered root of unity ξ and lattice L, let $\mathcal{U}_\xi^L$, $\mathcal{U}_\xi^H$ and $\overline{\mathcal{U}}_\xi^L$ be the unrestricted, unrolled and small quantum groups, respectively (see Sect. 3). Let $\mathscr{C}_{\text{odd}}$ (resp. $\mathscr{C}_{\text{odd}}^H$, resp. $\overline{\mathscr{C}}_{\text{odd}}$) be the category of $\mathcal{U}_\xi^L$ (resp. $\mathcal{U}^H$, resp. $\overline{\mathcal{U}}_\xi^L$) weight modules. The usual construction of quantum invariants do not directly apply to these categories because of the following obstructions: the categories are not semi-simple and have vanishing quantum dimensions. Partial results overcoming these obstructions have been obtained

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in [10,18–21,23]. In this paper we generalize some of these results using a new concept called generically semi-simple (loosely meaning the category is graded and semi-simple on a dense portion of the graded pieces). In [4,7], the results of this paper will be used in future work to construct topological invariants. Remark that since the first version of this paper, interesting new generalizations of unrolled quantum groups have been studied in the recent preprints [1,25].

The first part of this paper contains a general theory with two main theorems which extend properties of generic simple objects to general properties in the full category. The first (Theorem 2) loosely says that if a category is generically semi-simple, pivotal and braided such that there is a twist for every generic simple object then the full category has a twist and so the category is ribbon. The second (Theorem 3) loosely says that if a category is generically semi-simple and pivotal with a right trace on its projective objects whose modified dimension satisfies $\mathsf{d}(V) = \mathsf{d}(V^*)$ for all generic simple objects V then the right trace is a (two-sided) trace.

We apply these theorems to the categories of modules over the different quantum groups mentioned above. It was previously known that $\mathscr{C}^H_{\text{odd}}$ is a braided pivotal category and $\mathscr{C}_{\text{odd}}$ is a pivotal category with a right trace, see [19] and [20], respectively. Here we remark that $\mathscr{C}^H_{\text{odd}}$ and $\mathscr{C}_{\text{odd}}$ are generically semi-simple (see Sect. 4.3). In Sect. 4.4 we show that each generic simple module in $\mathscr{C}^H_{\text{odd}}$ has a twist and thus Theorem 2 implies $\mathscr{C}^H_{\text{odd}}$ is ribbon. Section 4.5 contains a proof that the modified dimension satisfies $\mathsf{d}(V) = \mathsf{d}(V^*)$ for all generic simple objects V of $\mathscr{C}_{\text{odd}}$ and so Theorem 3 implies the unique right trace (up to global scalar) on the ideal of projective modules of $\mathscr{C}_{\text{odd}}$ is a trace. This unique trace (up to global scalar) induces a trace on $\overline{\mathscr{C}}_{\text{odd}}$.

The main theorems in this paper about $\mathscr{C}^H_{\text{odd}}$ and $\mathscr{C}_{\text{odd}}$ use deep results developed by De Concini, Kac, Procesi, Reshetikhin and Rosso in the series of papers [11–14]. In particular, we use the quantum coadjoint action. In general, these results hold for odd ordered roots of unity. However, in Sect. 5 we show directly that in the case of $\mathfrak{sl}(2)$ the results discussed above hold for even and odd ordered roots of unity. In particular, we show $\mathscr{C}_{\mathfrak{sl}(2)}$ is braided and $\mathscr{C}^H_{\mathfrak{sl}(2)}$ and $\overline{\mathscr{C}}_{\mathfrak{sl}(2)}$ have unique traces. Finally, in Sect. 6 we conjecture that Sects. 4.4 and 4.5 generalize to even ordered root of unity.

1 Tensor categories

In this section we give the basic definitions for tensor categories and the functors induced by graphic calculus.

1.1 Pivotal and ribbon categories

We recall the definition of pivotal and ribbon tensor categories, for more details see for instance, [2]. In this paper, we consider strict tensor categories with tensor product $\otimes$ and unit object $\mathbb{I}$. Let $\mathscr{C}$ be such a category. The notation $V \in \mathscr{C}$ means that V is an object of $\mathscr{C}$.

The category $\mathscr{C}$ is a *pivotal category* if it has duality morphisms

$\overrightarrow{\mathrm{coev}}_V:\ \mathbb{I}\to V\otimes V^*,\quad \overrightarrow{\mathrm{ev}}_V:\ V^*\otimes V\to\mathbb{I},\quad \overleftarrow{\mathrm{coev}}_V:\ \mathbb{I}\to V^*\otimes V$ and $\overleftarrow{\mathrm{ev}}_V:\ V\otimes V^*\to\mathbb{I}$

which satisfy compatibility conditions (see for example [2,19]). In particular, the left dual and right dual of a morphism $f\colon V\to W$ in $\mathscr{C}$ coincide:

$$\begin{aligned} f^* &= \left(\overrightarrow{\mathrm{ev}}_W\otimes \mathrm{Id}_{V^*}\right)(\mathrm{Id}_{W^*}\otimes f\otimes \mathrm{Id}_{V^*})\left(\mathrm{Id}_{W^*}\otimes \overrightarrow{\mathrm{coev}}_V\right)\\ &= \left(\mathrm{Id}_{V^*}\otimes \overleftarrow{\mathrm{ev}}_W\right)(\mathrm{Id}_{V^*}\otimes f\otimes \mathrm{Id}_{W^*})\left(\overleftarrow{\mathrm{coev}}_V\otimes \mathrm{Id}_{W^*}\right):\ W^*\to V^*. \end{aligned}$$

Then there is a natural notion of right categorical (partial) trace in $\mathscr{C}$: for any $V,W\in\mathscr{C}$,

$$\begin{aligned} \mathrm{tr}_R : \mathrm{End}_{\mathscr{C}}(V) &\to \Bbbk\\ f &\mapsto \overleftarrow{\mathrm{ev}}_V(f\otimes \mathrm{Id}_{V^*})\overrightarrow{\mathrm{coev}}_V \end{aligned}\quad \text{and}$$

$$\begin{aligned} \mathrm{ptr}_R : \mathrm{End}_{\mathscr{C}}(V\otimes W) &\to \mathrm{End}_{\mathscr{C}}(V)\\ f &\mapsto (\mathrm{Id}_V\otimes \overleftarrow{\mathrm{ev}}_W)(f\otimes \mathrm{Id}_{W^*})(\mathrm{Id}_V\otimes \overrightarrow{\mathrm{coev}}_W) \end{aligned}$$

and analogous left categorical (partial) trace tr_L (resp. ptr_L) defined using $\overrightarrow{\mathrm{ev}}$ and $\overleftarrow{\mathrm{coev}}$.

A *braiding* on $\mathscr{C}$ consists of a family of natural isomorphisms $\{c_{V,W}: V\otimes W\to W\otimes V\}$ satisfying the Hexagon Axiom:

$$\begin{aligned} c_{U,V\otimes W} &= (\mathrm{Id}_V\otimes c_{U,W})\circ(c_{U,V}\otimes \mathrm{Id}_W)\quad\text{and}\\ c_{U\otimes V,W} &= (c_{U,W}\otimes \mathrm{Id}_V)\circ(\mathrm{Id}_U\otimes c_{V,W}) \end{aligned}$$

for all $U,V,W\in\mathscr{C}$. We say $\mathscr{C}$ is *braided* if it has a braiding. If $\mathscr{C}$ is pivotal and braided, one can define a family of natural automorphisms

$$\theta_V=\mathrm{ptr}_R(c_{V,V}):V\to V.$$

We say that $\mathscr{C}$ is *ribbon* and the morphism θ is a *twist* if

$$\theta_{V^*}=(\theta_V)^* \tag{1}$$

for all $V\in\mathscr{C}$.

Remark 1 An equivalent definition of a ribbon category is a braided left rigid balanced category. Left rigid means that there are left duals $(V,V^*,\overrightarrow{\mathrm{ev}}_V,\overrightarrow{\mathrm{coev}}_V)$ and the balance θ is a natural automorphism of the identity functor satisfying $\theta_{U\otimes V}=(\theta_U\otimes\theta_V)\circ c_{V\otimes U}\circ c_{U\otimes V}$ and Eq. (1).

1.2 $\Bbbk$-categories

Let $\Bbbk$ be a integral domain. A $\Bbbk$-category is a category $\mathscr{C}$ such that its hom-sets are left $\Bbbk$-modules, the composition of morphisms is $\Bbbk$-bilinear, and the canonical $\Bbbk$-algebra map $\Bbbk \to \mathrm{End}_{\mathscr{C}}(\mathbb{I})$, $k \mapsto k\,\mathrm{Id}_{\mathbb{I}}$ is an isomorphism. A tensor $\Bbbk$-category is a tensor category $\mathscr{C}$ such that $\mathscr{C}$ is a $\Bbbk$-category and the tensor product of morphisms is $\Bbbk$-bilinear. An object V of $\mathscr{C}$ is *simple* if $\mathrm{End}_{\mathscr{C}}(V) = \Bbbk\mathrm{Id}_V$. Let V be an object in $\mathscr{C}$ and let $\alpha : V \to W$ and $\beta : W \to V$ be morphisms. The triple (V, α, β) (or just the object V) is a *retract* of W if $\beta\alpha = \mathrm{Id}_V$. An object W is a *direct sum* of the finite family $(V_i)_i$ of objects of $\mathscr{C}$ if there exist retracts (V_i, α_i, β_i) of W with $\beta_i\alpha_j = 0$ for $i \neq j$ and $\mathrm{Id}_W = \sum_i \alpha_i\beta_i$. An object which is a direct sum of simple objects is called *semi-simple*.

1.3 Traces on ideals in pivotal categories

Here we recall the definition of a (right) trace on an (right) ideal in a pivotal $\Bbbk$-category $\mathscr{C}$, for more details see [23]. By a *right ideal* of $\mathscr{C}$ we mean a full subcategory $\mathcal{I}$ of $\mathscr{C}$ such that:

1. If $V \in \mathcal{I}$ and $W \in \mathscr{C}$, then $V \otimes W \in \mathcal{I}$.
2. If $V \in \mathcal{I}$ and if $W \in \mathscr{C}$ is a retract of V, then $W \in \mathcal{I}$.

One defines similarly the notion of a *left ideal* by replacing in the above definition $V \otimes W \in \mathcal{I}$ by $W \otimes V \in \mathcal{I}$. A full subcategory $\mathcal{I}$ of $\mathscr{C}$ is an *ideal* if it is both a right and left ideal.

If $\mathcal{I}$ is a right ideal in $\mathscr{C}$ then a *right trace* on $\mathcal{I}$ is a family of linear functions

$$\{\mathsf{t}_V : \mathrm{End}_{\mathscr{C}}(V) \to \Bbbk\}_{V \in \mathcal{I}}$$

such that following two conditions hold:

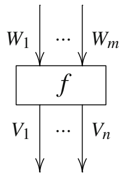
1. If $U, V \in \mathcal{I}$ then for any morphisms $f : V \to U$ and $g : U \to V$ in $\mathscr{C}$ we have
$$\mathsf{t}_V(gf) = \mathsf{t}_U(fg).$$
2. If $U \in \mathcal{I}$ and $W \in \mathscr{C}$ then for any $f \in \mathrm{End}_{\mathscr{C}}(U \otimes W)$ we have
$$\mathsf{t}_{U\otimes W}(f) = \mathsf{t}_U\big(\mathrm{ptr}_R(f)\big).$$

The notion of a left trace on a left ideal is obtained by replacing (2) in the above definition with $\mathsf{t}_{W\otimes U}(f) = \mathsf{t}_U\big(\mathrm{ptr}_L(f)\big)$ for all $f \in \mathrm{End}_{\mathscr{C}}(W \otimes U)$. A family $\mathsf{t} = \{\mathsf{t}_V\}_{V\in\mathcal{I}}$ is a *trace* if $\mathcal{I}$ is an ideal and t is both a left and right trace.

The class of projective modules Proj in $\mathscr{C}$ is an ideal. In a pivotal category projective and injective objects coincide (see [23]). The ideal Proj is an important example which we will consider later in this paper.

1.4 Colored ribbon graph invariants

Let $\mathscr{C}$ be a pivotal category. A morphism $f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m$ in $\mathscr{C}$ can be represented by a box and arrows (we use Turaev's convention ([30]) for orientations of diagrams):



Boxes as above are called *coupons*. By a ribbon graph in an oriented manifold Σ, we mean an oriented compact surface embedded in Σ which decomposed into elementary pieces: bands, annuli, and coupons (see [30]) and is the thickening of an oriented graph. In particular, the vertices of the graph lying in $\mathrm{Int}\,\Sigma = \Sigma \backslash \partial\Sigma$ are thickened to coupons. A $\mathscr{C}$-colored ribbon graph is a ribbon graph whose (thickened) edges are colored by objects of $\mathscr{C}$ and whose coupons are colored by morphisms of $\mathscr{C}$. The intersection of a $\mathscr{C}$-colored ribbon graph in Σ with $\partial\Sigma$ is required to be empty or to consist only of vertices of valency 1. When Σ is a surface the ribbon graph is just a tubular neighborhood of the graph.

A $\mathscr{C}$-colored ribbon graph in $\mathbb{R}^2$ is called *planar*. A $\mathscr{C}$-colored ribbon graph in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ is called *spherical*. A $\mathscr{C}$-colored ribbon graph in $\mathbb{R}^3$ or $\mathbb{R}^2 \times [0, 1]$ are called *spatial*.

For $i \in \{2, 3\}$, the $\mathscr{C}$-colored ribbon graphs in $\mathbb{R}^{i-1} \times [0, 1]$ form a category $\mathrm{Gr}^i_{\mathscr{C}}$ as follows: objects of $\mathrm{Gr}^i_{\mathscr{C}}$ are finite sequences of pairs (X, ε), where $X \in \mathscr{C}$ and $\varepsilon = \pm$. Morphisms of $\mathrm{Gr}^i_{\mathscr{C}}$ are isotopy classes of $\mathscr{C}$-colored ribbon graphs in $\mathbb{R}^{i-1} \times [0, 1]$. By a *(1,1)-ribbon graph* in $\mathrm{Gr}^i_{\mathscr{C}}$ we mean a $\mathscr{C}$-colored ribbon graph which is an endomorphism of an object $(V, +)$ in $\mathrm{Gr}^i_{\mathscr{C}}$. Let $\mathsf{F} : \mathrm{Gr}^i_{\mathscr{C}} \to \mathscr{C}$ be the Reshetikhin-Turaev $\Bbbk$-linear functor (see [22]). If $\mathscr{C}$ is a ribbon category, the functor on planer graphs $\mathsf{F} : \mathrm{Gr}^2_{\mathscr{C}} \to \mathscr{C}$ extends to the functor on spatial graphs $\mathsf{F} : \mathrm{Gr}^3_{\mathscr{C}} \to \mathscr{C}$.

1.5 Renormalized colored ribbon graph invariants

The motivation of this paper is to provide the underpinnings for the construction of topological invariants. With this in mind, in this subsection, we recall the notion of *renormalized* colored ribbon graph invariants introduced and studied in [18,20–23]. This subsection is independent of the rest of the paper. The theory of renormalized invariants produces nontrivial invariants in some situations when the standard approaches fail. In particular, these invariants can be nontrivial when quantum dimension vanish. The renormalized invariant of closed $\mathscr{C}$-colored ribbon graph can be computed in three steps: (1) cut a special edge of a closed $\mathscr{C}$-colored ribbon graph, (2) apply F to the resulting graph to obtain an endomorphism and (3) apply to the endomorphism a $\Bbbk$-linear functional to obtain a number. If this functional has certain properties then the number is an invariant of the ribbon graph. Here is a more precise definition:

Let $\mathscr{C}$ be a pivotal (resp., ribbon) category. Let $\mathcal{T}_{\text{adm}}$ be a class of planar (resp. spatial) $\mathscr{C}$-colored (1,1)-ribbon graphs. We denote an element of $\mathcal{T}_{\text{adm}}$ by T_V where V is the object of $\mathscr{C}$ which colors the open edge of the (1,1)-ribbon graph. We call V the section of T_V. Given such a graph T_V the right braid closure gives a well defined equivalence class of a closed ribbon graph $\widehat{T_V}$ in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$). Let $\mathcal{L}_{\text{adm}}$ be the class of right braid closures of elements of $\mathcal{T}_{\text{adm}}$. Let $\mathsf{t} = \{\mathsf{t}_V : \operatorname{End}_{\mathscr{C}}(V) \to \Bbbk\}_V$ be a family of linear maps where V runs over all the sections of elements $T_V \in \mathcal{T}_{\text{adm}}$. Suppose that t satisfies the condition:

If $T_V, T'_W \in \mathcal{T}_{\text{adm}}$ such that $\widehat{T_V}$ is isotopic to $\widehat{T'_W}$ then $\mathsf{t}_V(\mathsf{F}(T_V)) = \mathsf{t}_W\big(\mathsf{F}(T'_W)\big)$.

We call the function

$$\mathsf{F}' : \mathcal{L}_{\text{adm}} \to \Bbbk \text{ defined by } \mathsf{F}'\big(\widehat{T_V}\big) = \mathsf{t}_V(\mathsf{F}(T_V))$$

the *renormalized invariant* associated with $\mathcal{L}_{\text{adm}}$ and t.

We will now give some examples of renormalized invariants.

Example 1 T-ambi pair Let A be a class of simple objects in a pivotal $\Bbbk$-category $\mathscr{C}$. The classes $\mathcal{T}_{\text{adm}}$ and $\mathcal{L}_{\text{adm}}$ are formed by the trivalent ribbon planar graphs whose edges are colored by elements of A. The family $\mathsf{t} = (\mathsf{t}_V)_{V\in\mathsf{A}}$ is determined by a mapping $\mathsf{d} : \mathsf{A} \to \Bbbk^\times$ constant on isomorphism classes of objects. Then the map t_V is determined by $\mathsf{t}_V(\lambda \operatorname{Id}_V) = \lambda\mathsf{d}(V)$. If F' is invariant by isotopy in the sphere S^2 (here we consider an isomorphism $S^2 \simeq \mathbb{R}^2 \cup \{\infty\}$) then we say that (A, d) is a *trivalent-ambidextrous pair* or *t-ambi* for short. A (modified) $6j$-symbol is the value of a tetrahedron under F'. These $6j$-symbols are the elementary algebraic ingredients of a renormalized Turaev–Viro-type invariant of 3-manifolds defined by state sums on triangulations [20,22].

Example 2 Right trace Let t be a right trace on a right ideal $\mathcal{I}$ in a pivotal $\Bbbk$-category $\mathscr{C}$, see Sect. 1.3. Let $\mathcal{T}_{\text{adm}}$ be all the $\mathscr{C}$-colored (1,1)-ribbon planar graphs whose sections are in $\mathcal{I}$. Then F' is a invariant of planar isotopy but in general it is not an invariant of isotopy in the sphere S^2. Nevertheless, for $V \in \mathcal{I}$ one can set $\mathsf{d}(V) = \mathsf{t}_V(\operatorname{Id}_V)$ then Corollary 7 of [23] implies that for $\mathsf{A} = \{V \in \mathcal{I} : V \text{ is simple}, V^* \in \mathcal{I}, \mathsf{d}(V) = \mathsf{d}(V^*)\}$, (A, d) is a trivalent-ambidextrous pair and the restriction of F' to A-colored graphs is an invariant of isotopy in the sphere S^2.

Example 3 (Two-sided) trace Let $\mathscr{C}$ be a pivotal (resp. ribbon) $\Bbbk$-category and let t be a trace on an ideal $\mathcal{I}$. Again, let $\mathcal{T}_{\text{adm}}$ be all $\mathscr{C}$-colored (1,1)-ribbon planar (resp. spatial) graphs whose sections are in $\mathcal{I}$. By setting $\mathsf{d}(V) = \mathsf{t}_V(\operatorname{Id}_V)$ for $V \in \mathcal{I}$ then Theorem 5 of [23] implies F' is an invariant of spherical (resp. spatial) ribbon graphs. Moreover, $\mathcal{L}_{\text{adm}}$ is formed by the $\mathscr{C}$-colored ribbon graphs with at least one edge in $\mathcal{I}$.

1.6 $\mathcal{G}$-graded and generically $\mathcal{G}$-semi-simple categories

We now fix a group $\mathcal{G}$.

Definition 1 Grading: A pivotal $\Bbbk$-category is $\mathcal{G}$-graded if for each $g \in \mathcal{G}$ we have a nonempty full subcategory $\mathscr{C}_g$ of $\mathscr{C}$ stable by retract such that

1. $\mathscr{C} = \bigoplus_{g\in\mathcal{G}} \mathscr{C}_g$,

2. if $V \in \mathscr{C}_g$, then $V^* \in \mathscr{C}_{g^{-1}}$,
3. if $V \in \mathscr{C}_g$, $V' \in \mathscr{C}_{g'}$ then $V \otimes V' \in \mathscr{C}_{gg'}$,
4. if $V \in \mathscr{C}_g$, $V' \in \mathscr{C}_{g'}$ and $\mathrm{Hom}_{\mathscr{C}}(V, V') \neq 0$, then $g = g'$.

For a subset $\mathcal{X} \subset \mathcal{G}$ we say:

1. $\mathcal{X}$ is *symmetric* if $\mathcal{X}^{-1} = \mathcal{X}$,
2. $\mathcal{X}$ is *small* in $\mathcal{G}$ if the group $\mathcal{G}$ can not be covered by a finite number of translated copies of $\mathcal{X}$, in other words, for any $g_1, \ldots, g_n \in \mathcal{G}$, we have $\bigcup_{i=1}^n (g_i\mathcal{X}) \neq \mathcal{G}$.

Definition 2 Semi-simplicity:

1. A $\Bbbk$-category $\mathscr{C}$ is *semi-simple* if all its objects are semi-simple.
2. A $\Bbbk$-category $\mathscr{C}$ is *finitely semi-simple* if it is semi-simple and has finitely many isomorphism classes of simple objects.
3. A $\mathcal{G}$-graded category $\mathscr{C}$ is a generically $\mathcal{G}$-semi-simple category (resp. generically finitely $\mathcal{G}$-semi-simple category) if there exists a small symmetric subset $\mathcal{X} \subset \mathcal{G}$ such that for each $g \in \mathcal{G}\backslash\mathcal{X}$, $\mathscr{C}_g$ is semi-simple (resp. finitely semi-simple). We call $\mathcal{X}$ the *singular locus* of $\mathscr{C}$. By a *generic simple* object we mean a simple object of $\mathscr{C}_g$ for some $g \in \mathcal{G}\backslash\mathcal{X}$.

Remark 2 For a generically $\mathcal{G}$-semi-simple category $\mathscr{C}$ with singular locus $\mathcal{X}$, its ideal Proj of projective objects contains all objects of $\mathscr{C}_g$ for $g \in \mathcal{G}\backslash\mathcal{X}$. In particular, generic simple objects of $\mathscr{C}$ are projective.

The notion of generically $\mathcal{G}$-semi-simple categories appears in [20,22] through the following generalization of fusion categories (in particular, fusion categories satisfy the following definition when $\mathcal{G}$ is the trivial group, $\mathcal{X} = \emptyset$ and $\mathsf{d} = \mathsf{b} = \mathrm{qdim}_{\mathscr{C}}$ is the quantum dimension):

Definition 3 (Relative $\mathcal{G}$-spherical category) Let $\mathscr{C}$ be a generically finitely $\mathcal{G}$-semi-simple pivotal $\Bbbk$-category with singular locus $\mathcal{X} \subset \mathcal{G}$ and let A be the class of generic simple objects of $\mathscr{C}$. We say that $\mathscr{C}$ is $(\mathcal{X}, \mathsf{d})$-relative $\mathcal{G}$-spherical if

1. there exists a map $\mathsf{d} : \mathsf{A} \to \Bbbk^\times$ such that (A, d) is a t-ambi pair,
2. there exists a map $\mathsf{b} : \mathsf{A} \to \Bbbk$ such that $\mathsf{b}(V) = \mathsf{b}(V^*)$, $\mathsf{b}(V) = \mathsf{b}(V')$ for any isomorphic objects $V, V' \in \mathsf{A}$ and for any $g_1, g_2, g_1g_2 \in \mathcal{G}\backslash\mathcal{X}$ and $V \in \mathcal{G}_{g_1g_2}$ we have
$$\mathsf{b}(V) = \sum_{V_1 \in irr(\mathscr{C}_{g_1}),\, V_2 \in irr(\mathscr{C}_{g_2})} \mathsf{b}(V_1)\mathsf{b}(V_2) \dim_{\Bbbk}(\mathrm{Hom}_{\mathscr{C}}(V, V_1 \otimes V_2))$$
where $\mathrm{irr}(\mathscr{C}_{g_i})$ denotes a representing set of isomorphism classes of simple objects of $\mathscr{C}_{g_i}$.

We finish this section by recalling the following theorem and corollary which we use later to show certain algebras are semi-simple. For a proof of the theorem see [15].

Theorem 1 *(the Density Theorem) Let A be an algebra over an algebraically closed field k. Let $\{V_i\}_i$ be a set of irreducible pairwise nonisomorphic finite-dimensional modules over A. Then the map $\oplus_i \rho_i : A \to \bigoplus_i \mathrm{End}_k(V_i)$ is surjective.*

Corollary 1 *Let $\mathscr{C}$ be the category of finite-dimensional modules of the finite-dimensional algebra A. Let $\{V_i\}_i$ be a set of irreducible pairwise nonisomorphic finite-dimensional modules over A. If*

$$\dim(A) = \sum_i \dim(V_i)^2$$

then $\mathscr{C}$ is semi-simple.

Proof From the Density Theorem we know that $\oplus_i \rho_i : A \to \bigoplus_i End_k(V_i)$ is surjective. The assumption on the dimensions implies that this map is a injective. Thus, A is isomorphic to the direct sum of matrix algebras which is semi-simple and it follows that $\mathscr{C}$ is semi-simple. □

2 Extension of generic properties

Let $\mathscr{C}$ be a pivotal $\Bbbk$-category. In this section we present two theorems which extend properties observed for generic simple objects of $\mathscr{C}$ to general properties in the full category.

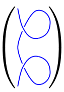
Proposition 1 *If $\mathscr{C}$ is a braided pivotal $\Bbbk$-category then the class of objects*

$$\{V \in \mathscr{C} : \theta_{V^*} = (\theta_V)^*\}$$

forms a full subcategory of $\mathscr{C}$ which is ribbon.

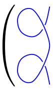
Proof For each object $V \in \mathscr{C}$, consider the automorphism

$$\mathscr{E}_V = \mathrm{ptr}_R(c_{V,V}^{-1}) \circ \mathrm{ptr}_R(c_{V,V}) = \mathsf{F}\left(\quad \right) \in \mathrm{End}_{\mathscr{C}}(V).$$



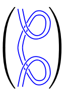
The family $(\mathscr{E}_V)_{V\in\mathscr{C}}$ defines a natural transformation (as $\mathscr{E}_*$ is an automorphism of the identity functor, its naturality is just the fact that $\mathscr{E}_V f = f\mathscr{E}_U$ for any $f : U \to V$). Its inverse is given by

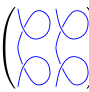
$$\mathscr{E}_V^{-1} = \mathsf{F}\left(\quad \right) = \mathrm{ptr}_L(c_{V,V}^{-1}) \circ \mathrm{ptr}_L(c_{V,V}) = \mathrm{ptr}_L(c_{V,V}) \circ \mathrm{ptr}_L(c_{V,V}^{-1}).$$



The naturality of $c_{V,V}$ implies that the image by F of a $\mathscr{C}$-colored diagram is invariant by Reidemester II and III moves so that for any $V, W \in \mathscr{C}$

$$\mathscr{E}_{V\otimes W} = \mathsf{F}\left(\quad \right) = \mathsf{F}\left(\quad \right) = \mathscr{E}_V \otimes \mathscr{E}_W\,, \tag{2}$$





showing that $\mathscr{E}_*$ is a monoidal transformation. Finally, the dual of the right partial trace is given by the left partial trace so

$$(\mathscr{E}_V)^* = \mathscr{E}_{V^*}^{-1}. \tag{3}$$

For any object $V \in \mathscr{C}$, $(\theta_V)^* = (\mathscr{E}_V)^* \circ \theta_{V^*}$ so $(\theta_V)^* = \theta_{V^*} \iff \mathscr{E}_V = \mathrm{Id}_V$ and the class in the proposition is clearly a subpivotal category of $\mathscr{C}$ because of properties (2), (3) of the natural automorphism $\mathscr{E}_*$. $\square$

The following theorem generalizes an argument of Ha (see [24, Proposition 3.7]).

Theorem 2 *Let $\mathscr{C}$ be a generically $\mathcal{G}$-semi-simple pivotal braided category. If $\theta(V)^* = \theta(V^*)$ holds for any generic simple object V, then $\mathscr{C}$ is a ribbon category, i.e., θ is a twist on the full category $\mathscr{C}$.*

Proof Consider the natural automorphism $\mathscr{E}_*$ of the proof of Proposition 1. By assumption we have $\mathscr{E}_V = \mathrm{Id}_V$ for any generic simple object V. By naturality, this property is stable by direct sums so it is also true for any semi-simple object that is a direct sum of generic simple objects. Now for any homogenous object $W \in \mathscr{C}_g$, let $h \in \mathcal{G}$ be such that $h, hg \notin \mathcal{X}$ and let $V \in \mathscr{C}_h$. Then $V \otimes W \in \mathscr{C}_{\mathrm{hg}}$ is semi-simple and

$$\mathrm{Id}_V \otimes \mathrm{Id}_W = \mathscr{E}_{V\otimes W} = \mathscr{E}_V \otimes \mathscr{E}_W = \mathrm{Id}_V \otimes \mathscr{E}_W$$

thus $\mathscr{E}_W = \mathrm{Id}_W$ and $\theta(W)^* = \theta(W^*)$. $\square$

For the second theorem we first recall the relation between (right) trace and duality:

Proposition 2 *(see [23, Lemmas 2 & 3])*

1. *If $\mathcal{I}$ is a right ideal then $\mathcal{I}^* = \{V \in \mathscr{C},\ V^* \in \mathcal{I}\}$ is a left ideal and*

$$\mathcal{I} \textit{ is an ideal} \iff \mathcal{I}^* = \mathcal{I}.$$

2. *If t is a right trace on the right ideal $\mathcal{I}$ then t^* is a left trace on $\mathcal{I}^*$ where by definition, $\mathsf{t}^*_V(f) = \mathsf{t}_{V^*}(f^*)$ for any $V \in \mathcal{I}$, any $f \in \mathrm{End}_{\mathscr{C}}(V)$ and*

$$\mathsf{t} \textit{ is a trace} \iff \mathsf{t}^* = \mathsf{t}.$$

Definition 4 1. A pseudo-monoid class in a pivotal category $\mathscr{C}$ is a class of objects stable by dual, retracts and tensor product.
2. Let t be a right trace on a right ideal $\mathcal{I}$. The horizontal part of $\mathcal{I}$ for t is the class

$$\mathcal{I}^{=} = \{V \in \mathcal{I} : V^* \in \mathcal{I} \text{ and } \mathsf{t}_V = \mathsf{t}^*_V\}.$$

Remark 3 An ideal of a pivotal category is a pseudo-monoid and the converse is partially true. Indeed one could define the idealizer of a pseudo-monoid $\mathcal{M}$ to be the full subcategory of $\mathscr{C}$ whose objects are

$$\mathbb{I}_{\mathscr{C}}(\mathcal{M}) = \{V \in \mathscr{C} : V \otimes \mathcal{M} \subset \mathcal{M} \supset \mathcal{M} \otimes V\}.$$

Then one can prove that $\mathbb{I}_{\mathscr{C}}(\mathcal{M})$ is a subpivotal category of $\mathscr{C}$ which contains $\mathcal{M}$ such that $\mathcal{M}$ is an ideal in $\mathbb{I}_{\mathscr{C}}(\mathcal{M})$.

Proposition 3 *Let* t *be a right trace on a right ideal* $\mathcal{I}$. *Then the horizontal part* $\mathcal{I}^{=}$ *of* $\mathcal{I}$ *is a pseudo-monoid. Furthermore,* $\mathcal{I}^{=}$ *is stable by direct sum in the following weak sense: if* $\{V_i\}_i$ *is a finite family of object of* $\mathcal{I}^{=}$ *and* $W \in \mathcal{I} \cap \mathcal{I}^*$ *is isomorphic to their direct sum, then* $W \in \mathcal{I}^{=}$.

Proof If V is a retract of $W \in \mathcal{I}^{=}$ then there are $\alpha : V \to W$, $\beta : W \to V$ with $\beta\alpha = \mathrm{Id}_V$. Then V^* is a retract of $W^* \in \mathcal{I}^{=}$ so $V^* \in \mathcal{I}$. For any $f \in \mathrm{End}_{\mathscr{C}}(V)$, by definition $\mathsf{t}^*_V(f) = \mathsf{t}_{V^*}(f^*)$ but

$$\mathsf{t}_V(f) = \mathsf{t}_V(f\beta\alpha) = \mathsf{t}_W(\alpha f \beta) = \mathsf{t}_{W^*}(\beta^* f^* \alpha^*) = \mathsf{t}_{V^*}(\alpha^*\beta^* f^*) = \mathsf{t}_{V^*}(f^*).$$

So $V \in \mathcal{I}^{=}$ and $\mathcal{I}^{=}$ is stable by retract (recall in particular that an isomorphic object is a retract).

Let $V \in \mathcal{I}^{=}$ and let $\phi_V : V \xrightarrow{\sim} V^{**}$ be the pivotal isomorphism. Then for any $f \in \mathrm{End}_{\mathscr{C}}(V)$, $\mathsf{t}_V(f) = \mathsf{t}_V(\phi_V^{-1} f^{**} \phi_V) = \mathsf{t}_{V^{**}}(f^{**}\phi_V\phi_V^{-1}) = \mathsf{t}_{V^{**}}(f^{**})$. Given $g \in \mathrm{End}_{\mathscr{C}}(V^*)$ then g is the dual of $f = \phi_V g^* \phi_V^{-1} \in \mathrm{End}_{\mathscr{C}}(V)$ so

$$\mathsf{t}_{V^*}(g) = \mathsf{t}_{V^*}(f^*) = \mathsf{t}_V(f) = \mathsf{t}_{V^{**}}(f^{**}) = \mathsf{t}^*_{V^*}(g)$$

and $(\mathcal{I}^{=})^* \subset \mathcal{I}^{=}$.

Let $V, W \in \mathcal{I}^{=}$ and let $f \in \mathrm{End}_{\mathscr{C}}(V \otimes W)$. Then $V^* \in \mathcal{I}$ and t is right ambidextrous by [23, Lemma 4] so $\mathsf{t}(\mathrm{ptr}_R(f)^*) = \mathsf{t}(\mathrm{ptr}_L(f')) = \mathsf{t}(\mathrm{ptr}_L(f))$ where $f' = (\phi_V \otimes \mathrm{Id}_W) f (\phi_V^{-1} \otimes \mathrm{Id}_W)$ so

$$\begin{aligned}\mathsf{t}(f) = \mathsf{t}(\mathrm{ptr}_R(f)) &= \mathsf{t}(\mathrm{ptr}_R(f)^*) = \mathsf{t}(\mathrm{ptr}_L(f)) \\ &= \mathsf{t}(\mathrm{ptr}_L(f)^*) = \mathsf{t}(\mathrm{ptr}_R(f^*)) = \mathsf{t}(f^*)\end{aligned}$$

thus $V \otimes W \in \mathcal{I}^{=}$.

If W is a direct sum of objects of $\mathcal{I}^{=}$ with associated projectors $p_i = \alpha_i\beta_i$, then so is W^* with projectors p_i^*. Now for any $f \in \mathrm{End}_{\mathscr{C}}(W)$, $f = \left(\sum_i p_i\right) f \left(\sum_i p_i\right) = \sum_{i,j} p_i f p_j$ so

$$\mathsf{t}(f) = \sum_i \mathsf{t}(p_i f p_i) + \sum_{i \neq j} \cancel{\mathsf{t}(f p_j p_i)} = \sum_i \mathsf{t}\big(\beta_i f \alpha_i \cancel{\beta_i \alpha_i}\big) = \sum_i \mathsf{t}\big(\alpha_i^* f^* \beta_i^*\big) = \mathsf{t}(f^*).$$

□

An object $V \in \mathscr{C}$ is nonzero if $\mathrm{Id}_V \neq 0$. For the next theorem, we will make the following assumption that is true for example if $\mathscr{C}$ is a category of finite-dimensional representations of a Hopf $\Bbbk$-algebra: We will assume that $\mathscr{C}$ is a Krull-Schmidt category and that for any nonzero $V \in \mathscr{C}$, $V \otimes \cdot$ and $\cdot \otimes V$ are faithful functors. The former implies that any object in $\mathscr{C}$ is isomorphic to an unique direct sum of indecomposable objects (an object is indecomposable if it is not isomorphic to a direct sum of two nonzero objects).

Theorem 3 *Let $\mathscr{C}$ be a generically $\mathcal{G}$-semi-simple pivotal category with the above assumption. Let* t *be a right trace on* Proj *and let* $\mathsf{d} = \{\mathsf{d}(V) \in \Bbbk\}_{V\in\mathsf{Proj}}$ *denote the associated modified dimension. If* $\mathsf{d}(V) = \mathsf{d}(V^*)$ *holds for any generic simple object* V*, then* t *is a trace on* Proj.

Proof By assumption we have that any generic simple object V is in $\mathsf{Proj}^=$ as $\mathrm{End}_{\mathscr{C}}(V) \simeq \Bbbk \simeq \mathrm{End}_{\mathscr{C}}(V^*)$ so the right trace on these endomorphisms is determined by $\mathsf{d}(V) = \mathsf{d}(V^*)$. Since $\mathsf{Proj}^=$ is stable by direct sums it contains any semi-simple object that is a direct sum of generic simple objects. Let W be a indecomposable projective object. Then there exists g such that $W \in \mathscr{C}_g$. Let $h \in \mathcal{G}$ be such that $h, h^{-1}g \notin \mathcal{X}$ (recall that $\mathcal{X}$ is small) and let $V \in \mathscr{C}_h$. Then $W \in \mathcal{I}_V = \mathsf{Proj}$ thus there exists $U \in \mathscr{C}$ such that W is a retract of $V \otimes U$. As $\mathscr{C} = \bigoplus_{g\in\mathcal{G}} \mathscr{C}_g$, we can assume up to replacing U by one of its summand that $U \in \mathscr{C}_{h^{-1}g}$. Then U is a semi-simple direct sum of generic simple objects and an element of $\mathsf{Proj}^=$. Since $\mathsf{Proj}^=$ is a pseudo-monoid then $V \otimes U \in \mathsf{Proj}^=$ and $W \in \mathsf{Proj}^=$ as W is a retract. In addition, since $\mathsf{Proj}^=$ is stable by direct sums we get that $W \in \mathsf{Proj}^=$ even if $W \in \mathsf{Proj}$ is not indecomposable. Thus, $\mathsf{Proj}^= = \mathsf{Proj}$ and $\mathsf{t} = \mathsf{t}^*$ on Proj. In other words, t is a trace. □

3 Quantum groups at roots of unity

In this section we first recall some of the deep results established by De Concini, Kac, Procesi, Reshetikhin and Rosso in the series of papers [11–14]. Then we observe that the twist θ and the modified dimension d are generically self-dual. As a consequence we get the existence of a ribbon structure associated with the unrolled quantum group and the existence of a trace on Proj for the categories of weight modules over the small, unrolled and unrestricted groups.

3.1 The small, the unrolled and the unrestricted

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra of rank n and dimension $2N + n$ with the following:

1. a Cartan subalgebra $\mathfrak{h}$,
2. a root system consisting in simple roots $\{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$,
3. a Cartan matrix $A = (a_{ij})_{1\le i,j\le n}$,
4. a set Δ^+ of N positive roots,
5. a root lattice $L_R = \bigoplus_i \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$,
6. a scalar product $\langle\cdot,\cdot\rangle$ on the real span of L_R given by its matrix $DA = \big(\langle\alpha_i, \alpha_j\rangle\big)_{ij}$ where $D = \mathrm{diag}(d_1, \dots, d_n)$ and the minimum of all the d_i is 1.

The Cartan subalgebra has a basis $\{H_i\}_{i=1\cdots n}$ determined by $\alpha_i(H_j) = a_{ji}$ and its dual basis of $\mathfrak{h}^*$ is the fundamental weights basis which generate the lattice of weights L_W. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha \in L_W$.

Let q be an indeterminate and for $i = 1, \dots, n$, let $q_i = q^{d_i}$. Let ℓ be an integer such that $\ell \ge 2$ (and $\ell \notin 3\mathbb{Z}$ if $\mathfrak{g} = G_2$). Let $\xi = \mathrm{e}^{2\sqrt{-1}\pi/\ell}$ and $r = \frac{2\ell}{3+(-1)^\ell}$. For $x \in \mathbb{C}$ and $k, l \in \mathbb{N}$ we use the notation:

$$\xi^x = \mathrm{e}^{\frac{2i\pi x}{\ell}}, \quad \{x\}_q = q^x - q^{-x},$$
$$[x]_q = \frac{\{x\}_q}{\{1\}_q}, \quad [k]_q! = [1]_q[2]_q \cdots [k]_q, \quad \begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{[k]_q!}{[l]_q![k-l]_q!}.$$

Let $\mathbb{K}_\ell$ be the subring of $\mathbb{C}(q)$ made of fractions that have no poles at ξ ($\mathbb{K}_\ell$ is a localization of $\mathbb{C}[q]$). A $\mathbb{K}_\ell$-module can be specialized at $q = \xi$ (the specialization is the tensor product with the $\mathbb{K}_\ell$-module $\mathbb{C}$ where q acts by ξ).

For each lattice L with $L_R \subset L \subset L_W$, there is an associated quantum group which contains the group ring of L: define $\mathcal{U}_q^L$ as the $\mathbb{K}_\ell$-algebra with generators $K_\beta,\ X_i,\ X_{-i}$ for $\beta \in L,\ i = 1, \ldots, n$ and relations

$$K_0 = 1, \quad K_\beta K_\gamma = K_{\beta+\gamma}, \quad K_\beta X_{\sigma i} K_{-\beta} = q^{\sigma\langle\beta,\alpha_i\rangle} X_{\sigma i}, \tag{4}$$
$$[X_i, X_{-j}] = \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}}, \tag{5}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} X_{\sigma i}^k X_{\sigma j} X_{\sigma i}^{1-a_{ij}-k} = 0, \text{ if } i \neq j \tag{6}$$

where $\sigma = \pm 1$. Drinfeld and Jimbo consider the quantum group corresponding to L_R. The algebra $\mathcal{U}_q^L$ is a Hopf algebra with coproduct Δ, counit ϵ and antipode S defined by

$$\begin{aligned}
&\Delta(X_i) = 1 \otimes X_i + X_i \otimes K_{\alpha_i}, \quad \Delta(X_{-i}) = K_{\alpha_i}^{-1} \otimes X_{-i} + X_{-i} \otimes 1, \\
&\Delta(K_\beta) = K_\beta \otimes K_\beta, \qquad \epsilon(X_i) = \epsilon(X_{-i}) = 0, \qquad \epsilon(K_{\alpha_i}) = 1, \\
&S(X_i) = -X_i K_{\alpha_i}^{-1}, \qquad S(X_{-i}) = -K_{\alpha_i} X_{-i}, \qquad S(K_\beta) = K_{-\beta}.
\end{aligned}$$

The *unrestricted quantum group* $\mathcal{U}_\xi^L$ is the $\mathbb{C}$-algebra obtained from $\mathcal{U}_q^L$ by specializing q to ξ.

The *unrolled quantum group* $\mathcal{U}^H = \mathcal{U}_\xi^H$ is the algebra generated by K_β, X_i, X_{-i}, H_i for $\beta \in L,\ i = 1, \ldots, n$ with Relations (4), (5), (6) where $q = \xi$ plus the relations

$$[H_i, X_{\sigma j}] = \sigma a_{ij} X_{\sigma j}, \qquad [H_i, H_j] = [H_i, K_\beta] = 0 \tag{7}$$

where $\sigma = \pm 1$. The algebra $\mathcal{U}^H$ is a Hopf algebra with coproduct Δ, counit ϵ and antipode S defined as above on $K_\beta,\ X_i,\ X_{-i}$ and defined on the elements H_i for $i = 1, \ldots, n$ by

$$\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1, \qquad \epsilon(H_i) = 0, \qquad S(H_i) = -H_i.$$

For a fixed choice of a convex order $\beta_* = (\beta_1, \ldots, \beta_N)$ of Δ^+ define recursively a convex set of root vectors $(X_{\pm\beta})_{\beta\in\beta_*}$ in $\mathcal{U}_\xi^L$ (see Sects. 8.1 and 9.1 of [8]). The *small* *quantum group* $\overline{\mathcal{U}}_\xi^L$ is a finite-dimensional quotient of $\mathcal{U}_\xi^L$. When ℓ is odd or $\mathfrak{g}$ is simply laced,[1] it is the quotient of $\mathcal{U}_\xi^L$ by the relations

$$X_{\pm\beta}^r = 0 \text{ and } K_\gamma^{2r} = 1 \text{ for all } \beta \in \Delta_+, \gamma \in L_R.$$

The dimension of $\overline{\mathcal{U}}_\xi^L$ is then $r^{2N}(2r)^n[L:L_R]$. From now on if it is clear we will not write the superscript L in $\mathcal{U}_\xi^L$ or $\overline{\mathcal{U}}_\xi^L$.

3.2 Pivotal structures

For any central group-like element c, the element $\phi = cK_{2\rho}$ in U where U is $\mathcal{U}_\xi, \mathcal{U}^H$ or $\overline{\mathcal{U}}_\xi$ is group-like and satisfies $S^2(x) = \phi x \phi^{-1}$. It follows that U is a pivotal Hopf algebra and the category of finite-dimensional U-modules is a pivotal $\mathbb{C}$-category, for details see [5,20]. Here the dual is given by the dual vector space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ equipped with the transpose action composed with antipodal morphism S. The duality morphisms $\overrightarrow{\mathrm{ev}}$ and $\overrightarrow{\mathrm{coev}}$ are the standard evaluation and coevaluation, whereas $\overleftarrow{\mathrm{ev}} = \overrightarrow{\mathrm{ev}} \circ \tau \circ (\phi \otimes 1)$ and $\overleftarrow{\mathrm{coev}} = \tau \circ (\phi^{-1} \otimes 1) \circ \overrightarrow{\mathrm{coev}}$ where τ is the flip.

Remark 4 The h-adic version of the quantum group $U_h(\mathfrak{g}) = U_h$ is the $\mathbb{C}[[h]]$-topological Hopf algebra with generators X_i, X_{-i}, H_i for $i = 1, \ldots, n$ and Relations (5), (6) and (7) where q and K_{α_i} are replaced by $\mathrm{e}^{h/2}$ and $q_i^{H_i}$, respectively. The category of finite-dimensional topologically free U_h-modules is a pivotal category where the pivotal element is normally given by $K_{2\rho}$. In this paper, for $\mathfrak{g} = \mathfrak{sl}_2$ or when the order ℓ of the root of unity is odd, we use $\phi = K_{2\rho}^{1-r}$ instead of $K_{2\rho}$ which also defines a pivotal structure. The reason we make this choice is because the pivotal structure for $\phi = K_{2\rho}^{1-r}$ is compatible with the braiding and trace consider in future subsections. The pivotal structure corresponding to $K_{2\rho}$ does not have such compatibilities: the left and right twist differ by a square and the modified dimensions of V and V^* are not equal.

4 Quantum groups at odd ordered roots of unity

In this section we consider the case when ℓ is an odd integer. In this case, $\ell = r$. To make the notation clear we will write $\mathscr{C}_{\mathrm{odd}}$, $\mathscr{C}_{\mathrm{odd}}^H$ and $\overline{\mathscr{C}}_{\mathrm{odd}}$ for the categories in this section.

4.1 Weight modules

Let Z_0 be the subalgebra of $\mathcal{U}_\xi^L$ generated by $\{X_{\pm\beta}^r, K_\gamma^r : \beta \in \Delta_+, \gamma \in L\}$. Then Z_0 is a sub-Hopf algebra contained in the center of $\mathcal{U}_\xi^L$. Moreover, Z_0 is isomorphic to

[1] Simon Lentner pointed to us that the non simply laced case with even root of unity is more tricky (see [26, Sect. 8.2.]).

the Hopf algebra of regular functions of a group $\mathcal{G} = \mathrm{Spec}(Z_0)$ which is Poisson-Lie dual to a complex simple Lie group with Lie algebra $\mathfrak{g}$ (see [12]). Here $\mathrm{Spec}(Z_0)$ is the algebraic variety of algebra homomorphisms from Z_0 to $\mathbb{C}$.

By a $\mathcal{U}_\xi$-*weight module* we mean a finite-dimensional module over $\mathcal{U}$ which restrict to a semi-simple module over Z_0. Since $X^r_{\pm\beta} = 0$ and $K^{2r}_\gamma = 1$ in $\overline{\mathcal{U}}_\xi$ we say all finite-dimensional $\overline{\mathcal{U}}_\xi$-modules are *weight modules*. Finally, a finite-dimensional $\mathcal{U}^H$-module V is a $\mathcal{U}^H$-*weight module* if it is a semi-simple module over the subalgebra generated by $\{H_i : i = 1 \cdots n\}$ and for any $\gamma = \sum_i c_i\alpha_i \in L$ then $q^{\sum_i d_i c_i H_i} = K_\gamma$ as operators on V. Let $\mathscr{C}_{\mathrm{odd}}$ (resp. $\mathscr{C}^H_{\mathrm{odd}}$, resp. $\overline{\mathscr{C}}_{\mathrm{odd}}$) be the category of $\mathcal{U}_\xi$ (resp. $\mathcal{U}^H$, resp. $\overline{\mathcal{U}}_\xi$) weight modules. As the action of each K_β is determined by the action of the H_i's it follows that $\mathscr{C}^H_{\mathrm{odd}}$ does not depend on the lattice L.

For $g \in \mathcal{G} = \mathrm{Hom}_{\mathrm{alg}}(Z_0, \mathbb{C})$ let $\mathscr{C}_g$ be the full subcategory of $\mathscr{C}_{\mathrm{odd}}$ whose objects are modules where each $z \in Z_0$ act by $g(z)$. For $g \in \mathcal{G}$ with $g(X^r_{\pm\beta}) = 0$ for all $\beta \in \Delta_+$ we say g is *diagonal* and the modules of $\mathscr{C}_g$ are *nilpotent*. Similarly, $\mathscr{C}^H_{\mathrm{odd}}$ is graded by the group $\mathcal{D}$ of diagonal elements of $\mathcal{G}$.

The dual of a weight module is a weight module. This implies that the category $\mathscr{C}_{\mathrm{odd}}$ (resp. $\mathscr{C}^H_{\mathrm{odd}}$, resp. $\overline{\mathscr{C}}_{\mathrm{odd}}$) is a subpivotal $\mathbb{C}$-category of the category of all finite-dimensional modules.

4.2 The braiding on $\mathscr{C}^H_{\mathrm{odd}}$

In this subsection we briefly recall the existence of a braiding for $\mathscr{C}^H_{\mathrm{odd}}$. We refer to Theorem 41 of [20] for details.

Recall the h-adic quantum group $U_h(\mathfrak{g}) = U_h$ defined in Remark 4. For a root $\beta \in L_R$, let $q_\beta = q^{\langle\beta,\beta\rangle/2}$. Let $\exp_q(x) = \sum_{i=0}^{\infty} \frac{(1-q)^i x^i}{(1-q^i)(1-q^{i-1})\cdots(1-q)}$. Consider the following elements of the h-adic version of the quantum group U_h:

$$HH^h = q^{\sum_{i,j} d_i (A^{-1})_{ij} H_i \otimes H_j}, \quad \check{R}^h = \prod_{\beta\in\beta_*} \exp_{q_\beta^{-2}}\left(\left(q_\beta - q_\beta^{-1}\right) X_\beta \otimes X_{-\beta}\right) \tag{8}$$

where the product is ordered by the convex order β_* of Δ^+. It is well known that $R^h = HH^h \check{R}^h$ defines a quasi-triangular structure on U_h (see for example [27]). Let $\exp^<_q(x) = \sum_{i=0}^{r-1} \frac{(1-q)^i x^i}{(1-q^i)(1-q^{i-1})\cdots(1-q)}$. Replacing the occurrences of $\exp_{q_\beta^{-2}}$ with $\exp^<_{q_\beta^{-2}}$ in the formula of $\check{R}^h$ gives an element of $\mathcal{U}_q \otimes \mathcal{U}_q$ (seen as a subalgebra of $U_h[h^{-1}]^{\otimes 2}$) called the truncated quasi-R-matrix. Specializing for $q = \xi$ we get an element $\check{R}^< \in \mathcal{U}_\xi \otimes \mathcal{U}_\xi$. The action of $R^< = HH^h \check{R}^<$ determines a braiding on $\mathscr{C}^H$ given by $c_{V,W}(v \otimes w) = \tau(R^<(v \otimes w))$ where τ is the flip. Summarizing we have:

Proposition 4 *The category $\mathscr{C}^H_{\mathrm{odd}}$ is a braided pivotal category.*

4.3 Semi-simple

Here we explain why $\mathscr{C}_{\mathrm{odd}}$ and $\mathscr{C}^H_{\mathrm{odd}}$ are generically semi-simple categories. The g-th graded piece of $\mathscr{C}_{\mathrm{odd}}$ can be described using ideals of the algebra: $\mathscr{C}_g$ is the category of weight module whose annihilator contains the two-sided ideal I_g generated by $\{z - g(z) : z \in Z_0\}$. Hence $\mathscr{C}_g$ is identified with the category of finite-dimensional modules of the finite-dimensional algebra $\mathcal{U}_\xi / I_g$. For generic g, we will prove that the algebra $\mathcal{U}_\xi / I_g$ is semi-simple which will imply the category $\mathscr{C}_g$ is semi-simple.

A diagonal element of $g \in \mathcal{G}$ is *regular* if $g(K^r_{\pm\beta}) \neq \pm 1$ for all $\beta \in \Delta_+$. For regular g, the category $\mathscr{C}_g$ has r^n nonisomorphic highest weight irreducible modules V_i. These modules are also modules over $\mathcal{U}_\xi^{L_R}$ which is a subset of $\mathcal{U}_\xi^L$. Then Corollary 3.2 of [11] implies that these modules have dimension r^N. Therefore, we have $\sum_i \dim(V_i)^2 = r^{2N+n}$. The PBW theorem implies $\dim(\mathcal{U}_\xi^L) = \sum_i \dim(V_i)^2$. Thus, Corollary 1 implies that for regular g, the category $\mathscr{C}_g$ is semi-simple.

De Concini and Kac consider certain derivations $\underline{e}_i$ and $\underline{f}_i$ for $i = 1, ..., n$. For each $t \in \mathbb{C}$, in [11, Section 3.5], it is shown these derivations give automorphisms $\exp t\underline{e}_i$ and $\exp t\underline{f}_i$ of (a completion of) the algebra $\mathcal{U}_\xi$. Let $\widetilde{G}$ be the group generated by all these automorphisms. This group leaves Z_0 invariant and acts as holomorphic transformations on the algebraic variety $\mathrm{Spec}(Z_0)$. The action of $\widetilde{G}$ on $\mathrm{Spec}(Z_0)$ is called the quantum coadjoint action. Theorem 6.1 of [12] considers the orbits of this action. In particular, part (d) of this theorem says that the union of all $\widetilde{G}$ orbits which contain at least one regular element of $\mathcal{G}$ is Zariski open and dense in $\mathcal{G}$. This can be reformulated as there exist a set $\mathcal{X}$ with the following two properties: (1) $\mathcal{G}\backslash\mathcal{X}$ is a Zariski dense open subset of $\mathcal{G}$ and (2) for each $g \in \mathcal{G}\backslash\mathcal{X}$ there is a regular d and an outer automorphism of $\mathcal{U}_\xi$ inducing an isomorphism of algebras $\mathcal{U}_\xi / I_g \to \mathcal{U}_\xi / I_d$. Therefore, for each $g \in \mathcal{G}\backslash\mathcal{X}$ the algebra $\mathcal{U}_\xi / I_g$ is semi-simple. By replacing $\mathcal{X}$ with $\mathcal{X} \cup \mathcal{X}^{-1}$ we can assume $\mathcal{X}$ is symmetric. Thus, we have shown $\mathscr{C}_{\mathrm{odd}}$ is a generically finitely $\mathcal{G}$-semi-simple category with singular locus $\mathcal{X}$.

Remark 5 By definition, every $\widetilde{G}$ orbit in $\mathcal{G}$ that contains an element of $\mathcal{G}\backslash\mathcal{X}$ also contains a regular element and thus a diagonal element of $\mathcal{G}$.

Recall $\mathscr{C}^H_{\mathrm{odd}}$ is graded by the group $\mathcal{D}$. Let $\mathcal{X}_\mathcal{D}$ be all the nonregular elements of $\mathcal{D}$. It follows from Lemma 7.1 of [9] that $\mathscr{C}^H_{\mathrm{odd}}$ is a generically $\mathcal{D}$-semi-simple category with the singular locus $\mathcal{X}_\mathcal{D}$.

The results described here and in Sect. 3 imply the following proposition.

Proposition 5 *The category $\mathscr{C}^H_{\mathrm{odd}}$ is a generically $\mathcal{D}$-semi-simple braided pivotal category.*

4.4 The ribbon structure in $\mathscr{C}^H_{\mathrm{odd}}$.

Let V be a generic simple object of $\mathscr{C}^H_{\mathrm{odd}}$. We will show that $\theta_{V^*} = (\theta_V)^*$. Since V is simple, the morphism θ_V is a scalar times the identity of V. Moreover, $(\theta_V)^*$ is the same scalar times Id_{V^*}. The module V is determined by its highest weight which is of the form $\lambda + (r-1)\rho$. Its dual V^* has highest weight $-\lambda + (r-1)\rho$. To compute the twist on V notice that HH^h is the only part of the R-matrix contributing to the following computation:

$$\theta_V(v) = \mathrm{ptr}_R(c_{V,V})(v) = q^{\langle\lambda,\lambda\rangle-(r-1)^2\langle\rho,\rho\rangle}v$$

where v is a highest weight vector of V (see [20] for more details). As $\langle-\lambda,-\lambda\rangle = \langle\lambda,\lambda\rangle$ then a similar calculation shows that the scalar determining θ_{V^*} is equal to the scalar determining both θ_V and $(\theta_V)^*$. Thus, Proposition 5 and the results of this subsection imply that $\mathscr{C}^H_{\mathrm{odd}}$ satisfies the hypothesis of Theorem 2 and so we have:

Theorem 4 *The category $\mathscr{C}^H_{\mathrm{odd}}$ is a ribbon category.*

Note that, in [20] we showed that a subcategory of $\mathscr{C}^H_{\mathrm{odd}}$ is ribbon.

4.5 The trace on the ideal of projective $\mathcal{U}_\xi$-modules

From Theorem 4.7.1 of [19] the ideal Proj of projective $\mathcal{U}_\xi$-modules in $\mathscr{C}_{\mathrm{odd}}$ admits a unique nontrivial right trace which we denote by $\{\mathsf{t}_V\}_{V\in\mathsf{Proj}}$. In this subsection we use Theorem 3 to show this right trace is a trace.

To apply the theorem we need to show $\mathsf{d}(V) = \mathsf{d}(V^*)$ for any generic simple $\mathcal{U}_\xi$-module V. We now explain how we reduce this equality to a computation related to open Hopf links. Let V_0 be the weight $\mathcal{U}_\xi$-module determined by the highest weight vector v_0 such that $X_iv_0 = 0$ for $i = 1, ..., n$ and $K_\gamma v_0 = q^{\sum_i d_ic_i\langle(r-1)\rho,\alpha_i\rangle}v_0$ for $\gamma = \sum_i c_i\alpha_i \in L$. The module V_0 is isomorphic to V_0^*. To simplify notation, in the following computations we identify these modules with an isomorphism. Given a simple module and a morphism $g : W \to W$ define $\langle g\rangle \in \mathbb{C}$ as $g = \langle g\rangle\,\mathrm{Id}_W$. Suppose $f : V_0 \otimes V \to V_0 \otimes V$ is a morphism in $\mathscr{C}$. Then

$$\begin{aligned}\mathsf{d}(V)\big\langle \mathrm{ptr}_L(f)\big\rangle &= \mathsf{t}_V(\mathrm{ptr}_L(f)) = \mathsf{t}_{V_0^*}((\mathrm{ptr}_R(f))^*)\\ &= \mathsf{d}(V_0^*)\big\langle(\mathrm{ptr}_R(f))^*\big\rangle = \mathsf{d}(V_0)\big\langle\mathrm{ptr}_R(f)\big\rangle \end{aligned}\tag{9}$$

where the second equality is from Lemma 4(b) of [23] and the final equality holds because V_0 is isomorphic to V_0^*. Therefore, to compute $\mathsf{d}(V)$ we need to give a morphism $f : V_0 \otimes V \to V_0 \otimes V$ and compute its left and right partial trace. We will do this now using the braiding of $U_h(\mathfrak{g})$.

Recall that the algebra U_h and its R-matrix given in Sect. 4.2. Let V_0^h be the simple U_h-weight module with highest weight $(r-1)\rho$. So V_0^h has dimension r^N and a highest weight vector v_0 such that $H_iv_0 = \langle(r-1)\rho,\alpha_i\rangle v_0$ for $i = 1, ..., n$. The module $V_0^h \otimes \mathbb{C}[h^{-1}]$ contains its $\mathcal{U}_q$ analog $V_0^q = \mathcal{U}_q.v_0$ which specialize at $q = \xi$ to the $\mathcal{U}_\xi$-module V_0.

We consider the U_h-module $V_0^h \otimes U_h$. Let $W_0 \subset L_R$ be the set of weights of V_0^h and p^λ the projector of V_0^h on the weight space of weight λ. In general, if v and w are vectors of weight λ and μ, respectively then $HH^h(v\otimes w) = q^{\langle\lambda,\mu\rangle}v\otimes w = v\otimes K_\lambda w$. So the action of HH^h on $V_0^h \otimes U_h$ is given by

$$HH^h = \sum_{\lambda \in W_0} p^\lambda \otimes K_\lambda. \tag{10}$$

Consider the endomorphism f_{U_h} of $V_0^h \otimes U_h$ given by the square of the braiding. In other words, f_{U_h} is the morphism determined by the action of $R_{21}R_{12}$.

Since $X^r_{\pm\alpha}$ vanishes on V_0^h the only portion of the quasi R-matrix contributing to f_{U_h} is terms of the truncated quasi R-matrix. Then Eqs. (8) and (10) imply f_{U_h} determines an element $f_{\mathcal{U}_q} \in \mathrm{Aut}_{\mathbb{K}_\ell}(V_0^q) \otimes \mathcal{U}_q$. Moreover, these equations imply that q can be specialized at ξ to obtain an element $f_{\mathcal{U}_\xi} \in \mathrm{Aut}_{\mathbb{C}}(V_0) \otimes \mathcal{U}_\xi$. Then for any representation $\rho_V : \mathcal{U}_\xi \to \mathrm{End}(V)$, $f_V = (\mathrm{Id} \otimes \rho_V)(f_{\mathcal{U}_\xi})$ is an automorphism of the $\mathcal{U}_\xi$-module $V_0 \otimes V$ (f_* defines a natural automorphism of the functor $V_0 \otimes *$).

Lemma 1 *For any generic simple $\mathcal{U}_\xi$-module V we have* $\mathrm{ptr}_R(f_V) = r^N \,\mathrm{Id}_{V_0}$.

Proof Consider a highest weight vector v_0 of V_0. Then Eq. (10) implies that

$$\check{R}^<(v_0 \otimes x) = v_0 \otimes K_{2(r-1)\rho} x + \left(\text{terms} \in V_0' \otimes \mathcal{U}_\xi\right)$$

where $x \in \mathcal{U}_\xi$ and V_0' is the sum of weight spaces of V_0 of weight strictly less than $(r-1)\rho$. Therefore,

$$\begin{aligned}
\mathrm{ptr}_R(f_V)(v_0) &= \mathrm{ptr}_R(p^{(r-1)\rho} \otimes \rho_V(K_{2(r-1)\rho}))(v_0)\\
&= v_0 \left(\sum_i x_i^* \left(K_{2\rho}^{1-r} K_{2(r-1)\rho} x_i\right)\right)\\
&= r^N v_0
\end{aligned}$$

where x_i is a basis of V and x_i^* is its dual basis. Since V_0 is simple then $\mathrm{ptr}_R(f_V) = r^N \,\mathrm{Id}_V$. □

Let V be a simple module of $\mathscr{C}_g$ where $g \in \mathcal{G}\backslash\mathcal{X}$. Then there exists an outer automorphism $\gamma \in \widetilde{G}$ such that $\rho_V \circ \gamma$ is a nilpotent representation of $\mathcal{U}_\xi$ (see Remark 5). We say $\rho_V \circ \gamma$ is a *nilpotent deformation* of V.

Lemma 2 *The element $\delta_q = \mathrm{ptr}_L(f_{\mathcal{U}_q})$ belongs to the center of $\mathcal{U}_q$ and can be specialized to obtain an element δ_ξ in the center of $\mathcal{U}_\xi$. Then for any generic simple $\mathcal{U}_\xi$-module V we have $\mathrm{ptr}_L(f_V) = \rho_V(\delta_\xi)$. Moreover,*

$$\mathrm{ptr}_L(f_V) = \prod_{\alpha \in \Delta^+} \frac{\{r\langle \mu, \alpha\rangle\}}{\{\langle \mu, \alpha\rangle\}} \,\mathrm{Id}_V \tag{11}$$

where $\mu + (r-1)\rho$ is the highest weight of a nilpotent deformation of V.

Proof Recall that the morphism $f_{U_h} : V_0^h \otimes U_h \to V_0^h \otimes U_h$ is given by multiplication on the left by a truncated part of $R_{21}R_{12}$. Using Eqs. (8), (10) and the definition of partial trace, we have that $\mathrm{ptr}_L(f_{U_h})$ is a U_h-endomorphism given by left multiplication of an element δ_h of U_h. Thus, δ_h is an element of the center of U_h. Moreover, δ_h is an element of $\mathcal{U}_q$ seen as a subalgebra of $U_h[h^{-1}]$ because it is equal to $\delta_q = \mathrm{ptr}_L(f_{\mathcal{U}_q})$. Therefore, δ_q is central since δ_h is central.

Let Z_ξ be the specialization at $q = \xi$ of the center of $\underset{\sim}{\mathcal{U}_q}$. Then by definition of the derivations $\underline{e}_i$ and $\underline{f}_i$ it follows that the automorphisms of $\tilde{G}$ act trivially on Z_ξ. Since $\delta_\xi \in Z_\xi$, it acts as the same scalar endomorphism on V and on any of its nilpotent deformation. Hence the general formula in Eq. (11) can be deduced from the case when V is the nilpotent module with highest weight $\mu + (r-1)\rho$ which was computed in Proposition 45 of [20]. □

Theorem 5 *The right trace $\{\mathsf{t}_V\}_{V\in\mathsf{Proj}}$ on Proj in $\mathscr{C}_{\mathrm{odd}}$ is a trace. This trace is unique up to a global scalar since this is true for the right trace.*

Proof Let V be a generic simple $\mathcal{U}_\xi$-module. Combining Equation (9) and Lemmas 1 and 2 we have

$$\mathsf{d}(V) = \frac{\mathsf{d}(V_0)\big\langle \mathrm{ptr}_R(f)\big\rangle}{\big\langle \mathrm{ptr}_L(f)\big\rangle} = \mathsf{d}(V_0) r^N \prod_{\alpha\in\Delta^+} \frac{\{\langle\mu,\alpha\rangle\}}{\{r\langle\mu,\alpha\rangle\}} \tag{12}$$

where $\mu + (r-1)\rho$ is the highest weight of a nilpotent deformation of V. The dual of such a nilpotent module has highest weight $-\mu + (r-1)\rho$. Thus, Eq. (12) implies $\mathsf{d}(V) = \mathsf{d}(V^*)$. Now from Sects. 3.2 and 4.3, $\mathscr{C}_{\mathrm{odd}}$ is a generically $\mathcal{G}$-semi-simple pivotal category so the theorem follows from Theorem 3. □

Corollary 2 *There exists a unique trace (up to global scalar) on the projective modules of $\overline{\mathscr{C}}_{\mathrm{odd}}$.*

Proof Consider the projection morphism $\mathcal{U}_\xi \to \overline{\mathcal{U}}_\xi$. Using this morphism each projective $\overline{\mathcal{U}}_\xi$ weight module becomes a projective $\mathcal{U}_\xi$ weight module. In fact we have $\overline{\mathscr{C}}_{\mathrm{odd}} \cong \bigoplus_{g\in\mathcal{G}_0} \mathscr{C}_g$ where $\mathcal{G}_0 = \{g \in \mathcal{G} : g(X^r_{\pm\beta}) = 0,\ g(K^r_\beta) = \pm 1,\ \beta \in \Delta_+\}$.

It follows that the trace of Theorem 5 induces a trace on the idea of projective $\overline{\mathcal{U}}_\xi$-module. This trace is unique because from Theorem 4.7.1. of [19] there exists unique nontrivial right trace on the projective ideal of $\overline{\mathcal{U}}_\xi$-modules. □

5 The case of $\mathfrak{sl}(2)$

In the previous subsections we considered three versions of the quantum groups associated with any Lie algebra $\mathfrak{g}$, associated with different lattices L when q was a root of unity of odd order. The main reason the we do not treat even ordered roots of unity above is because we use results on the quantum coadjoint action which at this point requires odd order roots of unity (see [12]). In this section we treat the case of *all* orders of roots of unity for the Drinfeld-Jimbo quantization of $\mathfrak{sl}(2)$ (i.e., the root lattice L_R). In this case we can prove the needed results involving the quantum coadjoint action directly using the Casimir element.

Let $\ell \geq 2$, $\xi = e^{2i\pi/\ell}$ and $r = \frac{2\ell}{3+(-1)^\ell}$. As above we consider the three versions of quantum $\mathfrak{sl}(2)$ associated with the root lattice:

$$\mathcal{U}_\xi(\mathfrak{sl}(2)) = \mathcal{U} = \left\langle E, F, K^{\pm1} | KE - \xi^2 EK = FK - \xi^2 KF = [E, F] - \frac{K - K^{-1}}{\xi - \xi^{-1}} = 0 \right\rangle,$$
$$\mathcal{U}_\xi^H(\mathfrak{sl}(2)) = \mathcal{U}^H = \langle \mathcal{U}, H | [H, K] = [H, E] - 2E = [H, F] + 2F = 0 \rangle \quad \text{and}$$
$$\overline{\mathcal{U}} = \mathcal{U}/(E^r, F^r, K^{2r} - 1).$$

As above, let $\mathscr{C}_{\mathfrak{sl}(2)}$, $\mathscr{C}^H_{\mathfrak{sl}(2)}$ and $\overline{\mathscr{C}}_{\mathfrak{sl}(2)}$ be their categories of weight modules, respectively. These categories are graded as follows: Let Z_0 be the subalgebra generated by E^r, F^r and K^r. The center of $\mathcal{U}$ is generated by Z_0 with the Casimir element

$$\Omega = \{1\}^2 EF + K\xi^{-1} + K^{-1}\xi = \{1\}^2 FE + K\xi + K^{-1}\xi^{-1}$$

which satisfies the polynomial equation $\mathsf{C}_r(\Omega) = \{1\}^{2r} E^r F^r - (-1)^\ell (K^r + K^{-r})$ where C_r is the renormalized rth Chebyshev polynomial (determined by $\mathsf{C}_r(2\cos\theta) = 2\cos(r\theta)$).

The set

$$\mathcal{G} = \left\{ M(\kappa, \varepsilon, \varphi) = \left(\begin{pmatrix} 1 & \varepsilon \\ 0 & \kappa \end{pmatrix}, \begin{pmatrix} \kappa & 0 \\ \varphi & 1 \end{pmatrix} \right) : \varepsilon, \varphi \in \mathbb{C}, \kappa \in \mathbb{C}^* \right\}$$

is a group where the multiplication is given by componentwise matrix multiplication. Also, as above $\mathrm{Hom}_{\mathrm{alg}}(Z_0, \mathbb{C})$ is a group where the multiplication is given by $g_1 g_2 = g_1 \otimes g_2 \circ \Delta$. Then the morphism $\mathrm{Hom}_{\mathrm{alg}}(Z_0, \mathbb{C}) \to \mathcal{G}$ given by

$$M(\kappa, \varepsilon, \varphi)(K^r) = \kappa,$$
$$M(\kappa, \varepsilon, \varphi)(E^r) = \{1\}^{-r}\varepsilon \quad \text{and}$$
$$M(\kappa, \varepsilon, \varphi)(F^r) = (-1)^\ell \{1\}^{-r} \varphi \kappa^{-1}.$$

is a group homomorphism. We use this morphism to identify $\mathcal{G}$ and $\mathrm{Hom}_{\mathrm{alg}}(Z_0, \mathbb{C})$.

For each $g \in \mathcal{G}$ let $\mathscr{C}_g$ be the full subcategory of $\mathscr{C}_{\mathfrak{sl}(2)}$ whose objects are modules where each $z \in Z_0$ act by $g(z)$. As above, for $g \in \mathcal{G}$ with $g(E^r) = g(F^r) = 0$ we say g is *diagonal* and the modules of $\mathscr{C}_g$ are *nilpotent*. Also, a diagonal element g is *regular* if $g(K^r) \neq \pm 1$. The category $\mathscr{C}^H_{\mathfrak{sl}(2)}$ is graded by the group $\mathcal{D} \simeq (\mathbb{C}/2\mathbb{Z}, +) \simeq \mathfrak{h}^*/L_R \simeq (\mathbb{C}^*, \times)$ of diagonal elements of $\mathcal{G}$. Finally $\overline{\mathscr{C}}_{\mathfrak{sl}(2)}$ is $\mathbb{Z}/2\mathbb{Z}$-graded as a full subcategory of $\mathscr{C}_{\mathfrak{sl}(2)}$.

As above it is known that the element $\phi = K_{2\rho}^{1-r}$ in $\mathcal{U}, \mathcal{U}^H$ and $\overline{\mathcal{U}}$ satisfies $S^2(x) = \phi x \phi^{-1}$. It follows that $\mathscr{C}_{\mathfrak{sl}(2)}$, $\mathscr{C}^H_{\mathfrak{sl}(2)}$ and $\overline{\mathscr{C}}_{\mathfrak{sl}(2)}$ are all pivotal $\mathbb{C}$-categories. Moreover, $\mathcal{U}^H$ is braided, see [20] for the odd case and [21] for the even.

Lemma 3 *The category $\mathscr{C}_{\mathfrak{sl}(2)}$ is generically $\mathcal{G}$-semi-simple category with singular locus*

$$\mathcal{X} = \left\{ M(\kappa, \varepsilon, \varphi) : \kappa + \frac{1}{\kappa} - \frac{\varepsilon\varphi}{\kappa} = \pm 2 \right\}.$$

Proof A direct calculation shows $\mathcal{X}$ is small and symmetric. We consider the quantum coadjoint action $\widetilde{G}$ on $\mathcal{G}$, see [11] and above. Let $B \subset G$ be the union of all $\widetilde{G}$-orbits which contain a regular element. To prove the lemma we will show that the category $\mathscr{C}_h$ is semi-simple for each $h \in B$ and that $\mathcal{G}\backslash\mathcal{X} \subset B$. To do this we follow an argument similar to the one in Sect. 4.3: direct construction shows that for any regular g the category $\mathscr{C}_g$ has r nonisomorphic highest weight modules of dimension r (also see Corollary 3.2 of [11]). Thus, the PBW Theorem and Corollary 1 imply that for regular g, the category $\mathscr{C}_g$ semi-simple.

Now let $h = M(\kappa_h, \varepsilon, \varphi) \in B$ then by definition of B there exists regular $g = M\big(\kappa_g, 0, 0\big) \in \mathcal{G}$ such that g and h are in the same $\widetilde{G}$-orbit. Then the $\widetilde{G}$-action gives an outer automorphism of $\mathcal{U}$ inducing an isomorphism of algebras $\mathcal{U}/I_g \to \mathcal{U}/I_h$ and so $\mathscr{C}_h$ is semi-simple. Finally, the element $\Omega_q = \{1\}^2 EF + Kq^{-1} + K^{-1}q$ is in the center of the $\mathbb{K}_\ell$-algebra $\mathcal{U}_q$ and specializes to Ω. The quantum coadjoint action is constant on $\Omega = \Omega_\xi$ because as explained above this action is constant on the subalgebra of $\mathcal{U}$ corresponding to the center of $\mathcal{U}_q$. Therefore, if V and W are modules of $\mathscr{C}_g$ and $\mathscr{C}_h$, respectively, then $\mathsf{C}_r(\Omega)$ acts on both V and W by

$$-(-1)^\ell(\kappa_g + \kappa_g^{-1}) = (-1)^\ell \varepsilon\varphi\kappa_h^{-1} - (-1)^\ell(\kappa_h + \kappa_h^{-1}).$$

Since g is regular then $\kappa_g \neq \pm 1$, so $-\varepsilon\varphi\kappa_h^{-1} + \kappa_h + \kappa_h^{-1} \neq \pm 2$. Thus, $h \in \mathcal{G}\backslash\mathcal{X}$. $\square$

Corollary 3 *The category $\mathscr{C}^H_{\mathfrak{sl}(2)}$ is ribbon.*

Proof Let $\mathcal{X}_H = \mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}/2\mathbb{Z}$ and let $d \in \mathbb{C}/2\mathbb{Z}\backslash\mathcal{X}_H$. The proof of Lemma 3 and the forgetful functor $\mathscr{C}^H_{\mathfrak{sl}(2)} \to \mathscr{C}_{\mathfrak{sl}(2)}$ imply $\mathscr{C}_d$ has r nonisomorphic highest weight modules of dimension r (one can also see this by direct construction). Then the PBW Theorem and Corollary 1 imply that $\mathscr{C}_d$ is semi-simple. Combining this with the results at the beginning of this subsection we have $\mathscr{C}^H_{\mathfrak{sl}(2)}$ is generically $\mathcal{G}$-semi-simple pivotal braided category with singular locus $\mathcal{X}_H$. It is known that for a generic simple object V that $\theta(V)^* = \theta(V^*)$, see [20] for the odd case and [21] for the even. The corollary follows from Theorem 2. $\square$

Note when ℓ is ever Corollary 3 was previously known, see [21,28,29]. When ℓ is odd it was known that $\mathscr{C}^H_{\mathfrak{sl}(2)}$ contained a subcategory which is ribbon, see [20].

Next we prove that $\mathscr{C}_{\mathfrak{sl}(2)}$ has a trace and show the modified dimensions only depend on the action of the Casimir element. First, the projective cover of the trivial module is self-dual, see [16,17] for the even case and [19] for the odd. Therefore, Corollary 3.2.1 of [19] implies the ideal Proj of projective $\mathcal{U}_\xi$-modules admits a unique (up to global scalar) nontrivial right trace which we denote by $\{\mathsf{t}_V\}_{V\in\mathsf{Proj}}$. We will show this right trace is a trace. As above let $U_h = U_h(\mathfrak{sl}(2))$ be the h-adic quantum group generated by E, F and H. Let V_0^h be the highest weight module of U_h with a highest weight vector v_0 such that $Hv_0 = (r-1)v_0$. Let V_0^q be the highest weight $\mathcal{U}_q$-module with a highest weight vector v such that $Kv_0 = q^{r-1}v_0$. Let V_0 be the highest weight $\mathcal{U}$-module which is the specialization of V_0^q. Let W_0 be the set of weights of V_0^h. Then $W_0 \subset L_R$ if ℓ is odd and $W_0 \subset \frac{1}{2}L_R$ if ℓ is even. Let f_{U_h} and $f_{\mathcal{U}_q}$ be as in Sect. 4.5. As above in the odd case q can be specialized at ξ to obtain an element $f_{\mathcal{U}_\xi} \in \mathrm{Aut}_{\mathbb{C}}(V_0) \otimes \mathcal{U}$ and any for representation $\rho_V : \mathcal{U} \to \operatorname{End}(V)$, $f_V = (\operatorname{Id} \otimes \rho_V)(f_{\mathcal{U}_\xi})$ is an automorphism of the $\mathcal{U}$-module $V_0 \otimes V$.

Lemma 4 *For any generic simple $\mathcal{U}_\xi(\mathfrak{sl}(2))$-module V we have* $\operatorname{ptr}_R(f_V) = r \operatorname{Id}_{V_0}$.

Proof The proof of Lemma 1 applies here for general ℓ. □

Lemma 5 *There exists a polynomial $P(X) \in \mathbb{C}[X]$ such that for any generic simple $\mathcal{U}_\xi(\mathfrak{sl}(2))$-module V we have* $\operatorname{ptr}_L(f_V) = P(\omega) \operatorname{Id}_V$ *where* $\omega = \langle \rho_V(\Omega) \rangle$.

Proof As in the proof of Lemma 2 the morphism $\operatorname{ptr}_L(f_{U_h})$ is given by left multiplication of a central element δ_h of $U_h(\mathfrak{sl}(2))$. The center of $U_h(\mathfrak{sl}(2))$ is generated by the Casimir element Ω_h. So there exists a polynomial $P(X) \in \mathbb{C}[X]$ such that $\delta_h = P(\Omega_h)$. Here the partial closure is taken in $U_h(\mathfrak{sl}(2))$. On the other hand, the partial trace of the central element $\delta_q = \operatorname{ptr}_L(f_{\mathcal{U}_q})$ is taken in $\mathcal{U}_q(\mathfrak{sl}(2))$-mod. Recall that the pivotal structure of $U_h(\mathfrak{sl}(2))$-mod and $\mathcal{U}_q(\mathfrak{sl}(2))$-mod differ by K^{-r}, see Remark 4. In particular, taking the left closer with respect to V_0^h and V_0^q differ by the constant $(-1)^{r-1}$ determined by $K^{-r}v_0 = q^{-r(r-1)}v_0 = (-1)^{r-1}v_0$. Therefore, δ_h viewed as an element of $\mathcal{U}_q$ (i.e., seen as the subalgebra of $U_h[h^{-1}]$) is equal to the element $(-1)^{r-1}\delta_q$. Now the morphism $\operatorname{ptr}_L(f_V)$ is given by left multiplication by δ_ξ which is specialization a of δ_q. Thus, $\operatorname{ptr}_L(f_V) = (-1)^{r-1}P(\omega)\operatorname{Id}_V$ where $\omega = \langle \rho_V(\Omega) \rangle$. □

Corollary 4 *Let V be a generic simple $\mathcal{U}_\xi(\mathfrak{sl}(2))$-module. Then there exists $\alpha \in (\mathbb{C}\backslash\mathbb{Z}) \cup r\mathbb{Z}$ such that $\langle \rho_V(\Omega) \rangle = (-1)^r \left(q^\alpha + q^{-\alpha}\right)$ and*

$$\begin{aligned} \mathsf{d}(V) &= (-1)^{r-1} \prod_{j=1}^{r-1} \frac{\{j\}}{\{\alpha + r - j\}} \\ &= (-1)^{r-1} \frac{r\,\{\alpha\}}{\{r\alpha\}} = \frac{(-1)^{r-1} r}{q^{(1-r)\alpha} + \cdots + q^{(r-3)\alpha} + q^{(r-1)\alpha}}. \end{aligned} \tag{13}$$

Moreover, $\mathsf{d}(V) = \mathsf{d}(V^*)$. *Here we have fixed the global scalar of the right trace by defining* $\mathsf{d}(V_0) = \mathsf{t}_{V_0}(\operatorname{Id}_{V_0}) = (-1)^{r-1}$.

Proof From Eq. (9) we have $\mathsf{d}(V) = \frac{\mathsf{d}(V_0)\langle \operatorname{ptr}_R(f)\rangle}{\langle \operatorname{ptr}_L(f)\rangle}$. So by Lemmas 4 and 5 we have $\mathsf{d}(V) = \mathsf{d}(W)$ for any generic simple W such that $\langle \rho_W(\Omega) \rangle = \langle \rho_V(\Omega) \rangle$. The proof of Lemma 3 implies there exists a nilpotent simple module W in the quantum coadjoint orbit of V. Moreover, this proof implies that the action of Ω is the same on V and W. Since W is nilpotent there exists $\alpha \in (\mathbb{C}\backslash\mathbb{Z}) \cup r\mathbb{Z}$ such that Ω acts on W by $(-1)^r \left(q^\alpha + q^{-\alpha}\right)$. From [10] it is shown that the modified quantum dimension of W is given by the formula in Eq. (13). Thus, Eq. (13) follows.

To see the last statement of the corollary, notice $S(\Omega) = \Omega$ so $\langle \rho_V(\Omega) \rangle = \langle \rho_{V^*}(\Omega) \rangle$ and $\mathsf{d}(V) = \mathsf{d}(V^*)$. □

Theorem 3, Lemma 3 and Corollary 4 imply the following corollary.

Corollary 5 *The right trace* $\{\mathsf{t}_V\}_{V \in \mathsf{Proj}}$ *on* Proj *in* $\mathscr{C}_{\mathfrak{sl}(2)}$ *is a trace.*

Corollary 6 *There exists a unique (up to global scalar) trace on the projective modules of* $\overline{\mathscr{C}}_{\mathfrak{sl}(2)}$.

6 Conjectures for quantum groups at even ordered roots of unity

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra with data $\mathfrak{h}$, Δ^+, L_R, L_W, ... discussed in Sect. 3.1. Let ℓ be an even integer such that $\ell \geq 2$ (and $\ell \notin 3\mathbb{Z}$ if $\mathfrak{g} = G_2$). Let $\xi = e^{2\sqrt{-1}\pi/\ell}$ and $r = \ell/2$. As above, for each lattice L, consider the three versions of the quantum groups: $\mathcal{U}_\xi^L(\mathfrak{g})$, $\mathcal{U}_\xi^H(\mathfrak{g})$ and $\overline{\mathcal{U}}_\xi^L(\mathfrak{g})$.

The case when ℓ is even the situation is not very well developed at this time and needs additional work to apply the results of this paper. For example, the subalgebra Z_0 of $\mathcal{U}_\xi^L(\mathfrak{g})$ generated by $\{X_{\pm\beta}^r, K_\gamma^r : \beta \in \Delta_+, \gamma \in L\}$ is not necessarily commutative: if $\langle \beta, \alpha_i \rangle = 1$ then Eq. (4) implies

$$K_\beta^r X_j = q^{r\langle \beta, \alpha_i \rangle} X_j K_\beta^r = -X_j K_\beta^r.$$

So one must first define an appropriate notion of weight modules for the algebras $\mathcal{U}_\xi^L(\mathfrak{g})$, $\mathcal{U}_\xi^H(\mathfrak{g})$ and $\overline{\mathcal{U}}_\xi^L(\mathfrak{g})$. Also, as mentioned above the quantum coadjoint action is not worked out for case when ℓ is even. Some work in this direction has been done for $\ell \in 4\mathbb{Z}$, see [3].

We are lead to the following question, "For even ordered ξ, can one define categories of weight modules $\mathscr{C}_{\text{even}}$, $\mathscr{C}_{\text{even}}^H$ and $\overline{\mathscr{C}}_{\text{even}}$ over $\mathcal{U}_\xi^L(\mathfrak{g})$, $\mathcal{U}_\xi^H(\mathfrak{g})$ and $\overline{\mathcal{U}}_\xi^L(\mathfrak{g})$, respectively, such that the following conjectures are true?"

Conjecture 1 For even ordered ξ, there exists a category $\mathscr{C}_{\text{even}}^H$ of modules over $\mathcal{U}_\xi^H(\mathfrak{g})$ which is ribbon.

Conjecture 2 For even ordered ξ, there exists a category $\mathscr{C}_{\text{even}}$ of weight modules over $\mathcal{U}_\xi^L(\mathfrak{g})$ such that there exists a unique (up to global scalar) two-sided trace on the ideal of projective modules of $\mathscr{C}_{\text{even}}$.

Conjecture 3 For even ordered ξ, there exists a category $\overline{\mathscr{C}}_{\text{even}}$ of modules over $\overline{\mathcal{U}}_\xi^L(\mathfrak{g})$ such that there exists a unique (up to global scalar) two-sided trace on the ideal of projective modules of $\overline{\mathscr{C}}_{\text{even}}$.

The conjectures in this section are motivated by applications in low-dimensional topology. In particular, when $\ell = 4$ and $\mathfrak{g} = \mathfrak{sl}(2)$ in [6] it is shown that the closed 3-manifold invariant of [9] associated with the category $\mathscr{C}_{\mathfrak{sl}(2)}^H$ are a canonical normalization of Reidemeister torsion defined by Turaev which gives rise to a Topological Quantum Field Theory (TQFT). It would be interesting to see what properties the analogous topological invariants have for other Lie algebras at similar level. The first step in defining such invariants is a proof of Conjecture 1. Also, Conjecture 3 would also be interesting in generalizing the work of [4].

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