Distribution Results for a Multi-Rank Version of the Reed-Yu Detector

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***Abstract*—In this paper we revisit a detector first derived by Reed and Yu [1], generalized by Bliss and Parker [2], and recently studied by Hiltunen, Loubaton, and Chevalier [3], [4]. The problem is to detect a known signal transmitted over an unknown MIMO channel of unknown complex gains and unknown additive noise covariance. The probability distribution of a CFAR detector for this problem was first derived for the SIMO channel in [1]. We generalize this distribution for the case of a MIMO channel, and show that the CFAR detector statistic is distributed as the product of independent scalar beta random variables under the null. Our results, based on the theory of beta distributed random matrices, hold for M symbols transmitted from p transmitters and received at L receivers. The asymptotic results of [3], [4] are based on large random matrix theory, which assumes L and M to be unbounded.**

I. Problem Statement

Consider a subspace signal-plus-noise model $\mathbf{x}_m = \mathbf{H}\mathbf{s}_m + \mathbf{n}_m$, for $m = 1, \ldots, M$. The signal component lies in an unknown p dimensional subspace $\langle \mathbf{H} \rangle$, with unknown basis $\mathbf{H} \in \mathbb{C}^{L\times p}$. For each time sample, its location in this subspace is determined by the vector of signals $\mathbf{s}_m \in \mathbb{C}^p$. The noise snapshots $\mathbf{n}_m$ are proper and independent for $m = 1, \ldots, M$, and distributed as $\mathbf{n}_m \sim \mathcal{CN}_L[\mathbf{0}, \mathbf{\Sigma}]$, with $\mathbf{\Sigma}$ a positive definite covariance matrix. Thus, the measurements $\mathbf{x}_m$ are independent and distributed as $\mathbf{x}_m \sim \mathcal{CN}_L[\mathbf{H}\mathbf{s}_m, \mathbf{\Sigma}]$, with $\mathbf{H}$ and $\mathbf{\Sigma}$ unknown; the signal sequence $\{\mathbf{s}_m, m = 1, 2, \ldots, M\}$ is known.

This data model corresponds to one channel use of a multiple input multiple output (MIMO) transmission system with p transmitting antennas, L receiving antennas, and M symbol transmissions, when the transmitting and receiving antennas are perfectly synchronized [2]. We are interested in the following binary hypothesis test:

$$H_0 : \mathbf{s}_m = \mathbf{0} \quad \text{versus} \quad H_1 : \mathbf{s}_m \neq \mathbf{0} \quad \text{for} \quad m = 1, \ldots, M. \tag{1}$$

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Define $\mathbf{S}^H = [\mathbf{s}_1, \ldots, \mathbf{s}_M]$, $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_M]$, and $\mathbf{N} = [\mathbf{n}_1, \ldots, \mathbf{n}_M]$. Then, the data matrix $\mathbf{X} = \mathbf{H}\mathbf{S}^H + \mathbf{N}$ is distributed as

$$f(\mathbf{X}, \mathbf{H}, \mathbf{S}, \mathbf{\Sigma}) = \frac{1}{\pi^{LM} (\det \mathbf{\Sigma})^M} \times \exp\{-\mathrm{tr}[(\mathbf{X} - \mathbf{H}\mathbf{S}^H)^H \mathbf{\Sigma}^{-1} (\mathbf{X} - \mathbf{H}\mathbf{S}^H)]\} \tag{2}$$

In this paper, we assume $\mathbf{S}$ is known, but the channel gains $\mathbf{H}$ and the noise covariance $\mathbf{\Sigma}$ are unknown. For $p = 1$, the Generalized Likelihood Ratio Test (GLRT) for this measurement model has been derived in [1], for a problem of optical pattern detection with unknown spectral distribution. For $p \geq 1$ Bliss and Parker [2] generalized this result for synchronization in a MIMO channel. The analysis in [1] considers the special case of a real measurement model for the rank one signal model $p = 1$. The analysis in [2] assumes complex measurements and a rank-p signal model. We follow the approaches of [1], [2] to identify Maximum Likelihood (ML) estimates of the unknown parameters, and use these estimates to form the GLRT for the hypothesis test (1). Then we derive the distribution of the detector statistic.

II. Generalized Likelihood Ratio Test and Its Distribution

To find the GLRT for this problem, we need the ML estimate of $\mathbf{\Sigma}$ under H_0 and H_1, and the ML estimate of $\mathbf{H}$ under H_1. Under H_0 the ML estimate of the covariance matrix $\mathbf{\Sigma}$ is

$$\hat{\mathbf{\Sigma}}_0 = \frac{1}{M} \mathbf{X}\mathbf{X}^H \tag{3}$$

Similarly, under H_1, the ML estimates of $\mathbf{\Sigma}$ and $\mathbf{H}$ are

$$\hat{\mathbf{\Sigma}}_1 = \frac{1}{M} (\mathbf{X} - \hat{\mathbf{H}}\mathbf{S}^H)(\mathbf{X} - \hat{\mathbf{H}}\mathbf{S}^H)^H, \tag{4}$$

and

$$\hat{\mathbf{H}} = \mathbf{X}\mathbf{S}(\mathbf{S}^H\mathbf{S})^{-1}. \tag{5}$$

The GLR $\ell = \frac{f_{H_0}(\mathbf{X}, \mathbf{S}, \hat{\mathbf{\Sigma}}_0)}{f_{H_1}(\mathbf{X}, \hat{\mathbf{H}}, \mathbf{S}, \hat{\mathbf{\Sigma}}_1)}$ is then

$$\ell^{1/M} = \frac{\det(\mathbf{X}(\mathbf{I}_M - \mathbf{P_S})\mathbf{X}^H)}{\det(\mathbf{X}\mathbf{X}^H)}. \tag{6}$$

The distribution of (6) has been derived for finite L and M in [1] for the case $p = 1$. This is the original result we generalize in this paper. Hiltunen, et al. [3] show that in the case where the number of receiving antennas L and the number of snapshots M are large and of the same order of magnitude, but the number of transmitting antennas p remains fixed, -logdet(ℓ) converges to a normal distribution under H_0 and H_1. Then, pragmatic approximations for the distribution are derived for large p. In [4], it is shown that asymptotically in L, M, and p, logdet(ℓ) is distributed as a normal random variable (See Theorem 1 of [3]).

We study the exact distribution of (6) under H_0 and H_1, and in the non-asymptotic case. The distribution of ℓ is invariant to the transformation $\mathbf{X} \to \mathbf{\Sigma}^{-1/2}\mathbf{X}$, thus we may assume $\mathbf{X}$ to be proper, and distributed as $\mathcal{CN}_{L\times M}(\mathbf{\Sigma}^{-1/2}\mathbf{H}\mathbf{S}^H, \mathbf{I}_L \otimes \mathbf{I}_M)$.

Let $\mathbf{P_S} = \mathbf{U_S}\mathbf{U_S}^H$, $\mathbf{U} = [\mathbf{U_S}, \mathbf{U_{S^\perp}}]$, $\mathbf{Y}_1 = \mathbf{X}\mathbf{U_S}$, and $\mathbf{Y}_2 = \mathbf{X}\mathbf{U_{S^\perp}}$, then

$$\ell^{1/M} = \frac{\det(\mathbf{Y}_2\mathbf{Y}_2^H)}{\det(\mathbf{Y}_1\mathbf{Y}_1^H + \mathbf{Y}_2\mathbf{Y}_2^H)}, \tag{7}$$

which is the same as Wilks's statistic [5]. Here, $\mathbf{Y}_1 \sim \mathcal{CN}_{L\times p}(\mathbf{M}_1, \mathbf{I}_L \otimes \mathbf{I}_p)$, $\mathbf{Y}_2 \sim \mathcal{CN}_{L\times(M-p)}(\mathbf{0}, \mathbf{I}_L \otimes \mathbf{I}_p)$, are independent, $\mathbf{M}_1 = \mathbf{\Sigma}^{-1/2}\mathbf{H}\mathbf{S}^H\mathbf{U_S}$, and $\mathbf{Y}_2\mathbf{Y}_2^H \sim \mathcal{CW}(L, M-p, \mathbf{I}_L)$. The statistic in (7) may be written as

$$\begin{aligned}\ell^{1/M} &= \frac{\det(\mathbf{Y}_2\mathbf{Y}_2^H)}{\det(\mathbf{Y}_1\mathbf{Y}_1^H + \mathbf{Y}_2\mathbf{Y}_2^H)}\\ &= \det\left[\left(\mathbf{I}_L + (\mathbf{Y}_2\mathbf{Y}_2^H)^{-1/2}\mathbf{Y}_1\mathbf{Y}_1^H(\mathbf{Y}_2\mathbf{Y}_2^H)^{-H/2}\right)^{-1}\right]\\ &= \det\left[\left(\mathbf{I}_p + \mathbf{Y}_1^H(\mathbf{Y}_2\mathbf{Y}_2^H)^{-1}\mathbf{Y}_1\right)^{-1}\right]\\ &= \det\left[(\mathbf{I}+\mathbf{F})^{-1}\right]\\ &= \det(\mathbf{B}). \end{aligned}\tag{8}$$

The distribution of $\mathbf{F} = \mathbf{Y}_1^H(\mathbf{Y}_2\mathbf{Y}_2^H)^{-1}\mathbf{Y}_1$ is given in [6] as

$$\begin{aligned}& e^{-\mathrm{tr}(\mathbf{M}_1^H\mathbf{M}_1)}{}_1\tilde{F}_1(M; L; \mathbf{M}_1^H\mathbf{M}_1(\mathbf{I}+\mathbf{F}^{-1})^{-1})\times\\ & \frac{\tilde{\Gamma}_p(M)}{\tilde{\Gamma}_p(M-L)\tilde{\Gamma}_p(L)}\frac{\det(\mathbf{F})^{L-p}}{\det(\mathbf{I}+\mathbf{F})^M}\end{aligned}\tag{9}$$

which is a noncentral matrix $\mathbf{F}$ distribution. Using the transformation $\mathbf{B} = (\mathbf{I}_p + \mathbf{F})^{-1}$, the pdf of $\mathbf{B}$ may be written as

$$\begin{aligned}& e^{-\mathrm{tr}(\mathbf{M}_1^H\mathbf{M}_1)}{}_1\tilde{F}_1(M; L; \mathbf{M}_1^H\mathbf{M}_1(\mathbf{I}-\mathbf{B}))\frac{\tilde{\Gamma}_p(M)}{\tilde{\Gamma}_p(M-L)\tilde{\Gamma}_p(L)}\times\\ & \det(\mathbf{B})^{M-L-p}\det(\mathbf{I}-\mathbf{B})^{L-p},\end{aligned}\tag{10}$$

which may be considered a complex noncentral matrix variate Beta distribution $\mathbb{C}\mathbf{B}_p(M-L, L, \mathbf{M}_1\mathbf{M}_1^H)$. Under H_0 ($\mathbf{S} = \mathbf{0}$), det($\mathbf{B}$) is distributed as the product of independent beta random variables. That is,

$$\ell^{1/M} = \det(\mathbf{B}) \sim \prod_{i=1}^{p} b_i; \quad b_i \sim \beta(M-L-i+1, L). \tag{11}$$

The probability density function of the product of independent Beta random variables has been studied in [7].

Under H_1, we derive the MGF of $\log W$ for $W = \det(\mathbf{B})$. Using the pdf of $\mathbf{B}$ in (10), $M_{\log W}(h) = E(W^h)$ can be derived as

$$\begin{aligned}M_{\log W}(h) =& \frac{\tilde{\Gamma}_p(M)\tilde{\Gamma}_p(M-L+h)}{\tilde{\Gamma}_p(M+h)\tilde{\Gamma}_p(M-L)}e^{-\mathrm{tr}(\mathbf{M}_1^H\mathbf{M}_1)}\times\\ & {}_1\tilde{F}_1(M; M+h; \mathbf{M}_1^H\mathbf{M}_1)\end{aligned}\tag{12}$$

We may now use the inverse Mellin transform to determine the density of W.

With these results, the likelihood ratio statistic $\ell^{1/M}$ may be simulated for the setting of detection thresholds to control false alarms, or the characteristic function of the product of betas may be used with saddlepoint integration [8] to accurately approximate false alarm probability.

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