INFINITELY MANY MONOTONE LAGRANGIAN TORI IN $\mathbb{R}^6$

DENIS AUROUX

Abstract. We construct infinitely many families of monotone Lagrangian tori in $\mathbb{R}^6$, no two of which are related by Hamiltonian isotopies (or symplectomorphisms). These families are distinguished by the (arbitrarily large) numbers of families of Maslov index 2 pseudo-holomorphic discs that they bound.

1. Introduction

The study and classification of Lagrangian submanifolds in symplectic manifolds is a central topic of modern symplectic topology; in spite of spectacular advances in the last few decades, it remains poorly understood, even in very simple symplectic manifolds such as the standard symplectic vector space $(\mathbb{R}^{2d}, \omega_0)$.

By a celebrated result of Gromov, there are no closed exact Lagrangian submanifolds in $\mathbb{R}^{2d}$, and in fact any closed Lagrangian in $\mathbb{R}^{2d}$ must bound some pseudo-holomorphic discs of non-zero area [10]. (This is in sharp contrast with the situation for immersed Lagrangians, see e.g. [6].) Thus, the nicest condition that one could impose on a closed Lagrangian submanifold $L \subset \mathbb{R}^{2d}$ is for it to be *monotone*, i.e. that the symplectic area of discs with boundary on L is (positively) proportional to their *Maslov index*.

The simplest examples of monotone Lagrangians in $\mathbb{R}^{2d}$ are the tori obtained as products of d circles of equal radius, $L = S^1(r) \times \cdots \times S^1(r)$. In the early 1990s Chekanov found the first examples of Lagrangian tori in $\mathbb{R}^{2d}$ that cannot be related to product tori by Hamiltonian isotopies (or symplectomorphisms) [3] (see also [7]). Subsequent work of Chekanov and Schlenk has led to more examples, the so-called *monotone twist tori* [4]; the number of tori produced by this construction grows exponentially with the dimension, but remains finite for all d.

More recently, Renato Vianna's thesis [12] shows that $\mathbb{CP}^2$ contains at least one new kind of monotone Lagrangian torus besides product and Chekanov tori; this result was recently improved to show that $\mathbb{CP}^2$ contains *infinitely many* non-isotopic monotone Lagrangian tori [9, 13].

In this paper, we construct infinitely many families of monotone Lagrangian tori in $\mathbb{R}^6$, no two of which are related by symplectomorphisms. Specifically, the invariants

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that we use to distinguish these tori are the algebraic counts of Maslov index 2 pseudo-holomorphic discs whose boundary passes through a given point (see §3.1); these invariants were already used by Eliashberg-Polterovich to distinguish the Chekanov torus in $\mathbb{R}^4$ [7] and in much of the subsequent work [4, 12, 13].

Theorem 1. *For each integer $n \geq 0$, and for any choice of monotonicity constant, there exists a monotone Lagrangian torus $L \subset (\mathbb{R}^6, \omega_0)$ such that there are $n+2$ distinct Maslov index 2 classes in $\pi_2(\mathbb{R}^6, L)$ for which the algebraic count of pseudo-holomorphic discs passing through a point of L is non-zero (and the sum of these counts is $2^n + 1$). Therefore, for different n these tori cannot be related by symplectomorphisms.*

Remark.

(1) Taking the product of these tori with circles of the appropriate radius, we also obtain infinitely many examples in $\mathbb{R}^{2d}$ for all $2d \geq 6$ (similarly distinguished by counts of Maslov index 2 pseudo-holomorphic discs).
(2) For $n = 1$ our tori are most likely symplectomorphic to standard product tori. For $n = 0$ they can be shown to be symplectomorphic to the product of a circle in $\mathbb{R}^2$ with the monotone Chekanov torus in $\mathbb{R}^4$.
(3) Vianna's recent result concerning the existence of infinitely many monotone Lagrangian tori in $\mathbb{CP}^2$ ([13], see also [9]) should also imply a result similar to Theorem 1, by considering the preimages of these tori under the natural projection map from the unit sphere $S^5 \subset \mathbb{R}^6$ to $\mathbb{CP}^2$. However, the construction we give here is substantially simpler.
(4) Monotonicity plays a key role in the construction. Indeed, after arbitrarily small Lagrangian isotopies (not preserving monotonicity), our tori become Hamiltonian isotopic to standard product tori.
(5) The least elementary part of our argument is the discussion of orientations of moduli spaces. The reader unwilling to delve into these should be content to work with mod 2 counts of holomorphic discs; the number of Maslov index 2 classes for which the algebraic count of discs is non-zero mod 2, and the number of integer points in their convex hull inside $\pi_2(\mathbb{R}^6, L) \simeq \mathbb{Z}^3$, are in fact sufficient to distinguish the monotone tori we construct for different n.

Acknowledgements. While the methods of this paper are elementary, some of the key conceptual ideas come from the joint work of the author with Mohammed Abouzaid and Ludmil Katzarkov [1], and from Renato Vianna's thesis [12] (see §5). I thank all three of them for helping shape my thoughts on this subject. I also thank Felix Schlenk and the anonymous referees for their careful comments.

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2. Kähler reduction and Moser flow on the reduced space

Our main object of study is the manifold

$$X = \{(x, y, z, w) \in \mathbb{C}^4 \mid xy = h(z, w)\}, \tag{1}$$

where for $n \geq 0$,

$$h(z, w) = cz^n + c^{-1}w - 1, \tag{2}$$

for $c \gg 1$ a constant (e.g., $c = 10$). As a complex manifold X is isomorphic to $\mathbb{C}^3$ via projection to the coordinates (x, y, z), as $w = c(xy+1) - c^2 z^n$. We equip X with the Kähler form

$$\omega_X = \frac{i}{2} dz \wedge d\bar{z} + \frac{i}{2} dw \wedge d\bar{w} + \kappa \left(\frac{i}{2} dx \wedge d\bar{x} + \frac{i}{2} dy \wedge d\bar{y} \right), \tag{3}$$

where $\kappa > 0$ is a small positive constant to be determined below. We note that up to a rescaling of the x and y coordinates ω_X is simply the restriction to X of the standard Kähler form of $\mathbb{C}^4$.

The action of S^1 on X by

$$e^{i\theta} \cdot (x, y, z, w) = (e^{i\theta} x, e^{-i\theta} y, z, w) \tag{4}$$

is Hamiltonian, with moment map

$$\mu_X = \frac{\kappa}{2}(|x|^2 - |y|^2). \tag{5}$$

We will consider the reduced space

$$X_{red} = \mu_X^{-1}(0)/S^1. \tag{6}$$

As a complex manifold, X_{red} can be naturally identified with $\mathbb{C}^2$ via projection to the coordinates (z, w). Indeed, for fixed (z, w) the part of the conic $xy = h(z, w)$ where $|x| = |y|$ consists of a single S^1-orbit; the reduced space is therefore naturally a smooth complex manifold, even though $\mu_X^{-1}(0)$ is singular at the fixed points of the S^1-action, i.e. where $h(z, w) = 0$ and $x = y = 0$.

Lemma 1. *The reduced Kähler form on $X_{red} \simeq \mathbb{C}^2$ is given by*

$$\omega_{red} = \frac{i}{2} dz \wedge d\bar{z} + \frac{i}{2} dw \wedge d\bar{w} + \frac{i\kappa}{4} \frac{dh \wedge d\bar{h}}{|h|} = \omega_0 + \frac{\kappa}{2} dd^c(|h|). \tag{7}$$

(As expected this form is singular along the complex curve $h(z, w) = 0$.)

Proof. Given any point of X_{red} where $h(z, w) \neq 0$, we choose a local square root of h, and observe that a local section of the quotient map from $\mu_X^{-1}(0)$ to X_{red} is given by setting $x = y = h(z, w)^{1/2}$. By definition, the reduced Kähler form ω_{red} agrees with the pullback of ω_X under this local section map. Setting $x = y = h^{1/2}$, we find that

$$dx \wedge d\bar{x} + dy \wedge d\bar{y} = 2d(h^{1/2}) \wedge d(\bar{h}^{1/2}) = \frac{1}{2|h|} dh \wedge d\bar{h}.$$

The first part of (7) follows immediately by substitution into (3). The second equality follows from the observation that

$$dd^c(|h|) = 2i\partial\bar{\partial}(h^{1/2} \cdot \bar{h}^{1/2}) = \frac{i}{2|h|} dh \wedge d\bar{h}.$$

□

Next we recall the following explicit form of Moser's lemma in the Kähler case.

Lemma 2. *Let ω_0 and $\omega_1 = \omega_0 + dd^c\varphi$ be two Kähler forms on a complex manifold. Denote by $g_t = (1-t)g_0 + tg_1$ the Kähler metric corresponding to the Kähler form $\omega_t = \omega_0 + t\, dd^c\varphi$ for $t \in [0,1]$, by $\xi_t = -\nabla_{g_t}(\varphi)$ the gradient of φ with respect to g_t, and by ψ_t the isotopy generated by ξ_t wherever it is well-defined. Then $\psi_t^*(\omega_t) = \omega_0$. Moreover, when $\omega_0 = d\theta_0$ is exact, setting $\theta_t = \theta_0 + td^c\varphi$, the pullback $\psi_t^*(\theta_t)$ differs from θ_0 by an exact form.*

Proof. The result follows from Moser's trick and the observation that

$$\omega_t(\xi_t, \cdot) = -g_t(\xi_t, J\,\cdot) = d\varphi(J\,\cdot).$$

Thus, $\iota_{\xi_t}\omega_t = -d^c\varphi$, and

$$\tfrac{d}{dt}(\psi_t^*\omega_t) = \psi_t^*(\tfrac{d}{dt}\omega_t + L_{\xi_t}\omega_t) = \psi_t^*(dd^c\varphi + d\iota_{\xi_t}\omega_t) = 0.$$

Similarly, in the exact case,

$$\tfrac{d}{dt}(\psi_t^*\theta_t) = \psi_t^*(\tfrac{d}{dt}\theta_t + L_{\xi_t}\theta_t) = \psi_t^*(d^c\varphi + \iota_{\xi_t}(d\theta_t) + d(\iota_{\xi_t}\theta_t)) = \psi_t^*(d\iota_{\xi_t}\theta_t)$$

is exact as claimed. □

Applying this to the case at hand, we obtain:

Lemma 3. *Let U be the complement of an arbitrarily small neighborhood of $h^{-1}(0)$ inside an arbitrarily large ball in $\mathbb{C}^2$. Then there exists a constant $\kappa_0 > 0$ (depending on U) and an isotopy $(\psi_\kappa)_{\kappa\in[0,\kappa_0]}$ defined on U, $\psi_0 = \mathrm{id}$, such that for all $\kappa \in (0, \kappa_0)$, ψ_κ gives an exact symplectomorphism between $U \subset (\mathbb{C}^2, \omega_0)$ and $\psi_\kappa(U) \subset (X_{red}, \omega_{red})$.*

Proof. Let Ω be a compact subset of $\mathbb{C}^2 \setminus h^{-1}(0)$ whose interior contains the closure of U. On Ω, the function $|h|$ is smooth and has bounded derivatives, and the Kähler metric g_κ associated to $\omega_{red} = \omega_0 + \frac{\kappa}{2} dd^c(|h|)$ is bounded between fixed multiples of the standard metric g_0 for all $\kappa \in [0,1]$. Thus, the vector field $\xi_\kappa = -\frac{1}{2}\nabla_{g_\kappa}|h|$ is smooth and has bounded norm on Ω. Applying Lemma 2, the isotopy ψ_κ generated by ξ_κ is well-defined on U for small enough κ and gives the desired symplectomorphisms. □

3. Monotone tori in X_{red} and X

3.1. **An enumerative invariant of monotone Lagrangians.** Before proceeding with our construction, we recall some basic facts about holomorphic discs and the invariant we use to distinguish our tori. (See also [7, 2, 12].)

Let L be a closed oriented spin Lagrangian submanifold in a symplectic manifold (M^{2d}, ω) equipped with a compatible almost-complex structure J. When M is non-compact we always assume that ω is convex at infinity (in our case, this follows from the properness and strict plurisubharmonicity of the Kähler potential).

Given a J-holomorphic map $u : (D^2, \partial D^2) \to (M, L)$, the Maslov index $\mu([u]) \in 2\mathbb{Z}$ is the homotopy class of the loop of Lagrangian spaces given by TL along the boundary of u (relative to a trivialization of u^*TM). The deformation of u as a J-holomorphic map is governed by a Cauchy-Riemann type operator (in the integrable case, an honest $\bar\partial$ operator) on the space of sections of u^*TM taking values in u^*TL along the boundary. The index of this operator is $\mathrm{ind}(\bar\partial) = d + \mu([u])$, and when it is surjective (i.e., u is *regular*) the space of pseudo-holomorphic maps is locally a smooth manifold of this dimension.

Assume now that L is monotone, and fix a homotopy class $\beta \in \pi_2(M, L)$ with $\mu(\beta) = 2$. We consider the moduli space of J-holomorphic discs with one boundary marked point $1 \in \partial D^2$, i.e. the quotient

$$\mathcal{M}_1(L, \beta, J) = \big\{u : (D^2, \partial D^2) \to (M, L) \,|\, \bar\partial_J u = 0,\ u_*[D^2] = \beta\big\}/\mathrm{Aut}(D^2, 1). \quad (8)$$

Since $\mu(\beta) = 2$ takes the smallest possible positive value, and the monotonicity of L guarantees that the symplectic area of discs is positively proportional to their Maslov index, discs in the class β have the smallest possible symplectic area. Therefore, bubbling can be excluded *a priori*. Moreover, all J-holomorphic discs in the class β are somewhere injective, and so a generic choice of J ensures their regularity. $\mathcal{M}_1(L, \beta, J)$ is then a smooth compact manifold of dimension $d + \mu(\beta) - 2 = d$.

Fix an orientation and a spin structure on L. The spin structure determines an orientation of $\mathcal{M}_1(L, \beta, J)$ (cf. [8, 5]), and the degree of the evaluation map

$$ev : \mathcal{M}_1(L, \beta, J) \to L, \qquad [u] \mapsto u(1)$$

is then a well-defined integer – essentially, a signed count of J-holomorphic discs in the class β whose boundary passes through a given point of L. Moreover, a generic path between two regular almost-complex structures J_0 and J_1 determines an oriented cobordism between $\mathcal{M}_1(L, \beta, J_0)$ and $\mathcal{M}_1(L, \beta, J_1)$, which shows that the degree of the evaluation map is independent of the chosen regular J. We denote its value by $n(L, \beta) \in \mathbb{Z}$.

Definition. We call $n(L, \beta) \in \mathbb{Z}$ the *algebraic count* of pseudo-holomorphic discs in the class β passing through a point of L.

By the same cobordism argument, the algebraic counts $n(L,\beta)$ are invariant under isotopies of L among monotone Lagrangian submanifolds; and they are also invariant under simultaneous deformations of the symplectic form on M and of the Lagrangian submanifold L, as long as convexity at infinity and monotonicity are preserved. Another invariance property concerns symplectomorphisms of M: if $L' = \phi(L)$ for some symplectomorphism ϕ, then $\mathcal{M}_1(L,\beta,J) \simeq \mathcal{M}_1(L',\phi_*\beta,\phi_*J)$, and so (with compatible choices of orientations and spin structures) we have $n(L,\beta) = n(L',\phi_*\beta)$.

As pointed out in the introduction, the reader unwilling to deal with spin structures and orientations of moduli spaces should be content to work with $n(L,\beta)$ mod 2.

3.2. **A monotone torus in X_{red}.** Let $T_{std} = \{(z,w),\ |z| = |w| = 1\}$ be the standard product torus in $(\mathbb{C}^2,\omega_0)$ equipped with the standard Kähler form and the standard complex structure. The following is well-known (see e.g. [5]; we sketch the proof for completeness):

Lemma 4. *T_{std} is a monotone Lagrangian torus in $(\mathbb{C}^2,\omega_0)$. There are two families of holomorphic discs of Maslov index 2 with boundary on T_{std}, which can be parametrized by the maps $u_\alpha : z \mapsto (z, e^{i\alpha})$ and $v_\alpha : z \mapsto (e^{i\alpha}, z)$ for $e^{i\alpha} \in S^1$. These discs are all regular, and for a suitable choice of spin structure on T_{std} the algebraic count of discs passing through a point of T_{std} is $+1$ for each of the two families.*

Proof. The maps u_α and $v_\alpha : (D^2,\partial D^2) \to (\mathbb{C}^2, T_{std})$ obviously define holomorphic discs. To calculate their Maslov index, we note that the pullback bundle $u_\alpha^*(T\mathbb{C}^2)$ can be identified with the direct sum of two trivial holomorphic line bundles in such a way that, at a point $e^{i\theta} \in \partial D^2$, the pullback of TT_{std} splits into the direct sum of the real lines $\ell_1 = e^{i\theta}\mathbb{R} \subset \mathbb{C}$ in the first factor and $\ell_0 = \mathbb{R} \subset \mathbb{C}$ in the second factor.

Thus, the Maslov index of u_α is equal to the sum of the Maslov indices of the two families of lines ℓ_1 and ℓ_0 in $\mathbb{C}$, namely $2 + 0 = 2$. Furthermore, the regularity of u_α follows from the surjectivity of the $\bar\partial$ operator for complex-valued functions on the disc with boundary conditions in ℓ_1 (resp. ℓ_0) (as follows e.g. from the reflection principle). Similarly for v_α.

To see that these are the only Maslov index 2 discs, we observe that $\beta_1 = [u_\alpha]$ and $\beta_2 = [v_\alpha]$ generate $\pi_2(\mathbb{C}^2, T_{std}) \simeq \pi_1(T_{std}) = \mathbb{Z}^2$, so by linearity the Maslov index of a disc with boundary on T_{std} is equal to twice its algebraic intersection number with the union of the coordinate axes. For holomorphic discs, positivity of intersection implies that a Maslov index 2 disc in $(\mathbb{C}^2, T_{std})$ intersects only one of the two coordinate axes $z = 0$ and $w = 0$, transversely, and at a single point.

If for example the holomorphic disc $u : (D^2,\partial D^2) \to (\mathbb{C}^2, T_{std})$ is disjoint from the line $w = 0$, then applying the maximum principle to the projection to the w coordinate, we find that $w \circ u : (D^2,\partial D^2) \to (\mathbb{C}^*, S^1)$ must take some constant value $e^{i\alpha}$. Meanwhile, the projection to the z coordinate has a single zero of order 1, which means that $z \circ u : (D^2,\partial D^2) \to (\mathbb{C}, S^1)$ is a biholomorphism from the unit disc to itself, i.e. the identity map up to reparametrization. Thus u is equivalent to u_α up to reparametrization. Similarly for the other case where the disc is disjoint from $z = 0$ and intersects $w = 0$ once.

Finally, the moduli space $\mathcal{M}_1(L, \beta_1, J_0)$ consists of reparametrizations of the discs u_α, e.g. the maps $z \mapsto (e^{i\beta}z, e^{i\alpha})$ for $(e^{i\beta}, e^{i\alpha}) \in S^1 \times S^1$. Thus $\mathcal{M}_1(L, \beta_1, J_0) \simeq T^2$, and the evaluation map to T_{std} is a diffeomorphism; choosing the "standard" spin structure ensures that this diffeomorphism is orientation-preserving [5], hence $n(L, \beta_1) = +1$. Similarly for the other class β_2. □

Next we observe that T_{std} lies away from the complex curve

$$C = h^{-1}(0) = \{(z, w) \in \mathbb{C}^2 \mid cz^n + c^{-1}w - 1 = 0\}, \tag{9}$$

and that the disc u_α intersects C transversely at n distinct points, where the z coordinate takes the values

$$z_k = e^{2\pi ik/n}c^{-1/n}(1 - c^{-1}e^{i\alpha})^{1/n},$$

while v_α is disjoint from C.

The regularity of the discs u_α and v_α implies that they deform smoothly under small isotopies of T_{std}. Thus, for small enough values of the constant κ, denoting by ψ_κ the isotopy constructed in Lemma 3, the Lagrangian torus

$$T_{red} = \psi_\kappa(T_{std})$$

in (X_{red}, ω_{red}) again bounds two families of Maslov index 2 holomorphic discs u'_α and v'_α, representing the homotopy classes $\beta'_1 = (\psi_\kappa)_*(\beta_1)$ and $\beta'_2 = (\psi_\kappa)_*(\beta_2)$. We obtain:

Lemma 5. *For $\kappa > 0$ small enough, (X_{red}, ω_{red}) contains a monotone Lagrangian torus T_{red}, disjoint from $C = h^{-1}(0)$, which bounds exactly two families of Maslov index 2 holomorphic discs, representing classes β'_1, β'_2 that span $\pi_2(X_{red}, T_{red}) \simeq \mathbb{Z}^2$. These discs are all regular, and for a suitable spin structure their algebraic counts are $n(T_{red}, \beta'_1) = n(T_{red}, \beta'_2) = +1$. Moreover, the discs in the class β'_1 intersect C transversely in n distinct points, while those in the class β'_2 are disjoint from C.*

Remark. While ω_{red} is singular along C, it can still be integrated over a disc that intersects C transversely, so the notion of monotonicity still makes sense. In fact, symplectic area can also be defined as the integral of the Liouville form

$$\theta_{red} = d^c(\tfrac{1}{4}|z|^2 + \tfrac{1}{4}|w|^2 + \tfrac{\kappa}{2}|h|)$$

along the boundary of a disc. Perhaps even better, we can modify ω_{red} in a neighborhood of C (disjoint from T_{red}) by a small exact deformation so as to cure its lack of smoothness; this can be achieved simply by replacing $|h|$ by a smooth function $\rho(|h|)$ in the expression for the Kähler potential (taking $\rho : [0, \infty) \to [0, \infty)$ to be any smooth, convex function which agrees with identity outside of $[0, \epsilon]$ and has vanishing odd derivatives at the origin). This modification does not affect the properties of the isotopy ψ_κ away from C, nor the symplectic areas of holomorphic discs.

Proof of Lemma 5. The existence and regularity for small κ of the two families of holomorphic discs u'_α and v'_α with boundary on $T_{red} = \psi_\kappa(T_{std})$ representing the classes β'_1 and β'_2, obtained as smooth deformations of the discs u_α and v_α under the isotopy, is a direct consequence of the regularity of the latter discs.

Since the isotopy is exact ($\psi_\kappa^*(\theta_{red})$ agrees with the standard Liouville form θ_0 up to an exact term), the symplectic areas of the discs are preserved, which proves the monotonicity of T_{red}. Moreover, Gromov compactness implies that T_{red} does not bound any other Maslov index 2 holomorphic discs: if such discs existed for arbitrarily small κ, taking the limit of a subsequence with $\kappa \to 0$ would yield a contradiction.

Finally, because the discs u_α and v_α deform smoothly under the isotopy of T_{std} to T_{red}, for small κ the discs u'_α and v'_α continue to intersect C transversely, and the algebraic counts remain unchanged (in fact the evaluation maps $ev : \mathcal{M}_1(T_{red}, \beta'_i, J_0) \to T_{red}$ remain diffeomorphisms). □

3.3. **A monotone torus in X.** From now on we fix the value of the constant $\kappa > 0$ so that the conclusion of Lemma 5 holds. We then construct a Lagrangian torus T in (X, ω_X) by lifting T_{red} to $\mu_X^{-1}(0)$:

Definition. We denote by T the preimage of T_{red} under the projection map from $\mu_X^{-1}(0) \subset X$ to X_{red}, i.e.

$$T = \{(x, y, z, w) \in X \mid (z, w) \in T_{red} \text{ and } |x| = |y|\}. \tag{10}$$

We also denote by $\pi : X \to X_{red}$ the projection to the (z, w) coordinates,

$$\pi(x, y, z, w) = (z, w). \tag{11}$$

Lemma 6. *T is a monotone Lagrangian torus in (X, ω_X).*

Conceptually, this follows from the observation that T is the image of T_{red} under the monotone Lagrangian correspondence between X_{red} and X induced by $\mu_X^{-1}(0)$. A more elementary argument is as follows.

Proof. Since the restriction of ω_X to $\mu_X^{-1}(0)$ agrees with the pullback of ω_{red} via the projection map π, $\omega_X|_T$ is the pullback of $\omega_{red}|_{T_{red}}$ under the projection from $T \subset \mu_X^{-1}(0)$ to $T_{red} \subset X_{red}$, i.e. it vanishes, and T is Lagrangian.

Let $u : (D^2, \partial D^2) \to (X, T)$ be a disc with boundary on T (not necessarily holomorphic), and denote by $\gamma : S^1 \to T$ its boundary loop. Perturbing u if necessary, we can assume that it avoids the fixed point set $F = \{x = y = 0\}$ (which has real codimension 4). In terms of the Liouville form

$$\theta_X = d^c(\tfrac{1}{4}|z|^2 + \tfrac{1}{4}|w|^2 + \tfrac{\kappa}{4}|x|^2 + \tfrac{\kappa}{4}|y|^2), \tag{12}$$

the symplectic area of u is given by the integral of θ_X along the boundary loop γ. However, along $\mu_X^{-1}(0)$ we have $|x|^2 = |y|^2 = |h|$, and $|x|^2 + |y|^2$ achieves its fiberwise minimum so its derivative vanishes in all directions tangent to the fibers of π.

Therefore, at every point of $\mu_X^{-1}(0)$ the 1-form θ_X coincides with

$$\pi^*\theta_{red} = d^c(\tfrac{1}{4}|z|^2 + \tfrac{1}{4}|w|^2 + \tfrac{\kappa}{2}|h|).$$

Denoting by $u_{red} = \pi \circ u : (D^2, \partial D^2) \to (X_{red}, T_{red})$ and $\gamma_{red} = \pi \circ \gamma : S^1 \to T_{red}$ the projections of u and γ, we conclude that

$$\int_{D^2} u^*\omega_X = \int_{S^1} \gamma^*\theta_X = \int_{S^1} \gamma^*(\pi^*\theta_{red}) = \int_{S^1} \gamma_{red}^*(\theta_{red}) = \int_{D^2} u_{red}^*(\omega_{red}), \tag{13}$$

i.e. the disc u and its projection u_{red} have the same symplectic areas. Meanwhile, away from the fixed point locus F, denote by

$$\mathcal{L}_{\mathbb{R}} = \mathbb{R} \cdot (ix, -iy, 0, 0) \quad \text{and} \quad \mathcal{L} = \mathbb{C} \cdot (ix, -iy, 0, 0) \tag{14}$$

the real and complex spans of the vector field generating the S^1-action. Then $\mathcal{L}$ is a trivial holomorphic subbundle of TX, and $TX/\mathcal{L} \simeq \pi^*TX_{red}$, i.e. away from F we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{L} \longrightarrow TX \xrightarrow{d\pi} \pi^*TX_{red} \longrightarrow 0. \tag{15}$$

Along T, we have a similar short exact sequence of real subbundles,

$$0 \longrightarrow \mathcal{L}_{\mathbb{R}} \longrightarrow TT \xrightarrow{d\pi} \pi^*TT_{red} \longrightarrow 0. \tag{16}$$

Since the trivial subbundles $(u^*\mathcal{L}, \gamma^*\mathcal{L}_{\mathbb{R}})$ do not contribute to the Maslov index, $\mu([u])$ can be computed by considering the quotient bundles $(u^*(TX/\mathcal{L}), \gamma^*(TT/\mathcal{L}_{\mathbb{R}})) \simeq (u_{red}^*(TX_{red}), \gamma_{red}^*(TT_{red}))$. In other terms,

$$\mu([u]) = \mu([u_{red}]). \tag{17}$$

Comparing (13) and (17), we find that the proportionality between Maslov index and symplectic area for discs in X_{red} with boundary on T_{red} implies the same proportionality for discs in X with boundary on T. ☐

Lemma 7. *The projection $u_{red} = \pi \circ u : (D^2, \partial D^2) \to (X_{red}, T_{red})$ of a holomorphic disc $u : (D^2, \partial D^2) \to (X, T)$ is a holomorphic disc, and $\mu([u_{red}]) = \mu([u])$.*

Conversely, let $u_{red} : (D^2, \partial D^2) \to (X_{red}, T_{red})$ be a holomorphic disc that intersects $C = h^{-1}(0)$ transversely in k points, and fix a point $p_0 \in T$ such that $\pi(p_0) = u_{red}(1)$. Then there are exactly 2^k holomorphic discs $u : (D^2, \partial D^2) \to (X, T)$ such that $\pi \circ u = u_{red}$ and $u(1) = p_0$. Moreover, if u_{red} is regular then all these discs are regular.

Proof. The first statement follows immediately from the holomorphicity of π and the Maslov index calculation in the proof of Lemma 6 (equation (17)).

For the second part, let u_{red} be a holomorphic disc in X_{red} that intersects C transversely, with $u_{red}^{-1}(C) = \{t_1, \dots, t_k\} \subset D^2$, and let u be a lift of u_{red} to a disc in X with boundary on T. Along the holomorphic disc u, the product $xy = h(z, w)$ has simple zeroes at $t_1, \dots, t_k$, i.e. u intersects $\pi^{-1}(C) = \{x = 0\} \cup \{y = 0\}$ transversely at the k points $u(t_1), \dots, u(t_k)$. The quotient $q = x/y$ then defines a meromorphic function on the disc, which has either a simple zero or a simple pole at each of $t_1, \dots, t_k$, and no other zeroes or poles. Moreover, on the boundary we have $|x| = |y|$, so q maps the unit circle to itself.

Given any function $\varepsilon : \{1, \dots, k\} \to \{\pm 1\}$, set

$$\vartheta_\varepsilon(z) = \prod_{j=1}^{k} \left(\frac{z - t_j}{1 - \bar{t}_j z} \right)^{\varepsilon(j)}, \tag{18}$$

which is a meromorphic function on the unit disc, mapping the unit circle to itself, and with simple zeroes (resp. poles) at all t_j such that $\varepsilon(j) = +1$ (resp. -1).

Thus, choosing $\varepsilon(j) = \mathrm{ord}_{t_j}(q)$ according to the poles and zeroes of $q = x/y$ along the disc u, we find that ϑ_ε and q have the same zeroes and poles on the unit disc, and their ratio defines a nowhere vanishing holomorphic function on the unit disc, taking values in the unit circle at the boundary. By the maximum principle this function is constant, i.e. there exists $e^{i\theta} \in S^1$ such that $q = e^{i\theta} \vartheta_\varepsilon$.

By construction the holomorphic functions $(h \circ u_{red}) \vartheta_\varepsilon^{\pm 1}$ only have double zeroes, and so we can choose square roots

$$\zeta_\pm = \left((h \circ u_{red})\, \vartheta_\varepsilon^{\pm 1} \right)^{1/2},$$

with $\zeta_+/\zeta_- = \vartheta_\varepsilon$ and $\zeta_+ \zeta_- = h \circ u_{red}$. We obtain that along the disc u the coordinates x and y are given by

$$x = e^{i\theta/2} \zeta_+ \quad \text{and} \quad y = e^{-i\theta/2} \zeta_-,$$

for some $e^{i\theta/2} \in S^1$. Conversely, these formulas determine holomorphic lifts of u_{red} for all $\varepsilon : \{1, \dots, k\} \to \{\pm 1\}$ and for all $e^{i\theta/2} \in S^1$, and the condition that $u(1) = p_0$ determines the normalization factor $e^{i\theta/2}$ uniquely for given ε. Hence there are 2^k lifts of u_{red} as claimed, determined by the choice of whether x or y vanishes at each point where u_{red} intersects C.

Finally, we note that none of the lifts u pass through the fixed point locus of the S^1-action (since x and y do not vanish simultaneously). Thus, pulling back the exact sequences (15) and (16) along u, we find that the holomorphic vector bundle u^*TX admits a trivial holomorphic line subbundle $u^*\mathcal{L}$, with a trivial real subbundle at the boundary $u^*_{|S^1}\mathcal{L}_\mathbb{R}$. Since the $\bar\partial$ operator for complex-valued functions on the disc with the trivial real boundary condition $\mathbb{R} \subset \mathbb{C}$ on the unit circle is surjective, the surjectivity of the $\bar\partial$ operator on sections of u^*TX with boundary conditions $u^*_{|S^1}(TT)$ is equivalent to that of the $\bar\partial$ operator on the quotient bundle $u^*TX/u^*\mathcal{L} \simeq u^*_{red}TX_{red}$ with boundary conditions $u^*_{|S^1}(TT)/u^*_{|S^1}(\mathcal{L}_\mathbb{R}) \simeq u^*_{red|S^1}(TT_{red})$. Thus, the regularity of u is equivalent to that of u_{red} as claimed. □

Corollary 8. *There are $n+2$ distinct Maslov index 2 classes in $\pi_2(X, T)$ for which the algebraic count of pseudo-holomorphic discs is non-zero, and for a suitable choice of spin structure the sum of these counts is $2^n + 1$.*

Proof. By Lemma 7, the holomorphic discs of Maslov index 2 bounded by T are lifts of those bounded by T_{red} in X_{red}, which are determined by Lemma 5.

The discs representing the class $\beta_2' \in \pi_2(X_{red}, T_{red})$ are disjoint from C, hence they admit a unique lift up to the S^1-action. Denoting by $\hat{\beta}_2 \in \pi_2(X,T)$ the class of these lifts, the moduli space $\mathcal{M}_1(T, \hat{\beta}_2, J_0)$ is an S^1-bundle over $\mathcal{M}_1(T_{red}, \beta_2', J_0)$, and the evaluation map to T is equivariant with respect to the S^1-action; thus the evaluation map $ev : \mathcal{M}_1(T, \hat{\beta}_2, J_0) \to T$ is again a diffeomorphism, and its degree is ± 1.

Meanwhile, the discs representing the class $\beta_1' \in \pi_2(X_{red}, T_{red})$ intersect C transversely in n points (cf. Lemma 5), so by Lemma 7 they can be lifted in 2^n different ways up to the S^1-action. Observe that elements of $\pi_2(X,T) \simeq \mathbb{Z}^3$ are determined by their intersection numbers with the three hypersurfaces $x = 0$, $z = 0$, and $w = 0$. Thus, the lifts live in $n+1$ different classes $\hat{\beta}_{1,\ell} \in \pi_2(X,T)$, $\ell = 0, \dots, n$, depending on the intersection number of the lifted disc with the hypersurface $x = 0$; each value of ℓ is achieved by $\binom{n}{\ell}$ of the 2^n lifts. The moduli space $\mathcal{M}_1(T, \hat{\beta}_{1,\ell}, J_0)$ then projects to $\mathcal{M}_1(T_{red}, \beta_1', J_0)$ with fiber a union of $\binom{n}{\ell}$ circles. The evaluation map $ev : \mathcal{M}_1(T, \hat{\beta}_{1,\ell}, J_0) \to T$ is thus an unramified $\binom{n}{\ell}$-sheeted covering.

To determine the orientations, we briefly recall the construction in [8, Chapter 8] (see also [5, Prop. 5.2] for a simpler presentation that suffices for the case at hand). A spin structure on T determines a trivialization of its tangent bundle along the boundary of a holomorphic disc u. Using this trivialization, the $\bar{\partial}$ operator can be deformed to the direct sum of a complex linear operator and a $\bar{\partial}$ operator for sections of a trivialized complex vector bundle with trivial real boundary condition (namely, the tangent bundles to X and T along the boundary of u, with the trivialization determined by the spin structure). Since the kernel of the latter operator can be identified with the tangent space to T at the marked point, an orientation of T then determines an orientation of the tangent space to the moduli space at u.

In our case, we choose the spin structure on T to be standard along the orbits of the S^1-action and consistent under the splitting (16) with that previously chosen on T_{red}. Thus, the preferred trivialization of TT along the boundary of a holomorphic disc u agrees with that induced via (16) by the trivialization of TT_{red} along the boundary of $u_{red} = \pi \circ u$ and the natural trivialization of the trivial line bundle $\mathcal{L}_{\mathbb{R}}$. The orientation at u of the moduli space of holomorphic discs in (X,T) then agrees with that induced by the orientation at u_{red} of the moduli space of holomorphic discs in (X_{red}, T_{red}) and the chosen orientation of the orbits of the S^1-action. With this understood, the orientation-preserving nature of the evaluation maps for discs in (X_{red}, T_{red}) implies that the evaluation maps for discs in (X,T) are also orientation-preserving, i.e. the degrees are positive. □

(For the reader working mod 2, we note that the odd values of $n(T,\beta)$ are achieved for $\hat{\beta}_2$ and those $\hat{\beta}_{1,\ell}$ for which $\binom{n}{\ell}$ is odd, including the extremal cases $\hat{\beta}_{1,0}$ and $\hat{\beta}_{1,n}$.)

4. Proof of Theorem 1

In light of Corollary 8 and the invariance properties of the algebraic counts $n(T,\beta)$, the only thing that remains to be done is to construct an isotopy between the Kähler form ω_X on (a bounded subset of) $X \simeq \mathbb{C}^3$ and the standard Kähler form. We will again rely on Moser's trick (Lemma 2). We denote by

$$\Phi_1 = \tfrac{\kappa}{4}|x|^2 + \tfrac{\kappa}{4}|y|^2 + \tfrac{1}{4}|z|^2 \tag{19}$$

the Kähler potential for the standard (up to rescaling) Kähler form on $\mathbb{C}^3$,

$$\omega_1 = dd^c\Phi_1 = \tfrac{i}{2}dz\wedge d\bar{z} + \kappa(\tfrac{i}{2}dx\wedge d\bar{x} + \tfrac{i}{2}dy\wedge d\bar{y}).$$

The Kähler potential for ω_X is

$$\Phi_X = \Phi_1 + \tfrac{1}{4}|w|^2,$$

where we recall that w is determined as a function of the coordinates (x,y,z) by

$$w = c(xy+1) - c^2 z^n. \tag{20}$$

The estimate that ensures the existence of the Moser flow is the following:

Lemma 9. *Given any bounded subset $B \subset \mathbb{C}^3$, there exist positive constants C and M such that the real-valued function $\varphi = C\Phi_1 - \Phi_X$ is bounded above by M on B, and the connected component Ω of $\varphi^{-1}((-\infty, M])$ which contains B is compact.*

Proof. We equip $\mathbb{C}^3$ with the Euclidean metric for which the positive definite quadratic form Φ_1 is the square of the distance to the origin (i.e., a rescaling of the usual metric).

Let $R > 0$ be such that B is contained within the ball $B(0,R)$ of radius R (for this metric), denote by K the supremum of $\frac{1}{4}|w|^2/\Phi_1$ in $B(0,2R)\setminus B(0,R)$, and set $C = 2K+1$. Then in $B(0,2R)\setminus B(0,R)$ we have

$$K\Phi_1 \le \varphi = (C-1)\Phi_1 - \tfrac{1}{4}|w|^2 \le 2K\Phi_1,$$

and the upper bound continues to hold inside $B(0,R)$.

Then inside $B(0,R)$ we have $\varphi \le 2K\Phi_1 \le 2KR^2$, while in $B(0,2R)\setminus B(0,\sqrt{3}R)$ we have $3KR^2 \le K\Phi_1 \le \varphi$. Thus, setting $M = \frac{5}{2}KR^2$, there is a connected component Ω of $\varphi^{-1}((-\infty, M])$ for which $B(0,R)\subset\Omega\subset B(0,\sqrt{3}R)$. □

Choosing B to be a polydisc in $\mathbb{C}^3$ large enough to contain T, and taking C as in Lemma 9, we now apply Lemma 2 to the Kähler forms ω_X and $C\omega_1$, to construct an exact isotopy ψ_t such that $\psi_1^*(C\omega_1) = \omega_X$. Because the isotopy is generated by the negative gradient of $\varphi = C\Phi_1 - \Phi_X$ (with respect to a varying family of Kähler metrics), the values of φ decrease along the flow. Thus, the compact subset $\Omega \supset B$ constructed in Lemma 9 is preserved, so the isotopy is well-defined everywhere in it, and in particular in B.

Since the isotopy is exact, $\psi_t(T)$ is a monotone Lagrangian torus in $\mathbb{C}^3$ equipped with the Kähler form $\omega_t = Ct\omega_1 + (1-t)\omega_X$, and the algebraic counts of Maslov index 2 holomorphic discs remain constant along the isotopy. For $t = 1$ we obtain a monotone Lagrangian torus in $(\mathbb{C}^3, C\omega_1)$ with the desired properties. Rescaling the coordinate axes by suitable constant factors, we obtain a monotone Lagrangian torus in $\mathbb{C}^3$ equipped with the standard Kähler form, and by further rescaling we obtain tori with arbitrary monotonicity constants and the same algebraic counts of pseudo-holomorphic discs.

5. Comments on the construction

Our construction is inspired by ideas from mirror symmetry, and more precisely the Strominger-Yau-Zaslow (SYZ) conjecture, whereby the mirror of a given Kähler manifold is constructed geometrically from a Lagrangian torus fibration on the complement of a complex hypersurface. The numbers of Maslov index 2 discs bounded by the fibers exhibit discontinuities across a set of *walls* which separate the fibration into *chambers*, each with its own enumerative behavior; each chamber corresponds to a distinguished coordinate chart on the mirror (cf. [1, §2] and [2]).

In a given Lagrangian fibration, the vast majority of fibers are not monotone, and the counts of Maslov index 2 discs are not invariant under Hamiltonian isotopies. However, by deforming the fibration suitably it is often possible to arrange the existence of a monotone fiber in any given chamber. For example, the complement of a smooth cubic in $\mathbb{CP}^2$ admits a Lagrangian torus fibration with three singular fibers and infinitely many chambers; Vianna's constructions in [12, 13] can be understood as modifying the fibration to place the monotone fiber in a prescribed chamber.

The construction of Theorem 1 relies on the fact that $\mathbb{C}^3$ can be presented as a conic bundle $\{xy = h(z, w)\}$ over $\mathbb{C}^2$ with a discriminant curve $h^{-1}(0) \subset \mathbb{C}^2$ of arbitrarily large degree. SYZ mirror symmetry for conic bundles over toric varieties has been studied in detail in [1], where it was shown that the chamber structure is governed by the tropical geometry of $h^{-1}(0)$ (or, in more classical terms, by the various manners in which product tori in $\mathbb{C}^2$ can be linked with $h^{-1}(0)$). Thus, by increasing the degree of h we can exhibit Lagrangian torus fibrations on open dense subsets of $\mathbb{C}^3$ (namely, those points where z and w are nonzero) with arbitrarily many chambers. Choosing the coefficients of h suitably ensures the existence of monotone fibers in the "most interesting" chambers. (In fact, choosing h to be analytic rather than algebraic one could obtain a single fibration with infinitely many chambers, with monotone representatives corresponding to all the values of n in our main construction at once.)

Another perspective on the construction comes from singularity theory: projecting the conic bundle $X \simeq \mathbb{C}^3$ to the coordinate w presents it as an unfolding of the A_{n-1} singularity $xy = cz^n$. The A_{n-1} Milnor fiber contains non-displaceable monotone Lagrangian tori (cf. [1, Corollary 9.1] and [11]). The examples of Theorem 1 can be obtained by transporting these tori along a circle in the w coordinate; even though the unfolding makes the ambient manifold contractible and the tori displaceable, the distinctive enumerative features of the tori in the fibers persist.

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Department of Mathematics, UC Berkeley, Berkeley CA 94720-3840, USA

E-mail address: auroux@math.berkeley.edu