Spectrum-Based Comparison of Multivariate Complex Random Signals of Unequal Lengths

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***Abstract*—We consider the problem of comparing two complex multivariate random signal realizations of unequal lengths, to ascertain whether they have identical power spectral densities. A binary hypothesis testing approach is formulated and a generalized likelihood ratio test (GLRT) is derived. An asymptotic analytical solution for calculating the test threshold is provided. The results are illustrated via computer simulations. Past work on this problem is limited to either complex or real signals of equal lengths, or to real-valued scalar signals of unequal lengths. The proposed test has applications in diverse areas including user authentication in wireless networks with multiantenna receivers.**

I. Introduction

This paper is concerned with comparison of two realizations (sample functions) of some random signals (time series) to assess if they are realizations of the same random signal. The realizations are not necessarily of the same length (unequal sample sizes). A motivation for such problems is application to two areas: (i) user authentication for wireless network security enhancement at the physical layer [1], [2], and (ii) spectrum sensing (looking for presence/absence of primary users (PUs) in spectral bands) in cognitive radio (CR) networks (based on the two-window approach of [3]). Recent general articles on comparison of random signals include [4], [5], [6], and references therein, where a variety of applications have been mentioned: earthquake-explosion discrimination [7], financial portfolio management, clustering of environmental data [6], analysis of photometry data [4], and development of climate reference stations [5].

In [8] we have investigated comparison of complex-valued random signals of equal lengths using their power spectral densities (PSD). In this paper we extend the theory of [8] to random signals of unequal lengths (sample sizes). In [9] real-valued scalar signals of unequal lengths have been considered.

Notation: $|\mathbf{A}|$ and $\mathrm{etr}(\mathbf{A})$ denote the determinant and the exponential of the trace of the square matrix $\mathbf{A}$, respectively; i.e. $\mathrm{etr}(\mathbf{A}) = \exp(\mathrm{tr}(\mathbf{A}))$. $\mathbf{B}_{ij}$ denotes the ijth element of the matrix $\mathbf{B}$ and $\mathbf{I}$ is the identity matrix. The superscripts $*$ and H denote the complex conjugate and the Hermitian (conjugate transpose) operations, respectively. The notation $y = \mathcal{O}(g(x))$ means that there exists some finite real number $b > 0$ such that $\lim_{x\to\infty} |y/g(x)| \le b$. Given a column vector $\mathbf{x}$, $\mathrm{diag}\{\mathbf{x}\}$ denotes a square matrix with elements of $\mathbf{x}$ along its main diagonal and zeros everywhere else. $\delta(\tau)$ denotes the Kronecker delta, i.e. $\delta(\tau) = 1$ if $\tau = 0$, $= 0$ otherwise. Given two random vectors $\mathbf{x}$ and $\mathbf{y}$, we define $\mathrm{cov}(\mathbf{x}, \mathbf{y}) := E\{\mathbf{x}\mathbf{y}^H\} - E\{\mathbf{x}\}E\{\mathbf{y}^H\}$.

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II. System Model

We consider two zero-mean (proper) complex multivariate (dimension p) stationary random signals $\{\mathbf{x}(t)\}$ and $\{\mathbf{y}(t)\}$ with $p \times p$ PSD matrices $\mathbf{S}_x(f)$ and $\mathbf{S}_y(f)$, respectively. We observe $\{\mathbf{x}(t)\}$ for $t = 0, 1, \cdots, N_x - 1$ and $\{\mathbf{y}(t)\}$ for $t = 0, 1, \cdots, N_y - 1$. We employ multivariate spectral analysis to test if the two sets of observation are realizations of two random signals with identical PSDs. We will assume that both $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are stationary with all bounded moments so that some asymptotic results from [10] regarding PSD estimators can be invoked; the time series need not be Gaussian. See also [8]. We will use $W_C(p, K, \mathbf{S}(f))$ to denote the complex Wishart distribution of dimension p, degrees of freedom K, and mean $K\mathbf{S}(f)$.

We will use the Daniell method (unweighted smoothing in frequency-domain) for PSD estimation. Consider a set of M normalized frequencies $\tilde{f}_k$, in increasing order with k, over $[0, 1]$, $k = 1, 2, \cdots, M$. Define

$$f_l^{(xk)} = \left[\tilde{f}_k N_x\right] + l, \quad f_l^{(yk)} = \left[\tilde{f}_k N_y\right] + l, \tag{1}$$

$$\mathbf{d}_x(f_l^{(xk)}) = \sum_{t=0}^{N_x-1} \mathbf{x}(t) \exp\left(-j2\pi f_l^{(xk)} t\right), \tag{2}$$

$$\mathbf{d}_y(f_l^{(xk)}) = \sum_{t=0}^{N_y-1} \mathbf{y}(t) \exp\left(-j2\pi f_l^{(yk)} t\right), \tag{3}$$

where $[x]$ denotes rounding of x to an integer. The Daniell estimators of the PSDs of $\mathbf{x}(t)$ and $\mathbf{y}(t)$ at normalized frequency $\tilde{f}_k$, denoted by $\mathbf{X}_k$ and $\mathbf{Y}_k$, respectively, are given by

$$\mathbf{X}_k = \frac{1}{K_x} \sum_{l=-m_{tx}}^{m_{tx}} N^{-1} \mathbf{d}_x(f_l^{(xk)}) \mathbf{d}_x^H(f_l^{(xk)}), \tag{4}$$

$$\mathbf{Y}_k = \frac{1}{K_y} \sum_{l=-m_{ty}}^{m_{ty}} N^{-1} \mathbf{d}_y(f_l^{(yk)}) \mathbf{d}_y^H(f_l^{(yk)}), \tag{5}$$

where $K_x = 2m_{tx}+1$ and $K_y = 2m_{ty}+1$ are smoothing window sizes. We ensure that $\tilde{f}_k$, m_{tx}, m_{ty} and M are such that

$$\begin{aligned} &\left\{f_l^{(xk)},\ l=-m_{tx},\cdots,m_{tx}\right\}\cap \\ &\qquad\left\{f_l^{(x(k+1))},\ l=-m_{tx},\cdots,m_{tx}\right\} \\ &\quad=\emptyset,\ \ k=1,2,\cdots,M, \end{aligned} \tag{6}$$

$$f_l^{(x1)}-m_{tx}\ge 0,\quad f_l^{(xM)}+m_{tx}<N_x, \tag{7}$$

$$\begin{aligned} &\left\{f_l^{(yk)},\ l=-m_{ty},\cdots,m_{ty}\right\}\cap \\ &\qquad\left\{f_l^{(y(k+1))},\ l=-m_{ty},\cdots,m_{ty}\right\} \\ &\quad=\emptyset,\ \ k=1,2,\cdots,M, \end{aligned} \tag{8}$$

$$f_l^{(y1)}-m_{ty}\ge 0,\quad f_l^{(yM)}+m_{ty}<N_y. \tag{9}$$

This ensures that none of the smoothing windows overlap, and all frequencies stay within the normalized range $[0,1)$. For given N_x, N_y and pre-selected m_{tx} and m_{ty}, one way to satisfy the above requirements is to first pick

$$\Delta=\max\left(\frac{2m_{tx}+1}{N_x},\frac{2m_{ty}+1}{N_y}\right),$$

and then set $\tilde{f}_k=(\Delta+0.01):\Delta:(1-(\Delta/2))$ (in MATLAB notation of picking reals from $(\Delta+0.01)$ and $(1-(\Delta/2))$ in steps of Δ).

As shown in [8] exploiting [10], as $N_x, N_y\to\infty$, $\mathbf{X}_k$ and $\mathbf{Y}_k$ approach complex Wishart matrix distributions

$$\mathbf{X}_k \sim W_C\left(p,K_x,K_x^{-1}\mathbf{S}_x(\tilde{f}_k)\right), \tag{10}$$

$$\mathbf{Y}_k \sim W_C\left(p,K_y,K_y^{-1}\mathbf{S}_y(\tilde{f}_k)\right) \tag{11}$$

and the two are assumed to be independent. Since the smoothing windows used in (4) do not overlap, $\mathbf{X}_k$ and $\mathbf{Y}_k$ are (asymptotically) independent over the M frequencies $\tilde{f}_k$, $k=1,2,\cdots,M$.

If $\mathbf{X}\sim W_C\,(p,K,\mathbf{S}(f))$, then by [10, Sec. 4.2], $E\{\mathbf{X}\}=K\mathbf{S}(f)$, $\text{cov}\,\{\mathbf{X}_{jk},\mathbf{X}_{lm}\}=K\mathbf{S}_{jl}(f)\mathbf{S}^*_{km}(f)$ and the probability density function (pdf) of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{X})=\frac{1}{\Gamma_p(K)}\frac{1}{|\mathbf{S}(f)|^K}|\mathbf{X}|^{K-p}\,\text{etr}\{-\mathbf{S}^{-1}(f)\mathbf{X}\} \tag{12}$$

where, as is common practice, we do not distinguish between the random matrix (vector) and the values taken by it, the pdf (12) is defined for positive-definite Hermitian $\mathbf{X}$ and is otherwise zero, and

$$\Gamma_p(K):=\pi^{p(p-1)/2}\prod_{j=1}^{p}\Gamma(K-j+1) \tag{13}$$

where $\Gamma(n)$ denotes the (complete) Gamma function $\Gamma(z):=\int_0^\infty t^{z-1}e^{-t}\,dt$.

III. PSD-Based GLRT

The binary hypothesis testing problem now is

$$\begin{aligned} \mathcal{H}_0 &: \ \mathbf{S}_y(\tilde{f}_k)=\mathbf{S}_x(\tilde{f}_k)=\mathbf{S}(\tilde{f}_k)\ \forall\tilde{f}_k \\ \mathcal{H}_1 &: \ \mathbf{H}_0^c = \text{complement of } \mathcal{H}_0 \end{aligned} \tag{14}$$

given the "data" $\mathbf{X}_k$ and $\mathbf{Y}_k$, where the unknown parameters are $\mathbf{S}_y(\tilde{f}_k)$ and $\mathbf{S}_y(\tilde{f}_k)$, $k=1,2,\cdots,M$. We follow the GLRT approach.

Under $\mathcal{H}_0$, the joint pdf of $\mathbf{X}_k$ and $\mathbf{Y}_k$ for $k=1,2,\cdots,M$, is given by

$$\begin{aligned} &f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_0,\mathbf{S}(\tilde{f}_k))= \\ &\prod_{k=1}^{M}\frac{K_x^{K_x}K_y^{K_y}}{\Gamma_p(K_x)\Gamma_p(K_y)}\frac{|\mathbf{X}_k|^{K_x-p}|\mathbf{Y}_k|^{K_y-p}}{|\mathbf{S}(\tilde{f}_k)|^{K_x+K_y}} \\ &\qquad\times\,\text{etr}\{-\mathbf{S}^{-1}(\tilde{f}_k)(K_x\mathbf{X}_k+K_y\mathbf{Y}_k)\}. \end{aligned} \tag{15}$$

The above joint pdf is maximized w.r.t. the Hermitian matrix $\mathbf{S}(f_k)$ for $\hat{\mathbf{S}}(f_k)=\alpha_x\mathbf{X}_k+\alpha_y\mathbf{Y}_k$, leading to the optimized joint pdf

$$\begin{aligned} &f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_0,\hat{\mathbf{S}}(\tilde{f}_k))= \\ &\prod_{k=1}^{M}\frac{K_x^{K_x}K_y^{K_y}}{\Gamma_p(K_x)\Gamma_p(K_y)}\frac{|\mathbf{X}_k|^{K_x-p}|\mathbf{Y}_k|^{K_y-p}}{|\alpha_x\mathbf{X}_k+\alpha_y\mathbf{Y}_k|^{K_x+K_y}} \\ &\qquad\times\,\exp\{-p(K_x+K_y)\}. \end{aligned} \tag{16}$$

where

$$\alpha_x=\frac{K_x}{K_x+K_y},\ \alpha_y=\frac{K_y}{K_x+K_y}. \tag{17}$$

Under $\mathcal{H}_1$, the joint pdf of $\mathbf{X}_k$ and $\mathbf{Y}_k$ for $k=1,2,\cdots,M$, is given by

$$\begin{aligned} &f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_1,\mathbf{S}_x(\tilde{f}_k),\mathbf{S}_y(\tilde{f}_k))= \\ &\prod_{k=1}^{M}\frac{K_x^{K_x}K_y^{K_y}}{\Gamma_p(K_x)\Gamma_p(K_y)}\frac{|\mathbf{X}_k|^{K_x-p}|\mathbf{Y}_k|^{K_y-p}}{|\mathbf{S}_x(\tilde{f}_k)|^{K_x}\,|\mathbf{S}_y(\tilde{f}_k)|^{K_y}} \\ &\qquad\times\text{etr}\{-K_x\mathbf{S}_x^{-1}(\tilde{f}_k)\mathbf{X}_k-K_y\mathbf{S}_y^{-1}(C_k)\mathbf{Y}_k\}. \end{aligned} \tag{18}$$

The above joint pdf is maximized w.r.t. the Hermitian matrices $\mathbf{S}_x(\tilde{f}_k)$ and $\mathbf{S}_y(\tilde{f}_k)$ for $\mathbf{S}_x(\tilde{f}_k)=\mathbf{X}_k$ and $\mathbf{S}_y(\tilde{f}_k)=\mathbf{Y}_k$, leading to the optimized joint pdf

$$\begin{aligned} &f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_1,\hat{\mathbf{S}}_x(\tilde{f}_k),\hat{\mathbf{S}}_y(\tilde{f}_k))= \\ &\prod_{k=1}^{M}\frac{K_x^{K_x}K_y^{K_y}}{\Gamma_p(K_x)\Gamma_p(K_y)}\frac{|\mathbf{X}_k|^{K_x-p}|\mathbf{Y}_k|^{K_y-p}}{|\mathbf{X}_k|^{K_x}\,|\mathbf{Y}_k|^{K_y}} \\ &\qquad\times\,\exp\{-p(K_x+K_y)\}. \end{aligned} \tag{19}$$

Using (16) and (19) one gets the GLRT

$$\begin{aligned} \mathcal{L}:=&\frac{f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_1,\hat{\mathbf{S}}_x(\tilde{f}_k),\hat{\mathbf{S}}_y(\tilde{f}_k))}{f(\mathbf{X}_k,\mathbf{Y}_k,k\in[1,M]|\mathcal{H}_0,\hat{\mathbf{S}}(\tilde{f}_k))} \\ =&\prod_{k=1}^{M}\frac{|\alpha_x\mathbf{X}_k+\alpha_y\mathbf{Y}_k|^{K_x+K_y}}{|\mathbf{X}_k|^{K_x}\,|\mathbf{Y}_k|^{K_y}}\ \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}}\ \tau\,, \end{aligned} \tag{20}$$

where the threshold τ is picked to achieve a given probability of false alarm $P_{fa} = P\{\mathcal{L} \geq \tau \,|\, \mathcal{H}_0\}$. This requires pdf of $\mathcal{L}$ under $\mathcal{H}_0$. To this end we can establish Theorem 1.

First some notation and definitions. Let $B_r(h)$ denote the Bernoulli polynomial of degree r and order unity. Define

$$\nu = Mp^2, \tag{21}$$

$$\rho = 1 - \frac{2p^2-1}{6p}\left[K_x^{-1} + K_y^{-1} - (K_x+K_y)^{-1}\right], \tag{22}$$

$$\begin{aligned}\omega_r = \frac{(-1)^{r+1}M}{r(r+1)} \sum_{l=1}^{p} \Big[& \frac{B_{r+1}((1-\rho)K_x + 1 - l)}{(\rho K_x)^r} \\ & + \frac{B_{r+1}((1-\rho)K_y + 1 - l)}{(\rho K_y)^r} \\ & - \frac{B_{r+1}((1-\rho)(K_x+K_y) + 1 - l)}{(\rho(K_x+K_y))^r}\Big]\end{aligned} \tag{23}$$

and

$$\begin{aligned}\ln(\mathcal{L}) = \sum_{k=1}^{M} \Big(& (K_x+K_y)\, \ln|\alpha_x \mathbf{X}_k + \alpha_y \mathbf{Y}_k| \\ & - K_x \ln|X_k| - K_y \ln|Y_k| \Big).\end{aligned} \tag{24}$$

Our main result is stated below and proved in the Appendix. **Theorem 1** : The GLRT for the binary hypothesis testing problem (14) is given by

$$2\rho\, \ln(\mathcal{L}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \tau \tag{25}$$

where ρ and $\ln(\mathcal{L})$ are given by (21) and (24), respectively. The threshold τ is picked to achieve a pre-specified probability of false alarm $P_{fa} = P\{2\rho \ln(\mathcal{L}) > \tau \,|\, \mathcal{H}_0\} = 1 - P\{2\rho \ln(\mathcal{L}) \leq \tau \,|\, \mathcal{H}_0\}$. With χ_n^2 denoting a random variable with central chi-square distribution with n degrees of freedom (as well as the distribution itself),

$$\begin{aligned}P\{2\rho\, \ln(\mathcal{L}) \leq z \,|\, \mathcal{H}_0\} &= P\{\chi_\nu^2 \leq z\} + \omega_2\left[P\{\chi_{\nu+4}^2 \leq z\}\right. \\ &\left. - P\{\chi_\nu^2 \leq z\}\right] + \omega_3\left[P\{\chi_{\nu+6}^2 \leq z\} - P\{\chi_\nu^2 \leq z\}\right] \\ &+ \Big\{\omega_4\left[P\{\chi_{\nu+8}^2 \leq z\} - P\{\chi_\nu^2 \leq z\}\right] + \frac{1}{2}\omega_2^2\left[P\{\chi_{\nu+8}^2 \leq z\}\right. \\ &\left. - 2P\{\chi_{\nu+4}^2 \leq z\} + P\{\chi_\nu^2 \leq z\}\right]\Big\} + \mathcal{O}(M/(\min(K_x,K_y))^5).\end{aligned} \tag{26}$$

IV. Simulation Examples

We generate stationary $\mathbf{x}(t)$ and $\mathbf{y}(t) \in \mathbb{C}^p$, $p = 3$, as $\mathbf{x}(t) = \mathbf{n}(t)$, $\mathbf{y}(t) = \mathbf{n}(t)$ under $\mathcal{H}_0$, and $\mathbf{x}(t) = \mathbf{n}(t)$, $\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{n}(t)$ under $\mathcal{H}_1$, where $\{\mathbf{n}(t)\}$ is spatially uncorrelated, colored, proper complex Gaussian noise, and $\{\mathbf{s}(t)\}$ is the signal sequence. Noise sequences $\{\mathbf{n}(t)\}$ under $\mathcal{H}_0$ and $\mathcal{H}_1$ are independent of each other, but identically distributed. The noise sequence $\{\mathbf{n}(t)\}$ is generated as

$$\mathbf{n}(t) = \mathbf{n}_c(t) + \mathbf{n}_w(t)\,, \tag{27}$$

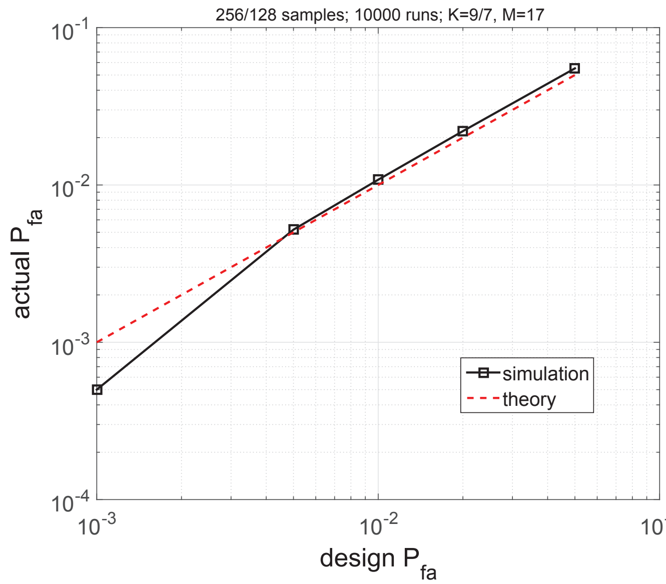

Fig. 1. *Actual P_{fa} vs design P_{fa} based on 10,000 runs, $p = 3$*

where $\mathbf{n}_w(t) \sim \mathcal{N}_c(\mathbf{0}, \sigma_w^2 \mathbf{I})$ is i.i.d., and $\mathbf{n}_c(t)$ is generated as follows. The various components of $\mathbf{n}_c(t)$ are i.i.d., and each component is generated by filtering an i.i.d. scalar sequence, distributed as $\mathcal{N}_c(0, \sigma_c^2)$, through a linear filter with impulse response $\{0.2762, 0.9206, 0.2762\}$, where $\sqrt{(0.2762)^2 + (0.9206)^2 + (0.2762)^2} = 1$. Therefore, $\mathbb{E}\{\|\mathbf{n}_c(t)\|^2\} = p\sigma_c^2$, leading to $\mathbb{E}\{\|\mathbf{n}(t)\|^2\} = p(\sigma_c^2 + \sigma_w^2) = p\sigma_n^2$. We pick $\sigma_w^2 = 0.2\sigma_n^2$ for a given value of σ_n^2. The signal $\{\mathbf{s}(t)\}$ is a filtered digital communications signal generated by passing an information sequence through a frequency-selective Rayleigh fading channel as follows:

$$\mathbf{s}(t) = \sum_{l=0}^{4} \mathbf{h}(l) d(t-l) \tag{28}$$

where $d(t)$ is a scalar i.i.d. QPSK sequence, filtered through a random time-invariant, frequency-selective Rayleigh fading $p \times 1$ vector channel $\mathbf{h}(l)$ with 5 taps, equal power delay profile, mutually independent components, which are identically distributed zero-mean proper complex Gaussian random variables. For different ls, $\mathbf{h}\,(l)$s are mutually independent and identically distributed as $\mathbf{h}(l) \sim \mathcal{N}_c(\mathbf{0}, \sigma_h^2 \mathbf{I})$. The signal $\mathbf{s}(t)$ was scaled to achieve a given SNR $\mathbb{E}\{\|\mathbf{s}(t)\|^2\}/\mathbb{E}\{\|\mathbf{n}(t)\|^2\}$.

We picked N_x=256 or 128 with K_x=9 or 7, respectively, and N_y=128 with $K_y = 7$, and $M = 17$.

A. Threshold Calculation

First we investigate the efficacy of Theorem 1 in computing the GLRT threshold for a given P_{fa}. In Fig. 1, we compare actual P_{fa} and design P_{fa} based on 10,000 runs for N_x=256 and N_y=128, where Theorem 1 was used to calculate the test threshold. It is seen that Theorem 1 is effective in calculating the test threshold.

B. Detection Performance

In Fig. 2, we show the ROC curves under two different sample sizes: equal and unequal. Given unequal sample sizes, one can reduce the larger sample size to smaller sample size by discarding "extra" data, and then apply the results of [8]. This is shown in Fig. 2. But we can also apply the proposed detector with unequal sample sizes where we do not discard any data. It is seen from Fig. 2 that we obtain an improved performance (higher probability of detection for a given false alarm rate).

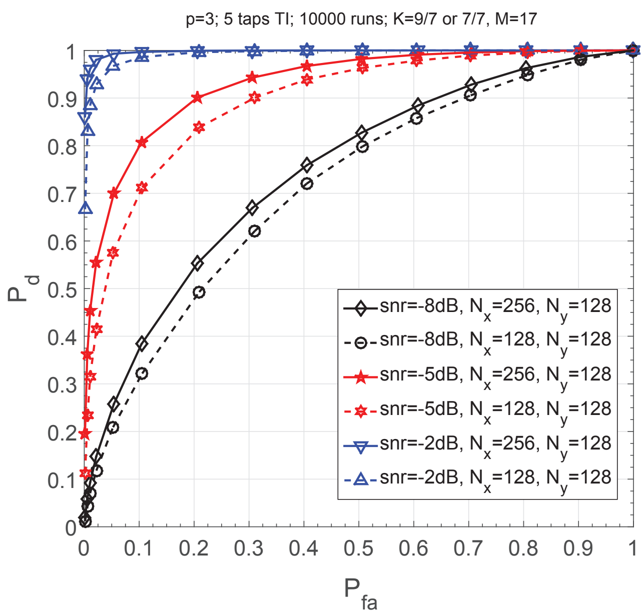

Fig. 2. *ROC curves based on 10,000 runs, $p = 3$*

APPENDIX

The proof of Theorem 1 relies on the following result which is based on [11, Sec. 8.5.1], [12, Sec. 8.2.4] (see [13, Lemma 9] for details, also [8, Lemma 2]).

Lemma 1 : Consider a random variable W $(0 \leq W \leq 1)$ with hth moment

$$E\{W^h\} = C\left(\frac{\prod_{j=1}^{b} y_j^{y_j}}{\prod_{k=1}^{a} x_k^{x_k}}\right)^h \frac{\prod_{k=1}^{a}\Gamma(x_k(1+h)+\xi_k)}{\prod_{j=1}^{b}\Gamma(y_j(1+h)+\eta_j)} \quad (29)$$

where a and b are integers, C is a constant such that $E\{W^0\} = 1$ and $\sum_{k=1}^{a} x_k = \sum_{j=1}^{b} y_j$. Define

$$\nu = -2\Big[\sum_{k=1}^{a}\xi_k - \sum_{j=1}^{b}\eta_j - \frac{1}{2}(a-b)\Big], \quad (30)$$

$$\rho = 1 - \frac{1}{\nu}\Big[\sum_{k=1}^{a} x_k^{-1}\big(\xi_k^2 - \xi_k + \frac{1}{6}\big) - \sum_{j=1}^{b} y_j^{-1}\big(\eta_j^2 - \eta_j + \frac{1}{6}\big)\Big], \quad (31)$$

$$\beta_k = (1-\rho)x_k, \quad \epsilon_j = (1-\rho)y_j, \quad (32)$$

and

$$\omega_r = \frac{(-1)^{r+1}}{r(r+1)}\Big\{\sum_{k=1}^{a}\frac{B_{r+1}(\beta_k+\xi_k)}{(\rho x_k)^r} - \sum_{j=1}^{b}\frac{B_{r+1}(\epsilon_j+\eta_j)}{(\rho y_j)^r}\Big\}. \quad (33)$$

Then

$$\begin{aligned} P\{-2\rho\ln(W) \leq z\} &= P\{\chi_\nu^2 \leq z\} + \omega_2\left[P\{\chi_{\nu+4}^2 \leq z\} - P\{\chi_\nu^2 \leq z\}\right] \\ &+ \omega_3\left[P\{\chi_{\nu+6}^2 \leq z\} - P\{\chi_\nu^2 \leq z\}\right] \\ &+ \Big\{\omega_4\left[P\{\chi_{\nu+8}^2 \leq z\} - P\{\chi_\nu^2 \leq z\}\right] + \frac{1}{2}\omega_2^2 \\ &\times \left[P\{\chi_{\nu+8}^2 \leq z\} - 2P\{\chi_{\nu+4}^2 \leq z\} + P\{\chi_\nu^2 \leq z\}\right]\Big\} \\ &+ \sum_{k=1}^{a}\mathcal{O}(x_k^{-5}) + \sum_{j=1}^{b}\mathcal{O}(y_j^{-5}). \end{aligned} \quad (34)$$

In order to apply Lemma 1 to our problem, we establish Lemma 2.

Lemma 2 : Under $\mathcal{H}_0$, for any $\mathrm{Re}(h) \geq 0$,

$$\begin{aligned} E\{\mathcal{L}^{-h}\} = &\frac{1}{(\alpha_x^{K_x}\alpha_y^{K_y})^{Mph}} \\ &\times \left[\frac{\Gamma_p(K_x(1+h))\Gamma_p(K_y(1+h))}{\Gamma_p(K_x)\Gamma_p(K_y)}\right]^M \\ &\times \left[\prod_{\ell=1}^{p}\frac{\Gamma(K_x+K_y-\ell+1)}{\Gamma((K_x+K_y)(1+h)-\ell+1)}\right]^M. \end{aligned} \quad (35)$$

Proof: Define

$$\mathcal{L}_k = \frac{|\alpha_x\mathbf{X}_k + \alpha_y\mathbf{Y}_k|^{K_x+K_y}}{|\mathbf{X}_k|^{K_x}\,|\mathbf{Y}_k|^{K_y}}, \quad \text{with} \quad \mathcal{L} = \prod_{k=1}^{M}\mathcal{L}_k. \quad (36)$$

Define

$$\check{\mathbf{X}}_k := K_x\mathbf{S}_x^{-1/2}(\tilde{f}_k)\mathbf{X}_k\mathbf{S}_x^{-1/2}(\tilde{f}_k), \quad (37)$$

$$\check{\mathbf{Y}}_k := K_y\mathbf{S}_y^{-1/2}(\tilde{f}_k)\mathbf{Y}_k\mathbf{S}_y^{-1/2}(\tilde{f}_k). \quad (38)$$

Then

$$\mathcal{L}_k = \frac{(\alpha_x^{K_x}\alpha_y^{K_y})^p\,|\check{\mathbf{X}}_k + \check{\mathbf{Y}}_k|^{K_x+K_y}}{|\check{\mathbf{X}}_k|^{K_x}\,|\check{\mathbf{Y}}_k|^{K_y}}, \quad (39)$$

and under $\mathcal{H}_0$,

$$\check{\mathbf{X}}_k \sim W_C(p, K_x, \mathbf{I}), \quad \check{\mathbf{Y}}_k \sim W_C(p, K_y, \mathbf{I}). \quad (40)$$

Using (39) and (40), we have

$$E\{1/\mathcal{L}_k^h \mid \mathcal{H}_0\} = (\alpha_x^{K_x}\alpha_y^{K_y})^{-ph} \times \int\int \frac{|\check{\mathbf{X}}_k|^{K_x h + K_x - p} |\check{\mathbf{Y}}_k|^{K_y h + K_y - p}}{|\check{\mathbf{X}}_k + \check{\mathbf{Y}}_k|^{(K_x+K_y)h}} \times \frac{1}{\Gamma_p(K_x)\Gamma_p(K_y)} \operatorname{etr}\{-\check{\mathbf{X}}_k - \check{\mathbf{Y}}_k\}\, d\check{\mathbf{X}}_k\, d\check{\mathbf{Y}}_k = (\alpha_x^{K_x}\alpha_y^{K_y})^{-ph} \frac{\Gamma_p(K_x(1+h))\Gamma_p(K_y(1+h))}{\Gamma_p(K_x)\Gamma_p(K_y)} \times E\{|\mathbf{X}_k' + \mathbf{Y}_k'|^{-h(K_x+K_y)}\}, \tag{41}$$

where $\mathbf{X}_k'$ and $\tilde{\mathbf{Y}}_k'$ are mutually independent with distributions given by

$$\mathbf{X}_k' \sim W_C(p, K_x(1+h), \mathbf{I}), \tag{42}$$

$$\mathbf{Y}_k' \sim W_C(p, K_y(1+h), \mathbf{I}). \tag{43}$$

Since $\mathbf{X}_k' + \tilde{\mathbf{Y}}_k' \sim W_C(p, (K_x+K_y)(1+h), \mathbf{I})$, by [14], we have

$$E\{|\mathbf{X}_k' + \tilde{\mathbf{Y}}_k'|^{-h(K_x+K_y)}\} = \prod_{\ell=1}^{p} \frac{\Gamma(K_x+K_y-\ell+1)}{\Gamma((K_x+K_y)(1+h)-\ell+1)}. \tag{44}$$

and therefore, from (36), (41), (44), and independence of $\mathcal{L}_k$ over the M frequencies $\tilde{f}_k$, $k = 1, 2, \cdots, M$, we have (35). □

Now we apply Lemma 1 to Lemma 2 to prove Theorem 1. In order to exploit Lemma 1, we need to establish that $0 \leq \mathcal{L}^{-1} \leq 1$. Since $\mathbf{X}_k$ and $\mathbf{Y}_k$ are both positive-definite, $\mathcal{L}_k^{-1} \geq 0$ follows immediately. By complex equivalent of [15, Sec. 17.9, Eqn. (17.115)], noting that $\alpha_y = 1 - \alpha_x$, $|\alpha_x \mathbf{X}_k + \alpha_y \mathbf{Y}_k)| \geq |\mathbf{X}_k|^{\alpha_x} \cdot |\mathbf{Y}_k|^{\alpha_y}$, hence, $|\alpha_x \mathbf{X}_k + \alpha_y \mathbf{Y}_k)|^{K_x+K_y} \geq |\mathbf{X}_k|^{\alpha_x(K_x+K_y)} \cdot |\mathbf{Y}_k|^{\alpha_y(K_x+K_y)} = |\mathbf{X}_k|^{K_x} |\mathbf{Y}_k|^{K_y}$, which implies $\mathcal{L}^{-1} \leq 1$.

Proof of Theorem 1 : Comparing (29) with (35), it can be shown that we have the correspondence

$$a = 2Mp,\ b = Mp, \tag{45}$$

$$x_k = \begin{cases} K_x, & k = 1, 2, \cdots, Mp \\ K_y, & k = Mp+1, Mp+2, \cdots, 2Mp \end{cases}, \tag{46}$$

$$\xi_k = 1 - k \bmod(p) \text{ for } k = 1, 2, \cdots, a, \tag{47}$$

$$y_j = K_x + K_y \text{ for } j = 1, 2, \cdots, b, \tag{48}$$

$$\eta_j = 1 - j \bmod(p) \text{ for } j = 1, 2, \cdots, b. \tag{49}$$

Using (32) and (48), we have

$$\beta_k = \begin{cases} (1-\rho)K_x, & k = 1, 2, \cdots, Mp \\ (1-\rho)K_y, & k = Mp+1, Mp+2, \cdots, 2Mp \end{cases}, \tag{50}$$

$$\epsilon_j = (1-\rho)(K_x+K_y)\ \forall j. \tag{51}$$

Using the above values ((45)-(51)), in (30), (31), (33) and (34), we obtain (21)-(23) and (26), thereby establishing Theorem 1. □

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