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Statistical inference of semidefinite programming

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Abstract In this paper we consider covariance structural models with which we associate semidefinite programming problems. We discuss statistical properties of estimates of the respective optimal value and optimal solutions when the 'true' covariance matrix is estimated by its sample counterpart. The analysis is based on perturbation theory of semidefinite programming. As an example we consider asymptotics of the so-called minimum trace factor analysis. We also discuss the minimum rank matrix completion problem and its SDP counterparts.

Keywords Semidefinite programming · Minimum trace factor analysis · Matrix completion problem · Minimum rank · Nondegeneracy · Statistical inference · Asymptotics

Mathematics Subject Classification 62F12 · 62F30 · 90C22

1 Introduction

Consider $p \times p$ symmetric matrices Σ_0 and A_i $i = 1, \ldots, n$, and the following Semidefinite Programming (SDP) problem

$$\min_{x \in \mathbb{R}^n} c^\top x \text{ subject to } \Sigma_0 + \mathcal{A}(x) \succeq 0, \tag{1.1}$$

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where $\mathcal{A} : \mathbb{R}^n \to \mathbb{S}^p$ is the linear mapping $\mathcal{A}(x) := \sum_{i=1}^n x_i A_i$. By $\mathbb{S}^p$ we denote the linear space of $p \times p$ symmetric matrices and $A \succeq 0$ means that matrix $A \in \mathbb{S}^p$ is positive semidefinite. We view $\Sigma_0 \in \mathbb{S}^p$ as a covariance matrix of a $p \times 1$ random vector Y and the mapping $\mathcal{A} : \mathbb{R}^n \to \mathbb{S}^p$ as a (linear) covariance structural model. Classical example is the Factor Analysis model where the (population) covariance matrix Σ_0 is decomposed into $(\Sigma_0 - \Psi) + \Psi$ with Ψ being a diagonal matrix and matrix $\Sigma_0 - \Psi \succeq 0$ having rank $r < p$. (Diagonal elements of matrix Ψ represent variances of the respective error terms and should be nonnegative, for the sake of simplicity we do not consider these constraints; if the population values of the diagonal elements of Ψ are positive, then this does not change the asymptotic analysis.) In that case $\mathcal{A}(x)$ can be defined as the respective diagonal matrix $X = \operatorname{diag}(x_1, \ldots, x_p)$, i.e., $A_i = \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0)$, $i = 1, \ldots, p$, is diagonal matrix with zero diagonal elements except i-th diagonal element equal one. If further $c := (1, \ldots, 1)^\top$ is vector of ones, denoted $\mathbf{1}$, then (1.1) becomes the so-called minimum trace factor analysis (MTFA) problem

$$\min_{x \in \mathbb{R}^p} \mathbf{1}^\top x \text{ subject to } \Sigma_0 + X \succeq 0. \tag{1.2}$$

In that problem $n = p$. Originally the MTFA problem was motivated by computation of a lower bound to a reliability coefficient used in Factor Analysis (cf., Bentler [3]).

From a statistical point of view the population covariance matrix Σ_0 is not known and is estimated by the sample covariance matrix

$$S = \frac{1}{N-1} \sum_{i=1}^N (Y_i - \bar{Y})(Y_i - \bar{Y})^\top$$

based on a sample $Y_1, \ldots, Y_N$, of size N, of the random vector Y. Consequently the 'true' problem (1.1) is approximated by the respective sample based problem

$$\min_{x \in \mathbb{R}^n} c^\top x \text{ subject to } S + \mathcal{A}(x) \succeq 0. \tag{1.3}$$

Of interest are statistical properties of the optimal value $\hat{\vartheta}_N$ and an optimal solution $\hat{x}_N$ of problem (1.3) considered as estimates of their counterparts of the true problem (1.1). To a certain extent statistical properties of the MTFA were already investigated in [14,21].

In this paper we present statistical inference of such SDP problems from a general point of view. The basic tool in our analysis will be the modern theory of sensitivity analysis of parameterized SDP problems (cf., Bonnans and Shapiro [4, Section 5.3]). In the next section we give a summary of relevant results from that theory. Section 3 is devoted to statistical inference of the SDP problem (1.3). In Sect. 4 we apply general results to the MTFA problem. Although some results of that type are already available and scattered in various publications, we give a unified statistical inference of the (linear) SDP problems. In Sect. 5 we discuss the so-called matrix completion problem, this could have an independent interest. Finally in Sect. 6 we give concluding remarks.

Such statistical inference could be useful in several ways. One is a qualitative type characterization of the considered estimators, e.g., under what conditions one could expect these estimators to have approximately normal distributions, how biased these estimators could be etc. Another standard application is construction of confidence intervals and hence evaluation of approximate errors of these estimators. It is also possible to use this for evaluation of critical regions in respective hypotheses testing.

We use the following notation and terminology throughout the paper. By $\dim(\mathbb{V})$ we denote dimension of a finite dimensional vector (linear) space $\mathbb{V}$. It is said that a property holds for *almost every* (a.e.) $v \in \mathbb{V}$, or *almost surely*, if it holds for all $v \in \mathbb{V}$ except on a set of Lebesgue measure zero. By I_m we denote the $m \times m$ identity matrix. By $\mathbb{S}^p_+$ and $\mathbb{S}^p_{++}$ we denote cones of symmetric positive semidefinite and positive definite $p \times p$ matrices, respectively. Trace $\mathrm{tr}(A)$ of square matrix A is the sum of its diagonal elements. For $A, B \in \mathbb{S}^p$ their scalar product is defined as $A \bullet B = \mathrm{tr}(AB)$. By $A \otimes B$ we denote Kronecker product of $p \times q$ matrix $A = [a_{ij}]$ and $r \times s$ matrix $B = [b_{ij}]$. That is, $A \otimes B$ is $pr \times qs$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ a_{21}B & \cdots & a_{2q}B \\ & \cdots & \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}.$$

For a $p \times q$ matrix A we denote by $\mathrm{vec}(A)$ the $pq \times 1$ vector obtained by stacking columns of matrix A. We use notation $\sigma = \mathrm{vec}(\Sigma)$ and $s = \mathrm{vec}(S)$ for vector counterparts of matrices Σ and S, etc. We use the following matrix identity for matrices A, B, C of appropriate order

$$(\mathrm{vec}(A))^\top (B \otimes C)(\mathrm{vec}(A)) = \mathrm{tr}(BA^\top CA). \tag{1.4}$$

In particular, $(\mathrm{vec}(A))^\top \mathrm{vec}(A) = \mathrm{tr}(A^\top A)$. By $A^\dagger$ we denote the Moore-Penrose pseudoinverse of matrix A. In particular if $A \in \mathbb{S}^p$ is matrix of rank r and $A = NDN^\top$ is its singular value (spectral) decomposition, i.e., $N^\top N = I_r$ and D is $r \times r$ diagonal matrix with diagonal entries given by nonzero eigenvalues of A, then $A^\dagger = ND^{-1}N^\top$. For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order we denote by $A \circ B = [a_{ij}b_{ij}]$ their term by term product (Hadamard product).

Directional derivative of a mapping (function) $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $x \in \mathbb{R}^n$ in direction $h \in \mathbb{R}^n$ is defined as

$$f'(x, h) := \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

If this limit exists for all $h \in \mathbb{R}^n$, then it is said that f is directionally differentiable at x. Furthermore it is said that f is directionally differentiable at x in the sense of Fréchet if

$$f(x + h) - f(x) = f'(x, h) + r(h),$$

where $r(h) = o(\|h\|)$, i.e., $\lim_{h\to 0} r(h)/\|h\| = 0$.

2 Perturbation analysis of SDP problems

In this section we discuss some basic results of sensitivity analysis of problem

$$\min_{x\in\mathbb{R}^n} c^\top x \text{ subject to } \Sigma + \mathcal{A}(x) \succeq 0, \tag{2.1}$$

viewed as an SDP problem parameterized by matrix $\Sigma \in \mathbb{S}^p$. In particular we investigate differentiability properties of the optimal value $\vartheta(\Sigma)$ and an optimal solution $\bar{x}(\Sigma)$ of problem (2.1) considered as functions of matrix $\Sigma \in \mathbb{S}^p$. We also use notation $\sigma := \mathrm{vec}(\Sigma)$, and $\vartheta(\sigma)$ and $\bar{x}(\sigma)$ for the respective optimal value and optimal solution.

The Lagrangian of problem (2.1) is

$$L(x, \Lambda) := c^\top x - \Lambda \bullet (\Sigma + \mathcal{A}(x)),\ (x, \Lambda) \in \mathbb{R}^n \times \mathbb{S}^p,$$

and the (Lagrangian) dual of problem (2.1) is the problem

$$\max_{\Lambda\in\mathbb{S}^p_+} \min_{x\in\mathbb{R}^n} L(x, \Lambda),$$

which can be written as

$$\max_{\Lambda\in\mathbb{S}^p_+} \Lambda \bullet \Sigma \ \text{ subject to } \Lambda \bullet A_i = c_i,\ i = 1, \ldots, n. \tag{2.2}$$

We refer to problems (2.1) and (2.2) as the primal (P) and dual (D) problems, respectively.

It is said that Slater condition holds for the primal problem if there exists $x^* \in \mathbb{R}^n$ such that $\Sigma + \mathcal{A}(x^*) \in \mathbb{S}^p_{++}$, i.e., $\Sigma + \mathcal{A}(x^*)$ belongs to the interior of cone $\mathbb{S}^p_+$. If Slater condition holds, then optimal values of problems (P) and (D) are equal to each other. Assuming that the optimal value of problem (P) is finite, Slater condition holds iff the set of optimal solutions of the dual problem is nonempty and bounded.

For a discussion of the following basic results we can refer to [4, section 5.3]. The tangent cone to $\mathbb{S}^p_+$ at matrix $A \in \mathbb{S}^p_+$ of rank $r = \mathrm{rank}(A)$ is given by

$$T_{\mathbb{S}^p_+}(A) = \{H \in \mathbb{S}^p : E^\top H E \succeq 0\} \tag{2.3}$$

where E is a $p \times (p - r)$ matrix of rank $p - r$ such that $AE = 0$. The lineality space of cone $T_{\mathbb{S}^p_+}$ is

$$\mathrm{lin}\big(T_{\mathbb{S}^p_+}(A)\big) = \{H \in \mathbb{S}^p : E^\top H E = 0\}. \tag{2.4}$$

Note that the matrix E is defined up to a transformation $E \mapsto ET$, where T can be any nonsingular $(p - r) \times (p - r)$ matrix, and that the right hand sides of (2.3) and (2.4) do not depend on a particular choice of such matrix E.

Matrices $A \in \mathbb{S}^p$ of $\operatorname{rank}(A) = r \leq p$ form a smooth manifold, denoted $\mathcal{W}_r$, of dimension

$$\dim(\mathcal{W}_r) = p(p+1)/2 - (p-r)(p-r+1)/2 = pr - r(r-1)/2 \tag{2.5}$$

(e.g., Helmke and Moore [11, Chapter 5, Proposition 1.1]). It could be noted that for $A \in \mathcal{W}_r \cap \mathbb{S}^p_+$,

$$\operatorname{lin}\big(T_{\mathbb{S}^p_+}(A)\big) = T_{\mathcal{W}_r}(A), \tag{2.6}$$

where $T_{\mathcal{W}_r}(A)$ is the tangent space of the manifold $\mathcal{W}_r$ at $A \in \mathcal{W}_r$.

Definition 1 It is said that $x^* \in \mathbb{R}^n$ is a nondegenerate point of mapping $x \mapsto \Sigma + \mathcal{A}(x)$ if for $\Upsilon := \Sigma + \mathcal{A}(x^*)$ and $r := \operatorname{rank}(\Upsilon)$ it follows that

$$\mathcal{A}(\mathbb{R}^n) + T_{\mathcal{W}_r}(\Upsilon) = \mathbb{S}^p, \tag{2.7}$$

otherwise point x^* is said to be degenerate.

That is, x^* is nondegenerate if mapping $x \mapsto \Sigma + \mathcal{A}(x)$ intersects the smooth manifold $\mathcal{W}_r$ transversally at $\Sigma + \mathcal{A}(x^*) \in \mathcal{W}_r$. Note that in this definition of nondegeneracy we do not require for matrix $\Sigma + \mathcal{A}(x^*)$ to be positive semidefinite. If $\Sigma + \mathcal{A}(x^*) \succeq 0$, then by (2.6) condition (2.7) becomes

$$\mathcal{A}(\mathbb{R}^n) + \operatorname{lin}\big(T_{\mathbb{S}^p_+}(\Upsilon)\big) = \mathbb{S}^p. \tag{2.8}$$

Transversality concept is borrowed from differential geometry. For a detail study of transversality concept and relevant references we can refer, e.g., to Golubitsky and Guillemin [10]. In eigenvalue and semidefinite programming the above definition of nondegeneracy was suggested in Shapiro and Fan [20] and Shapiro [17]. For optimization problems subject to polyhedral cone constraints an analogue of equation (2.8) was introduced in Robinson [12] as definition of nondegeneracy. For general cone constrained problems definition (2.8) was used in [4, section 4.6.1] where it was also related to the concept of cone reducibility. A definition of nondegeneracy in SDP, formulated in an equivalent algebraic form, was used in Alizadeh et al. [1].

If the primal problem has a nondegenerate optimal solution, then the set of the optimal solutions of the dual problem is a singleton. Also there is the following algebraic characterization of nondegeneracy (cf., [4, Proposition 5.71],[17, Proposition 6]).

Proposition 1 *For $\Sigma + \mathcal{A}(x) \in \mathbb{S}^p_+$ suppose that $\operatorname{rank}(\Sigma + \mathcal{A}(x)) = r$ and let $e_1, \ldots, e_{p-r}$ be a basis of the null space of the matrix $\Sigma + \mathcal{A}(x)$. Then x is a nondegenerate point of mapping $\Sigma + \mathcal{A}(\cdot)$ iff vectors $v_{ij} := (e_i^\top A_1 e_j, \ldots, e_i^\top A_n e_j)^\top$, $1 \leq i \leq j \leq p-r$, are linearly independent.*

Transversality is a generic property in the following sense. Those Σ such that the corresponding mapping $\Sigma + \mathcal{A}(\cdot)$ has a degenerate point form a set of Lebesgue measure zero in the space $\mathbb{S}^p$. In other words for almost every $\Sigma \in \mathbb{S}^p$ the mapping $\Sigma + \mathcal{A}(\cdot)$ does not possess degenerate points. Since matrices A_i are linearly independent we have that $\dim(\mathcal{A}(\mathbb{R}^n)) = n$. Together with (2.5) this implies the following generic result (cf., [17, p.309]).

Proposition 2 *For almost every $\Sigma \in \mathbb{S}^p$ it follows that*

$$(p - r_x)(p - r_x + 1)/2 \leq n, \quad \forall x \in \mathbb{R}^n, \tag{2.9}$$

where $r_x := \operatorname{rank}(\Sigma + \mathcal{A}(x))$.

This means that for almost every $\Sigma \in \mathbb{S}^p$ the rank r_x of $\Sigma + \mathcal{A}(x)$ cannot be reduced below the bound

$$r_x \geq \frac{2p + 1 - \sqrt{8n + 1}}{2}. \tag{2.10}$$

Another way of saying this is that if matrix $\Sigma \in \mathbb{S}^p$ is random having continuous distribution, then the reduced rank inequality (2.10) holds with probability one (w.p.1). For recent studies of genericity in conic programming we can refer to Dür et al. [8] and references therein.

Let us discuss now differentiability properties of the optimal value function $\vartheta(\cdot)$ and an optimal solution $\bar{x}(\cdot)$. By $\operatorname{Sol}(P)$ we denote the set of optimal solutions of the *reference (true)* problem (1.1), and by $\operatorname{Sol}(D)$ the set of optimal solutions of its dual problem (2.2) for $\Sigma = \Sigma_0$. By the classical convex analysis the function $\vartheta(\cdot)$ is convex and we have the following result (e.g., [18, Theorem 4.1.9]).

Proposition 3 *Suppose that Slater condition holds for the reference problem* (1.1) *and its optimal value $\vartheta(\Sigma_0)$ is finite. Then the set* $\operatorname{Sol}(D)$ *is nonempty, convex and compact and the optimal value function $\vartheta(\cdot)$ is continuous and Fréchet directionally differentiable at Σ_0 with*

$$\vartheta'(\Sigma_0, H) = \sup_{\Lambda \in \operatorname{Sol}(D)} \Lambda \bullet H. \tag{2.11}$$

We assume in the remainder of this section that *Slater condition holds for the reference problem.* By the first order optimality conditions we have that for $x^* \in \operatorname{Sol}(P)$ and $\Lambda \in \operatorname{Sol}(D)$ the following complementarity condition follows

$$(\Sigma_0 + \mathcal{A}(x^*)) \bullet \Lambda = 0. \tag{2.12}$$

Note that since $\Sigma_0 + \mathcal{A}(x^*) \succeq 0$ and $\Lambda \succeq 0$, this complementarity condition is equivalent to $(\Sigma_0 + \mathcal{A}(x^*))\Lambda = 0$ and hence $\operatorname{rank}(\Lambda) \leq p - r$, where

$$r := \operatorname{rank}(\Sigma_0 + \mathcal{A}(x^*)). \tag{2.13}$$

It is said that the *strict complementarity* condition holds at $\Lambda \in \operatorname{Sol}(D)$ if $\operatorname{rank}(\Lambda) = p - r$.

The critical cone at $x^* \in \operatorname{Sol}(P)$ is defined as

$$C(x^*) := \left\{ h \in \mathbb{R}^n : \mathcal{A}(h) \in T_{\mathbb{S}^p_+}(\Upsilon),\ c^\top h = 0 \right\},$$

where $\Upsilon := \Sigma_0 + \mathcal{A}(x^*)$. Because of (2.3) and since $\Lambda \bullet A_i = c_i$, $i = 1, \ldots, n$, for any $\Lambda \in \mathrm{Sol}(D)$, it can be written as

$$C(x^*) = \left\{ h \in \mathbb{R}^n : E^\top \mathcal{A}(h) E \succeq 0,\ \Lambda \bullet \mathcal{A}(h) = 0 \right\},$$

where $r = \mathrm{rank}(\Upsilon)$, E is a $p \times (p-r)$ matrix of rank $p-r$ such that $\Upsilon E = 0$.

Suppose further that the strict complementarity condition holds at some $\Lambda \in \mathrm{Sol}(D)$. Then $\Lambda \succeq 0$ has rank $p-r$ and hence $\Lambda = EE^\top$ for some $p \times (p-r)$ matrix $E = [e_1, \ldots, e_{p-r}]$ of rank $p-r$. Consequently the critical cone can be written in the form

$$C(x^*) = \left\{ h \in \mathbb{R}^n : \sum_{i=1}^n h_i E^\top A_i E = 0 \right\} \tag{2.14}$$

$$= \left\{ h \in \mathbb{R}^n : v_{ij}^\top h = 0,\ 1 \le i \le j \le p-r \right\}, \tag{2.15}$$

where

$$v_{ij} := (e_i^\top A_1 e_j, \ldots, e_i^\top A_n e_j)^\top,\ 1 \le i \le j \le p-r. \tag{2.16}$$

Note that replacing matrix E in (2.14) with matrix ET, for any nonsingular $(p-r) \times (p-r)$ matrix T, does not change the corresponding equations. Hence matrix E in (2.14) can be any $p \times (p-r)$ matrix of rank $p-r$ such that $(\Sigma_0 + \mathcal{A}(x^*))E = 0$.

It follows that under the assumption of strict complementarity the critical cone $C(x^*)$ is a linear space and

$$\dim(C(x^*)) \ge n - (p-r)(p-r+1)/2. \tag{2.17}$$

Moreover, if x^* is a nondegenerate point of $\Sigma_0 + \mathcal{A}(\cdot)$, then vectors v_{ij}, $1 \le i \le j \le p-r$, are linearly independent (see Proposition 1), and hence by (2.15) the equality in (2.17) holds.

Now consider points $x^* \in \mathrm{Sol}(P)$, $\Lambda \in \mathrm{Sol}(D)$ and $n \times n$ matrix $\mathcal{H} = \mathcal{H}(x^*, \Lambda)$ with elements

$$[\mathcal{H}]_{ij} := \mathrm{tr}\left[\Lambda A_i \Upsilon^\dagger A_j \right],\ i, j = 1, \ldots, n, \tag{2.18}$$

where $\Upsilon := \Sigma_0 + \mathcal{A}(x^*)$. By matrix identity (1.4) this matrix can be written in the form $\mathcal{H} = A^\top [\Lambda \otimes \Upsilon^\dagger] A$, where $A := [\mathrm{vec}(A_1), \ldots, \mathrm{vec}(A_n)]$ is $p^2 \times n$ matrix. Consider the following second order conditions

$$h^\top \mathcal{H} h > 0,\ \ \forall h \in C(x^*) \backslash \{0\}. \tag{2.19}$$

These conditions imply the so-called second order growth condition at the point x^*, and if moreover $\mathrm{Sol}(D) = \{\Lambda\}$ is a singleton are necessary and sufficient for the second order growth condition (cf., [18, Theorem 4.1.8]). In particular conditions (2.19) imply that x^* is the unique optimal solution of the reference problem. Assuming strict complementarity the following converse statement holds (cf., Scheinberg [13, Theorem 4.2.1]).

Proposition 4 *Suppose that* $\mathrm{Sol}(P) = \{x^*\}$ *is a singleton and the strict complementarity condition holds at some* $\Lambda \in \mathrm{Sol}(D)$. *Then the second order conditions* (2.19) *follow.*

Proof Let $\Upsilon := \Sigma_0 + \mathcal{A}(x^*)$, $r := \mathrm{rank}(\Upsilon)$ and $\Upsilon = NDN^\top$ be the spectral decomposition of matrix Υ, and hence $\Upsilon^\dagger = ND^{-1}N^\top$. Because of the strict complementarity we have that $\Lambda = FF^\top$ for some $p \times (p-r)$ matrix F of rank $p-r$ such that $N^\top F = 0$. Then

$$h^\top \mathcal{H} h = \sum_{i,j=1}^{n} h_i h_j [\mathcal{H}]_{ij} = \mathrm{tr}\left[\Lambda\, \mathcal{A}(h)\Upsilon^\dagger \mathcal{A}(h)\right] = \mathrm{tr}\left[MD^{-1}M^\top\right],$$

where $M := \sum_{i=1}^n h_i F^\top A_i N$. Therefore (2.19) holds iff $M \neq 0$ for any $h \in C(x^*) \setminus \{0\}$. Now for $h \in C(x^*)$ and $K := [F, N]$ we have by (2.14) that $M = 0$ iff $\sum_{i=1}^n h_i F^\top A_i K = 0$. Note that matrix K is nonsingular, and hence $\sum_{i=1}^n h_i F^\top A_i K = 0$ iff $\sum_{i=1}^n h_i F^\top A_i = 0$. Since matrices $A_1, \ldots, A_n$ are linearly independent, we obtain that (2.19) holds iff $F^\top \mathcal{A}(h) = 0$ (or equivalently $\mathcal{A}(h)\Lambda = 0$) implies that $h = 0$ for any $h \in C(x^*)$.

We argue now by a contradiction. Suppose that there is $\bar{h} \neq 0$ such that $\mathcal{A}(\bar{h})\Lambda = 0$. Then point $\bar{x} := x^* + t\bar{h}$ is also an optimal solution of the reference problem (1.1) for some $t > 0$. Indeed

$$(\Sigma_0 + \mathcal{A}(\bar{x}))\Lambda = (\Sigma_0 + \mathcal{A}(x^*))\Lambda = 0.$$

Because of the strict complementarity it follows that $\mathrm{rank}(\Sigma_0 + \mathcal{A}(\bar{x})) = \mathrm{rank}(\Sigma_0 + \mathcal{A}(x^*))$ for $t > 0$ small enough. Let us note that if $A \in \mathbb{S}^p$ is a positive semidefinite matrix of rank r, then any matrix $B \in \mathbb{S}^p$ of rank r in a sufficiently small neighborhood of A is also positive semidefinite. It follow that $\Sigma_0 + \mathcal{A}(\bar{x}) \succeq 0$ for $t > 0$ small enough. By the first order optimality conditions this implies that $\bar{x} := x^* + t\bar{h} \in \mathrm{Sol}(P)$, which contradicts the assumption of uniqueness of the optimal solution x^*. □

Let us discuss now differentiability properties of an optimal solution $\bar{x}(\sigma)$ of problem (2.1) considered as a function of $\sigma = \mathrm{vec}(\Sigma)$. Suppose that $\mathrm{Sol}(P) = \{x^*\}$ and that x^* is a nondegenerate point of $\Sigma_0 + \mathcal{A}(\cdot)$, and hence $\mathrm{Sol}(D) = \{\Lambda\}$ is a singleton. Suppose also that the strict complementarity condition holds. Let $\Upsilon = NDN^\top$ be the spectral decomposition of matrix $\Upsilon = \Sigma_0 + \mathcal{A}(x^*)$, and $\Lambda = E\Theta E^\top$ be the spectral decomposition of matrix Λ. Recall that because of the strict complementarity, $\mathrm{rank}(\Lambda) = p - r$. Hence E is a $p \times (p-r)$ matrix of rank $p-r$ with orthonormal columns, i.e., $E^\top E = I_{p-r}$, and such that $N^\top E = 0$.

Consider the following optimization problem[1]

[1] As it was pointed before, the constraints in (2.20) are invariant with respect to replacing matrix E by matrix ET for an arbitrary nonsingular $(p-r) \times (p-r)$ matrix T. Therefore, unless stated otherwise, for the sake of computational convenience we assume that matrix E has *orthonormal columns*.

$$\begin{aligned} &\min_{h\in\mathbb{R}^n} \ \operatorname{tr}\Big[\Lambda(\mathcal{A}(h)+\Delta)\Upsilon^{\dagger}(\mathcal{A}(h)+\Delta)\Big] \\ &\text{s.t.} \ \ E^{\top}\mathcal{A}(h)E + E^{\top}\Delta E = 0 \end{aligned} \tag{2.20}$$

depending on $\Delta \in \mathbb{S}^p$. This is a problem of minimization of quadratic function subject to linear constraints

$$v_{ij}^{\top} h + e_i^{\top}\Delta e_j = 0, \quad 1 \le i \le j \le p-r, \tag{2.21}$$

with vectors v_{ij} defined in (2.16). By using matrix identity (1.4) the objective function of problem (2.20) can be written in the following form

$$\begin{aligned} \operatorname{tr}\Big[\Lambda(\mathcal{A}(h)+\Delta)\Upsilon^{\dagger}(\mathcal{A}(h)+\Delta)\Big] &= [\operatorname{vec}(\mathcal{A}(h)+\Delta)]^{\top}[\Lambda\otimes\Upsilon^{\dagger}][\operatorname{vec}(\mathcal{A}(h)+\Delta)] \\ &= h^{\top}\underbrace{A^{\top}[\Lambda\otimes\Upsilon^{\dagger}]A}_{\mathcal{H}}h + 2h^{\top}A^{\top}[\Lambda\otimes\Upsilon^{\dagger}]\delta \\ &\quad +\delta^{\top}[\Lambda\otimes\Upsilon^{\dagger}]\delta, \end{aligned} \tag{2.22}$$

where $A := [\operatorname{vec}(A_1), \ldots, \operatorname{vec}(A_n)]$ is $p^2 \times n$ matrix and $\delta := \operatorname{vec}(\Delta)$.

Recall that since x^* is assumed to be nondegenerate, vectors v_{ij}, $1 \le i \le j \le p-r$, are linearly independent. Hence because of (2.19) problem (2.20) has a unique optimal solution $\bar{h}(\delta)$, which is a linear function of $\delta = \operatorname{vec}(\Delta)$, and the optimal value of (2.20) is a quadratic function of δ. That is $\bar{h}(\delta) = J^{\top}\delta$, where J is the corresponding $p^2 \times n$ matrix, and the optimal value of problem (2.20) can be written as $\delta^{\top}Q\delta$ with Q being $p^2 \times p^2$ positive semidefinite matrix. By Proposition 4 we have the following result which is a particular case of a general result [4, Theorem 5.95 and eq. (5.238)].

Theorem 1 *Suppose that* $\operatorname{Sol}(P) = \{x^*\}$ *is a singleton, and that* x^* *is a nondegenerate point of* $\Sigma_0 + \mathcal{A}(\cdot)$ *and the strict complementarity condition holds. Then* $\bar{x}(\cdot)$ *is differentiable at* $\sigma_0 = \operatorname{vec}(\Sigma_0)$ *and*

$$\bar{x}(\sigma) = \bar{x}(\sigma_0) + J^{\top}(\sigma - \sigma_0) + o(\|\sigma - \sigma_0\|), \tag{2.23}$$

where $J^{\top}\delta$ *is the optimal solution of problem* (2.20) *with* Λ *being the optimal solution of the dual problem and* E *being a matrix whose columns are orthonormal and generate the null space of the matrix* $\Sigma_0 + \mathcal{A}(x^*)$.

Moreover

$$\vartheta(\sigma) = \vartheta(\sigma_0) + \Lambda \bullet (\Sigma - \Sigma_0) + (\sigma - \sigma_0)^{\top}Q(\sigma - \sigma_0) + o(\|\sigma - \sigma_0\|^2), \tag{2.24}$$

where $\delta^{\top}Q\delta$ *is the optimal value of problem* (2.20).

That is, J is the Jacobian matrix of $\bar{x}(\sigma)$ and $2Q$ is the Hessian matrix of $\vartheta(\sigma)$ at $\sigma = \sigma_0$.

Remark 1 In particular, if $n = (p-r)(p-r+1)/2$, then under the assumptions of Theorem 1, $\dim(C(x^*)) = 0$, i.e., $C(x^*) = \{0\}$. In that case the constraints of problem (2.20),

$$E^\top \mathcal{A}(h)E + E^\top \Delta E = 0, \tag{2.25}$$

define unique feasible point $\bar{h}(\delta)$ which is the optimal solution of problem (2.20). The constraints (2.25) can be written as the system of linear equations (2.21). Consider matrix V with columns

$$v_{ij} = (e_i^\top A_1 e_j, \ldots, e_i^\top A_n e_j)^\top, \ 1 \le i \le j \le p-r.$$

Since $n = (p-r)(p-r+1)/2$, matrix V an $n \times n$ matrix and because of the nondegeneracy assumption is nonsingular (see Proposition 1). Then the system (2.21) can be written as $V^\top h + U(\delta) = 0$, where $U(\delta)$ is $n \times 1$ vector valued linear function of $\delta = \mathrm{vec}(\Delta)$ with respective elements $e_i^\top \Delta e_j = \delta^\top \mathrm{vec}(e_j e_i^\top)$. Hence in that case

$$\bar{h}(\delta) = -(V^{-1})^\top U(\delta). \tag{2.26}$$

3 Statistical inference

In this section we assume that matrix Σ_0 is estimated by the sample covariance matrix S, based on a sample of size N. The optimal value $\hat{\vartheta}_N := \vartheta(S)$ and an optimal solution $\hat{x}_N := \bar{x}(S)$ of the sample counterpart of the 'true' (population) problem (1.1) are viewed as estimates of their population counterparts ϑ^* and x^*. In this section we investigate statistical properties of the estimates $\hat{\vartheta}_N$ and $\hat{x}_N$.

- We assume that the population distribution has finite fourth order moments.

It follows then by the Central Limit Theorem that $N^{1/2}(s-\sigma_0)$ converges in distribution to multivariate normal with mean vector zero and $p^2 \times p^2$ covariance matrix Γ. We write this as $N^{1/2}(s-\sigma_0) \Rightarrow \mathcal{N}(0, \Gamma)$, with " $\Rightarrow$" denoting convergence in distribution. Note that $p^2 \times 1$ vector $s = \mathrm{vec}(S)$ has only $p(p+1)/2$ nonduplicated elements, and hence $\mathrm{rank}(\Gamma) \le p(p+1)/2$.

In particular if the sample is drawn from a normally distributed population, then the element in row ij and column kl of matrix Γ is

$$[\Gamma]_{ij,k\ell} = \sigma_{ik}\sigma_{j\ell} + \sigma_{i\ell}\sigma_{jk},$$

where $\sigma_{ij} = [\Sigma_0]_{ij}$. This can be written in the following matrix form (cf., Browne [5])

$$\Gamma = 2M_p(\Sigma_0 \otimes \Sigma_0), \tag{3.1}$$

where M_p is $p^2 \times p^2$ symmetric matrix with $[M_p]_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where $\delta_{ik} = 1$ if $i = k$, and $\delta_{ik} = 0$ if $i \ne k$. This matrix has the following properties: $M_p^2 = M_p$, $\mathrm{rank}(M_p) = p(p+1)/2$, $M_p(\Sigma_0 \otimes \Sigma_0) = (\Sigma_0 \otimes \Sigma_0)M_p$ and $M_p(\mathrm{vec}(A)) = \mathrm{vec}(A)$ for any $A \in \mathbb{S}^p$. It follows that in case of normally distributed population, $\mathrm{rank}(\Gamma) = p(p+1)/2$.

We use notation $o_p(\cdot)$ and $O_p(\cdot)$ for statistical analogues of $o(\cdot)$ and $O(\cdot)$. That is, for two sequences X_k and Y_k of random variables, $X_k = o_p(Y_k)$ means that X_k/Y_k tends in probability to zero, and $X_k = O_p(Y_k)$ means that X_k/Y_k is bounded in probability. In particular $X_k = o_p(1)$ means that X_k converges in probability to zero. This notation is standard in Statistics literature.

Theorem 2 *Suppose that the optimal value $\vartheta^* = \vartheta(\Sigma_0)$ is finite and Slater condition for the true problem holds. Then*

$$\hat{\vartheta}_N - \vartheta^* = \sup_{\Lambda \in \mathrm{Sol}(D)} \Lambda \bullet (S - \Sigma_0) + o_p(N^{-1/2}), \tag{3.2}$$

$$N^{1/2}(\hat{\vartheta}_N - \vartheta^*) \Rightarrow \sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top Z, \tag{3.3}$$

where Z is a random vector having multivariate normal distribution $\mathcal{N}(0, \Gamma)$.

Moreover, if $\mathrm{Sol}(D) = \{\Lambda_0\}$ is a singleton, then $N^{1/2}(\hat{\vartheta}_N - \vartheta^*)$ converges in distribution to normal with zero mean and variance $\sigma^2 = \lambda_0^\top \Gamma \lambda_0$, where $\lambda_0 := \mathrm{vec}(\Lambda_0)$. In particular, if Γ is of the form (3.1), then the asymptotic variance $\sigma^2 = 2\mathrm{tr}(\Sigma_0 \Lambda_0 \Sigma_0 \Lambda_0)$.

Proof Because of Slater condition we have that the primal and dual problems have the same optimal value and the set of optimal solutions of the dual problem is nonempty and compact. Formula (3.2) follows from (2.11). Indeed by Proposition 3 we have

$$\hat{\vartheta}_N - \vartheta^* = \sup_{\Lambda \in \mathrm{Sol}(D)} \Lambda \bullet (S - \Sigma_0) + o(\|s - \sigma_0\|).$$

Since $N^{1/2}(s - \sigma_0)$ converges in distribution, it is bounded in probability. It follows that $\|s - \sigma_0\| = O_p(N^{-1/2})$. Hence by the standard calculus of $o_p(\cdot)$ and $O_p(\cdot)$ it follows that $o(\|s - \sigma_0\|) = o_p(N^{-1/2})$, and thus (3.2) follows.

By (3.2) we have that

$$N^{1/2}(\hat{\vartheta}_N - \vartheta^*) = \sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top [N^{1/2}(s - \sigma_0)] + o_p(1). \tag{3.4}$$

By the continuous mapping theorem we have that the first term in the right hand side of (3.4) converges in distribution to $\sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top Z$. Together with Slutsky's theorem this implies (3.3). In particular if $\mathrm{Sol}(D) = \{\Lambda_0\}$ is a singleton, then it follows by (3.3) that

$$N^{1/2}(\hat{\vartheta}_N - \vartheta^*) \Rightarrow \mathcal{N}(0, \sigma^2), \tag{3.5}$$

where $\sigma^2 = \lambda_0^\top \Gamma \lambda_0$. The above derivations basically is the classical (finite dimensional) Delta Theorem.

Now suppose that matrix Γ is of the form (3.1). Then since for $\lambda = \mathrm{vec}(\Lambda)$ we have that $M_p \lambda = \lambda$ and using matrix identity (1.4) we can write

$$\lambda_0^\top \Gamma \lambda_0 = 2\,\mathrm{vec}(\Lambda_0)(\Sigma_0 \otimes \Sigma_0)\mathrm{vec}(\Lambda_0) = 2\mathrm{tr}[\Sigma_0 \Lambda_0 \Sigma_0 \Lambda_0].$$

This completes the proof. □

Consider set

$$\mathfrak{F} := \left\{ \Lambda \in \mathbb{S}^p_+ : \Lambda \bullet A_i = c_i,\ i = 1, \ldots, n \right\}$$

of feasible solutions of the dual problem . Assuming Slater condition we have that $\vartheta(S) = \sup_{\Lambda \in \mathfrak{F}} \Lambda \bullet S$. Since S is an unbiased estimate of Σ_0, i.e., $\mathbb{E}[S] = \Sigma_0$, we can write

$$\mathbb{E}[\hat{\vartheta}_N] = \mathbb{E}\left[\sup_{\Lambda \in \mathfrak{F}} \Lambda \bullet S \right] \geq \sup_{\Lambda \in \mathfrak{F}} \mathbb{E}[\Lambda \bullet S] = \sup_{\Lambda \in \mathfrak{F}} \Lambda \bullet \Sigma_0 = \vartheta^*.$$

That is, $\hat{\vartheta}_N$ is an upward biased estimate of ϑ^*. Under mild regularity conditions it follows from the convergence in distribution (3.3) that the expected value of $N^{1/2}(\hat{\vartheta}_N - \vartheta^*)$ converges to the expected value of $\sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top Z$ (see, e.g., [19, Remark 57]). That is

$$\mathbb{E}[\hat{\vartheta}_N] - \vartheta^* = N^{-1/2} \mathbb{E}\left[\sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top Z \right] + o(N^{-1/2}), \tag{3.6}$$

where $Z \sim \mathcal{N}(0, \Gamma)$. If the dual problem has more than one optimal solution, i.e., $\mathrm{Sol}(D)$ is not a singleton, then the term $\mathbb{E}\left[\sup_{\lambda \in \mathrm{Sol}(D)} \lambda^\top Z\right]$ is positive. In that case the asymptotic bias of $\hat{\vartheta}_N$ is of order $O(N^{-1/2})$. In case $\mathrm{Sol}(D)$ is a singleton, the asymptotic bias of $\hat{\vartheta}_N$ typically is of order $O(N^{-1})$ (see Remark 3 below).

Theorem 3 *Suppose that* $\mathrm{Sol}(P) = \{x^*\}$ *is a singleton, and that* x^* *is a nondegenerate point of* $\Sigma_0 + \mathcal{A}(\cdot)$ *and the strict complementarity condition holds. Then* $N^{1/2}(\hat{x}_N - x^*)$ *converges in distribution to normal* $\mathcal{N}(0, J^\top \Gamma J)$*, where* J *is the* $p^2 \times n$ *matrix such that* $J^\top \delta$ *is the optimal solution of problem* (2.20)*.*

Proof By (2.23) we have

$$\hat{x}_N - x^* = J^\top (s - \sigma_0) + o(\|s - \sigma_0\|).$$

In the same way as in the proof of Theorem 2 we can conclude that $N^{1/2}(\hat{x}_N - x^*)$ converges in distribution to $J^\top Z$, where $Z \sim \mathcal{N}(0, \Gamma)$. Since $J^\top Z \sim \mathcal{N}(0, J^\top \Gamma J)$ we obtain that $N^{1/2}(\hat{x}_N - x^*) \Rightarrow \mathcal{N}(0, J^\top \Gamma J)$. □

Remark 2 In particular suppose that $n = (p - r)(p - r + 1)/2$ (see Remark 1). Then by formula (2.26) we have that $N^{1/2}[U(s - \sigma_0)] \Rightarrow \mathcal{N}(0, \Omega)$ and hence $J^\top \Gamma J = (V^{-1})^\top \Omega V^{-1}$, with

$$[\Omega]_{ij,k\ell} = [\mathrm{vec}(e_j e_i^\top)]^\top \Gamma [\mathrm{vec}(e_\ell e_k^\top)]. \tag{3.7}$$

If moreover Γ is of the form (3.1), then by using the matrix identity (1.4) we have

$$[\Omega]_{ij,k\ell} = 2[\mathrm{vec}(e_j e_i^\top)]^\top (\Sigma_0 \otimes \Sigma_0)[\mathrm{vec}(e_\ell e_k^\top)] = 2(e_i^\top \Sigma_0 e_k)(e_j^\top \Sigma_0 e_\ell). \tag{3.8}$$

In particular, if $\Sigma_0 = \sigma^2 I_p$ for some $\sigma^2 > 0$, then $[\Omega]_{ij,k\ell} = 2\sigma^4$ for $(i, j) = (k, \ell)$, and $[\Omega]_{ij,k\ell} = 0$ for $(i, j) \neq (k, \ell)$, i.e., in that case $\Omega = 2\sigma^4 I_n$.

Remark 3 By using (3.6) we can write, under assumptions of Theorem 3, the following second order expansion of the optimal value

$$\hat{\vartheta}_N = \vartheta^* + \Lambda \bullet (S - \Sigma_0) + (s - \sigma_0)^\top Q(s - \sigma_0) + o_p(N^{-1}). \tag{3.9}$$

Here Λ is the unique optimal solution of the dual of the true problem, $\delta^\top Q \delta$ is the optimal value of problem (2.20) and the term $o_p(N^{-1})$ is obtained from the term $o(\|\sigma - \sigma_0\|^2)$ in expansion (3.6). Since S is an unbiased estimate of Σ_0, it follows that $\mathbb{E}[\Lambda \bullet (S - \Sigma_0)] = 0$. Hence the asymptotic bias of $\hat{\vartheta}_N$ here is defined by the term $(s - \sigma_0)^\top Q(s - \sigma_0)$. Note that for $Z \sim \mathcal{N}(0, \Gamma)$ the expectation $\mathbb{E}[Z^\top QZ] = \mathrm{tr}(Q\Gamma)$. We obtain that under assumptions of Theorem 3 the asymptotic bias of $\hat{\vartheta}_N$ is $N^{-1}\mathrm{tr}(Q\Gamma) + o(N^{-1})$.

4 Factor analysis

In this section we apply general results of Sect. 3 to the Factor Analysis model and in particular to the MTFA problem (1.2). We denote by $\mathbb{D}^p$ the space of $p \times p$ diagonal matrices. The corresponding mapping here is $X \mapsto \Sigma + X$ from $\mathbb{D}^p$ into $\mathbb{S}^p$ (we use notation $x = \mathrm{diag}(X)$ for $X \in \mathbb{D}^p$). By Proposition 2 we have the following almost sure lower bound for the reduced rank of covariance matrix (cf., [14]).

Proposition 5 *For almost every $\Sigma \in \mathbb{S}^p$ it follows that*

$$\mathrm{rank}(\Sigma + X) \geq \frac{2p + 1 - \sqrt{8p + 1}}{2}, \quad \forall X \in \mathbb{D}^p. \tag{4.1}$$

By Proposition 1 we have that $X^* \in \mathbb{D}^p$ is a nondegenerate point of the mapping $X \mapsto \Sigma + X$ iff vectors $e_i \circ e_j$, $1 \leq i < j \leq p - r$, are linearly independent, where $r := \mathrm{rank}(\Sigma + X^*)$ and $e_1, \ldots, e_{p-r}$ is a basis of the null space of the matrix $\Sigma + X^*$. It follows that if X^* is nondegenerate, then $(p - r)(p - r + 1)/2 \leq p$. This leads to the almost sure bound (4.1).

Consider now the MTFA problem (1.2). It is interesting to note that the MTFA solution may not coincide with the minimum rank solution even if the minimal rank is equal to one. In this respect there is the following result (cf., [15, Theorem 3.1]).

Proposition 6 *Suppose that $\Sigma = \gamma\gamma^\top + \Psi$ for some $p \times 1$ nonzero vector γ and $\Psi \in \mathbb{D}^p$, and let X^* be the optimal solution of the corresponding MTFA problem. Then $X^* = -\Psi$ iff the following condition holds*

$$|\gamma_i| \leq \sum_{j \neq i} |\gamma_j|, \ i = 1, \ldots, p. \tag{4.2}$$

Otherwise, if condition (4.2) is not satisfied, then $\mathrm{rank}(\Sigma + X^*) = p - 1$.

The Lagrangian dual of problem (1.2) is

$$\max_{\Lambda \in \mathbb{S}^p_+} \mathrm{tr}(\Lambda \Sigma_0) \text{ subject to } \mathrm{diag}(\Lambda) = \mathbf{1}, \tag{4.3}$$

where $\mathbf{1}$ is $p \times 1$ vector of ones. Optimal value of the primal problem (1.2) is finite and Slater condition always holds here. Hence the dual problem (4.3) has a nonempty convex compact set of optimal solutions, denoted $\mathrm{Sol}(D)$, and there is no duality gap between the primal and dual problems. The set of optimal solutions of the primal problem (1.2) is a singleton, i.e., $\mathrm{Sol}(P) = \{X^*\}$ (cf., Ten Berge et al. [22]). Indeed, let X^* and $\bar{X}$ be optimal solutions of the MTFA problem (1.2) and Λ be an optimal solution of the dual problem. Then by the first order optimality conditions we have that $(X^* - \bar{X})\Lambda = 0$. Since $\mathrm{diag}(\Lambda) = \mathbf{1}$, it follows that $X^* - \bar{X} = 0$.

Consider now the estimates $\hat{\vartheta}_N$ and $\hat{x}_N$ of the optimal value and optimal solution of the MTFA problem (1.2). The asymptotics of $\hat{\vartheta}_N$ are given in Theorem 2 (cf., [14, Theorem 4.2]). Assuming that the nondegeneracy and strict complementarity conditions hold, asymptotics of $\hat{x}_N$ can be derived from general results of Theorem 3.

By (2.18) and (2.22) we have here that

$$\mathcal{H} = \Lambda \circ \Upsilon^\dagger \ \text{ and } \ h^\top A^\top [\Lambda \otimes \Upsilon^\dagger]\delta = \mathrm{tr}\left[H\Lambda\Delta\Upsilon^\dagger\right], \tag{4.4}$$

where $h = \mathrm{diag}(H)$, $\Upsilon = \Sigma_0 + X^*$, $\Delta \in \mathbb{S}^p$ and $\delta = \mathrm{vec}(\Delta)$. Therefore problem (2.20) takes here the following form (with the last term in (2.22) omitted)

$$\begin{aligned} &\min_{h \in \mathbb{R}^p} \ h^\top(\Lambda \circ \Upsilon^\dagger)h + 2\mathrm{tr}\left[H\Lambda\Delta\Upsilon^\dagger\right] \\ &\text{s.t.} \ \ E^\top H E + E^\top \Delta E = 0. \end{aligned} \tag{4.5}$$

By Theorem 3 and Remark 2 we have the following results.

Theorem 4 *Let x^* be the optimal solution of the MTFA problem (1.2), $\Upsilon = \Sigma_0 + X^*$ and $r = \mathrm{rank}(\Upsilon)$. Suppose that the point x^* is nondegenerate and the strict complementarity condition holds. Then $N^{1/2}(\hat{x}_N - x^*)$ converges in distribution to normal $\mathcal{N}(0, J^\top \Gamma J)$, where J is the $p^2 \times p$ matrix such that $J^\top \delta$ is the optimal solution of problem (4.5).*

In particular, if $p = (p-r)(p-r+1)/2$, then $J^\top \Gamma J = (V^{-1})^\top \Omega V^{-1}$, where Ω is given in (3.7) (and in (3.8) for normally distributed population), and V is the $p \times p$ matrix with columns $e_i \circ e_j$, $1 \le i \le j \le p - r$.

It is also possible to evaluate the asymptotic bias of the estimator $\hat{\vartheta}_N$. By the discussion of section 3 we have that this asymptotic bias is of order $O(N^{-1/2})$ if $\mathrm{Sol}(D)$ is not a singleton, and is of order $O(N^{-1})$ under the assumptions of Theorem 4.

5 Matrix completion

Consider the problem of recovering an $m_1 \times m_2$ data matrix of low rank when observing a small number $\mathfrak{m}$ of its entries. This can be written as the following optimization problem (cf., Candés and Recht [6])

$$\min_{Y \in \mathbb{R}^{m_1 \times m_2}} \mathrm{rank}(Y) \text{ subject to } Y_{ij} = M_{ij}, \ (i,j) \in \iota, \tag{5.1}$$

where $\iota \subset \{1, \ldots, m_1\} \times \{1, \ldots, m_2\}$ is an index set of cardinality $\mathfrak{m}$ and M_{ij} are specified values. We refer to (5.1) as the minimum rank matrix completion (MRMC) problem.

The MRMC problem (5.1) can be formulated in the following equivalent form (see Remark 4 below)

$$\min_{X \in \mathbb{V}_\tau} \operatorname{rank}(\Xi + X) \text{ subject to } \Xi + X \succeq 0, \tag{5.2}$$

where $\Xi \in \mathbb{S}^p$, $p = m_1 + m_2$, is symmetric matrix of the form $\Xi = \begin{bmatrix} 0 & M \\ M^\top & 0 \end{bmatrix}$, M is the $m_1 \times m_2$ matrix with entries M_{ij} at $(i, j) \in \iota$, and all other entries equal zero. Minimization in (5.2) is performed over matrices $X \in \mathbb{S}^p$ which are complement to Ξ in the sense of having zero entries at all places corresponding to the specified values M_{ij}, $(i, j) \in \iota$. That is $X \in \mathbb{V}_\tau$, where τ is the symmetric index set associated with the index set[2] ι and

$$\mathbb{V}_\tau := \{X \in \mathbb{S}^p : X_{ij} = 0, \ (i, j) \in \tau\} \tag{5.3}$$

is the respective linear subspace of $\mathbb{S}^p$.

Remark 4 Consider a feasible point Y of problem (5.1), i.e., $Y_{ij} = M_{ij}$, $(i, j) \in \iota$, of rank r. It can be written as $Y = VW^\top$ (singular value decomposition), where V and W are matrices of the respective order $m_1 \times r$ and $m_2 \times r$ and common rank r. Consider

$$X := UU^\top - \Xi, \text{ where } U := \begin{bmatrix} V \\ W \end{bmatrix}, \text{ i.e., } X = \begin{bmatrix} VV^\top & Y - M \\ (Y - M)^\top & WW^\top \end{bmatrix}. \tag{5.4}$$

This matrix X is a feasible point of problem (5.2) and $X + \Xi$ has rank r. It follows that the optimal value of problem (5.2) is less than or equal to the optimal value of problem (5.1). Conversely let X be a feasible point of problem (5.2) and $r := \operatorname{rank}(X + \Xi)$. Then $X + \Xi = UU^\top$ for some $p \times r$ matrix U of rank r. By partitioning U as in (5.4), we obtain that $Y := VW^\top$ is a feasible point of problem (5.1) and $\operatorname{rank}(Y) \leq r$. It follows that the optimal value of problem (5.1) is less than or equal to the optimal value of problem (5.2), and hence these two problems have the same optimal value denoted r^*.

Let us note that if X is a feasible point (of the form (5.4)) of problem (5.2) with $r = \operatorname{rank}(X + \Xi)$, then for any matrix $T \in \mathbb{S}^r_{++}$, the matrix $\begin{bmatrix} VTV^\top & Y - M \\ (Y - M)^\top & WT^{-1}W^\top \end{bmatrix}$ is also a feasible point of problem (5.2) of the same rank r. Therefore to any solution of (5.1) of rank r corresponds a manifold of dimension $r(r + 1)/2$ of solutions of problem (5.2).

[2] The index set $\tau \subset \{1, \ldots, p\} \times \{1, \ldots, p\}$ is symmetric in the sense that if $(i, j) \in \tau$, then $(j, i) \in \tau$, and is formed by such (i, j) that $(i, j - n_1) \in \iota$ for $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n_1 + n_2$, and the respective (j, i) otherwise.

As a heuristic it was suggested in Fazel [9] (see also [6]) to approximate problem (5.2) by the following problem

$$\min_{X \in \mathbb{V}_\tau} \operatorname{tr}(X) \text{ subject to } \Xi + X \succeq 0. \tag{5.5}$$

Of course problem (5.5) can be considered as a particular case of the SDP problem (1.1). Note that $\operatorname{tr}(\Xi + X) = \operatorname{tr}(X)$ for any $X \in \mathbb{V}_\tau$ and $\Xi \in \mathbb{V}_{\bar{\tau}}$.

In this section we consider a slightly more general minimum rank problem of the form (5.2) in that we allow the index set τ to be a general symmetric subset of $\{1, \ldots, p\} \times \{1, \ldots, p\}$, and such that if $(i, j) \in \tau$, then $i \neq j$. By $\bar{\tau} \subset \{1, \ldots, p\} \times \{1, \ldots, p\}$ we denote the symmetric complement index set of τ, i.e., if $(i, j) \in \bar{\tau}$, then $(j, i) \in \bar{\tau}$; and $(i, j) \in \bar{\tau}$, $i \leq j$, iff $(i, j) \notin \tau$. Note that $\mathbb{V}_\tau \cap \mathbb{V}_{\bar{\tau}} = \{0\}$ and $\mathbb{V}_\tau + \mathbb{V}_{\bar{\tau}} = \mathbb{S}^p$. The matrix $\Xi \in \mathbb{V}_{\bar{\tau}}$ is supposed to have specified entries Ξ_{ij}, $(i, j) \in \tau$, and all other entries equal zero. Together with the minimum rank problem (5.2) we consider the following SDP problem

$$\min_{X \in \mathbb{V}_\tau} C \bullet X \text{ subject to } \Xi + X \succeq 0, \tag{5.6}$$

for some matrix $C \in \mathbb{V}_\tau$. Note that $C \bullet \Xi = 0$ for any $\Xi \in \mathbb{V}_{\bar{\tau}}$. For $C := I_p$, problem (5.6) coincides with problem (5.5). The MTFA can be also considered in this framework by defining the set τ to be the symmetric set of all indexes (i, j) such that $i \neq j$, and taking $C := I_p$. Also for this index set τ the respective minimum rank problem (5.2) becomes the so-called Minimum Rank Factor Analysis (MRFA) problem. As it was discussed in Proposition 6, even if the MRFA problem has solution of rank one the corresponding MTFA problem may have solution of rank $p - 1$.

Define the respective mapping $\mathcal{A} : \mathbb{V}_\tau \to \mathbb{S}^p$ as $\mathcal{A}(X) := X$. Then similar to Definition 1 we say that $X \in \mathbb{V}_\tau$ is a *nondegenerate* point of problem (5.2) (of problem (5.6)) if for $r := \operatorname{rank}(\Xi + X)$ it follows that

$$\mathbb{V}_\tau + T_{\mathcal{W}_r}(\Xi + X) = \mathbb{S}^p. \tag{5.7}$$

Otherwise we say that the point X is degenerate. We have that the following is a necessary condition for a point $X \in \mathbb{V}_\tau$ to be nondegenerate

$$\dim(\mathbb{V}_\tau) + \dim(\mathcal{W}_r) \geq \dim(\mathbb{S}^p). \tag{5.8}$$

By formula (2.5) for dimension of $\mathcal{W}_r$, condition (5.8) can be written as (compare with (2.9) and (2.10))

$$(p - r_X)(p - r_X + 1)/2 \leq \dim(\mathbb{V}_\tau), \tag{5.9}$$

or equivalently

$$r_X \geq \frac{2p + 1 - \sqrt{8 \dim(\mathbb{V}_\tau) + 1}}{2}, \tag{5.10}$$

where $r_X := \operatorname{rank}(\Xi + X)$.

Theorem 5 *For a.e. $\Xi \in \mathbb{V}_{\bar{\tau}}$ it follows that problem (5.2) does not have degenerate points and the inequalities (5.9) and (5.10) hold for all $X \in \mathbb{V}_\tau$.*

Proof Note that we cannot apply here the result of Proposition 2 in a direct way since projection of a set of measure zero onto a subspace of lower dimension does not need to have measure zero. However, the same type of arguments can be used in the proof. Consider mapping $\mathcal{G}(X, \Xi) := \Xi + X$, $X \in \mathbb{V}_\tau$, $\Xi \in \mathbb{V}_{\bar{\tau}}$. Since $\mathbb{V}_\tau + \mathbb{V}_{\bar{\tau}} = \mathbb{S}^p$, the image

$$\mathrm{Im}(\mathcal{G}) = \{\Xi + X : X \in \mathbb{V}_\tau,\ \Xi \in \mathbb{V}_{\bar{\tau}}\}$$

of this mapping coincides with the whole space $\mathbb{S}^p$. It follows that mapping $\mathcal{G}$ intersects every $\mathcal{W}_r$ transversally, i.e., $\mathrm{Im}(\mathcal{G}) + T_{\mathcal{W}_r}(\Upsilon) = \mathbb{S}^p$ for every $\Upsilon \in \mathcal{W}_r$. Viewing Ξ as a parameter vector, by the classical result of differential geometry (e.g., [10]) we have then that for a.e. $\Xi \in \mathbb{V}_{\bar{\tau}}$ the mapping $G_\Xi(\cdot) := \mathcal{G}(\cdot, \Xi)$ intersects $\mathcal{W}_r$ transversally. That is, for a.e. $\Xi \in \mathbb{V}_{\bar{\tau}}$ and every $\mathcal{W}_r$ it follows that for any $X \in \mathbb{V}_\tau$ either $G_\Xi(X) \notin \mathcal{W}_r$ or $G_\Xi(X) \in \mathcal{W}_r$ and condition (5.7) holds. Since there is a finite number of manifolds $\mathcal{W}_r$ it follows that for for a.e. $\Xi \in \mathbb{V}_{\bar{\tau}}$ all points of problem (5.2) are nondegenerate, and hence conditions (5.9) and (5.10) hold for all $X \in \mathbb{V}_\tau$. □

A natural question is how tight the bound (5.9), or equivalently the bound (5.10), for the respective minimal rank. As it was already pointed, when the index set τ consists of all indexes such that $i \neq j$, i.e., $\mathbb{V}_\tau = \mathbb{D}^p$, then problem (5.2) becomes the Minimum Rank Factor Analysis (MRFA), and the bound (5.10) becomes the bound (2.10) with $n = p$. It is well known that in that case the bound (2.10) is tight in the sense that the set of matrices for which it is attained is not of measure zero.

For the MRMC problem (5.1) the above bound (5.9) can be improved by recalling that to any solution of problem (5.1) corresponds a manifold of dimension $r(r+1)/2$ of solutions of problem (5.2) (see Remark 4). This leads to the following bound for the rank r,

$$\dim(\mathbb{V}_\tau) + \dim(\mathcal{W}_r) - r(r+1)/2 \geq \dim(\mathbb{S}^p). \tag{5.11}$$

Here $\dim(\mathbb{V}_\tau) = p(p+1)/2 - \mathfrak{m}$. Consequently using formula (2.5) for dimension of $\mathcal{W}_r$ we can write condition (5.11) in the following form (cf., [6,7])

$$r(m_1 + m_2 - r) \geq \mathfrak{m}. \tag{5.12}$$

Note that $r \leq \min\{m_1, m_2\}$ and bound (5.12) can be written in the following equivalent form

$$r \geq \mathfrak{R}(m_1, m_2, \mathfrak{m}), \tag{5.13}$$

where

$$\mathfrak{R}(m_1, m_2, \mathfrak{m}) := (m_1 + m_2)/2 - \sqrt{(m_1 + m_2)^2/4 - \mathfrak{m}}. \tag{5.14}$$

For the MRMC problem (5.1) bound (5.12) can be derived in a direct way. As before we say that a property holds for a.e. choice of the constants M_{ij}, $(i, j) \in \iota$, if

the set of vectors $[M_{ij}]_{(i,j)\in\iota}$ for which it does not hold has Lebesgue measure zero in the respective linear space of dimension $\mathfrak{m}$. Consider

$$\mathcal{M}_r := \left\{Y \in \mathbb{R}^{m_1\times m_2} : \operatorname{rank}(Y) = r\right\}$$

the set of $m_1 \times m_2$ matrices of rank r. It is well known that $\mathcal{M}_r$ is a smooth manifold with

$$\dim(\mathcal{M}_r) = r(m_1 + m_2 - r) \tag{5.15}$$

(e.g., [2]). By the same arguments as in the proof of Theorem 5 we have that the (affine) space

$$\mathcal{F}_M := \left\{Y \in \mathbb{R}^{m_1\times m_2} : Y_{ij} = M_{ij},\ (i,j) \in \iota\right\} \tag{5.16}$$

intersects $\mathcal{M}_r$ transversally for a.e. choice of the constants M_{ij}, $(i,j) \in \iota$. Note that $\mathcal{F}_M$ is the space of feasible points of problem (5.1).

Proposition 7 *For almost every choice of the constants M_{ij}, $(i,j) \in \iota$, the following holds:* (i) *for every feasible point Y of problem* (5.1) *it follows that*

$$\operatorname{rank}(Y) \geq \mathfrak{R}(m_1, m_2, \mathfrak{m}), \tag{5.17}$$

(ii) if the number $\mathfrak{R}(m_1, m_2, \mathfrak{m})$ is not an integer, then problem (5.1) *has multiple (more than one) optimal solutions.*

Proof Consider the space $\mathcal{F}_M$ of feasible points of problem (5.1). Of course this space depends on the choice of constants M_{ij}. As it was discussed above the transversality condition holds almost surely, i.e., for a.e. choice of the constants M_{ij}, $(i,j) \in \iota$. Since $\dim(\mathcal{F}_M) = m_1m_2 - \mathfrak{m}$ and because of (5.15), the transversality condition implies the bounds (5.12) and (5.13) for $r = \operatorname{rank}(Y)$. This completes the proof of assertion (i).

Suppose that $\mathcal{F}_M$ intersects $\mathcal{M}_r$ transversally at a point $Y^* \in \mathcal{M}_r$. Then $\mathcal{M}_r \cap \mathcal{F}_M$ forms a smooth manifold near Y^* with the tangent space to this manifold at Y^* given by

$$T_{\mathcal{M}_r\cap\mathcal{F}_M}(Y^*) = T_{\mathcal{M}_r}(Y^*) \cap \mathcal{F}_M.$$

The dimension of this manifold cannot be zero unless $\dim(\mathcal{M}_r) + \dim(\mathcal{F}_M)$ is equal to the dimension m_1m_2 of the whole space, which is equivalent to the condition $\operatorname{rank}(Y^*) = \mathfrak{R}(m_1, m_2, \mathfrak{m})$. If the manifold $\mathcal{M}_r \cap \mathcal{F}_M$ has a positive dimension in a neighborhood of an optimal solution Y^*, then all points of this manifold are optimal in that neighborhood, and hence problem (5.1) has multiple solutions. Of course if $\mathfrak{R}(m_1, m_2, \mathfrak{m})$ is not an integer we cannot have the equality $\operatorname{rank}(Y^*) = \mathfrak{R}(m_1, m_2, \mathfrak{m})$. Since the transversality holds almost surely this completes the proof of (ii). □

Remark 5 Consider the optimal value r^* of the MRMC problem (5.1), which is also the optimal value of problem (5.2). Of course for a given index set ι the minimal rank r^* depends on the specified values M_{ij}, $(i,j) \in \iota$. By Proposition 7 we have that if

values M_{ij}, $(i, j) \in \iota$, are observed with noise having a continuous nondegenerate distribution, then $r^* \geq \mathfrak{R}(m_1, m_2, \mathfrak{m})$ w.p.1. Another way of looking at this is that if $r^* < \mathfrak{R}(m_1, m_2, \mathfrak{m})$, then the respective MRMC problem is unstable in the sense that an arbitrary small change of values of M_{ij} can result in that the respective value r^* of the rank cannot be achieved. On the other hand if $\mathfrak{R}(m_1, m_2, \mathfrak{m})$ is not an integer, then almost surely $r^* > \mathfrak{R}(m_1, m_2, \mathfrak{m})$ and problem (5.2) has more than one optimal solution. We came to the conclusion that the MRMC problem can be both stable and generically have unique optimal solution only if $r^*(m_1+m_2-r^*) = \mathfrak{m}$ or equivalently $r^* = \mathfrak{R}(m_1, m_2, \mathfrak{m})$, which can happen only if $\mathfrak{R}(m_1, m_2, \mathfrak{m})$ is an integer.

Consider now the SDP problem (5.6). We assume that the matrix $C \in \mathbb{V}_\tau$ is *positive definite*. The (Lagrangian) dual of problem (5.6) is the problem

$$\max_{\Lambda \succeq 0} \min_{X \in \mathbb{V}_\tau} C \bullet X - \Lambda \bullet (\Xi + X). \tag{5.18}$$

For $\Lambda = C - \Theta$, with $\Theta \in \mathbb{V}_{\bar{\tau}}$, problem (5.18) can be written (recall that $C \bullet \Xi = 0$ for $\Xi \in \mathbb{V}_{\bar{\tau}}$) as

$$\max_{\Theta \in \mathbb{V}_{\bar{\tau}}} \Theta \bullet \Xi \text{ subject to } C - \Theta \succeq 0. \tag{5.19}$$

Note that Slater condition always holds for the primal problem (5.6), this can be seen by taking $X = \alpha I_p$ for sufficiently large $\alpha > 0$. Also since C is positive definite Slater condition holds for the dual problem (5.19), just take $\Theta = 0$. Hence there is no duality gap between problems (5.6) and (5.19), i.e., these problems have the same finite optimal value. It also follows that both problems have nonempty and bounded sets of optimal solutions.

Proposition 8 *The following holds: (i) given $\Xi \in \mathbb{V}_{\bar{\tau}}$, it follows that for a.e. positive definite matrix $C \in \mathbb{V}_\tau$ problem* (5.6) *has unique optimal solution, (ii) given positive definite matrix $C \in \mathbb{V}_\tau$, it follows that for a.e. $\Xi \in \mathbb{V}_{\bar{\tau}}$ the dual problem* (5.19) *has unique optimal solution.*

Proof Consider the set of positive definite matrices $C \in \mathbb{V}_\tau$. This set is nonempty convex and open. On this set the optimal value of problem (5.6), considered as a function of $C \in \mathbb{V}_\tau$ for given (fixed) Ξ, is finite valued and concave as the minimum of linear functions. It follows by Danskin's theorem (e.g., [4, Theorem 4.13]) that this function is differentiable at a point $C \in \mathbb{V}_\tau$ iff the corresponding set of optimal solutions of problem (5.6) is a singleton. By Rademacher's theorem a locally Lipschitz function is differentiable almost everywhere. This implies the statement (i) (cf., [16, Theorem 5.2]).

There is a symmetry between problems (5.6) and (5.19). By applying the same arguments to problem (5.19) we obtain (ii). □

This shows that unlike the MRCM problem, the SDP problem (5.6) typically has unique optimal solution. Recall that the MTFA problem always has unique optimal solution.

6 Conclusion

We showed in Sect. 3 that under certain regularity conditions optimal solutions of the sample based SDP problem have asymptotically normal distribution. We argued that the required regularity conditions, specified in Theorem 3, in a certain sense are generic. The assumption that the population distribution has finite fourth order moments is natural in the considered context, otherwise the matrix Γ is not defined. The required values of the components of matrix Γ and the Jacobian matrix J are functions of the respective population parameters. Since the population parameters are unknown the standard practice is to estimate these values by replacing the true covariance matrix Σ_0 by the sample covariance matrix[3] S and optimal solution x^* by the computed $\hat{x}_N$. These type of numerical experiments were performed in [21] for a problem related (but with a different objective function) to the MTFA with moderate values of dimension p.

In principle it is also possible to analyse asymptotics of the optimal solutions when the cost vector c is estimated, by considering the dual problem. However, it seems that this would be of a lesser interest.

The discussion of the minimum rank matrix completion problem in Sect. 5 has an independent interest. We argued in Remark 5 that, unless the number $\mathfrak{R}(m_1, m_2, \mathfrak{m})$ is an integer, such problem is either unstable or generically has multiple solutions. The corresponding SDP problem typically has a unique optimal solution (see Proposition 8). However, for large values of $p = m_1 + m_2$ it would be practically impossible to have reliable estimates of the components of the Jacobian matrix J. The situation is simplified considerably when the minimal rank is equal to the lower bound (see Remark 2). This could be a subject of future investigation.

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[3] In case of normally distributed population the covariance matrix Γ is completely defined by the respective matrix Σ_0 (see (3.1)). In general, components of Γ can be estimated by the respective estimates of fourth order moments.

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