Asia-Pacific Journal of Operational Research
Vol. 34, No. 5 (2017) 1750025 (16 pages)

DOI: 10.1142/S0217595917500257

A Computational Algorithm for Equilibrium Asset Pricing Under Heterogeneous Information and Short-Sale Constraints

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Received
Accepted
Published

We propose an efficient algorithm for computing the equilibrium of a capital asset pricing model with heterogeneous investors and short-sale constraints. We show that the equilibrium prices of the risky assets in the model are proportional to the Lagrangian multipliers of an equivalent dual formulation of the problem. Based on this observation, we derive sufficient conditions to guarantee the existence and uniqueness of equilibrium and prove the convergence of the algorithm. Numerical examples are also provided to illustrate the algorithm.

Keywords: Equilibrium pricing; aggregate utility function; convex optimization; tâtonnement.

1. Introduction

The capital asset pricing model (CAPM) proposed in Sharpe (1964), Lintner (1965), Mossin (1966) provides a useful instrument for computing asset prices. In its standard formulation, investors are assumed to have homogeneous beliefs (i.e., having the same expectation and covariance on future payoffs of risky assets) and select portfolios based on the mean–variance framework of Markowitz (1952). It is also assumed that the market is efficient and trading is frictionless. These assumptions, however, are easily violated in realistic financial markets. For example, heterogeneous private information and incomplete knowledge of prices may cause investors to hold different beliefs; moreover, regulators may also impose capital and/or quantitative limits on a temporary basis to restrict or prohibit short sales.

In an attempt to relax these idealistic assumptions, a variety of modified versions of the CAPM model have been proposed in the literature; see, e.g., Diamond and Verrecchia (1987), Fama and French (2007), Hong and Stein (2003), Jarrow (1980), among many others. Much of this work is carried out from a theoretical perspective, focusing primarily on investigating the impact of heterogeneous beliefs and/or short-sale constraints on the market equilibrium. In this paper, we propose a computational algorithm for efficiently determining the equilibrium of a CAPM model. Our model has the same structure as that of Jarrow (1980), which involves both heterogeneous investors and short-sale constraints. However, to account for the trading limitation that might arise in a partially restricted market, we generalize the model of Jarrow (1980) by assuming that the holding of each asset is confined within a given convex set. This allows us to specify whether the short selling of a particular asset is permissible as well as a quantitative limit on its trading. Unfortunately, this generalization leads to additional constraints on the underlying portfolio selection problems, rendering an analytical solution to the problem infeasible.

Our proposed algorithm for finding the equilibrium is inspired by the simple intuition that the price of an asset should be raised (reduced) whenever there is an excess demand (supply) on the market. Thus, at each iteration of the algorithm, an approximation of the market equilibrium is computed by adjusting the price of each asset in the direction of the difference between its demand and supply. The process continues until a set of market-clearing prices is obtained. Our algorithm is similar in spirit to an iterative price updating scheme called the tâtonnement process, which has been proposed in Walras (1954) and studied extensively in general equilibrium theory (see, e.g., Arrow *et al.*, 1959; Uzawa, 1960; Ginsburgh and Waelbroeck, 1979). However, since the tâtonnement process is primarily applied in general exchange economies, its convergence is often analyzed under simplifying assumptions tailored to economics research. Many of these assumptions, e.g., the weak axiom of revealed preference at the equilibrium (Uzawa, 1960), are either difficult to verify or fail to hold in our setting due to the lack of an analytical solution to the portfolio selection problem and the correlations among risky assets in our pricing model. To the best of our knowledge, little research in the current literature has addressed the use of tâtonnement to examine equilibrium prices in CAPM-type of problems. Thus, in a sense, this work can be viewed as an extension of the application of tâtonnement to financial engineering.

We prove the convergence of the algorithm under mild regularity conditions. The idea is to transform the equilibrium problem into an equivalent optimization problem through the aggregation of utility functions. Note that a similar approach has also been used in Eisenberg (1961) and Chen *et al.* (2007) to study competitive economy equilibria; however, since the utility function employed in our model violates the homogeneous property, the results of Eisenberg (1961) and Chen *et al.* (2007) cannot be directly applied to our case. In addition, our model allows short selling of assets, whereas the settings considered in Eisenberg (1961) and Chen *et al.* (2007) require the bundle of goods purchased by consumers to be non-negative. By analyzing the structural properties of the model, we show that the equilibrium asset prices are the optimal Lagrangian multipliers of the market-clearing constraints normalized by the price of the riskless asset. This implies that our algorithm is essentially a subgradient approach for searching the optimal solution to the dual formulation of the problem (cf., e.g., Bonnans *et al.*, 2006; Bertsekas, 2008). Consequently, its theoretical properties, including both convergence and convergence rate, can be investigated using existing results on subgradient methods.

In addition to providing an algorithm for asset pricing, we give a simple proof for the existence of equilibrium and provide sufficient conditions to guarantee its uniqueness. The existence of equilibrium in the CAPM has been previously discussed in Nielsen (1989), Nielsen (1990), Allingham (1991). Their results assume all investors have homogeneous expectations on return distributions and are primarily based on deriving sufficient conditions to rule out satiation caused by unbounded choice sets that may lead to nonexistence of equilibrium. In contrast, we consider the setting where the choice sets are bounded and investors may have heterogeneous beliefs on expected returns and covariance matrices. Consequently, our proof technique differs significantly from previous studies based on satiation and relies on exploiting the connection between equilibrium prices and the Lagrangian multipliers of the dual problem. Sufficient conditions on the uniqueness of the equilibrium in the CAPM have also been derived in, e.g., Nielsen (1988), Hens *et al.* (2002), but again it is still not clear under what conditions a restricted market with heterogenous investors possesses a unique equilibrium.

The rest of this paper is organized as follows. We begin with a description of our model in Sec. 2. In Sec. 3, we introduce the proposed algorithm. Its convergence properties are analyzed in Sec.4. Some preliminary numerical results are reported in Sec. 5. Finally, we conclude this paper in Sec. 6.

2. The Model

Consider a market consisting of K investors, J risky assets, and a riskfree asset. The investors are indexed by $k \in \{1, \ldots, K\}$ and assets are indexed by $j \in \{0, \ldots, J\}$, where $j = 0$ represents the riskfree asset. We consider a two-period model ($t = 0, 1$). Let p_j be the price of asset j at time $t = 0$ and the random variable X_j be its price at $t = 1$. Let r denote the riskfree interest rate. Initially, at time $t = 0$, each investor k is assumed to be endowed with n_j^k units of asset j. Thus, the initial wealth of investor k, denoted by $W^k(0)$, can be expressed as

$$W^k(0) = \sum_{j=1}^{J} n_j^k p_j + n_0^k p_0.$$

At $t = 0$, investors can rebalance the position of their assets according to their own preferences. Denoted by ϕ_j^k the position of asset j held by investor k after the rebalancing. Throughout this paper, we assume that the holding of an asset ϕ_j^k is constrained within a given (nonempty) closed convex set Ω_j^k for all $j = 1, \ldots, J$. This permits us to impose quantitative limits on asset allocations. For example, if the short selling of asset j is prohibited, then one can simply set $\Omega_j^k = [0, \infty)$. It is easy to see that the wealth of investor k at $t = 1$ can be expressed as

$$W^k(1) = \sum_{j=1}^{J} \phi_j^k X_j + \phi_0^k,$$

where we have normalized X_0 to 1 for simplicity (this is equivalent to setting $p_0 = 1/(1+r)$).

As in the standard CAPM, the preference of an investor is measured by the mean-variance utility function (Markowitz, 1952)

$$U^k(W^k(1)) = E^k[W^k(1)] - \frac{\alpha^k}{2}\mathrm{Var}^k[W^k(1)], \tag{1}$$

where $\alpha^k > 0$ is a constant that measures the degree of risk aversion, and $E^k[\cdot]$ and $\mathrm{Var}^k[\cdot]$ are the expectation and variance taken with respect to the distribution of investor k's belief regarding asset payoffs, which may differ across investors. Thus, by substituting $W^k(1)$ into (1), the optimal portfolio selection problem of investor k can be stated as follows:

$$(O_1) \quad \max_{\phi_0^k, \ldots, \phi_J^k} \quad \sum_{j=1}^{J} \phi_j^k E^k[X_j] + \phi_0^k - \frac{\alpha^k}{2} \sum_{i=1}^{J}\sum_{j=1}^{J} \phi_j^k \phi_i^k \sigma_{ji}^k \tag{2}$$

$$\text{subject to} \quad \sum_{j=1}^{J} \phi_j^k p_j + \phi_0^k p_0 = \sum_{j=1}^{J} n_j^k p_j + n_0^k p_0, \tag{3}$$

$$\phi_j^k \in \Omega_j^k, \quad j = 1, \ldots, J,$$

where σ_{ji}^k signifies investor k's belief of the covariance between assets i and j. We assume throughout this paper that the covariance matrix $\Sigma^k = [\sigma_{ji}^k]_{J \times J}$ is positive definite for all $k = 1, \ldots, K$. Note that the left-hand side of the equality constraint (3) represents the total wealth of investor k right after asset rebalancing, and this should be the same as his/her initial wealth.

Let $P = (p_1, \ldots, p_J)^\intercal$, $X = (X_1, \ldots, X_J)^\intercal$, $\Phi^k = (\phi_1^k, \ldots, \phi_J^k)^\intercal$, and $N^k = (n_1^k, \ldots, n_J^k)^\intercal$. Similar to Sun (2003), the following definition provides a useful characterization of the market equilibrium price.

AQ: Not present in Ref. list.

Definition 2.1. A vector $P^* \in \Re^J$ is called an equilibrium price of the market if there exist $\Phi^{k*} \in \Re^J$ for $k = 1, \ldots, K$ such that

(1) Φ^{k*} solves the optimization problem (O_1) at $P = P^*$ for $k = 1, \ldots, K$, and

(2) $\sum_{k=1}^{K} \Phi^{k*} = \sum_{k=1}^{K} N^k$.

Condition (2) in the above definition is often called the market-clearing condition or conservation equation for market equilibrium in economics.

3. The Algorithm

Note that ϕ_0^k can be expressed in terms of $\phi_1^k, \ldots, \phi_J^k$ based on (3). By substituting (3) into (2) and removing terms that are constants with respect to decision variables, we obtain the following optimization problem equivalent to (O_1):

$$(O_2) \max_{\phi_j^k \in \Omega_j^k, j=1,\ldots,J} \sum_{j=1}^{J} \left(E^k[X_j] - \frac{p_j}{p_0} \right) \phi_j^k - \frac{\alpha^k}{2} \sum_{i=1}^{J} \sum_{j=1}^{J} \sigma_{ji}^k \phi_j^k \phi_i^k.$$

Consider a recursive procedure that generates a sequence of price vectors $\{P_n\}_{n=0}^{\infty}$, where $P_n = (p_{n,1}, \ldots, p_{n,J})^\intercal$ is an approximation of the equilibrium price P^* obtained at the nth iteration. Let $\Phi_n^{k*} = (\phi_{n,1}^{k*}, \ldots, \phi_{n,J}^{k*})^\intercal$ be the solution to the optimal portfolio selection problem (O_2) when P is replaced with P_n. Intuitively, $\sum_{k=1}^{K} \phi_{n,j}^{k*}$ can be viewed as the market demand for asset j under price $p_{n,j}$ whereas $\sum_{k=1}^{K} n_j^k$ is the total supply of asset j. It is reasonable to speculate that if $p_{n,j}$ is lower than the equilibrium price, there will be an excess demand on the market, i.e., $\sum_{k=1}^{K} \phi_{n,j}^{k*} - \sum_{k=1}^{K} n_j^k > 0$. Similarly, a price $p_{n,j}$ that is higher than the equilibrium would result in excess supply, leading to $\sum_{k=1}^{K} \phi_{n,j}^{k*} - \sum_{k=1}^{K} n_j^k < 0$. Thus, to enforce $p_{n,j}$ to stay close to the equilibrium, its value should be adjusted depending on the sign of the direction of the difference $\sum_{k=1}^{K} \phi_{n,j}^{k*} - \sum_{k=1}^{K} n_j^k$. In vector form, this suggests the following iterative formula for updating asset prices:

$$P_{n+1} = P_n + a_n \frac{\sum_{k=1}^{K} \Phi_n^{k*} - \sum_{k=1}^{K} N^k}{\|\sum_{k=1}^{K} \Phi_n^{k*} - \sum_{k=1}^{K} N^k\|}, \tag{4}$$

where $\|\cdot\|$ is the Euclidean norm and $a_n \in (0,1)$ is a step size/gain parameter that controls the amount of adjustment at each step. We assume that the step size a_n satisfies

$$\sum_{n=0}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} a_n^2 < \infty, \tag{5}$$

which is a standard condition used in analyzing the convergence of gradient-like descent algorithms (see, e.g., Spall, 2003).

Our proposed algorithm is conceptually very simple and is stated below.

An algorithm for equilibrium asset pricing in CAPM:

Step 0: Specify an initial price vector P_n, a gain sequence $\{a_n\}_{n=0}^{\infty}$, and a tolerance level $\varepsilon > 0$. Set the iteration counter $n = 0$.

Step 1: Solve the portfolio selection problem (O_2) for $P = P_n$ and obtain Φ_n^{k*} for all k.

Step 2: Update the price vector according to (4).

Step 3: If $\|\sum_{k=1}^K \Phi_n^{k*} - \sum_{k=1}^K N^k\| \leq \varepsilon$, then terminate; otherwise set $n = n+1$ and go to Step 1.

At Step 1 of the algorithm, a number of standard numerical methods can be applied to solve the portfolio optimization problem. In particular, since problem (O_2) is convex, we have had success with using the convex programming (CVX) package provided in MATLAB. Note that (O_2) needs to be solved for all investors. So, the complexity of Step 1 grows linearly with the number of investors. For large problems, the computation can be expedited using a straightforward parallel implementation of the algorithm.

4. Convergence Analysis

In this section, we show that the sequence of price vectors $\{P_n\}$ generated by the algorithm converges to an equilibrium price P^*. This result is established based on an interesting connection between P^* and the optimal Lagrangian multipliers of the dual problem of an equivalent formulation of (O_1).

Theorem 4.1. *Assume an equilibrium price P^* exists. The portfolio selection problems (O_1) at $P = P^*$ for $k = 1, \ldots, K$ have the same optimal solutions as the following problem*:

$$(O_3) \quad \max_{\phi_1^1,\ldots,\phi_J^K} \quad \sum_{k=1}^K \left(\sum_{j=1}^J \phi_j^k E^k[X_j] - \frac{\alpha^k}{2} \sum_{i=1}^J \sum_{j=1}^J \sigma_{ji}^k \phi_j^k \phi_i^k \right)$$

$$\text{subject to} \quad \sum_{k=1}^K \phi_j^k = \sum_{k=1}^K n_j^k, \quad j = 1, \ldots, J$$

$$\phi_j^k \in \Omega_j^k, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K.$$

Proof. Since the portfolio selection problems (O_1) are solved independently for each investor, their solutions jointly solve the following (equivalent) optimization problem:

$$\max \quad \sum_{k=1}^K \left(\sum_{j=1}^J \phi_j^k E^k[X_j] + \phi_0^k - \frac{\alpha^k}{2} \sum_{i=1}^J \sum_{j=1}^J \sigma_{ji}^k \phi_j^k \phi_i^k \right) \tag{6}$$

$$\text{subject to} \quad \sum_{j=1}^J \phi_j^k p_j + \phi_0^k p_0 = \sum_{j=1}^J n_j^k p_j + n_0^k p_0, \quad k = 1, \ldots, K, \tag{7}$$

$$\phi_j^k \in \Omega_j^k, \quad j = 1, \ldots, J, \quad k = 1, \ldots, K.$$

Now consider the case when the market is at equilibrium ($P = P^*$). By Definition 2.1, the optimal solutions Φ^{k*} for $k = 1, \ldots, K$ must satisfy the market-clearing condition

$$\sum_{k=1}^{K} \Phi^{k*} = \sum_{k=1}^{K} N^k. \tag{8}$$

In addition, we have from (7) that

$$\phi_0^{k*} = \frac{1}{p_0}\left(\sum_{j=1}^{J} n_j^k p_j^* + n_0^k p_0 - \sum_{j=1}^{J} \phi_j^{k*} p_j^*\right). \tag{9}$$

Summing (9) over $k = 1, \ldots, K$ and using Eq. (8) we have

$$\sum_{k=1}^{K} \phi_0^{k*} = \sum_{k=1}^{K} n_0^k. \tag{10}$$

Finally, substituting (10) into (6) and dropping the constant term $\sum_{k=1}^{K} n_0^k$, we obtain the following equivalent optimization problem at $P = P^*$:

$$\max_{\phi_j^k \in \Omega_j^k, j=1,\ldots,J, k=1,\ldots,K} \sum_{k=1}^{K}\left(\sum_{j=1}^{J} \phi_j^k E^k[X_j] - \frac{\alpha^k}{2}\sum_{i=1}^{J}\sum_{j=1}^{J} \sigma_{ji}^k \phi_j^k \phi_i^k\right)$$

$$\text{subject to} \quad \sum_{k=1}^{K} \Phi^k = \sum_{k=1}^{K} N^k,$$

which completes the proof. □

Observe that (O_3) only depends on the means and variances of returns but not explicitly on P^*. Therefore, it is natural to look at the connection between P^* and (O_3). To this end, we introduce the Lagrangian multipliers $\lambda = (\lambda_1, \ldots, \lambda_J)^\intercal$ for the equality constraints in (O_3). The Lagrangian function is thus given by

$$\begin{aligned} L(\Phi^1, \ldots, \Phi^K, \lambda) &= \sum_{k=1}^{K}\left(\sum_{j=1}^{J} \phi_j^k E^k[X_j] - \frac{\alpha^k}{2}\sum_{i=1}^{J}\sum_{j=1}^{J} \sigma_{ji}^k \phi_j^k \phi_i^k\right) \\ &\quad + \sum_{j=1}^{J} \lambda_j \left(\sum_{k=1}^{K} \phi_j^k - \sum_{k=1}^{K} n_j^k\right) \\ &= \sum_{k=1}^{K}\left(\sum_{j=1}^{J} \phi_j^k E^k[X_j] - \frac{\alpha^k}{2}\sum_{i=1}^{J}\sum_{j=1}^{J} \sigma_{ji}^k \phi_j^k \phi_i^k \right.\\ &\quad \left. + \sum_{j=1}^{J} \lambda_j \phi_j^k - \sum_{j=1}^{J} \lambda_j n_j^k\right). \end{aligned}$$

Denote by $(\Phi^{1*}, \ldots, \Phi^{K*})$ the optimal solution to (O_3) and λ^* the optimal solution to its dual problem

$$\min_{\lambda \in \Re^J} \max_{\Phi^k \in \Omega^k, k=1,\ldots,K} L(\Phi^1, \ldots, \Phi^K, \lambda), \tag{11}$$

where $\Omega^k \doteq \Omega_1^k \times \cdots \times \Omega_J^k$. The next theorem gives an explicit formula relating P^* to the optimal Lagrangian multipliers λ^* and provides sufficient conditions for P^* to be unique.

Theorem 4.2. *There exists an equilibrium price vector given by $P^* = -p_0\lambda^*$. Furthermore, if $\Omega_j^k = [0, \infty)$ and $\sum_{k=1}^K n_j^k \neq 0$ for all $j = 1, \ldots, J$, then P^* is unique.*

Proof. Since Σ^k are assumed to be positive definite, (O_3) is a concave optimization problem, and its dual formulation (11) is always convex (cf., e.g., Bertsekas, 2008). Their optimal solutions are given by $(\Phi^{1*}, \ldots, \Phi^{K*})$ and λ^*, respectively. Thus, according to the primal-dual principle, Φ^{k*} must also solve the following problem:

$$\max_{\{\phi_j^k \in \Omega_j^k\}_{j,k=1}^{J,K}} \sum_{k=1}^K \left(\sum_{j=1}^J \phi_j^k E^k[X_j] - \frac{\alpha^k}{2} \sum_{i=1}^J \sum_{j=1}^J \sigma_{ji}^k \phi_j^k \phi_i^k + \sum_{j=1}^J \lambda_j^* \phi_j^k - \sum_{j=1}^J \lambda_j^* n_j^k \right).$$

Note that in the above problem, both the objective function and constraints are separable. Therefore, the optimization can be carried out by solving K independent subproblems

$$\max_{\{\phi_j^k \in \Omega_j^k\}_{j=1}^J} \sum_{j=1}^J \phi_j^k (E^k[X_j] + \lambda_j^*) - \frac{\alpha^k}{2} \sum_{i=1}^J \sum_{j=1}^J \sigma_{ji}^k \phi_j^k \phi_i^k, \quad k = 1, \ldots, K, \tag{12}$$

where we have removed the constant term. Comparing (12) with (O_2), we conclude that there exist a set of equilibrium prices given by $p_j^* = -p_0\lambda_j^*$ for all j.

Let $\tilde{P}^*$ be any equilibrium price vector. By Theorem 4.1, $\Phi^{k*}, k = 1, \ldots, K$ are also optimal solutions to (O_2) at $\tilde{P}^*$. Define $M = \{1, \ldots, J\}$. Under the condition $\Omega_j^k = [0, \infty)$ for all j, let $I_k = \{j | \phi_j^{k*} = 0\}$ be the set of indices of active constraints and $\hat{I}_k = M \backslash I_k$. Clearly, $\hat{I}_k = \emptyset$ indicates that $\Phi^{k*} = 0$. If $\hat{I}_k \neq \emptyset$, then $\phi_j^{k*}, j \in \hat{I}_k$ must solve the following degenerated version of (O_2) by setting $\phi_j^k = 0$ for all $j \in I_k$:

$$\max_{\phi_j^k, j \in \hat{I}_k} \sum_{j \in \hat{I}_k} \left(E^k[X_j] - \frac{\tilde{p}_j^*}{p_0} \right) \phi_j^k - \frac{\alpha^k}{2} \sum_{i,j \in \hat{I}_k} \sigma_{ji}^k \phi_j^k \phi_i^k.$$

Since ϕ_j^{k*} lies in the interior of Ω_j^k for all $j \in \hat{I}_k$, the first order necessary condition for optimality shows that

$$\Phi_{\hat{I}_k}^{k*} = \frac{1}{\alpha^k p_0} (\Sigma_{\hat{I}_k}^k)^{-1} (p_0 E^k[X_{\hat{I}_k}] - \tilde{P}_{\hat{I}_k}^*), \tag{13}$$

where $X_{\hat{I}_k} = (X_j, j \in \hat{I}_k)^\intercal$, $\tilde{P}_{\hat{I}_k}^* = (\tilde{p}_j^*, j \in \hat{I}_k)^\intercal$, and $\Phi_{\hat{I}_k}^{k*} = (\phi_j^{k*}, j \in \hat{I}_k)^\intercal$ are the respective subvectors of X, $\tilde{P}^*$, and Φ^{k*} with indices belonging to $\hat{I}_k$, and $\Sigma_{\hat{I}_k}^k =$

$[\sigma^k_{ji}]_{i,j\in I_k}$ is the corresponding principal submatrix of Σ^k. Rearranging (13), we get

$$\tilde{P}^*_{\hat{I}_k} = p_0\big(E^k[X_{\hat{I}_k}] - \alpha^k \Sigma^k_{\hat{I}_k} \Phi^{k*}_{\hat{I}_k}\big). \tag{14}$$

Finally, since $\sum_{k=1}^K n^k_j \neq 0$ for $j = 1, \dots, J$ by assumption, it is not difficult to see that $\bigcup_{k=1}^k \hat{I}_k = M$. Thus by (14), the entire price vector $\tilde{P}^*$ is uniquely determined by $\Phi^{k*}, k = 1, \dots, K$. Because $\Phi^{k*}, k = 1, \dots, K$ is the unique optimal solution to (O_3), $\tilde{P}^*$ must also be unique. □

We have the following convergence theorem for the proposed algorithm.

Theorem 4.3. *If the gain sequence* $\{a_n\}$ *satisfies* (5), *then the sequence of price vectors* $\{P_n\}_{n=0}^\infty$ *generated by the algorithm converges to an equilibrium price vector* P^* *at a sub-linear rate.*

Proof. Denote $g(\lambda)$ as the objective function in the dual formulation (11), i.e.,

$$g(\lambda) = \max_{\Phi^k \in \Omega^k, k=1,\dots,K} L(\Phi^1, \dots, \Phi^K, \lambda). \tag{15}$$

The following subgradient algorithm (see, e.g., Bonnans *et al.*, 2006) can be applied to search for the optimal λ^* of (11):

$$\lambda_{n+1} = \lambda_n - \tilde{a}_n \frac{d_n}{\|d_n\|}, \tag{16}$$

where λ_n is an estimate of λ^* at the nth iteration, $\tilde{a}_n = a_n/p_0$ is a step size satisfying (5), and d_n is a subgradient of $g(\lambda)$ at λ_n. From (15) and according to Proposition 6.1.1 in Bertsekas (2008), an obvious choice of d_n is given by

$$d_n = \sum_{k=1}^K \Phi^k_n - \sum_{k=1}^K N^k, \tag{17}$$

where $\Phi^k_n, k = 1, \dots, K$ is the optimal solution to (15) at $\lambda = \lambda_n$.

Now set $P_n = -p_0\lambda_n$. We see that Φ^k_n also solves (O_2) at P_n for $k = 1, \dots, K$. Substituting $\lambda_n = -\frac{P_n}{p_0}$ and (17) into (16), we obtain

$$P_{n+1} = P_n + a_n \frac{\sum_{k=1}^K \Phi^k_n - \sum_{k=1}^K N^k}{\|\sum_{k=1}^K \Phi^k_n - \sum_{k=1}^K N^k\|}.$$

This implies that our proposed algorithm is identical to a subgradient method for minimizing the function $g(\lambda)$. From the results in subgradient optimization theory (see, e.g., Bonnans *et al.*, 2006), the sequence $\{\lambda_n\}_{n=1}^\infty$ generated by (16) converges to λ^* at a sub-linear rate. This observation, together with the result of Theorem 4.2, implies that $\{P_n\}$ will converge to $-\lambda^* p_0$ at the same rate. □

5. Numerical Examples

In this section, we provide some examples to illustrate the performance of our algorithm. In all examples, we set the riskfree interest rate $r = 0.1$, $p_0 = 1$, and $X_0 = 1 + r$. In the implementation of the algorithm, the initial price vector P_0 is set to the arithmetic average of the expected returns of all investors, the gain parameter is taken to be of the form $a_n = a/(n + A)^\beta$ recommended in Spall (2003), where, unless otherwise specified, we set $a = 1$, $A = 100$ and $\beta = 0.51$. The portfolio selection problem (O_2) is solved at each iteration by applying the convex programming package (CVX) in MATLAB. All computational results were obtained on a Intel Dual-Core 2.5GHz CPU Windows machine with 4GB of RAM.

5.1. *Testing convergence*

Example 1. This example is directly taken from Jarrow (1980). Suppose the market consists of two investors and two stocks. Both individuals have identical risk aversion coefficients given by $\alpha^1 = \alpha^2 = 1$. At $t = 0$, the two investors have endowed shares $N^1 = (1, 0)^\intercal$ and $N^2 = (0, 1)^\intercal$ and the following beliefs:

$$E^1[X] = (2, 1)^\intercal, \quad E^2[X] = (1, 3)^\intercal, \quad \Sigma^1 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \Sigma^2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

We consider both cases when short sales are allowed and when short sales are prohibited. Table 1 shows the equilibrium prices and optimal asset allocations obtained by our algorithm under these two respective cases (with error tolerance $\varepsilon = 0.001$). These results conform very well with the analytical solutions reported in Jarrow (1980).[a]

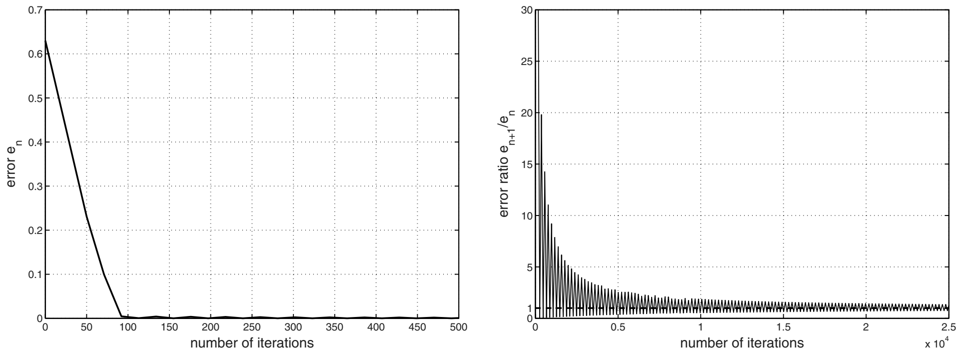
To illustrate the convergence behavior of the algorithm, we take the case when short sales are prohibited and consider the approximation error $e_n \doteq \|P_n - P^*\|$ obtained at successive iterations of (4), where P^* is the true equilibrium price vector given by $(10/11, 20/11)^\intercal$ (see Jarrow (1980)). Figure 1(a) plots the error e_n as a function of the number of algorithm iterations. The figure clearly shows the convergence of the algorithm to P^*, with the approximation error e_n approaching zero as the number of iteration increases.

A test was also performed to try to empirically observe the established asymptotic rate of the algorithm. Note that since this is a small-sized problem, we have used a (different) fast decay rate $\beta = 1$ in the test and run the algorithm for a large number of iterations to better highlight its asymptotic behavior. Figure 1(b) shows the ratio of errors e_{n+1}/e_n versus the number of algorithm iterations. From the figure, we see that the error ratio oscillates from one iteration to another. We believe that this oscillatory behavior is primarily due to the choice of the gain sequence. In

[a]The difference between our results and those in Jarrow (1980) is because that Jarrow assumes $p_0 = 1/(1 + r)$ and $X_0 = 1$, whereas we have used $p_0 = 1$, $X_0 = 1 + r$ in our experiment. If we set $p_0 = 1/(1 + r)$ instead, then the results will be identical.

Table 1. Two investors and two stocks.

Case		Investor 1	Investor 2	Equilibrium price
With short sales	stock 1	1.667	−0.667	1.061
	stock 2	−0.833	1.833	1.667
Without short sales	stock 1	1.000	0.000	0.909
	stock 2	0.000	1.000	1.818



(a) (b)

Fig. 1. Convergence of the algorithm.

particular, when the search is close to the equilibrium price, the magnitude of the gain sequence $\{a_n\}$ becomes large relative the magnitude of $\frac{\sum_{k=1}^{K}\Phi_n^k - \sum_{k=1}^{K}N^k}{\|\sum_{k=1}^{K}\Phi_n^k - \sum_{k=1}^{K}N^k\|}$ (see Eq. (4)), which causes the estimate P_n to bounce around in a small neighborhood of P^*. Nevertheless, we see that as the gain a_n diminishes (when the number of iterations gets large), the amplitude of the oscillation decreases and the error ratio gradually approaches one. This supports the sub-linear convergence rate claimed in Theorem 4.3.

Example 2. We consider the market consists of three investors and four stocks. All investors have identical risk aversion coefficients $\alpha^1 = \alpha^2 = \alpha^3 = 1$. They have initial endowments $N^1 = (1,1,0,0)^\intercal$, $N^2 = (0,0,1,0)^\intercal$, $N^3 = (0,0,0,1)^\intercal$ and beliefs $E^1[X] = (3,4,1,4)^\intercal$, $E^2[X] = (1,2,3,3)^\intercal$, $E^3[X] = (2,1,4,2)^\intercal$, and

$$\left(\Sigma^1 \,\vdots\, \Sigma^2 \,\vdots\, \Sigma^3\right) = \left(\begin{array}{cccc:cccc:cccc} 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 \end{array}\right).$$

Note that in contrast to Example 1, this is a comparatively high dimensional (three investors and four stocks) problem that can no longer be easily solved analytically using the KKT condition (see, e.g., Jarrow (1980)). From the results reported in Table 2, we observe that the prices of risky assets may either rise (stocks 2 and 3)

Table 2. Three investors and four stocks.

Case		Investor 1	Investor 2	Investor 3	Equilibrium
With short sales	stock 1	1.584	−0.591	0.007	0.962
	stock 2	1.271	0.089	−0.361	0.716
	stock 3	−1.409	1.220	1.190	1.706
	stock 4	0.496	0.405	0.098	0.969
Without short sales	stock 1	0.857	0.000	0.143	0.779
	stock 2	0.857	0.143	0.000	0.909
	stock 3	0.000	0.143	0.857	1.948
	stock 4	0.429	0.571	0.000	0.909

or fall (stocks 1 and 4) due to short-sale restrictions. This is also consistent with the findings of Jarrow (1980).

5.2. *Comparison results*

In this subsection, we apply our algorithm to larger problem instances and compare its performance with those of the branch-and-bound method and the interior-point quadratic programming algorithm.

The branch-and-bound algorithm is popular in handling equilibrium problems that can be transformed into mathematical programming problems with complementarity constraints. To apply the algorithm, we first formulate our model as a complementarity-constrained problem of the form

$$\min_{P,\{\Phi^k,\lambda^k\}_{k=1}^K} \left\| \sum_{k=1}^K \Phi^k - \sum_{k=1}^K N^k \right\|$$

$$\text{subject to } \alpha^k \Sigma^k \Phi^k - \left(\mu^k - \frac{1}{p_0} P\right) - \lambda^k = 0,$$

$$\lambda_i^k \phi_i^k = 0, \quad i = 1, \ldots, J; \quad k = 1, \ldots, K$$

$$P \in \Re^J, \quad \Phi^k \geq 0, \quad \lambda^k \geq 0, \quad k = 1, \ldots, K$$

and then transform it into the following equivalent mixed integer programming problem (see Hu *et al.*, 2012):

$$\min_{P,\{\Phi^k,\lambda^k\}_{k=1}^K} \left\| \sum_{k=1}^K \Phi^k - \sum_{k=1}^K N^k \right\|$$

$$\text{subject to } \alpha^k \Sigma^k \Phi^k - \left(\mu^k - \frac{1}{p_0} P\right) - \lambda^k = 0,$$

$$0 \leq \lambda_i^k \leq \bar{\lambda}_i^k z_i^k,$$

$$0 \leq \phi_i^k \leq (1 - z_i^k) \bar{\phi}_i^k,$$

$$z_i^k \in \{0,1\},$$

$$P \in \Re^J, \quad \Phi^k \geq 0, \lambda^k \geq 0, \quad k = 1, \dots, K,$$

where $\bar{\lambda}_i^k$ and $\bar{\phi}_i^k$ represent the upper bounds on λ_i^k and ϕ_i^k. The problem can then be solved by using the standard branch-and-bound algorithm. In our numerical experiments, the bounding constants $\bar{\lambda}_i^k$ and $\bar{\phi}_i^k$ are set to 1,000 for all i and k, and we have used the commercial optimization package CPLEX to implement the algorithm.

The use of quadratic programming is motivated by the result of Theorem 4.2, which states that the equilibrium price vector P^* only differs from the optimal solution to the dual formulation (11) of (O_3) by a constant factor $-p_0$. Since (O_3) is a quadratic programming problem, the problem along with its dual (11) can be conveniently solved using the interior-point quadratic programming algorithm of the package CVX in MATLAB. The equilibrium price P^* can then be obtained by simply multiplying the resulting optimal solution λ^* to (11) by $-p_0$.

Table 3 shows the performance of the three comparison algorithms on four test cases with varying numbers of investors and risky assets. Note that since we have used a sequential implementation of our algorithm, the portfolio selection problem (O_2) needs to be solved K times in succession at each iteration. Therefore, we have used a relatively large error tolerance level $\varepsilon = 0.1$ to avoid excessive long running times of the algorithm when K is large.

From the table, we see that both branch-and-bound and the quadratic programming algorithm significantly outperform our algorithm when the market is small. In particular, in the $(5, 5)$ case, the two alternative algorithms are able to produce highly accurate solutions in under one second. However, since the worst case complexity of branch-and-bound is the same as that of exhaustive search, the algorithm may quickly become impractical as the problem size increases. We see that even in the $(10, 10)$ case, branch-and-bound fails to produce an acceptable solution within a reasonable amount of time. On the other hand, the quadratic programming algorithm is much more competitive and can be applied to solve relatively large problem instances with up to a hundred of investors and assets. The complexity of quadratic programming is generally polynomial in the number of decision variables. Thus, as the problem size increases from $(10, 10)$ to $(100, 100)$, the number of decision variables grows by a factor of 1,000, leading to a drastic increase in the running time

Table 3. Comparison results with branch-and-bound and quadratic programming.

	Our algorithm		Branch-and-bound		Quadratic programming	
(K, J)	Time (s)	Error	Time (s)	Error	Time (s)	Error
(5,5)	11.40	0.084	0.54	0	0.93	1.588×10^{-10}
(10,10)	24.26	0.083	N/A	N/A	0.54	4.209×10^{-11}
(100,100)	212.52	0.089	N/A	N/A	178.01	2.404×10^{-10}
(1,000,100)	4060.01	0.079	N/A	N/A	N/A	N/A

of the algorithm. Note that a straightforward implementation of the quadratic programming algorithm requires the explicit specification and storage of the positive definition matrix appearing in the objective function. This can be an issue for large problems. For example, the algorithm fails to run successfully on the (1,000, 100) case because the size of the matrix reaches $10^5 \times 10^5$, which exceeds the memory capacity of the computer. In contrast, Table 3 shows that the performance of our algorithm is less susceptible to the increases in problem dimension. We see that as the problem size increases from $(5, 5)$ to $(100, 100)$, the running time of the algorithm grows roughly linearly with K. In the (1,000, 100) case, our algorithm terminates in a few tens of iterations with an approximation error of 0.079. This suggests that the computational time of our algorithm is primarily dominated by the time required to solve the K portfolio selection problems. Consequently, the performance of the algorithm can be dramatically improved by employing a parallel implementation scheme that solves the K portfolio selection problems simultaneously.

6. Conclusion

In this paper, we have introduced an algorithm for computing the equilibrium of a capital asset pricing model with heterogeneous investors and market restrictions. The algorithm is not only conceptually simple but also effective and easy to implement. A central contribution of this paper is to prove that our algorithm is essentially a subgradient approach for a dual formulation of the problem. We show that the equilibrium prices in our model actually turn out to be the scaled versions of the Lagrangian multipliers of a dual formulation of the problem. This key result enables us to prove the existence of equilibrium, identify sufficient conditions to guarantee its uniqueness, and establish convergence properties of the algorithm.

Acknowledgments

The authors thank the associate editor and three anonymous reviewers for their many helpful comments and suggestions, which have led to a substantially improved paper. This work was supported by the National Natural Science Foundation of China (NSFC) Under Grants 71571048 and 71071040, by the Science and Technology Agency of Sichuan Province Under Grant 2014GZX0002, by the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institution of Higher Learning. The work of Jiaqiao Hu was supported by the U.S. National Science Foundation Under Grant CMMI-1634627.

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