Computers & Industrial Engineering 117 (2018) 181–190

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

A minimum deviation approach for improving the consistency of uncertain 2-tuple linguistic preference relations

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ARTICLE INFO

Keywords:
Uncertain linguistic preference relations
2-tuple linguistic
Consistency improving
Optimization

ABSTRACT

This paper proposes a novel approach for improving the consistency of uncertain 2-tuple linguistic preference relations (U2TLPRs). In particular, we introduce a new definition of consistency for U2TLPRs and show that the degree of consistency of a given U2TLPR can be measured explicitly by minimizing its deviation from a consistent U2TLPR. Based on this finding, we provide an iterative algorithm for repeatedly adjusting the consistency of a U2TLPR to a desired level while taking into account the initial preferences of decision makers. In addition, by analyzing the structural properties of the algorithm, we further present an improved version of the procedure for directly obtaining an acceptable U2TLPR without any iteration. Numerical results indicate that the proposed method is not only simple and efficient in calculation but also effective in preserving the original preference information provided by decision makers.

1. Introduction

Preference relations (PRs), also known as the judgment matrix or pairwise comparison matrix, are a popular and powerful tool to model decision makers' preferences in decision making. PRs facilitate the expression of decision makers' opinions by allowing them to focus on a pair of elements at a time (Herrera, Herrera-Viedma, & Chiclana, 2001), making these methods more accurate and preferable than many other preference modeling techniques (Millet, 1997). The most popular types of PRs are multiplicative preference relation (Saaty, 1980) and fuzzy preference relation (Tanino, 1984), the entries of which are numerical values. There are also PRs capable of handling uncertain and vague information. Examples of these PRs include fuzzy interval preference relation (Xu, 2004b), triangular fuzzy preference relation (Van Laarhoven & Pedrycz, 1983), intuitionistic fuzzy preference relation (Szmidt & Kacprzyk, 1998; Wu & Chiclana, 2014). In all cases, the consistency of PRs turns out to be an important aspect of decision making and needs to be carefully examined to avoid misleading conclusions.

The concepts of consistency were first introduced in Saaty (1980) and Tanino (1984) for classical multiplicative preference relations and fuzzy preference relations. Since then, the classical definitions of consistency have been evolved and extended to different fuzzy-valued preference relations (Wang & Chen, 2008; Wang & Tong, 2016; Xu & Chen, 2008b). In recent years, some new ideas on the consistency of fuzzy-valued preference relations have also emerged. For example, Dubois (2011) pointed out that the consistency of fuzzy-valued preference relations should not be defined directly based on the classical ones. Liu, Pedrycz, and Zhang (2017) and Liu, Pedrycz, Wang, and Zhang (2017) also proposed the concept of approximation-consistency for fuzzy-valued preference relations, which improves upon existing consistency definitions by incorporating the additive/multiplicative reciprocal property and is invariant with respect to permutations of alternatives.

When decision makers only have vague knowledge about alternatives and cannot express their preferences with numerical values, qualitative (as opposed to quantitative) descriptions are used for pairwise comparison and the PRs are often stated in terms of linguistic variables. To support group decision making (GDM) with linguistic preference relations (LPRs), a variety of consensus reaching models (Alonso, Pérez, Cabrerizo, & Herrera-Viedma, 2013; Cabrerizo, Alonso, & Herrera-Viedma, 2009; Cabrerizo, Pérez, & Herrera-Viedma, 2010; Dong, Xu, Li, & Feng, 2010; Gong, Forrest, & Yang, 2013; Herrera & Herrera-Viedma, 1996; Herrera-Viedma, Martinez, Mata, & Chiclana, 2005; Li, Dong, Herrera, & Herrera-Viedma, 2017; Mata, Martínez, & Herrera-Viedma, 2009; Xu & Wu, 2013) and aggregation-and-ranking approaches (Herrera, Herrera-Viedma, & Verdegay, 1996; Wang & Chen, 2010; Wu, Li, Li, & Duan, 2009; Xu, 2004a) have been developed. Moreover, a number of approaches have proposed in literature for measuring and improving the consistency of LPRs (Alonso, Cabrerizo,

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https://doi.org/10.1016/j.cie.2018.01.024
Received 19 June 2017; Received in revised form 6 December 2017; Accepted 27 January 2018
Available online 31 January 2018
0360-8352/

Chiclana, Herrera, & Herrera-Viedma, 2009; Cabrerizo, Heradio, Pérez, & Herrera-Viedma, 2010; Cabrerizo, Pérez et al., 2010; Dong, Hong, & Xu, 2013; Dong, Li, & Herrera, 2015; Dong, Xu, & Li, 2008; Jin, Ni, Chen, & Li, 2016; Wang & Xu, 2016).

Due to the uncertainty of decision environment and/or the lack of relevant experience with decision alternatives, the decision makers may sometimes prefer to use uncertain linguistic variables in specifying preference relations. For example, when presented with a pair of alternatives, a decision maker may use uncertain linguistic terms such as "between slightly good and good" to indicate his/her preferences about different alternatives. This uncertainty in the preference information has led to the research focusing on GDM with uncertain linguistic preference relations (ULPRs) (Chen & Lee, 2010; Chen, Zhou, & Han, 2011; Tapia-García, Del Moral, Martínez, & Herrera-Viedma, 2012; Xu, 2006; Xu & Wu, 2013; Zhang & Guo, 2014a; Zhou & Chen, 2013). The use of ULPRs for modeling expert preference implies the use of Computing with Words (CW). To facilitate this process, the 2-tuple linguistic representation model (Herrera & Martínez, 2000) has been widely used in literature due to its advantages over other linguistic models. In particular, through the use of such a representation model, one can obtain a U2TLPR derived from a ULPR (Zhang & Guo, 2014b). Consequently, when the decision maker uses ULPR to express his/her preference, U2TLPR will be an important tool for decision analysis. Unfortunately, despite significant progress over the past years on decision making methods with ULPRs/U2TLPRs, only a few attempts have been made to address the consistency issue of individual ULPR/U2TLPR. By applying the consistency of LPRs, Meng, An, and Chen (2016) introduced a definition of consistency for ULPRs. Although the models they proposed can be used to generate consistent ULPRs, the issue of how to improve the consistency of a given ULPR remains unaddressed. Zhang and Guo (2014b) defined the additive consistency of ULPRs and provided two algorithms to estimate the missing entries in incomplete ULPRs. Subsequently, they proposed an iterative consistency improving procedure for U2TLPRs (Zhang & Guo, 2016). In their algorithm, the computation of consistency index and the consistency improvement of a U2TLPR in each round are based on a consistent U2TLPR constructed by using only the $n-1$ preference values above the diagonal of the original U2TLPR provided by decision maker. Since their approach does not fully utilize the initial preference values provided by decision makers, the resulting U2TLPR, albeit greatly improved in its consistency, may share little resemblance to the original U2TLPR.

Note that some preference values in the initial U2TLPR provided by the decision maker will need to be adjusted in order to improve its consistency. However, if the improved U2TLPR deviates too much from the initial U2TLPR, then it may fail to adequately represent the real preference of the decision maker. Thus, there is often a tradeoff between improving the consistency level of a U2TLPR and preserving the initial preference of the decision maker. To address this tradeoff, we propose a novel optimization-based approach for constructing ULPRs with desired levels of consistency while effectively retaining the initial preference information of decision makers. As in Zhang and Guo (2016), we assume that ULPRs are expressed using a 2-tuple fuzzy linguistic model in this paper. We introduce a new consistency definition for U2TLPRs and propose an index to measure their consistency levels. A linear programming model is subsequently developed to construct a consistent U2TLPR from an inconsistent one, and we show that the consistency index of a given U2TLPR can be explicitly calculated in terms of the solution to the optimization model. In contrast to the idea employed in Zhang and Guo (2016), our linear programming model incorporates all the preference values in the original U2TLPR in searching for a consistent U2TLPR, which in turn leads to the good performance of the proposed consistency improving algorithm in preserving the initial preference information. We then propose a sequential procedure that relies on repeatedly solving a sequence of linear programming problems to increase the consistency of a U2TLPR to an acceptable level. In addition, by exploiting the theoretical properties of the procedure, we construct a much simplified version of the algorithm that only requires solving the optimization problem once in obtaining a U2TLPR with a prescribed consistency level. Note that similar to the algorithm developed in Zhang and Guo (2016), our proposed consistency improving model is also based on the construction of a certain consistent U2TLPR. However, we argue analytically that among the set of all consistent U2TLPRs, the consistent U2TLPR obtained by our method provides the best possible approximation to the initial U2TLPR and thus implicitly accounts for all the preference values provided by decision makers.

The rest of this paper is organized as follows. Section 2 summaries the basic concepts of the 2-tuple linguistic representation model and U2TLPRs. In Section 3, we introduce a new consistency definition and propose a consistency index for U2TLPRs. An optimization model is then developed, based on which an iterative consistency improving algorithm, along with its simplified non-iterative version, is proposed. Numerical examples and a comparison study are presented in Section 4 to illustrate the performance of the proposed method. Finally, we conclude the paper in Section 5.

2. Preliminaries

This section reviews the basic necessary knowledge on the 2-tuple linguistic representation model and uncertain 2-tuple linguistic preference relations.

2.1. 2-tuple linguistic representation model

In order to compute with words, Herrera and Martínez (2000) proposed a 2-tuple linguistic representation model based on the concept of symbolic translation. The model uses a 2-tuple (s_i,α_i) to represent linguistic information, where s_i is a linguistic term belonging to the predefined linguistic term set and $\alpha_i \in [-0.5,0.5)$ denotes the symbolic translation. Specifically, the 2-tuple linguistic representation model is defined as follows.

Definition 1 (*Herrera and Martínez, 2000*). Let $S = \{s_0,s_1,...,s_g\}$ be a linguistic term set with odd cardinality and $\beta \in [0,g]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta: [0,g] \mapsto S \times [-0.5,0.5)$$
$$\Delta(\beta) = (s_i,\alpha_i),$$

where $i = round(\beta)$ and $\alpha_i = \beta - i$. Note that "*round*" is the usual round operator, s_i has the closest index label to β, and α_i is the value of the symbolic translation.Obviously, a linguistic term $s_i \in S$ can be viewed as a 2-tuple linguistic $(s_i,0)$. Unless otherwise specified, we will use 2-tuple linguistic representations instead of linguistic terms throughout the paper.

Definition 2 (*Herrera and Martínez, 2000*). Let $S = \{s_0,s_1,...,s_g\}$ be as before and (s_i,α_i) be a 2-tuple, there exists a function

$$\Delta^{-1}: S \times [-0.5,0.5) \mapsto [0,g]$$
$$\Delta^{-1}((s_i,\alpha_i)) = i + \alpha_i = \beta$$

that uniquely transforms a 2-tuple into its equivalent numerical value $\beta \in [0,g]$.We also refer to Herrera and Martínez (2000) for the operations on linguistic 2-tuples without loss of information.

2.2. Uncertain 2-tuple linguistic variables and uncertain 2-tuple linguistic preference relations

Definition 3 (*Zhang and Guo, 2016*). Let $S = \{s_0,s_1,...,s_g\}$ be as before, then $[l^-,l^+]$ is called an uncertain 2-tuple linguistic variable if $l^- = (s^-,\alpha^-),l^+ = (s^+,\alpha^+) \in S \times [-0.5,0.5)$ and $s^- \leqslant s^+$.For the operations on uncertain 2-tuple linguistic variables, we refer to Zhang and Guo (2016). Based on the uncertain 2-tuple linguistic variables, Zhang and Guo (2016) proposed the following definition of uncertain 2-tuple linguistic preference relation.

Definition 4 (*Zhang and Guo, 2016*). Let $S = \{s_0,s_1,...,s_g\}$ be as before, then a matrix $L = (l_{ik})_{n\times n}$ is called an Uncertain 2-tuple Linguistic Preference Relation (U2TLPR) if $l_{ik} = [l_{ik}^-,l_{ik}^+],l_{ik}^- \leqslant l_{ik}^+,\Delta^{-1}(l_{ik}^-) + \Delta^{-1}(l_{ki}^+) = \Delta^{-1}(l_{ik}^+) + \Delta^{-1}(l_{ki}^-) = g,l_{ii}^- = l_{ii}^+ = (s_{g/2},0)$, where l_{ik} is an uncertain 2-tuple linguistic variable indicating the preference degree of the alternative x_i over x_k, and $l_{ik}^-,l_{ik}^+ \in S\times[-0.5,0.5),i,k\in N$.It is easy to see that the reciprocal property is implied in the definition of U2TLPRs. In addition, because linguistic terms are essentially a special case of 2-tuple linguistic terms, U2TLPRs can be viewed as a generalization of LPRs.

Definition 5 (*Zhang and Guo, 2016*). Let $S = \{s_0,s_1,...,s_g\}$ be as before, and let $P = ([p_{ik}^-,p_{ik}^+])_{n\times n}$ and $Q = ([q_{ik}^-,q_{ik}^+])_{n\times n}$ be two U2TLPRs defined on S. The deviation measure between P and Q is defined as

$$d(P,Q) = \frac{1}{gn(n-1)}\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(p_{ik}^-)-\Delta^{-1}(q_{ik}^-)| + |\Delta^{-1}(p_{ik}^+)-\Delta^{-1}(q_{ik}^+)|). \tag{1}$$

Note that the deviation measure satisfies $0\leqslant d(P,Q)\leqslant 1$. Moreover, a small value of $d(P,Q)$ indicates a high degree of similarity between P and Q, whereas a large (close to one) value of $d(P,Q)$ signals a strong discrepancy between P and Q. Thus, Eq. (1) can be used as an indicative measure to determine the resemblance between two U2TLPRs.

3. An optimization-based consistency improving approach

In this section, we begin by introducing a new definition of consistency for U2TLPRs. Then we show that the consistency index of a given U2TLPR can be measured by minimizing its deviation to a consistent U2TLPR. Finally, we propose an iterative algorithm for adjusting the consistency of a U2TLPR, analyze its theoretical properties, and present a non-iterative improved version of the algorithm.

3.1. The consistency of U2TLPRs

Generally speaking, the consistency of preference relations can be defined from two perspectives (Xu, Wan, Wang, Dong, & Zeng, 2016). One type of definitions is built upon the so-called transitivity property of preference relations (Alonso et al., 2009; Cabrerizo, Heradio et al., 2010), and the other type is based on the connections between the elements of preference relations and the priority weights (Jin et al., 2016; Xu & Chen, 2008b; Xu, Li, & Wang, 2014). According to the transitivity, some consistency definitions for U2TLPRs have been proposed in literature Meng et al. (2016) and Zhang and Guo (2014b). However, to the best of our knowledge, there is no consistency definition for U2TLPRs based on the priority weights. In this section, motivated by the definition of additive consistency for interval fuzzy preference relations (Xu & Chen, 2008b; Xu et al., 2014), we begin by introducing a new definition of consistency for U2TLPRs, which serves as a basis for the proposed consistency improving approach.

Definition 6. Let $L = ([l_{ik}^-,l_{ik}^+])_{n\times n}$ be a U2TLPR, if there exist a vector $W = (w_1,w_2,...,w_n)^T$ satisfying $\sum_{i=1}^n w_i = 1,w_i\geqslant 0(i = 1,2,...,n)$ and a scalar parameter $\beta>0$ such that

$$\Delta^{-1}(l_{ik}^-)\leqslant 0.5g + \beta(w_i-w_k)\leqslant\Delta^{-1}(l_{ik}^+) \qquad \forall\ i,k = 1,2,...,n, \tag{2}$$

then we call L an additively consistent U2TLPR.

As we can see from Definition 6, the additive consistency of U2TLPRs is defined by extending the additive consistency of interval-valued preference relations (Xu & Chen, 2008b). The basic idea is to characterize the consistency of U2TLPRs based on a consistent 2-tuple linguistic preference relation. We remark that there are some different ideas on the definition of consistency for interval-valued preference relations. Krejčí (2017) argued that for some additively consistent interval-valued preference relations, there may not exist a vector satisfying the normalization equation $\sum_{i=1}^n w_i = 1,w_i\geqslant 0(i = 1,2,...,n)$, and one should use the normalization condition $|w_i-w_j|\leqslant 1$ proposed by Tanino (1984) to define the additive consistency for interval-valued preference relations. Liu, Peng, Yu, and Zhao (2018) pointed out that interval-valued preference relations are inconsistent in essence and propose the concept of additive approximation-consistency of interval-valued preference relations to incorporate the additive property and permutations of alternatives.

We remark that for a consistent fuzzy preference relation $R = (r_{ik})_{n\times n}$, Xu and Chen (2008a) reasoned that the relationship between the elements of R and the corresponding priority vector $W = (w_1,w_2,...,w_n)^T$ can be represented by $r_{ik} = 0.5(w_i-w_k) + 0.5$. However, Shen, Chyr, Lee, and Lin (2009) later pointed out that this relation may not hold true in general. Instead, it has been shown in Xu, Da, and Wang (2011) that for an additively consistent fuzzy preference relation, there exists a constant $\beta>0$ such that relation can be expressed as $r_{ik} = \beta(w_i-w_k) + 0.5$. They suggested setting the value of β to either $n/2$ (Xu et al., 2011) or $(n-1)/2$ (Xu, Da, & Liu, 2009), both of which have been shown to yield better performance than the case $\beta = 0.5$. Consequently, for additively consistent U2TLPRs, we suggest setting $\beta = ng/2$ or $\beta = (n-1)g/2$ in Definition 6 (see also Section 4.1).

Note that since U2TLPR satisfies the reciprocal property, Eq. (2) is equivalent to the following equation:

$$\Delta^{-1}(l_{ik}^-)\leqslant 0.5g + \beta(w_i-w_k)\leqslant\Delta^{-1}(l_{ik}^+) \quad i<k. \tag{3}$$

Therefore, it is sufficient to only check the upper or lower triangular elements of a U2TLPR for additive consistency.

The following definition, motivated by Dong et al. (2008) and Dong, Li, Chiclana, and Herrera-Viedma (2016), provides a useful measure for characterizing the consistency degree of a U2TLPR.

Definition 7. Let $L = ([l_{ik}^-,l_{ik}^+])_{n\times n}$ be a U2TLPR and P_n be the set of all $n\times n$ consistent U2TLPRs, then we call

$$CI(L) = \min_{P\in P_n} d(L,P) \tag{4}$$

the consistency index of L.

In Definition 7, the consistency index of a U2TLPR is defined as its smallest deviation from the collection of all consistent U2TLPRs. Thus, the matrix P that solves (4) is a consistent U2TLPR that shares the strongest resemblance to L. Clearly, the smaller the value of $CI(L)$ is, the more consistent L will be, and if $CI(L) = 0$, then L itself will be consistent. Unfortunately, although this definition is intuitive and conceptually appealing, directly calculating the index in practice requires solving a high-dimensional nonlinear optimization problem, which can be very challenging. Below, we address this computational issue by formulating an auxiliary linear programming model and showing that the consistency index given by (4) can be conveniently obtained in terms of the solution to the linear programming problem.

The key observation is that an inconsistent U2TLPR $L = ([l_{ik}^-,l_{ik}^+])_{n\times n}$ can be made to satisfy the following relaxed version of Eq. (3) by introducing two additional deviation variables $d_{ik}^-,d_{ik}^+(i<k)$:

$$\Delta^{-1}(l_{ik}^-)-d_{ik}^-\leqslant 0.5g + \beta(w_i-w_k)\leqslant\Delta^{-1}(l_{ik}^+) + d_{ik}^+, \tag{5}$$

where d_{ik}^- and $d_{ik}^+(i<k)$ are both non-negative real numbers. Notice that (5) can be viewed as an extension of (3) and reduces to it when both d_{ik}^- and $d_{ik}^+(i<k)$ are set to zero. Thus, smaller values of the deviation variables signify a higher degree of conformity of $L = ([l_{ik}^-,l_{ik}^+])_{n\times n}$ to a consistent U2TLPR. This intuition gives rise to the following linear optimization model $(M-1)$:

$(M-1)$

$$Min\ Z = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d_{ik}^- + d_{ik}^+)$$

$$s.\ t.\ \begin{cases} 0.5g + \beta(w_i - w_k) \geqslant \Delta^{-1}(l_{ik}^-) - d_{ik}^-,\ i<k \\ 0.5g + \beta(w_i - w_k) \leqslant \Delta^{-1}(l_{ik}^+) + d_{ik}^+,\ i<k \\ \Delta^{-1}(l_{ik}^-) - d_{ik}^- \geqslant 0,\ \Delta^{-1}(l_{ik}^+) + d_{ik}^+ \leqslant g,\ i<k \\ w_1 + w_2 + \cdots + w_n = 1 \\ w_i \geqslant 0,\ i = 1,2,\ldots,n;\ d_{ik}^-, d_{ik}^+ \geqslant 0,\ i<k. \end{cases}$$

In model $(M-1)$, the objective is to minimize the sum of all deviation variables, so that the original preference information of the decision makers can be retained to the largest possible extent. The first two constraints are the restrictions on the weight vector and the deviation variables posed by Eq. (5). The third constraint stipulates that the adjusted preference values should lie in the domain of uncertain linguistic variables. Note that the model does not explicitly consider the participation of decision makers, who may provide additional preference information during the consistency improving process. Such information can usually be encoded in the form of constraints on the deviation variables $d_{ik}^{-,*}$, $d_{ik}^{+,*} (i<k)$ and thus be incorporated into model $(M-1)$.

Let $d_{ik}^{-,*}, d_{ik}^{+,*} (i<k)$ and $w_i^* (i = 1,2,\ldots,n)$ be the optimal solution to the model $(M-1)$ and Z^* be the corresponding optimal objective function value, i.e.,

$$Z^* = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d_{ik}^{-,*} + d_{ik}^{+,*}).$$

For a given U2TLPR $L = ([l_{ik}^-, l_{ik}^+])_{n\times n}$, a new U2TLPR $\overline{L} = ([\overline{l}_{ik}^-, \overline{l}_{ik}^+])_{n\times n}$ will be constructed based on the optimal solution to $(M-1)$:

$$\overline{l}_{ik}^- = \Delta(\Delta^{-1}(l_{ik}^-) - d_{ik}^{-,*}), \overline{l}_{ik}^+ = \Delta(\Delta^{-1}(l_{ik}^+) + d_{ik}^{+,*}), \quad i<k, \tag{6}$$

$$\overline{l}_{ki}^- = \Delta(g - \Delta^{-1}(\overline{l}_{ik}^+)), \overline{l}_{ki}^+ = \Delta(g - \Delta^{-1}(\overline{l}_{ik}^-)), i<k. \tag{7}$$

The following result shows that the constructed U2TLPR $\overline{L}$ is additively consistent.

Theorem 1. *For a given U2TLPR $L = ([l_{ik}^-, l_{ik}^+])_{n\times n}$, let Z^* be the optimal value of the model $(M-1)$. The U2TLPR $\overline{L} = ([\overline{l}_{ik}^-, \overline{l}_{ik}^+])_{n\times n}$ constructed according to (6) and (7) is additively consistent.*

Proof. Let $d_{ik}^{-,*}, d_{ik}^{+,*} (i<k)$ be the components of the optimal solution to model $(M-1)$. According to the optimality of the solution, there exist a vector $(w_1^*, w_2^*, \ldots, w_n^*)^T$ and $\beta > 0$ satisfying

$$\begin{cases} 0.5g + \beta(w_i^* - w_k^*) \geqslant \Delta^{-1}(l_{ik}^-) - d_{ik}^{-,*},\ i<k \\ 0.5g + \beta(w_i^* - w_k^*) \leqslant \Delta^{-1}(l_{ik}^+) + d_{ik}^{+,*},\ i<k \\ \Delta^{-1}(l_{ik}^-) - d_{ik}^{-,*} \geqslant 0,\ \Delta^{-1}(l_{ik}^+) + d_{ik}^{+,*} \leqslant g,\ i<k \\ w_1^* + w_2^* + \cdots + w_n^* = 1 \\ w_i^* \geqslant 0, i = 1,2,\ldots,n; d_{ik}^{-,*}, d_{ik}^{+,*} \geqslant 0,\ i<k. \end{cases}$$

Therefore, we have from Eqs. (6) and (7) that, $\forall\ i,\ k = 1,2,\ldots,n$,

$$\Delta^{-1}(\overline{l}_{ik}^-) \leqslant 0.5g + \beta(w_i^* - w_k^*) \leqslant \Delta^{-1}(\overline{l}_{ik}^+),$$

which means $\overline{L} = ([\overline{l}_{ik}^-, \overline{l}_{ik}^+])_{n\times n}$ is additively consistent by Definition 6. This completes the proof of Theorem 1. □

Theorem 1 not only provides a way to determine the consistency of a U2TLPR (when $Z^* = 0$), but also shows how to construct a consistent U2TLPR from an inconsistent one. The next result further indicates that among the set of all consistent U2TLPRs, the constructed $\overline{L}$ has the smallest deviation to the original L.

Theorem 2. *Let L be a U2TLPR and $\overline{L}$ be the corresponding U2TLPR constructed by (6) and (7), then*

$$d(L,\overline{L}) = \min_{P\in P_n} d(L,P). \tag{8}$$

Proof. By Theorem 1, we know that $\overline{L}$ is a consistent U2TLPR, i.e., $\overline{L} \in P_n$. Now proceed by contradiction and assume that $d(L,\overline{L}) > \min_{p\in P_n} d(L,P)$. This implies the existence of another consistent U2TLPR $P^0 = ([p_{ik}^{0-}, p_{ik}^{0+}])_{n\times n} \in P_n$ such that $d(L,P^0) < d(L,\overline{L})$. Thus, it follows from Definition 5 that

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| + |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(p_{ik}^{0+})|) < \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(\overline{l}_{ik}^-)| + |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(\overline{l}_{ik}^+)|).$$

Since $|\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(\overline{l}_{ik}^-)| = d_{ik}^{-,*}$ and $|\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(\overline{l}_{ik}^+)| = d_{ik}^{+,*}$, we have

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| + |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(p_{ik}^{0+})|) < \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d_{ik}^{-,*} + d_{ik}^{+,*}) = Z^*. \tag{9}$$

In addition, because $P^0 = ([p_{ik}^{0-}, p_{ik}^{0+}])_{n\times n}$ is consistent, there exists $W^0 = (w_1^0, w_2^0, \ldots, w_n^0)^T$ such that

$$0 \leqslant \Delta^{-1}(p_{ik}^{0-}) \leqslant 0.5g + \beta(w_i^0 - w_k^0) \leqslant \Delta^{-1}(p_{ik}^{0+}) \leqslant g, \tag{10}$$

where $i<k$, $\sum_{i=1}^n w_i^0 = 1, w_i^0 \geqslant 0 (i = 1,2,\ldots,n)$ and $\beta > 0$.

We can construct a feasible solution to model $(M-1)$ based on the weight vector W^0. Let

$$IK_{\geqslant}^- = \{(i,k) | \Delta^{-1}(l_{ik}^-) - |\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| \geqslant 0, i<k\},$$
$$IK_{<}^- = \{(i,k) | \Delta^{-1}(l_{ik}^-) - |\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| < 0, i<k\},$$

$$IK_{\leqslant}^+ = \{(i,k) | \Delta^{-1}(l_{ik}^+) + |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(p_{ik}^{0+})| \leqslant g, i<k\},$$
$$IK_{>}^+ = \{(i,k) | \Delta^{-1}(l_{ik}^+) + |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(p_{ik}^{0+})| > g, i<k\}.$$

Let $d_{ik}^{-,0}, d_{ik}^{+,0} (i<k)$ be chosen as follows:

$$d_{ik}^{-,0} = \begin{cases} |\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| & (i,k) \in IK_{\geqslant}^- \\ d_{ik}^{-,*} & (i,k) \in IK_{<}^-; \end{cases} \tag{11}$$

$$d_{ik}^{+,0} = \begin{cases} |\Delta^{-1}(l_{ik}^+) - \Delta^{-1}(p_{ik}^{0+})| & (i,k) \in IK_{\leqslant}^+ \\ d_{ik}^{+,*} & (i,k) \in IK_{>}^+, \end{cases} \tag{12}$$

where $d_{ik}^{-,*}, d_{ik}^{+,*} (i<k)$ solve the minimization problem $(M-1)$ for L.

For an index pair $(i,k) \in IK_{\geqslant}^-$, it is not difficult to see that $\Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0} = \Delta^{-1}(l_{ik}^-) - |\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| \leqslant \Delta^{-1}(p_{ik}^{0-})$. On the other hand, when $(i,k) \in IK_{<}^-$, because $\Delta^{-1}(p_{ik}^{0-}) \geqslant 0$ and the fact that $\Delta^{-1}(l_{ik}^-) - |\Delta^{-1}(l_{ik}^-) - \Delta^{-1}(p_{ik}^{0-})| < 0$, it is readily seen that $\Delta^{-1}(l_{ik}^-) \leqslant \Delta^{-1}(p_{ik}^{0-})$. This further implies $\Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0} = \Delta^{-1}(l_{ik}^-) - d_{ik}^{-,*} \leqslant \Delta^{-1}(l_{ik}^-) \leqslant \Delta^{-1}(p_{ik}^{0-})$. Therefore, we obtain

$$\Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0} \leqslant \Delta^{-1}(p_{ik}^{0-}), \quad i<k. \tag{13}$$

On the other hand, according to the definition of $IK_{\geqslant}^-$ and the optimality of $d_{ik}^{-,*}$, we know that

$$\Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0} \geqslant 0, \quad i<k. \tag{14}$$

Similarly, it can be derived that

$$\Delta^{-1}(l_{ik}^+) + d_{ik}^{+,0} \geqslant \Delta^{-1}(p_{ik}^{0+}), \quad i<k, \tag{15}$$

$$\Delta^{-1}(l_{ik}^+) + d_{ik}^{+,0} \leqslant g, \quad i<k. \tag{16}$$

By Eq. (10) and Eqs. (11)–(16), we have

$$\begin{cases} 0.5g + \beta(w_i^0 - w_k^0) \geqslant \Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0},\ i<k \\ 0.5g + \beta(w_i^0 - w_k^0) \leqslant \Delta^{-1}(l_{ik}^+) + d_{ik}^{+,0},\ i<k \\ \Delta^{-1}(l_{ik}^-) - d_{ik}^{-,0} \geqslant 0 \text{ and } \Delta^{-1}(l_{ik}^+) + d_{ik}^{+,0} \leqslant g,\ i<k \\ w_1^0 + w_2^0 + \cdots + w_n^0 = 1 \\ w_i^0 \geqslant 0, i = 1,2,\ldots,n; d_{ik}^{-,0}, d_{ik}^{+,0} \geqslant 0,\ i<k, \end{cases}$$

which implies that $d_{ik}^{-,0}, d_{ik}^{+,0} (i<k)$, together with $W^0 = (w_1^0, w_2^0, \ldots, w_n^0)^T$, constitute a feasible solution to model $(M-1)$.

For $(i,k) \in IK_{<}^{-}$, we know that $d_{ik}^{-,*} \leqslant \Delta^{-1}(l_{ik}^{-}) < |\Delta^{-1}(l_{ik}^{-})-\Delta^{-1}(p_{ik}^{0-})|$, and similarly $d_{ik}^{+,*} \leqslant g-\Delta^{-1}(l_{ik}^{+}) < |\Delta^{-1}(l_{ik}^{+})-\Delta^{-1}(p_{ik}^{0+})|$ for $(i,k) \in IK_{>}^{+}$. Therefore, the corresponding objective function value of the feasible solution $d_{ik}^{-,0},d_{ik}^{+,0}(i<k)$ can be written as

$$Z^0 = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d_{ik}^{-,0}+d_{ik}^{+,0}) = \sum_{(i,k)\in IK_{\geqslant}^{-}}|\Delta^{-1}(l_{ik}^{-})-\Delta^{-1}(p_{ik}^{0-})| + \sum_{(i,k)\in IK_{<}^{-}} d_{ik}^{-,*} + \sum_{(i,k)\in IK_{\leqslant}^{+}}|\Delta^{-1}(l_{ik}^{+})-\Delta^{-1}(p_{ik}^{0+})| + \sum_{(i,k)\in IK_{>}^{+}} d_{ik}^{+,*} < \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(l_{ik}^{-})-\Delta^{-1}(p_{ik}^{0-})| + |\Delta^{-1}(l_{ik}^{+})-\Delta^{-1}(p_{ik}^{0+})|). \quad (17)$$

Finally, by Eqs. (9) and (17), we obtain

$$Z^0 < \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(|\Delta^{-1}(l_{ik}^{-})-\Delta^{-1}(p_{ik}^{0-})| + |\Delta^{-1}(l_{ik}^{+})-\Delta^{-1}(p_{ik}^{0+})|) < \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d_{ik}^{-,*}+d_{ik}^{+,*}) = Z^*.$$

This contradicts with the fact that Z^* is the optimal value of model $(M-1)$. Hence, we conclude that $d(L,\overline{L}) = \min_{P\in P_n} d(L,P)$. This completes the proof of the theorem. □

As a direct consequence of Theorem 2, the consistency index of a U2TLPR L defined in Definition 7 can be calculated via the following equation:

$$CI(L) = d(L,\overline{L}) = \frac{Z^*}{gn(n-1)},$$

where $\overline{L}$ is the U2TLPR constructed based on (6) and (7), which has the smallest deviation from L.

3.2. An iterative consistency improving algorithm for U2TLPRs

In practice, it is obviously desirable to have an $\overline{L}$ that is additively consistent; however, note that when $CI(L)$ is large, this may come at the expense of losing the original preference information contained in L, resulting in non-informative or even misleading conclusions during decision making. Consequently, in this section we try to strike a balance between consistency improvement and preference preservation. Similar to Saaty (1980), the idea is not to insist on the absolute consistency of a U2TLPR, but rather to use a pre-specify threshold $\overline{CI}$ and consider a U2TLPR to be of an acceptable consistency level whenever its consistency index falls below $\overline{CI}$. This induces a tradeoff in choosing between small values of $\overline{CI}$ to enhance the consistency of a U2TLPR and large values of $\overline{CI}$ to prevent it from deviating too much from the original U2TLPR. An appropriate choice of the threshold can be determined based on the prior knowledge of the problem at hand. Once a given threshold $\overline{CI}$ is specified, we propose the following algorithm for obtaining a U2TLPR with the desired consistency level:

Algorithm 1. Input: A U2TLPR $L = ([l_{ik}^{-},l_{ik}^{+}])_{n\times n}$, a consistency index threshold $\overline{CI}$, an adjustment parameter $\lambda \in (0,1)$.

Output: the adjusted U2TLPR $\widetilde{L} = ([\widetilde{l}_{ik}^{-},\widetilde{l}_{ik}^{+}])_{n\times n}$ satisfying $CI(\widetilde{L}) \leqslant \overline{CI}$.

Step 1: Let $h = 0$ and $L_0 = ([l_{ik,0}^{-},l_{ik,0}^{+}])_{n\times n} = ([l_{ik}^{-},l_{ik}^{+}])_{n\times n}$.

Step 2: Solve the optimization model $(M-1)$ for U2TLPR L_h. Let $d_{ik,h}^{-,*},d_{ik,h}^{+,*}(i<k)$ be the optimal solution and Z_h^* be the corresponding optimal value. If $Z_h^* = 0$, let $CI(L_h) = 0$ and go to step 5; Otherwise calculate $CI(L_h)$ as follows:

$$CI(L_h) = \frac{Z_h^*}{gn(n-1)}. \quad (18)$$

Step 3: If $CI(L_h) \leqslant \overline{CI}$, go to step 5; Otherwise, construct a consistent U2TLPR $\overline{L}_h = [\overline{l}_{ik,h}^{-},\overline{l}_{ik,h}^{+}])_{n\times n}$, where for all $i<k$,

$$\overline{l}_{ik,h}^{-} = \Delta(\Delta^{-1}(l_{ik,h}^{-})-d_{ik,h}^{-,*}), \overline{l}_{ik,h}^{+} = \Delta(\Delta^{-1}(l_{ik,h}^{+})+d_{ik,h}^{+,*}), \quad (19)$$

$$\overline{l}_{ki,h}^{-} = \Delta(g-\Delta^{-1}(\overline{l}_{ik,h}^{+})), \overline{l}_{ki,h}^{+} = \Delta(g-\Delta^{-1}(\overline{l}_{ik,h}^{-})). \quad (20)$$

Step 4: Adjust L_h as follows:

$$l_{ik,h+1}^{-} = \Delta((1-\lambda)\Delta^{-1}(l_{ik,h}^{-}) + \lambda\Delta^{-1}(\overline{l}_{ik,h}^{-})), \quad (21)$$

$$l_{ik,h+1}^{+} = \Delta((1-\lambda)\Delta^{-1}(l_{ik,h}^{+}) + \lambda\Delta^{-1}(\overline{l}_{ik,h}^{+})). \quad (22)$$

Set $h = h+1$ and go to step 2.

Step 5: Let $\widetilde{L} = L_h, CI(\widetilde{L}) = CI(L_h)$. Return the adjusted preference relation $\widetilde{L}$ and $CI(\widetilde{L})$.

At each iteration of Algorithm 1, a new U2TLPR L_{h+1} is obtained in step 4 as the weighted sum of the current L_h and a consistent U2TLPR $\overline{L}_h$ constructed based on L_h. Intuitively, the use of the adjustment parameter λ forces L_{h+1} to stay close to the consistent U2TLPR $\overline{L}_h$ while, on the other hand, ensures that its difference from the current L_h is only incremental. Notice that an excessively large value of λ may render the algorithm to terminate prematurely, resulting in an "overly consistent" U2TLPR (i.e., with a consistency index value that is much smaller than the prescribed threshold $\overline{CI}$) that deviates significantly from the initial U2TLPR. Thus, in order to prevent too much initial preference information from being lost during the construction of U2TLPRs, the use of large values of λ in the algorithm is not recommended.

3.3. Properties of Algorithm 1

In this subsection, we discuss the properties of the proposed consistency improving algorithm.

Theorem 3. *Let $L = ([l_{ik}^{-},l_{ik}^{+}])_{n\times n}$ be a given U2TLPR. At each iteration h, let $d_{ik,h}^{-,*},d_{ik,h}^{+,*}(i<k)$ be the components of the optimal solution to the model $(M-1)$ and $\{\overline{L}_h\}$ be the consistent U2TLPR constructed at Step 3 of Algorithm 1. Then $(1-\lambda)d_{ik,h}^{-,*}$ and $(1-\lambda)d_{ik,h}^{+,*}, i<k$ are the components of the optimal solution to the model $(M-1)$ in the $(h+1)$th iteration. Moreover, we have*

$$\overline{L}_{h+1} = \overline{L}_h = \overline{L} = ([\overline{l}_{ik}^{-},\overline{l}_{ik}^{+}])_{n\times n} \quad \text{for all } h = 0,1,2,..., \quad (23)$$

where

$$\overline{l}_{ik}^{-} = \Delta(\Delta^{-1}(l_{ik}^{-})-d_{ik,0}^{-,*}), \overline{l}_{ik}^{+} = \Delta(\Delta^{-1}(l_{ik}^{+})+d_{ik,0}^{+,*}), \quad i<k.$$

Proof. Since $d_{ik,h}^{-,*}$, and $d_{ik,h}^{+,*}$ are the components of the optimal solution to optimization model $(M-1)$ obtained at iteration h, there exists a vector $(w_1^*,w_2^*,...,w_n^*)^T$ satisfying

$$\begin{cases} 0.5g + \beta(w_i^*-w_k^*) \geqslant \Delta^{-1}(l_{ik,h}^{-})-d_{ik,h}^{-,*}, \ i<k \\ 0.5g + \beta(w_i^*-w_k^*) \leqslant \Delta^{-1}(l_{ik,h}^{+})+d_{ik,h}^{+,*}, \ i<k \\ \Delta^{-1}(l_{ik}^{-})-d_{ik,h}^{-,*} \geqslant 0 \text{ and } \Delta^{-1}(l_{ik}^{+})+d_{ik,h}^{+,*} \leqslant g, \ i<k \\ w_1^*+w_2^*+\cdots+w_n^* = 1 \\ w_i^* \geqslant 0, \ i = 1,2,...,n; \ d_{ik,h}^{-,*},d_{ik,h}^{+,*} \geqslant 0, \ i<k. \end{cases} \quad (24)$$

On the other hand, we have from Eqs. (19) and (20) that

$$\begin{aligned}\Delta^{-1}(l_{ik,h+1}^{-}) &= (1-\lambda)\Delta^{-1}(l_{ik,h}^{-}) + \lambda\Delta^{-1}(\overline{l}_{ik,h}^{-}) \\ &= (1-\lambda)\Delta^{-1}(l_{ik,h}^{-}) + \lambda(\Delta^{-1}(l_{ik,h}^{-})-d_{ik,h}^{-,*}) \\ &= \Delta^{-1}(l_{ik,h}^{-})-\lambda d_{ik,h}^{-,*}.\end{aligned} \quad (25)$$

Similarly,

$$\Delta^{-1}(l_{ik,h+1}^{+}) = \Delta^{-1}(l_{ik,h}^{+}) + \lambda d_{ik,h}^{+,*}. \quad (26)$$

The following relation can then be obtained from (24)–(26):

$$\begin{cases} 0.5g + \beta(w_i^*-w_k^*) \geqslant \Delta^{-1}(l_{ik,h+1}^{-})-(1-\lambda)d_{ik,h}^{-,*}, \ i<k \\ 0.5g + \beta(w_i^*-w_k^*) \leqslant \Delta^{-1}(l_{ik,h}^{+})+(1-\lambda)d_{ik,h}^{+,*}, \ i<k. \end{cases} \quad (27)$$

Next define

$$d_{ik,h+1}^{-,*} = (1-\lambda)d_{ik,h}^{-,*}, \ d_{ik,h+1}^{+,*} = (1-\lambda)d_{ik,h}^{+,*}, \ (i<k). \quad (28)$$

We now show that $d^{-,*}_{ik,h+1}$, $d^{+,*}_{ik,h+1} \geqslant 0$ $(i<k)$, together with the weights vector $(w_1^*,w_2^*,...,w_n^*)^T$, constitutes an optimal solution to model (M-1) at iteration $h+1$.

Note that by Eqs. (24) and (27), $d^{-,*}_{ik,h+1}$, $d^{+,*}_{ik,h+1} \geqslant 0$ $(i<k)$, and the vector $(w_1^*,w_2^*,...,w_n^*)^T$ form a feasible solution to model $(M{-}1)$ at iteration $h+1$ with the corresponding objective function value

$$Z^*_{h+1} = \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^{-,*}_{ik,h+1}+d^{+,*}_{ik,h+1}) = (1-\lambda)\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^{-,*}_{ik,h}+d^{+,*}_{ik,h}) = (1-\lambda)Z^*_h. \quad (29)$$

Let $d^{-}_{ik,h+1}$, $d^{+}_{ik,h+1}$ $(i<k)$, and $(w_1,w_2,...,w_n)^T$ be the components of another arbitrary feasible solution to $(M{-}1)$ at iteration $h+1$ satisfying the following constraints:

$$\begin{cases} 0.5g+\beta(w_i-w_k) \geqslant \Delta^{-1}(l^-_{ik,h+1})-d^-_{ik,h+1},\ i<k \\ 0.5g+\beta(w_i-w_k) \leqslant \Delta^{-1}(l^+_{ik,h+1})+d^+_{ik,h+1},\ i<k \\ \Delta^{-1}(l^-_{ik,h+1})-d^-_{ik,h+1} \geqslant 0, \Delta^{-1}(l^+_{ik,h+1})+d^+_{ik,h+1} \leqslant g,\ i<k \\ w_1+w_2+\cdots+w_n = 1, \\ w_i \geqslant 0,\ i=1,2,...,n;\ d^-_{ik,h+1},d^+_{ik,h+1} \geqslant 0,\ i<k. \end{cases} \quad (30)$$

From (25) and (26), (30) can be equivalently written as

$$\begin{cases} 0.5g+\beta(w_i-w_k) \geqslant \Delta^{-1}(l^-_{ik,h})-\lambda d^{-,*}_{ik,h}-d^-_{ik,h+1}, \\ 0.5g+\beta(w_i-w_k) \leqslant \Delta^{-1}(l^+_{ik,h})+\lambda d^{+,*}_{ik,h}+d^+_{ik,h+1}, \\ \Delta^{-1}(l^-_{ik,h})-\lambda d^{-,*}_{ik,h}-d^-_{ik,h+1} \geqslant 0, \\ \Delta^{-1}(l^+_{ik,h})+\lambda d^{+,*}_{ik,h}+d^+_{ik,h+1} \leqslant g, \\ w_1+w_2+\cdots+w_n = 1, \\ w_i \geqslant 0,\ i=1,2,...,n;\ d^-_{ik,h},d^-_{ik,h+1},d^+_{ik,h},d^+_{ik,h+1} \geqslant 0,\ i<k, \end{cases}$$

which implies that $d^{-,0}_{ik,h} = \lambda d^{-,*}_{ik,h}+d^-_{ik,h+1}$, $d^{+,0}_{ik,h} = \lambda d^{+,*}_{ik,h}+d^+_{ik,h+1}$ $(i<k)$, and $(w_1,w_2,...,w_n)^T$ also form a feasible solution to $(M{-}1)$ at iteration h. In addition, since $d^{-,*}_{ik,h}$ and $d^{+,*}_{ik,h}(i<k)$ are both optimal at iteration h, it follows that

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}[(\lambda d^{-,*}_{ik,h}+d^-_{ik,h+1})+(\lambda d^{+,*}_{ik,h}+d^+_{ik,h+1})] \geqslant \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^{-,*}_{ik,h}+d^{+,*}_{ik,h}). \quad (31)$$

Rearranging the terms in (31) yields

$$\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^-_{ik,h+1}+d^+_{ik,h+1}) \geqslant (1-\lambda)\sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^{-,*}_{ik,h}+d^{+,*}_{ik,h}) = (1-\lambda)Z^*_h = Z^*_{h+1},$$

where the last equality follows from (29). This proves the optimality of $d^{-,*}_{ik,h+1}$, $d^{+,*}_{ik,h+1} \geqslant 0$ $(i<k)$, and $(w_1^*,w_2^*,...,w_n^*)^T$ at iteration $h+1$.

To show (23), note that we have from Eqs. (19) and (21) that

$$\begin{aligned}\bar{l}^-_{ik,h+1} &= \Delta(\Delta^{-1}(l^-_{ik,h+1})-d^{-,*}_{ik,h+1}) \\ &= \Delta(\Delta^{-1}(\Delta((1-\lambda)\Delta^{-1}(l^-_{ik,h})+\lambda\Delta^{-1}(\bar{l}^-_{ik,h})))-d^{-,*}_{ik,h+1}) \\ &= \Delta((1-\lambda)\Delta^{-1}(l^-_{ik,h})+\lambda\Delta^{-1}(\bar{l}^-_{ik,h})-d^{-,*}_{ik,h+1}).\end{aligned}$$

By Eqs. (19) and (28), we can further obtain that

$$\begin{aligned}\bar{l}^-_{ik,h+1} &= \Delta((1-\lambda)\Delta^{-1}(l^-_{ik,h})+\lambda\Delta^{-1}(\Delta(\Delta^{-}(l^-_{ik,h})-d^{-,*}_{ik,h}))-(1-\lambda)d^{-,*}_{ik,h}) \\ &= \Delta((1-\lambda)\Delta^{-1}(l^-_{ik,h})+\lambda(\Delta^{-1}(l^-_{ik,h})-d^{-,*}_{ik,h})-(1-\lambda)d^{-,*}_{ik,h}) \\ &= \Delta(\Delta^{-1}(l^-_{ik,h})-d^{-,*}_{ik,h}) \\ &= \bar{l}^-_{ik,h}.\end{aligned}$$

By using a similar argument, it can be seen that $\bar{l}^+_{ik,h+1} = \bar{l}^+_{ik,h}$. Consequently, we conclude that $\bar{L}_{h+1} = \bar{L}_h$.

Finally, by applying the definition of $\bar{L}_0$ in Algorithm 1, we have

$$\bar{l}^-_{ik,h+1} = \bar{l}^-_{ik,h} = \bar{l}^-_{ik,0} = \Delta(\Delta^{-1}(l^-_{ik})-d^{-,*}_{ik,0}),\ i<k,$$
$$\bar{l}^+_{ik,h+1} = \bar{l}^+_{ik,h} = \bar{l}^+_{ik,0} = \Delta(\Delta^{-1}(l^+_{ik})+d^{+,*}_{ik,0}),\ i<k,$$

which completes the proof of the second claim. □

Theorem 3 shows that the constructed consistent U2TLPR remains unchanged in successive iterations of Algorithm 1. Thus, the recursions in Eq. (28) imply that the optimal solution to model $(M{-}1)$ at iteration h can be directly obtained from the initial $d^{-,*}_{ik,0}$ and $d^{+,*}_{ik,0}$ as

$$d^{-,*}_{ik,h} = (1-\lambda)^h d^{-,*}_{ik,0},\ d^{+,*}_{ik,h} = (1-\lambda)^h d^{+,*}_{ik,0}(i<k). \quad (32)$$

Moreover, the improved U2TLPR $L_{h+1} = ([l^-_{ik,h+1},l^+_{ik,h+1}])$ at iteration h can also be determined from (25) and (26) as follows:

$$l^-_{ik,h+1} = \Delta(\Delta^{-1}(l^-_{ik})-[1-(1-\lambda)^{h+1}]d^{-,*}_{ik,0}),\ i<k$$
$$l^+_{ik,h+1} = \Delta(\Delta^{-1}(l^+_{ik})+[1-(1-\lambda)^{h+1}]d^{+,*}_{ik,0}),\ i<k. \quad (33)$$

Theorem 4. *At each iteration h of Algorithm 1, let $Z^*_h(h=0,1,...)$ be the optimal objective function value of model $(M{-}1)$. Then the following results hold:*

(i) $Z^*_h = (1-\lambda)^h Z^*_0$;

(ii) *the consistency index of the modified U2TLPR obtained at iteration h is*

$$CI(L_h) = \frac{(1-\lambda)^h}{gn(n-1)}Z^*_0; \quad (34)$$

(iii) *the number of algorithm iterations required to obtain a U2TLPR within a desired consistency level $\overline{CI}$ is given by*

$$h_{min} = \max\left(1,\left\lceil \log_{1-\lambda}\frac{\overline{CI}g(n-1)n}{Z^*_0}\right\rceil\right), \quad (35)$$

where $\lceil X \rceil$ is the smallest integer that is no less than X.

Proof. The first claim (i) follows directly from Eq. (32), in particular,

$$\begin{aligned}Z^*_h &= \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}(d^{-,*}_{ik,h}+d^{+,*}_{ik,h}) \\ &= \sum_{i=1}^{n-1}\sum_{k=i+1}^{n}[(1-\lambda)^h d^{-,*}_{ik,0}+(1-\lambda)^h d^{+,*}_{ik,0}] \\ &= (1-\lambda)^h Z^*_0.\end{aligned}$$

(ii) Using Eq. (18) and the result from part (i), we immediately arrive at

$$CI(L_h) = d(L_h,\bar{L}_h) = \frac{1}{gn(n-1)}Z^*_h = \frac{(1-\lambda)^h}{gn(n-1)}Z^*_0.$$

(iii) Applying part (ii) of the theorem, we have $CI(L_h) \leqslant \overline{CI}$ whenever $\frac{(1-\lambda)^h}{gn(n-1)}Z^*_0 \leqslant \overline{CI}$. Solving the latter inequality for h, we get $h \geqslant \log_{1-\lambda}\frac{\overline{CI}g(n-1)n}{Z^*_0}$. Since h is a positive integer, the minimum number of iterations required is thus given by $h_{min} = \max\left(1,\left\lceil \log_{1-\lambda}\frac{\overline{CI}g(n-1)n}{Z^*_0}\right\rceil\right)$. □

3.4. An improved non-iterative version of Algorithm 1

Theorem 4 shows that at any iteration $h = 1,2,...$ of Algorithm 1, the optimal solution to model $(M{-}1)$, the constructed consistent U2TLPR, and the improved U2TLPR along with its associated consistency index can all be found directly from the values of $\overline{CI},L,\lambda$ and the initial Z^*_0. Thus, repeatedly solving the optimization model $(M{-}1)$ at each iteration of Algorithm 1 is not needed and could in fact lead to unnecessary waste of computational effort. This naturally suggests the following non-iterative version of Algorithm 1, where the optimization model

$(M-1)$ is only solved once at the beginning:

Algorithm 2. Input: A U2TLPR $L=([l_{ik}^-,l_{ik}^+])_{n\times n}$, a consistency index threshold $\overline{CI}$, an adjustment parameter $\lambda\in(0,1)$.

Output: the adjusted U2TLPR $\widetilde{L}=([\widetilde{l}_{ik}^-,\widetilde{l}_{ik}^+])_{n\times n}$ satisfying $CI(\widetilde{L})\leqslant\overline{CI}$.

Step 1: Solve the optimization model $(M-1)$ for the given U2TLPR L. Let Z_0^* be the optimal objective function value to model $(M-1)$. If $Z_0^*=0$, set $\widetilde{L}=L$ and $CI(\widetilde{L})=0$, go to step 4; Otherwise, go to step 2.

Step 2: Calculate the required number of iterations h_{min} from Eq. (35).

Step 3: Set $h=h_{min}$ and calculate the improved U2TLPR $\widetilde{L}$ using Eq. (33). Calculate the consistency index of U2TLPR $\widetilde{L}$ using Eq. (34).

Step 4: Return the adjusted U2TLPR $\widetilde{L}$ and its consistency index $CI(\widetilde{L})$.

$$h_{min}=\max\left(1,\left\lceil\log_{0.7}\frac{0.03\times8\times(5-1)\times5}{9}\right\rceil\right)=2.$$

Thus, an adjusted U2TLPR L_2 with the desired consistency level can also be immediately obtained from (33). For example, the (1,2)th entry of L_2 can be computed as follows:

$$l_{12,2}^-=\Delta(\Delta^{-1}(l_{12}^-)-[1-(1-\lambda)^{h_{min}}]d_{12,0}^{-,*})=\Delta(1-[1-(1-0.3)^2]\times0)=\Delta(1);$$
$$=(s_1,0)$$
$$l_{12,2}^+=\Delta(\Delta^{-1}(l_{12}^+)+[1-(1-\lambda)^{h_{min}}]d_{12,0}^{+,*})=\Delta(2+[1-(1-0.3)^2]\times1.331).$$
$$=\Delta(2.679)=(s_3,-0.321)$$

The detailed steps for computing other preference values are similar and thus omitted. The output U2TLPR of Algorithms 2 is an adjusted U2TLPR $\widetilde{L}=L_2$ (shown in (37)).

$$\widetilde{L}=\begin{pmatrix}[(s_4,0),(s_4,0)] & [(s_1,0),(s_3,-0.321)] & [(s_5,-0.091),(s_7,0)] & [(s_4,0),(s_5,0)] & [(s_5,0),(s_6,0)]\\ [(s_5,0.321),(s_7,0)] & [(s_4,0),(s_4,0)] & [(s_3,0),(s_4,0.270)] & [(s_2,0),(s_4,0.361)] & [(s_6,-0.169),(s_7,0)]\\ [(s_1,0),(s_3,0.091)] & [(s_4,-0.270),(s_5,0)] & [(s_4,0),(s_4,0)] & [(s_6,-0.439),(s_7,0)] & [(s_4,0),(s_5,0.071)]\\ [(s_3,0),(s_4,0)] & [(s_4,-0.361),(s_6,0)] & [(s_1,0),(s_2,0.439)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_4,-0.490)]\\ [(s_2,0),(s_3,0)] & [(s_1,0),(s_2,0.169)] & [(s_3,-0.071),(s_4,0)] & [(s_4,0.490),(s_6,0)] & [(s_4,0),(s_4,0)]\end{pmatrix}\quad(37)$$

4. Illustrative example and comparative analysis

In this section, we begin with a simple example to illustrate the application of the proposed procedure, followed by a computational comparison analysis to highlight its benefits and advantages in preference preservation.

4.1. An illustrative example

Construction projects are often very complex and may be exposed to many risk sources due to the involvement of different contracting parties (e.g., owners, designers, contractors, and suppliers) and additional economic, social, and environmental concerns. Therefore, effective risk management in construction projects is critical in achieving project objectives. In particular, there is a need for developing a quantitative risk assessment process to manage all types of risks. The aim of this case study is to determine the relative importance of the following five risk factors involved in a construction project: quality (x_1), time (x_2), cost (x_3), safety (x_4), and environmental sustainability (x_5). During the process, the project manager is asked to evaluate the five factors through pairwise comparison by using the linguistic term set $S=\{s_0$: *extremelylow*;s_1: *verylow*;s_2: *low*;s_3: *slightly low*;s_4: *fair*;s_5: *slightly high*;s_6: *high*;s_7: *very high*;s_8: *extremely high*$\}$. Based on his judgment, a U2TLPR L shown in (36) is obtained. The proposed consistency improving model then is employed to solve this problem.

$$L=\begin{pmatrix}[(s_4,0),(s_4,0)] & [(s_1,0),(s_2,0)] & [(s_6,0),(s_7,0)] & [(s_4,0),(s_5,0)] & [(s_5,0),(s_6,0)]\\ [(s_6,0),(s_7,0)] & [(s_4,0),(s_4,0)] & [(s_3,0),(s_4,0)] & [(s_2,0),(s_3,0)] & [(s_6,0),(s_7,0)]\\ [(s_1,0),(s_2,0)] & [(s_4,0),(s_5,0)] & [(s_4,0),(s_4,0)] & [(s_6,0),(s_7,0)] & [(s_4,0),(s_5,0)]\\ [(s_3,0),(s_4,0)] & [(s_5,0),(s_6,0)] & [(s_1,0),(s_2,0)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_3,0)]\\ [(s_2,0),(s_3,0)] & [(s_1,0),(s_2,0)] & [(s_3,0),(s_4,0)] & [(s_5,0),(s_6,0)] & [(s_4,0),(s_4,0)]\end{pmatrix}\quad(36)$$

We apply Algorithm 2 to solve the problem. In this example, the following set of parameters is used: $\beta=(ng)/2=20$, $\lambda=0.3$, and $\overline{CI}=0.03$. Note that the optimization model $(M-1)$ will only need to be solved once at the beginning. We obtained the optimal positive deviations are $d_{12,0}^{+,*}=1.331$, $d_{13,0}^{-,*}=2.140$, $d_{23,0}^{+,*}=0.529$, $d_{24,0}^{+,*}=2.669$, $d_{25,0}^{-,*}=0.331$, $d_{34,0}^{-,*}=0.860$, $d_{35,0}^{+,*}=0.140$ and $d_{45,0}^{+,*}=1.000$. The corresponding optimal objective function value is $Z_0^*=9.000$, and the consistency index of L_0 is $CI(L_0)=0.056$. This allows us to directly compute the required number of algorithm iterations via (35):

The consistency index associated with $\widetilde{L}$ can be computed via (34) without resorting to the specific form of $\widetilde{L}$:

$$CI(\widetilde{L})=CI(L_2)=\frac{(1-0.3)^2}{8\times5\times(5-1)}=0.028.$$

As a part of the optimal solution to model $(M-1)$, the following weight vector is derived in this example: $W^*=(0.2119,0.2454,0.2189,0.1619,0.1619)$, which means the ranking of the risk factors is $x_2\succ x_3\succ x_1\succ x_4\sim x_5$.

To investigate the impact of the choice of the parameter β on the consistency results, a test is performed by running Algorithm 2 with different β values. Table 1 shows the weight vectors derived under different choices of β values. We observe from the table that when the value of β is very large (e.g., $\beta=10{,}000$), then all w_i's are close to $1/n(0.2)$. Moreover, the larger the value of β is, the smaller the differences among the components of the weight vector. Although the values of the derived weights are affected by the value of β, the ranking of the weights remains unchanged ($x_2\succ x_3\succ x_1\succ x_4\sim x_5$), except for those cases when β is either very small or very large. This shows that the ranking of the alternatives is not sensitive to the choice of β. In practice, we suggest setting the value of β either to $ng/2$ or $(n-1)g/2$, as recommended in Section 3.1.

4.2. Comparison analysis

To further illustrate the proposed consistency improving approach, we consider some computational experiments on an example taken from Zhang and Guo (2016) and compare the performance of our

Table 1
The derived weights corresponding to different β values.

β	w_1	w_2	w_3	w_4	w_5
4	0.2778	0.3536	0.2996	0.0278	0.0412
8	0.2308	0.2846	0.2730	0.1058	0.1058
16	0.2149	0.2567	0.2237	0.1524	0.1524
20	0.2119	0.2454	0.2189	0.1619	0.1619
24	0.2099	0.2379	0.2158	0.1682	0.1682
100	0.2024	0.2091	0.2038	0.1924	0.1924
1000	0.2000	0.2012	0.2004	0.1992	0.1992
10,000	0.2000	0.2001	0.2001	0.1999	0.1999

Table 2
Comparison of distances between the improved U2TLPRs and the initial U2TLPR corresponding to different $\overline{CI}$ values.

$\overline{CI}$	0.08	0.06	0.04	0.02	0.01	0.008	0.006	0.004
$d(\widetilde{L},L_{Zh})$	0.0000	0.0000	0.0000	0.0075	0.0164	0.0190	0.0208	0.0220
$d(\widetilde{L}_{Zh},L_{Zh})$	0.0880	0.0880	0.1078	0.1200	0.1310	0.1310	0.1340	0.1340

algorithms with that of the method proposed in Zhang and Guo (2016). Let $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$ be the set of six decision alternatives. The decision maker expressed the preference information using a U2TLPR which is given by L in Zhang and Guo (2016). Here, denote this original U2TLPR as L_{Zh}.

The algorithm implemented in Zhang and Guo (2016) uses the following parameters: $\overline{CI} = 0.08$ and $\lambda = 0.6$. A consistent U2TLPR, denoted by L^*_{Zh} (shown in (38)), is then constructed, based on which the algorithm terminates in two iterations, yielding an improved U2TLPR $\widetilde{L}_{Zh}$ (denoted as L_2 in Zhang & Guo (2016)) with the desired consistency level. Note that similar to the algorithm proposed in Zhang and Guo (2016), the main idea of our consistency improving approach is also based on iterating towards a consistent U2TLPR. However, our algorithm takes a different approach by solving the optimization problem (M−1), leading to a consistent U2TLPR L^* (shown in (39)) that differs significantly from L^*_{Zh}. One important issue to note here is the difference between $d(L^*,L_{Zh})$ and $d(L^*_{Zh},L_{Zh})$. Indeed, from the deviation measure (1), we see that $d(L^*,L_{Zh}) = 0.0250$, whereas $d(L^*_{Zh},L_{Zh}) = 0.1375$. This shows that, although both L^*_{Zh} and L^* are consistent U2TLPRs derived from the same initial L_{Zh}, the U2TLPR L^* obtained by our approach is closer to the initial L_{Zh}. As mentioned in Section 1, the construction of the consistent U2TLPR is conducted in Zhang and Guo (2016) by using only the $n-1$ preference values above the main diagonal of the initial U2TLPR L_{Zh}, and the rest $(n-1)(n-2)/2$ preference values in the upper triangular part of L_{Zh} are not considered. Thus, the resulting improved U2TLPR usually does not fully reflect the original preferences of the decision maker. In contrast, our approach implicitly takes into account all the preference values in the original U2TLPR by searching for a consistent U2TLPR L^* that provides the closest distance to L_{Zh}. Moreover, Theorem 2 further suggests that L^* minimizes the deviation between L_{Zh} and any consistent U2TLPR.

$$L^*_{Zh} = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_2,0),(s_4,0)] & [(s_2,0),(s_5,0)] & [(s_3,0),(s_7,0)] & [(s_3,0),(s_8,0)] \\ [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_3,0)] & [(s_2,0),(s_4,0)] & [(s_3,0),(s_6,0)] & [(s_3,0),(s_7,0)] \\ [(s_4,0),(s_6,0)] & [(s_5,0),(s_6,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_5,0),(s_7,0)] & [(s_5,0),(s_8,0)] \\ [(s_3,0),(s_6,0)] & [(s_4,0),(s_6,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_5,0),(s_6,0)] & [(s_5,0),(s_7,0)] \\ [(s_1,0),(s_5,0)] & [(s_2,0),(s_5,0)] & [(s_1,0),(s_3,0)] & [(s_2,0),(s_3,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] \\ [(s_0,0),(s_5,0)] & [(s_1,0),(s_5,0)] & [(s_0,0),(s_3,0)] & [(s_1,0),(s_3,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] \end{pmatrix} \quad (38)$$

$$L^* = \begin{pmatrix} [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_2,0),(s_4,0)] & [(s_4,0),(s_6,0)] & [(s_4,0),(s_6,0)] & [(s_2,0),(s_5,0)] \\ [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_2,0),(s_3,0)] & [(s_4,0),(s_8,0)] & [(s_2,0),(s_4,0)] & [(s_4,0),(s_5,0)] \\ [(s_4,0),(s_6,0)] & [(s_5,0),(s_6,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] & [(s_3,0),(s_5,0)] & [(s_6,0),(s_7,0)] \\ [(s_2,0),(s_4,0)] & [(s_0,0),(s_4,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_6,0)] & [(s_5,0),(s_6,0)] \\ [(s_2,0),(s_4,0)] & [(s_4,0),(s_6,0)] & [(s_3,0),(s_5,0)] & [(s_2,0),(s_4,0)] & [(s_4,0),(s_4,0)] & [(s_4,0),(s_5,0)] \\ [(s_3,0),(s_6,0)] & [(s_3,0),(s_4,0)] & [(s_1,0),(s_2,0)] & [(s_2,0),(s_3,0)] & [(s_3,0),(s_4,0)] & [(s_4,0),(s_4,0)] \end{pmatrix} \quad (39)$$

Since L^* minimizes the deviation between L_{Zh} and any consistent U2TLPR, our algorithm has better performance in preference preservation in comparison with the consistency improving algorithm proposed by Zhang and Guo (2016). To evaluate the performance of different algorithms in terms of preference preservation, we run both algorithms using different threshold values $\overline{CI}$. Table 2 shows the comparison results. As we can see from the table, with the same threshold $\overline{CI}$, the improved U2TLPR ($\widetilde{L}$) generated by our algorithm is always closer to the initial U2TLPR than that obtained by the algorithm of Zhang and Guo (2016). This empirically illustrates the advantage of the proposed algorithm in preference preservation. It is worth noting that when $\overline{CI} \geqslant 0.0250$, the U2TLPR produced by our algorithm coincides with the initial U2TLPR, which explains why $d(\widetilde{L},L_{Zh}) = 0$ for $\overline{CI} = 0.08,0.06$ and 0.04 in Table 2. This suggests that the parameter $\overline{CI}$ in the proposed algorithm should not be chosen too large. In practice, we recommend to set the value of $\overline{CI}$ between 0 and 0.06. With regard to the determination of the threshold $\overline{CI}$, we will conduct further research by using methods such as simulation analysis in future work.

The computational complexity is an important criterion to evaluate a consistency improving approach. In the algorithm of Zhang and Guo (2016), an improved U2TLPR is updated based on a consistent U2TLPR constructed at each iteration. So the computational complexity of their algorithm is proportional to the number of algorithm iterations. When the size of the problem is large and/or the parameters λ and $\overline{CI}$ are small, such an iterative procedure could be computationally expensive, as it may take many iterations to produce an acceptable U2TLPR. On the other hand, since our proposed algorithm (Algorithm 2) is non-iterative, its complexity is not susceptible to the choices of algorithm parameters, making the algorithm faster and more efficient in handling large problem instances.

5. Conclusions

Maintaining the satisfactory consistency of decision makers' preference relations is critical to ensure accurate and reliable conclusions in decision making. This paper presents a new optimization-based approach to address the consistency improving issue when preferences are expressed using U2TLPRs. In particular, we have introduced a new consistency definition for U2TLPRs and proposed an index to measure their degree of consistency. We have then presented an optimization model and shown that the solution to this model can not only be employed to determine whether a given U2TLPR is consistent but also be used to compute its consistency index by minimizing its deviation from the set of all consistent U2TLPRs. Since consistency improving will inevitably alter the initial preference values, maintaining the consistency of a U2TLPR often conflicts with the goal of preserving the original preference information of the decision maker. Thus, another major contribution of this paper is the development of an iterative algorithm that aims to balance between consistency improvement and preference preservation. The algorithm takes an unacceptable consistent U2TLPR as input and generates an adjusted U2TLPR with a desired consistency level while effectively retaining the preference information expressed by the decision maker. In addition, theoretical analysis of the algorithm shows that there are structural properties that can be further exploited to allow us to arrive at an equivalent yet much more efficient implementation of the algorithm. Computational results on two numerical examples conform well to our theoretical findings, indicating superior performance of the proposed algorithms over an existing method in preserving the preference information.

Although the proposed approach is advantageous in preserving the original preference information given by decision maker, it does not consider the additional preference information that the decision maker may provide in the consistency improving process. Therefore, how to make use of the additional preference information provided by the decision maker is an important research issue worthy of further investigation. Some other future research topics include: (1) the extension of the approach to the additive consistency of U2TLPRs based on the idea of approximation-consistency (Liu et al., 2018); (2) the development of a systematic approach to automatically determine appropriate consistency thresholds in the proposed procedure; and (3) the integration of the proposed consistency improving model and consensus reaching model (Cabrerizo et al., 2015; Xu, Cabrerizo, & Herrera-Viedma, 2017) in GDM.

Acknowledgements

The authors are very grateful to the editor and the anonymous reviewers for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work of Shengbao Yao was supported by the Humanities and Social Sciences Foundation of Ministry of Education of China (No. 14YJA630086), the Fundamental Research Funds for the Central Universities (No. 31541410811). The work of Jiaqiao Hu was supported by the U. S. National Science Foundation under Grant CMMI-1634627.

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