Journal of Intelligent & Fuzzy Systems 33 (2017) 3835–3852
DOI:10.3233/JIFS-17701
IOS Press

Combining comparative linguistic expressions and numerical information in multi-attribute group decision making—A simulation-based approach

Shengbao Yao[a,*] and Jiaqiao Hu[b]
[a]*School of Business Administration, Zhongnan University of Economics and Law, Wuhan, China*
[b]*Department of Applied Mathematics and Statistics, Stony Brook University, New York, USA*

Abstract. In multi-attribute group decision making (MAGDM) problems, the information about attribute weights and the performance ratings of alternatives usually cannot be accurately quantified. This issue has motivated the development of various MAGDM models based on the fuzzy sets theory. However, these fuzzy MAGDM models mostly rely on using the extreme or expected values, but ignore the intermediate occurrences in determining the best alternatives. In order to provide a complete understanding of decision makers' preference structure, this paper takes a stochastic perspective and proposes a simulation-based approach to facilitate MAGDM under uncertainty when both quantitative and qualitative attributes are involved. The approach not only accounts for the incomplete information about the attribute weights during decision making, but also allows for the use of comparative linguistic expressions to better capture the decision makers' hesitancy about linguistic expressions. We apply the proposed approach to electric vehicle charging station site selection problem and highlight its effectiveness and advantages through an in-depth comparative analysis with some of the existing methods.

Keywords: Multi-attribute group decision making, comparative linguistic expression, incomplete weights, random preference, Monte Carlo simulation

1. Introduction

Decision-making, which tries to find the best alternative(s) from a set of feasible alternatives, is widely applied in many fields for the purposes of evaluation, selection, and prioritization. The complexity of real world problems has necessitated the need to consider multiple points of view during the decision making process. This naturally gives rise to the so-called group decision making (GDM) involving two or more experts (each with his/her own perceptions, attitudes, and motivation, etc.) who recognize the existence of a common problem and attempt to reach a collective decision [1]. As an important category of GDM, multi-attribute GDM (MAGDM) deals with decision problems where several experts express their opinions on a set of possible alternatives with respect to multiple attributes and attempt to find a common solution.

In MADGM environment, the information about the attribute weights and the performance ratings of alternatives is often uncertain, primarily due to the following causes [2]: (i) unquantifiable information,

*Corresponding author. Shengbao Yao, School of Business Administration, Zhongnan University of Economics and Law, Wuhan 430073, China. Tel.: +86 2788386757; E-mail: ysb@znufe.edu.cn.

(ii) incomplete information, and (iii) non-obtainable information. Over the last few decades, a variety of models have been proposed to deal with MADGM under uncertainty. In most of these models, the fuzzy sets theory has been served as an important and useful mean for the decision making problems [3–7]. An alternative way to handle uncertainty in MAGDM is the stochastic approaches [8–13], where some or all of the input parameters of a given problem are expressed in the form of probability distribution. Under the framework of stochastic approaches, simulation can be applied to MADGM under uncertainty. As pointed out in [13], the simulation-based MAGDM methods can provide a more complete understanding of possible outcomes during the decision making process. However, despite the progress made on using simulation to model the uncertainty in MAGDM, there are still some important issues related to these models that remain to be resolved. First, the uncertainty issue such as hesitancy about linguistic expressions under qualitative setting has not been considered in existing simulation-based MAGDM models. Although subjective attributes are taken into account in [12], only single linguistic terms are used by decision makers to articulate their preferences. As pointed out by Rodriguez et al. [14], however, the experts may think of several terms or look for more complex linguistic terms that are not defined in the linguistic term sets. Second, incomplete information about attribute weights has not been incorporated into the models. In current studies, the attribute weight information is either assumed to be completely unknown [8] or completely provided by the decision makers [13]. In practice, however, it is very likely that this information is in-between the above two extremes [15, 16]. Hence, the uncertainty caused by incomplete information about attribute weights should also be considered in the decision making process.

Motivated by the above discussion, in this paper we propose a new simulation-based approach to address the inherent uncertainties in MAGDM problems. The proposed approach takes as inputs the decision matrices provided by individual decision maker and the incomplete information about attribute weights. For the evaluation of performance rating with respect to qualitative attributes, the approach allows the decision makers to use comparative linguistic expressions to facilitate and increase the flexibility in eliciting their linguistic judgements. Compared to existing simulation-based models, the main characteristics of the proposed approach that lead to its novelty are as follows: (1) it captures DMs' hesitancy about linguistic expressions by allowing the use of comparative linguistic expressions under a general simulation framework; (2) it samples realizations of attribute weights from a set of prespecified weight constraints and thus explicitly takes into account the incomplete attribute weight information during the decision making process; and (3) it is capable of solving MAGDM problems involving both comparative linguistic expressions and numerical information.

The remainder of this paper is organized as follows. Section 2 begins with a brief review of approaches in MAGDM under uncertainty. Section 3 describes the considered MAGDM problem and summaries the basic concepts that will be needed in the rest of the paper. In Section 4, we provide a detailed description of the proposed simulation based MAGDM approach. In Section 5, we demonstrate the performance of the proposed approach on an electric vehicle charging station site selection problem. A comparison study is also carried out to illustrate its effectiveness and benefits. Finally, we conclude the paper in Section 6.

2. Literature review

2.1. Fuzzy approach in MAGDM under uncertainty

Ever since its introduction by Zadeh [17], the fuzzy sets theory has been an powerful tool to deal with uncertainty in decision making. In particular, the theory has been integrated with many MAGDM techniques to address the inherent impreciseness and subjectiveness during decision making, leading to the emergence of fuzzy multi-attribute group decision-making (FMAGDM) methods [2, 6]. Generally speaking, the fuzzy approach is utilized when the input parameters of the MAGDM problem are subjective and vague by using linguistic terms and membership functions.

The past decades have witnessed many fruitful studies on MAGDM problems based on the fuzzy sets theory. For instance, Chen [3] presented an extension of the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method for MAGDM under a fuzzy environment. Kahraman et al. [4] proposed a fuzzy analytic hierarchy process (AHP) model for determining the weights of the main attributes in MAGDM. Wang and Lin [18] studied a method for FMAGDM to select configuration items in software development. Ölçer and Odabaşi [19] proposed a three-stage conceptual model for FMAGDM and presented a MAGDM technique under a fuzzy environment for propulsion and maneuvering system selection. Wu and Chen [20] presented a method for maximizing deviation for group multi-attributes decision-making in a linguistic environment. Pang et al. [21] focused on the consensus problem and proposed an adaptive method for MAGDM under uncertain linguistic environment. Chou et al. [22] provided a fuzzy simple additive weighting system (FSAWS) for solving facility location selection problems by using objective/subjective attributes under GDM conditions. Yeh and Chang [23] presented a fuzzy MAGDM approach for evaluating decision alternatives involving subjective judgements made by a group of decision makers. Li [24] proposed some different distance measures and developed a method for solving FMAGDM problems with non-homogeneous information. Liu et al. [25] presented a method based on ordered weighted harmonic averaging operators is presented to solve the multiple attribute group decision making problems in which the attribute values take the form of generalized interval-valued trapezoidal fuzzy numbers. Samvedi et al. [5] used both fuzzy TOPSIS and Fuzzy AHP techniques to quantify risks in a supply chain. It is also worth noting that during the past two decades, several generalizations of the classic fuzzy sets have been developed, including type-2 fuzzy sets [26], intuitionistic fuzzy sets (IFSs) [27, 28], and hesitant fuzzy sets [29], some of which have been applied to the field of FMAGDM [7, 30–36].

2.2. *Stochastic approach in MAGDM under uncertainty*

An alternative way to handle imprecision and uncertainty in MAGDM is through the use of stochastic approaches, where some or all of the input parameters of a given problem are expressed in the form of probability distribution. Lahdelma and Salminen [8] suggested a method called Stochastic Multicriteria Acceptability Analysis (SMAA), which considers the case where both the weight and attribute values are inaccurate. The method relies on exploring the weight space in order to describe the valuations that would make each alternative the preferred one. Inaccurate or uncertain criteria values are represented by probability distributions, based on which confidence factors describing the reliability of the analysis can be computed. Prato [9] presented a stochastic multiple attribute evaluation method for selecting land use policies, in which the stochastic attributes of outcomes are characterized by probability distributions. To rank land use policies for stakeholder, different preferences are considered based on interdependent attributes of the policy outcomes. Lafleur [10] proposed to use the triangular distribution to describe the weight impreciseness of attributes in the pairwise comparison matrix of AHP and employed Monte Carlo simulation to determine the preference probabilities of alternatives. Liu et al. [11] studied an extended TOPSIS method for MAGDM problems based on probability theory and uncertain linguistic variables. Mousavi et al. [12] proposed a fuzzy-stochastic MAGDM approach by aggregating group preferences into triangular fuzzy numbers. They used Monte Carlo simulation to obtain probability distributions representing the performance of alternatives with respect to attributes. Then, a ranking technique (VIKOR) for final prioritization of alternatives is used. Recently, Hüsamettin [13] presented a MAGDM technique based on simulation and the TOPSIS method. In this model, individual preferences about attribute weights and attribute values are aggregated into triangular distributions. In addition, the use of simulation also enables decision makers to incorporate some decision constraints into the decision-making process.

It is clear that both the fuzzy approach and the stochastic approach are viable ways for modeling imprecision and uncertainty in MAGDM. Regarding the comparison between these two general approaches, Buckly [37] highlighted that the stochastic approach considers all ways to conduct a task, whereas the fuzzy approach provides the most optimistic way to accomplish the task. More specifically, fuzzy MAGDM techniques usually consider either the worst, best, or the most probable values, which means that the intermediate occurrences are often overlooked in determining the best alternatives [13]. Just as Marinoni pointed out [38], assigning extreme realizations and observing the range of outcomes is not necessarily a solution as these extreme realizations are normally rare events with low probabilities of occurrence. In addition, the frequently used techniques for defuzing fuzzy numbers in FMAGDM may cause additional information loss and thus limits the ability of fuzzy techniques to deal with imprecision and uncertainty [39]. On the other hand, simulation based multi-attribute group decision making techniques can provide a more complete understanding of possible outcomes during the decision making process.

3. Preliminaries

3.1. Problem description

We consider the following MAGDM problem under uncertainty: Let $A = \{A_1, A_2, \cdots, A_m\}$ be a finite set of alternatives, $DM = \{DM_1, DM_2, \cdots, DM_D\}$ be a group of decision-makers (DMs), and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_D)$ be the weight vector of DMs, where both m and D are positive integers. We focus on problems with both qualitative and quantitative attributes (i.e., dealing with linguistic and numerical information) and assume that each alternative has n ($n \geq 2$) attributes, with n_1 ($0 < n_1 < n$) being the number of qualitative attributes and $n - n_1$ being the number of quantitative attributes. Without loss of generality, we denote by $C = \{C_1, \cdots, C_{n_1}, C_{n_1+1}, \cdots, C_n\}$ the finite set of attributes. Let $X^d = (x_{ij}^d)_{m\times n}$ ($d = 1, 2, \ldots, D$) be the decision matrix provided by decision maker $DM_d \in DM$, where x_{ij}^d represents the performance rating of alternative $A_i \in A$ with respect to attribute $C_j \in C$. Specifically, x_{ij}^d, $j \in \{1, 2, \cdots, n_1\}$ are comparative linguistic expressions ([40]) and x_{ij}^d, $j \in \{n_1 + 1, \cdots, n\}$ are crisp numbers.

Let $w = (w_1, w_2, \cdots, w_n)$ be the weight vector of attributes, where $w_j \geq 0$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$. The weight vector w is assumed to be incompletely/partially known in our model, and the set of the known weight information, denoted by H, can be expressed through constraints of one of the following forms [15, 16]: for $i \neq j$, **Form 1.** A weak ranking: $w_i \geq w_j$; **Form 2.** A strict ranking: $w_i - w_j \geq \alpha_i, \alpha_i > 0$; **Form 3.** A ranking of differences: $w_i - w_j \geq w_k - w_l$, for $j \neq k \neq l$; **Form 4.** A ranking with multiples: $w_i \geq \alpha_i w_j, 0 \leq \alpha_i \leq 1$; **Form 5.** A interval form: $\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i, 0 \leq \alpha_i < \alpha_i + \varepsilon_i \leq 1$. Park [41] provided a detailed interpretation as to when incomplete weight information could occur in practice. We remark that in the proposed model completely unknown information about attribute weights can be regarded as a special case of incomplete information by setting $H = \{w = (w_1, \ldots, w_n) \geq 0 | \sum_{i=1}^{n} w_i = 1\}$.

The aim of MAGDM is to select or prioritize these finite alternatives based on performance assessments of alternatives, incomplete attribute weights and overall group satisfaction.

3.2. Context-free grammar approach and hesitant fuzzy linguistic term sets

Most fuzzy linguistic approaches handle linguistic terms with defined priori and thus prevent DMs from utilizing flexible expressions to provide their preferences. However, in the presence of a high degree of uncertainty, DMs may hesitate among different linguistic terms and would prefer to use more complex linguistic expressions, which cannot be expressed through the building of classical linguistic approaches. Rodríguez et al. [14] proposed the use of context-free grammar and further extended it [40] to generate comparative linguistic expressions.

Definition 1. [40] Let G_H be context-free grammar and $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$$
\begin{aligned}
V_N = \{&\langle primaryterm\rangle, \langle compositeterm\rangle,\\
&\langle unaryterm\rangle, \langle binaryterm\rangle, \langle conjuction\rangle\},\\
V_T = \{&lowerthan, greaterthan, atleast, atmost,\\
&between, and, s_0, s_1, \cdots, s_g\},\\
&I \in V_N.\\
P = \{&I ::= \langle primaryterm\rangle \mid \langle compositeterm\rangle\\
&\langle compositeterm\rangle ::= \langle unaryterm\rangle\\
&\langle primaryterm\rangle \mid \langle binaryterm\rangle\\
&\langle primaryterm\rangle\langle conjuction\rangle\langle primaryterm\rangle\\
&\langle primaryterm\rangle ::= s_0 \mid s_1 \mid \cdots \mid s_g\\
&\langle unaryterm\rangle ::= lowerthan \mid greaterthan \mid\\
&\quad atleast \mid atmost\\
&\langle binaryterm\rangle ::= between\\
&\langle conjunction\rangle ::= and\}.
\end{aligned}
$$

Definition 2. [14] Let $S = \{s_0, s_1, \cdots, s_g\}$ be a linguistic term set. An HFLTS H_S on S is an ordered finite subset of consecutive linguistic terms in S.

A transformation function E_{G_H} capable of converting the comparative linguistic expressions generated by the extended context-free grammar G_H approach into HFLTSs is introduced below.

Definition 3. [40] Let E_{G_H} be a function that transforms the linguistic expression $ll \in S_{ll}$ obtained by G_H into an HFLTS. S is the linguistic term set used by G_H, and S_{ll} is the expression domain generated by G_H: $E_{G_H} : S_{ll} \to H_S$.

The linguistic expressions generated by G_H using the production rules can be transformed into an HFLTS through the following transformations:

- $E_{G_H}(s_i) = \{s_i \mid s_i \in S\}$
- $E_{G_H}(atmost\ s_i) = \{s_j \mid s_j \in S\ and\ s_j \leq s_i\}$
- $E_{G_H}(lowerthan\ s_i) = \{s_j \mid s_j \in S\ and\ s_j < s_i\}$
- $E_{G_H}(atleast\ s_i) = \{s_j \mid s_j \in S\ and\ s_j \geq s_i\}$
- $E_{G_H}(greaterthan\ s_i) = \{s_j \mid s_j \in S\ and\ s_j > s_i\}$
- $E_{G_H}(between\ s_i\ and\ s_j) = \{s_k \mid s_k \in S\ and\ s_i \leq s_k \leq s_j\}$.

3.3. 2-tuple linguistic representation model

In order to compute with words without loss of information, Herrera and Martínez [42] proposed a 2-tuple linguistic representation model based on the concept of symbolic translation. The model uses a 2-tuple (s_i, α_i) to represent linguistic information, where s_i is a linguistic term belonging to the predefined linguistic term set and $\alpha_i \in [-0.5, 0.5)$ denotes the symbolic translation. Specifically, the 2-tuple linguistic representation model is defined as follows:

Definition 4. [42] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$$\Delta : [0, g] \mapsto S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_i, \alpha_i),$$

where $i = round(\beta)$ and $\alpha_i = \beta - i$. Note that "*round*" is the usual rounding operator, s_i has the closest index label to β, and α_i is the value of the symbolic translation.

Based on the above definition, a linguistic term can be viewed as a 2-tuple linguistic by adding a value 0 to it as symbolic translation. That is, $s_i \in S \Rightarrow (s_i, 0)$. Unless otherwise specified, we will use 2-tuple linguistic representations instead of linguistic terms throughout the paper.

Definition 5. [42] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and (s_i, α_i) be a 2-tuple, there exists a function

$$\Delta^{-1} : S \times [-0.5, 0.5) \mapsto [0, g]$$
$$\Delta^{-1}((s_i, \alpha_i)) = i + \alpha_i = \beta$$

that uniquely transforms a 2-tuple into its equivalent numerical value $\beta \in [0, g]$.

Definition 6. [42] Let $x = \{(s_1, \alpha_1), (s_2, \alpha_2), \cdots, (s_n, \alpha_n)\}$ be a set of 2-tuples and $W = \{w_1, \cdots, w_n\}$ be their associated weights. The 2-tuple weighted average $\overline{x}$ is defined as

$$\overline{x} = \Delta\left(\frac{\sum_{i=1}^{n} \Delta^{-1}(s_i, \alpha_i) \cdot w_i}{\sum_{i=1}^{n} w_i}\right). \quad (1)$$

3.4. Random preference derived from HFLTS

This subsection introduces the concept of random preference for HFLTS [43], which will be used to handle HFLTSs in the proposed simulation-based framework.

In our MAGDM context, the linguistic assessments x_{ij}^d about qualitative attributes are articulated by comparative linguistic expressions (including single linguistic terms) generated by the context-free grammar. The linguistic assessment x_{ij}^d will be expressed in terms of HFLTS and denoted by H_{ij}^d, where $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n_1, d = 1, 2, \cdots, D$. The HFLTS, H_{ij}^d, is a subset of the linguistic term set S, which represents expert DM_d's uncertain judgment for alternative A_i with respect to attribute C_j. For simplicity, we use $\Omega(S)$ to denote the family of all HFLTSs defined over the linguistic term set S.

When expert DM_d provides his/her judgment for the performance of alternative A_i with respect to attribute C_j, a probability distribution of his/her opinion on $\Omega(S)$ can be derived as

$$p_{\Omega(S)}(H \mid A_i, C_j, DM_d) = \begin{cases} 1, & if\ H = H_{ij}^d; \\ 0, & otherwise, \end{cases}$$

where $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n_1$, and $d = 1, 2, \cdots, D$. Yan et al. [43] pointed out that the probability distribution $p_{\Omega(S)}(H \mid A_i, C_j, DM_d)$ is nothing but a basic probability assignment in the sense of Shafer [44]. We then can use the so-called pignistic transformation method [45] to obtain the least prejudiced distribution over the linguistic term set S for alternative A_i on attribute C_j under expert DM_d as follows:

$$p_S(s_k \mid A_i, C_j, DM_d) = \frac{p_{\Omega(S)}(H \mid A_i, C_j, DM_d)}{|H_{ij}^d|} = \begin{cases} 1/|H_{ij}^d|, & if\ s_k \in H_{ij}^d; \\ 0, & otherwise, \end{cases} \quad (2)$$

where $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n_1$, and $d = 1, 2, \cdots, D$.

Yan et al. [43] remarked that such a probability distribution can be viewed as the prior probability that the expert DM_d believes that the linguistic term $s_k \in S$ is appropriate enough to describe the performance of alternative A_i on attribute C_j. For notational convenience, $p_S(s_k \mid A_i, C_j, DM_d)$ will be denoted by $p_{ij}^d(s_k)$. Under such a formulation, for each alternative A_i, each expert DM_d generates a vector of n_1 individual random preferences, denoted by $(P_{i1}^d, P_{i2}^d, \cdots, P_{in_1}^d)$, with

$$P_{ij}^d = [p_{ij}^d(s_0), p_{ij}^d(s_1), \cdots, p_{ij}^d(s_g)], \quad (3)$$

where $i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n_1,\ d = 1, 2, \cdots, D$.

3.5. Triangular distribution

The triangular distribution is often used to express the uncertainty of outcomes when the actual distribution of a random variable cannot be determined [13]. It is intuitive to non-statistically minded DMs and appears well suited for GDM/MAGDM environments due to its simplicity in capturing multiple preferences [46]. The triangular distribution is specified by three parameters: minimum (L), maximum (H), and most probable (M) value. Given a random variate U drawn from the uniform distribution over the interval (0, 1), it can be shown that the random variate

$$T = \begin{cases} L + \sqrt{U(H-L)(M-L)}, & if \quad U < \frac{M-L}{H-L} \\ H - \sqrt{(1-U)(H-L)(H-M)}, & if \quad U > \frac{M-L}{H-L}. \end{cases} \quad (4)$$

has a triangular distribution with parameters L, M, and H.

4. A simulation based approach for MAGDM problems

4.1. The proposed framework

The proposed approach starts by collecting information about weights of DMs, weights of attributes and attribute performance of each alternative with respect to each attribute. As mentioned before, it is assumed in our model that the information about attribute weights is partially known. In addition, DMs may also provide their assessments about the performance of alternatives using comparative linguistic expressions for qualitative attributes.

We adopt a probability based interpretation of the MAGDM problem and solve it by employing Monte Carlo simulation. The framework of the proposed approach is depicted in Fig. 1. The proposed approach consists of three main stages: Transformation $\longrightarrow$ Simulation $\longrightarrow$ Selection. Since the MAGDM problem will be analyzed by Monte Carlo simulation, the information about attribute performance of alternatives provided by individual DM is aggregated into probability distributions in the first stage. Specifically, for the assessment of each alternative with respect to each qualitative attribute, comparative linguistic expressions provided by DMs are aggregated into random preferences, whereas numerical numbers provided by DMs are aggregated into triangular distributions. Then in the second stage, the Monte Carlo simulation is repeated N times. In each simulation, a random decision matrix is generated by sampling from the random preferences and the triangular distributions formed in stage 1. Additionally, a random weight vector of attributes is generated by uniformly sampling from the space characterized by the partial information available on attribute weights. Then, the alternatives are ranked by solving the generated multi-attribute decision making problem. At the end of each simulation replication run, the rankings of alternatives are recorded. Finally, based on the results of N simulation runs, the ultimate rankings of alternatives is determined by using a particular measure in the last stage. The detailed steps of the proposed approach are described in the following subsection.

4.2. Description of the proposed approach

Setting up the Problem

Step 1: Specify attributes, alternatives and DMs: In this step, a set of attributes $C = \{C_1, \cdots, C_n\}$, a set of alternatives $A = \{A_1, A_2, \cdots, A_m\}$ and a group of decision makers $DM = \{DM_1, DM_2, \cdots, DM_D\}$ with respect to the decision problem are determined. The attributes are application-dependent, and brainstorming exercises may be required to determine the attributes for a particular application. Furthermore, the type of each attribute, i.e., qualitative attribute

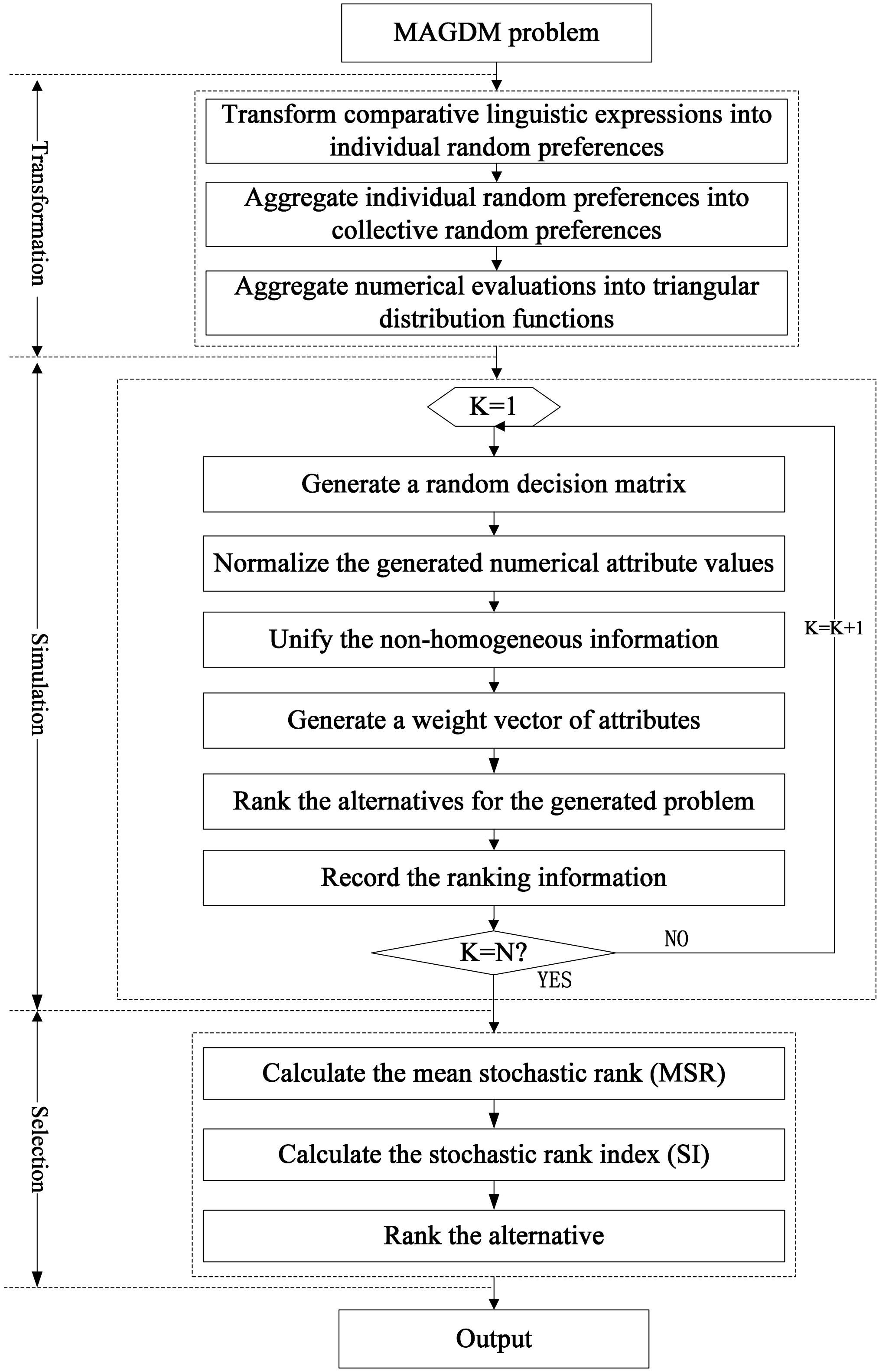

Fig. 1. Framework of the proposed approach.

Table 1
Description of variables and symbols

Variables/symbols	Description
X^d	decision matrix provided by decision maker D_d
x_{ij}^d	performance of A_i with respect to C_j provided by D_d
ll_{ij}^d	comparative linguistic expression provided by D_d for X_{ij}^d
H_{ij}^d	HFLTS derived from ll_{ij}^d
$P_{ij}^d = [p_{ij}^d(s_0, \ldots, p_{ij}^d(s_g)]$	individual random preference derived from H_{ij}^d
$P_{ij}^C = [p_{ij}^C(s_0, \ldots, p_{ij}^C(s_g)]$	collective random preference aggregated from P_{ij}^d
$T_{ij}^C = (t_{1ij}^C, t_{2ij}^C, t_{3ij}^C)$	parameters of the aggregated triangular distribution
G^t	decision matrix generated in the tth round
s_{ij}^t	single linguistic term generated for a qualitative attribute in G^t
r_{ij}^t	numerical number generated for a quantitative attribute in G^t
NG^t	normalized non-homogeneous decision matrix in the tth round
z_{ij}^t	normalized value of r_{ij}^t in NG^t
$(s_{ij}^t, \alpha_{ij}^t)$	linguistic 2-tuple obtained by transforming z_{ij}^t
H	the feasible weight space
$w^t = (w_1^t, w_2^t, \cdots, w_n^t)^T$	random weight vector by sampling from W in the tth round
$\overline{x}_i^t$	overall performance (a linguistic 2-tuple) of alternative A_i
β_i^t	the equivalent numerical value corresponding to $\overline{x}_i^t$
R_{is}	number of times that A_i is ranked as the sth best alternative
MSR_i	mean stochastic rank of alternative A_i
SI_i	stochastic rank index of alternative A_i

or quantitative attribute, should also be specified. In addition, the weight vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_D)$ of decision makers should also be determined in this step.

Step 2: Evaluate performance of alternatives: A linguistic term set $S = \{s_0, s_1, \cdots, s_g\}$ should be determined for evaluating the performance of alternatives with respect to qualitative attributes. Multi-attribute decision matrices $X^d = (x_{ij}^d)_{m\times n}(d = 1, 2, \cdots, D)$ are created based on the DMs' opinion. The element $x_{ij}^d(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; d = 1, 2, \cdots, D)$ in decision matrices X^d indicates the performance rating of alternative A_i with respect to attribute C_j provided by D_d. For qualitative attributes, the performance of alternatives is expressed using comparative linguistic expressions generated by the context-free grammar; for quantitative attributes, the performance of alternatives is evaluated by using crisp numbers.

Step 3: Articulate the incomplete information about attribute weights. In the third step, incomplete information about the weights of attributes is determined by DMs or experts. In practice, the set of known weight information can be constructed by the **Forms 1-5** given in subsection 3.1. Let $w = (w_1, w_2, \cdots, w_n) \in H$ be the weight vector of attributes, where $w_j \geq 0$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, H is the set of the known weight information. It is assumed that all DMs or experts agree on the incomplete information H about the weights of attributes.

Transforming information

Step 4: Aggregate individual evaluations into probability distributions: In this step, the individual evaluations about the performance of each alternative with respect to each attribute are aggregated into a probability distribution. This is carried out as follows:

Step 4.1: Aggregate comparative linguistic expressions into random preferences: Let ll_{ij}^d be the comparative linguistic expression provided by DM_d for the performance of the alternative A_i with respect to a qualitative attribute C_j. The transformation of ll_{ij}^d into a collective random preference P_{ij}^C consists of three steps (shown in Fig. 2).

In the first step, using the function E_{G_H}, the comparative linguistic expression ll_{ij}^d is transformed into HFLTS H_{ij}^d. Then, the individual random preference $P_{ij}^d = [p_{ij}^d(s_0), p_{ij}^d(s_1), \cdots, p_{ij}^d(s_g)]$ can be obtained from equation (2). At the last step, the individual random preferences are aggregated into a collective random preference $P_{ij}^C = [p_{ij}^C(s_0), p_{ij}^C(s_1), \cdots, p_{ij}^C(s_g)]$, where the kth element $p_{ij}^C(s_k)(k = 0, 1, \cdots, g)$ is determined by the following equation:

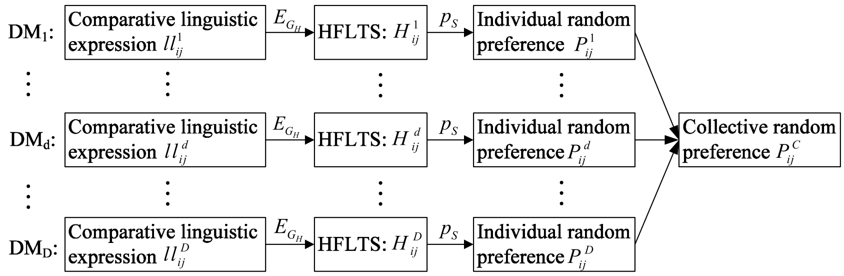

Fig. 2. Transforming comparative linguistic expressions into collective random preference.

$$p_{ij}^{C}(s_k) = \sum_{d=1}^{D} \lambda_d p_{ij}^{d}(s_k) \quad k = 0, 1, \cdots, g. \tag{5}$$

Step 4.2: Aggregate numerical numbers into triangular distributions: The individual evaluations of DMs for quantitative attributes are aggregated into triangular distributions. This is performed in a similar way as described in [13, 23]. Let $T_{ij}^{C} = (t_{1ij}^{C}, t_{2ij}^{C}, t_{3ij}^{C})$ be the parameters of the aggregated triangular probability density function representing the group performance rating of an alternative A_i $(i = 1, 2, \cdots, m)$ with respect to attribute C_j $(j = n_1 + 1, \cdots, n)$. The parameters of the triangular distribution can then be calculated as follows:

$$t_{1ij}^{C} = min_d x_{ij}^{d}, \tag{6}$$

$$t_{2ij}^{C} = \sqrt[n]{\prod_{d=1}^{d} x_{ij}^{d}}, \tag{7}$$

$$t_{3ij}^{C} = max_d x_{ij}^{d}. \tag{8}$$

Monte Carlo simulation

Step 5: Monte Carlo simulation: The simulation component consists of N rounds of simulation runs. In each round, a multi-attribute decision-making problem is generated and a ranking of alternative is derived by using a deterministic decision method. Based on the results of the N runs, the ultimate ranking of alternatives is determined.

Step 5.1: Generate a random decision matrix: At the beginning of each round, a multi-attribute decision matrix is generated by sampling from the aggregated probability distributions obtained in step 4. In the generated decision matrix, the performance of each alternative on each attribute is either a single linguistic term (for a qualitative attribute) or a crisp number (for a quantitative attribute). Let G^t be the generated decision matrix in the tth round:

$$G^t = \begin{pmatrix} s_{11}^t & \cdots & s_{1n_1}^t & r_{1(n_1+1)}^t & \cdots & r_{1n}^t \\ s_{21}^t & \cdots & s_{2n_1}^t & r_{2(n_1+1)}^t & \cdots & r_{2n}^t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{i1}^t & \cdots & s_{in_1}^t & r_{i(n_1+1)}^t & \cdots & r_{in}^t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{m1}^t & \cdots & s_{mn_1}^t & r_{m(n_1+1)}^t & \cdots & r_{mn}^t \end{pmatrix},$$

where $s_{ij}^t \in S, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n_1; r_{ij}^t$ is a crisp value, $i = 1, 2, \cdots, m, j = n_1 + 1, \cdots, n$. We remark that the single linguistic term s_{ij}^t $(j = 1, 2, \ldots, n_1)$ representing the value of alternative A_i $(i \in \{1, 2, \ldots, m\})$ with respect to a qualitative attribute C_j is generated by the collective random preference P_{ij}^{C} and the numerical value r_{ij}^t $(j = n_1 + 1, \ldots, n)$ is generated from the triangular distribution T_{ij}^{C}.

Step 5.2: Normalize the generated numerical attribute values: For the generated performance of alternatives with respect to the quantitative attributes, normalization is essential to eliminate the computational difficulties caused by incommensurable attributes. Let $J^b \subseteq \{n_1 + 1, \cdots, n\}$ be the index set of benefit attributes and $J^c \subseteq \{n_1 + 1, \cdots, n\}$ be the index set of cost attributes. For the generated attribute value $r_{ij}^t(j \in \{n_1 + 1, \cdots, n\})$ in the tth $(t = 1, 2, \cdots, N)$ round, we can obtain the normalized value $z_{ij}^t \in [0, 1]$ by the following $0 - 1$ normalization method:

$$z_{ij}^t = \frac{r_{ij}^t - min_i r_{ij}^t}{max_i r_{ij}^t - min_i r_{ij}^t} \quad i = 1, 2, \cdots, m; j \in J^b,$$

$$z_{ij}^t = \frac{max_i r_{ij}^t - r_{ij}^t}{max_i r_{ij}^t - min_i r_{ij}^t} \quad i = 1, 2, \cdots, m; j \in J^c. \tag{9}$$

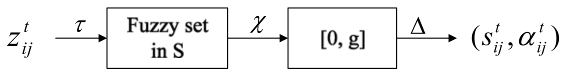

Fig. 3. Transforming numerical number into linguistic 2-tuple.

After normalization, the decision matrix is then converted into the following matrix:

$$NG^t = \begin{pmatrix} s_{11}^t & \cdots & s_{1n_1}^t & z_{1(n_1+1)}^t & \cdots & z_{1n}^t \\ s_{21}^t & \cdots & s_{2n_1}^t & z_{2(n_1+1)}^t & \cdots & z_{2n}^t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{i1}^t & \cdots & s_{in_1}^t & z_{i(n_1+1)}^t & \cdots & z_{in}^t \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_{m1}^t & \cdots & s_{mn_1}^t & z_{m(n_1+1)}^t & \cdots & z_{mn}^t \end{pmatrix},$$

where $s_{ij}^t \in S, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n_1; z_{ij}^t$ is numerical number in the interval $[0, 1]$, $i = 1, 2, \cdots, m, j = n_1 + 1, \cdots, n$.

Step 5.3: Unify the non-homogeneous information: In order to facilitate the ranking of the alternatives in the tth round, the non-homogeneous information in the decision matrix NG^t will first needs to be unified. Motivated by the idea proposed by Herrera and Martínez [47], we transform linguistic and numerical information into linguistic 2-tuples in our approach. For a single linguistic term $s_{ij}^t \in S$ $(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n_1)$ in NG^t, we can easily obtain the corresponding linguistic 2-tuple as $(s_{ij}^t, 0)$. The transformation from numerical values $z_{ij}^t (i = 1, 2, \cdots, m, j = n_1 + 1, \cdots, n) \in [0, 1]$ to linguistic 2-tuples can be realized by using the following equation:

$$\Delta(\chi(\tau(z_{ij}^t))) = (s_{ij}^t, \alpha_{ij}^t),$$

where the functions τ, χ and Δ are defined as in [47]. The transformation process is shown in Fig. 3.

Step 5.4: Generate a random weight vector of attributes. In this step, a random weight vector $w^t = (w_1^t, w_2^t, \cdots, w_n^t)^T$ is generated by sampling from the convex polytope of the feasible weight space H, which is characterized by linear constraints and the natural constraints $\sum_{j=1}^n w_j^t = 1, w_j^t \geq 0$ $(j = 1, 2, \cdots, n)$. Wand and Zionts [48] proposed a procedures for generating weights that satisfy general linear constraints. According to the procedure, we can obtain a random weight vector $w^t = (w_1^t, w_2^t, \cdots, w_n^t)^T$ by sampling from H in the tth round of the simulation. The details are given below:

(i) Enumerate the extreme points of the polytope of the feasible weight space H, $E_1, E_2, \cdots, E_L$, where E_l $(l = 1, 2, \cdots, L)$ is a real $n-$vector.
(ii) Generate an $L-$vector $(k_1^t, k_2^t, \cdots, k_L^t)$ uniformly distributed over $K = \{(k_1, k_2, \cdots, k_L) \mid k_l \geq 0, \sum_{l=1}^L k_l = 1\}$. To do this, $L - 1$ random numbers are first independently generated from the uniform distribution on [0, 1] and then ranked. Suppose the the ranked numbers are $1 \geq v_{L-1}^t \geq v_{l-2}^t \geq \cdots, v_2^t \geq v_1^t \geq 0$. We then let $k_1^t = v_1^t, k_2^t = v_2^t - v_1^t, \cdots, k_{L-1}^t = v_{L-1}^t - v_{L-2}^t, k_L^t = 1 - v_{L-1}^t$.
(iii) Generate the weight vector $w^t = \sum_{l=1}^L k_l^t E_l$.

Step 5.5: Calculate overall performance of each alternative: Let $\overline{x}_i^t$ be the overall performance of alternative A_i $(i = 1, 2, \cdots, n)$. Based on the weighted aggregation operator (Equation (1)), we can obtain $\overline{x}_i^t$ as follows:

$$\overline{x}_i^t = \Delta\left(\sum_{j=1}^n w_j^t \Delta^{-1}(s_{ij}^t, \alpha_{ij}^t)\right). \tag{10}$$

Note that $\overline{x}_i^t$ obtained by (10) is a linguistic 2-tuple that indicates the overall performance of alternative A_i. In this step, other operators or decision methods can also be used to aggregate the attribute values of each alternative.

Step 5.6: Rank the alternatives: In order to rank the alternatives, we transform the overall performance $\overline{x}_i^t$ into an equivalent numerical value β_i^t by the following equation:

$$\beta_i^t = \Delta^{-1}(\overline{x}_i^t). \tag{11}$$

The numerical value β_i^t $(i = 1, 2, \cdots, n)$ can be used as a ranking index of alternatives A_i $(i = 1, 2, \cdots, n)$ at the tth round. The larger the value β_i^t is, the better the alternative A_i will be. At the end of each simulation, the ranking of alternatives is recorded.

Selection of the best alternative

Step 6: Calculate rank indices: In order to determine the alternatives' ultimate preference order, a measure is used to handle the information about the ranking counts of the N simulation runs. We adopt the Mean Stochastic Rank (MSR) and Stochastic Rank

Index (SI) suggested by [49]. Let MSR_i and SI_i represent the MSR and SI of alternative A_i, respectively. Denote by R_{is} the number of times that alternative A_i is ranked as the s_{th} best alternative during the N simulation runs. Let MSR_{max} and MSR_{min} be the respective highest and lowest possible MSR values. Equations (12)-(13) give the explicit formulas for calculating these values.

$$MSR_i = \frac{1}{m}\sum_{i=1}^{n}(sR_{is}), \tag{12}$$

$$SI_i = \frac{MSR_i - MSR_{min}}{MSR_{max} - MSR_{min}}, \tag{13}$$

where $MSR_{max} = N$, $MSR_{min} = \frac{N}{m}$.

Step 7: Rank the alternatives: The ranking of the alternatives can be obtained according to the values of SI_i. The smaller the SI_i $(i = 1, 2, \cdots, n)$ values, the better the alternative A_i $(i = 1, 2, \cdots, n)$ is. In some cases, the difference between the SI values of two alternatives can be very small. As mentioned in [13, 49], this difference can be better distinguished by increasing the number of simulation replication runs.

5. Illustrative example and comparative analysis

5.1. An illustrative example

Electric vehicle charging station (EVCS) is a fundamental element in an infrastructure that provides the energy required for electric vehicles. Efficient, convenient and economic EVCS can enhance the consumers' willingness to buy electric vehicles and thus promote the success of the industry [50, 51]. As part of the EVCS construction plan, the EVCS site selection is very important and may have significant impact on the service quality and operational efficiency of EVCS. Therefore, it is necessary to employ proper method to determine the optimal EVCS site.

In this illustrative example, we consider a group of 10 DMs who are trying to select an appropriate site for EVCS. DMs have identified five alternative sites ($A_1 - A_5$) and six attributes ($C_1 - C_6$) to evaluate them. The six attributes are listed as follows: C_1: Traffic convenience; C_2: Harmonization of EVCS with the development planning of urban road network and power grid; C_3: Environment damage; C_4: Emission reduction; C_5: Construction cost; C_6: Annual operation and maintenance cost. Let $w = (w_1, w_2, w_3, w_4, w_5, w_6)$ be the weight vector of attributes, satisfying $w_i \geq 0$ $(i = 1, 2, \cdots, 6)$ and $\sum_{i=1}^{6} w_i = 1$. The attribute weights are not known with complete certainty and are characterized by the following constraints:

- $w_1, w_2, w_5, w_6 \geq 0.1; w_3, w_4 \geq 0.2;$
- $w_6 \leq w_5;$
- $w_5 + w_6 \leq w_1 + w_2 \leq w_3 + w_4.$

Thus, the incomplete information about these weights can be written as $H = \{(w_1, w_2, w_3, w_4, w_5, w_6) \mid w_1, w_2, w_5, w_6 \geq 0.1;\ w_3, w_4 \geq 0.2;\ w_6 \leq w_5; w_5 + w_6 \leq w_1 + w_2 \leq w_3 + w_4; w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 1\}$. Since no additional prior information is available, the weight vector w is assumed to be uniformly distributed on H in the Monte Carlo simulation.

The DMs are required to evaluate the performances of each alternative with respect to each attribute based on their personal judgments. It is not difficult to see that both quantitative attributes and qualitative attributes are involved in this multi-attribute decision-making problem. In particular, C_1, C_2, C_3 and C_4 are qualitative attributes, and C_5 and C_6 are quantitative. As mentioned before, the DMs declare their judgments by using comparative linguistic expressions generated by the context-free grammar for qualitative attributes. On the other hand, numerical numbers are used to evaluate the performance of quantitative attributes. Table 2 gives an example of the decision matrix provided by one DM. To save space, decision matrices provided by other DMs are omitted.

We apply the proposed simulation-based approach to rank the alternatives. All steps of the approach were implemented using MATLAB. Given the information provided by the DMs, the first stage is to transform the individual evaluations into probability distributions. According to the method presented in step 4.1 of the proposed approach, comparative linguistic expressions $ll_{ij}^d(d = 1, 2, \ldots, D)$ are transformed into individual random preference P_{ij}^d by using the function E_{G_H} and Equation (2). Further, applying Equation (5), we can aggregate the individual random preferences into collective random preferences which are shown in Table 3. In Table 3, the vector in each row represents a probability distribution over the linguistic term set $S = \{s_0, s_1, \cdots, s_g\}$, indicating the aggregated performance of the corresponding attribute value. For instance, the random preference in second row of the table, i.e., $P_{11}^C =$

Table 2
An example of one DM's Performance ratings of alternatives on attributes

x_{ij}^d	C_1	C_2	C_3	C_4	C_5	C_6
A_1	*greaterthan* s_3	*between* s_2 *and* s_4	s_1	*atleast* s_4	420	39
A_2	*lowerthan* s_4	s_3	*atleast* s_4	*greaterthan* s_2	380	36
A_3	*atleast* s_3	*greaterthan* s_2	s_4	*between* s_3 *and* s_4	440	51
A_4	s_3	s_4	*lessthan* s_3	*greaterthan* s_3	430	42
A_5	*between* s_3 *and* s_5	*greaterthan* s_3	s_2	*between* s_2 *and* s_4	385	40

Table 3
Aggregated random preferences for attributes C_1, C_2, C_3 and C_4

p_{ij}^C	s_0	s_1	s_2	s_3	s_4	s_5	s_6
P_{11}^C	0.000	0.000	0.100	0.100	0.533	0.133	0.133
P_{21}^C	0.100	0.233	0.233	0.283	0.050	0.050	0.050
P_{31}^C	0.100	0.100	0.100	0.400	0.100	0.100	0.100
P_{41}^C	0.080	0.180	0.180	0.480	0.080	0.000	0.000
P_{51}^C	0.100	0.100	0.160	0.227	0.227	0.127	0.060
P_{12}^C	0.075	0.075	0.215	0.215	0.340	0.040	0.040
P_{22}^C	0.100	0.100	0.100	0.400	0.300	0.000	0.000
P_{32}^C	0.075	0.075	0.175	0.425	0.150	0.050	0.050
P_{42}^C	0.000	0.000	0.000	0.367	0.467	0.167	0.000
P_{52}^C	0.040	0.040	0.207	0.207	0.307	0.100	0.100
P_{13}^C	0.100	0.100	0.400	0.400	0.000	0.000	0.000
P_{23}^C	0.000	0.000	0.133	0.233	0.300	0.167	0.167
P_{33}^C	0.000	0.000	0.300	0.350	0.250	0.050	0.050
P_{43}^C	0.133	0.433	0.433	0.000	0.000	0.000	0.000
P_{53}^C	0.075	0.075	0.407	0.207	0.127	0.060	0.060
P_{14}^C	0.000	0.000	0.133	0.183	0.450	0.117	0.117
P_{24}^C	0.100	0.100	0.200	0.200	0.200	0.100	0.100
P_{34}^C	0.000	0.000	0.300	0.133	0.233	0.233	0.100
P_{44}^C	0.090	0.090	0.090	0.090	0.507	0.067	0.067
P_{54}^C	0.000	0.000	0.200	0.600	0.200	0.000	0.000

(0.000, 0.000, 0.100, 0.100, 0.533, 0.133, 0.133), is a probability distribution over S, which represents the aggregated performance of alternative A_1 with respect to attribute C_1. We can see that $P_{11}^C(s_0) = P_{11}^C(s_1) = 0$, which means that the group does not give a rating of s_0 and s_1 for the performance of alternative A_1 on attribute C_1.

For quantitative attributes C_5 and C_6, individual numerical evaluations can be aggregated into triangular distributions. For example, suppose that attribute values of sites A_1 with respect to attribute C_5 assessed by 10 DMs are 420, 400, 430, 400, 420, 420, 440, 430, 415 and 430, respectively. By using Equations (6), (7) and (8), we have $t_{115} = 400$, $t_{215} = 420.32$ and $t_{315} = 440$. Therefore, the aggregated triangular distribution of A_1 with respect to C_5 is $T_{15}^C = (t_{115}, t_{215}, t_{315}) = (400.00, 420.32, 440.00)$. All the parameters of the aggregated triangular distributions are shown in Table 4.

Given the random preferences and triangular distributions, Monte Carlo simulation is then repeated N times at the second stage. In the tth simulation run, a decision matrix is randomly generated from the corresponding aggregated random preference and triangular distribution. A particular realization of the matrix is shown in Table 5. At the same time, a random weight vector is also generated by sampling from the space H following the procedure outlined in step 5.4. The generated weight vector in the tth run is given as

$$w^t = (0.163, 0.122, 0.276, 0.218, 0.112, 0.109).$$

A normalization is subsequently performed to eliminate computational problems caused by

Table 4
Parameters of aggregated triangular distributions for attributes C_5 and C_6

T_{ij}^C	t_{1i5}	t_{2i5}	t_{3i5}	t_{1i6}	t_{2i6}	t_{3i6}
A_1	400.00	420.32	440.00	39.00	40.48	43.00
A_2	360.00	373.39	390.00	35.00	36.93	42.00
A_3	410.00	436.62	460.00	38.00	45.92	51.00
A_4	400.00	417.40	430.00	40.00	44.06	46.00
A_5	365.00	386.87	400.00	37.00	40.36	43.00

Table 5
Random decision matrix G^t in round t

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	s_3	s_0	s_1	s_4	426.3380	39.5985
A_2	s_5	s_4	s_2	s_4	378.0720	37.3414
A_3	s_1	s_2	s_6	s_5	435.4156	43.4902
A_4	s_1	s_4	s_2	s_4	404.3874	44.8468
A_5	s_3	s_3	s_2	s_3	371.5935	41.6251

Table 6
The unified decision matrix in round t

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$(s_3, 0)$	$(s_0, 0)$	$(s_5, 0)$	$(s_4, 0)$	$(s_1, -0.1468)$	$(s_4, 0.1956)$
A_2	$(s_5, 0)$	$(s_4, 0)$	$(s_4, 0)$	$(s_4, 0)$	$(s_5, 0.3911)$	$(s_6, 0)$
A_3	$(s_1, 0)$	$(s_2, 0)$	$(s_0, 0)$	$(s_5, 0)$	$(s_0, 0)$	$(s_1, 0.0843)$
A_4	$(s_1, 0)$	$(s_4, 0)$	$(s_4, 0)$	$(s_4, 0)$	$(s_3, -0.0830)$	$(s_0, 0)$
A_5	$(s_3, 0)$	$(s_3, 0)$	$(s_4, 0)$	$(s_3, 0)$	$(s_6, 0)$	$(s_3, -0.4244)$

incommensurable attributes. Since the construction cost and annual operation and maintenance cost are both cost attributes, Equation (9) is used to normalize the generated performance of alternatives with respect to C_5 and C_6. Then, the non-homogeneous information in the normalized decision matrix is unified into linguistic 2-tuples by using the method described in step 5.3 of the proposed approach. The unified decision matrix in iteration t is given in Table 6. Using Equations (10) and (11), we can obtain the overall performance $\bar{x}_i^t$ and the corresponding numerical value β_i^t of each alternative. The computational details are reported in Table 7. From the values given in Table 7, we can see that the ranking of alternatives in tth iteration is: $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$. Accordingly, the rank counts $R_{21}, R_{52}, R_{13}, R_{44}, R_{35}$ are all increased by one in this iteration.

Based on the results of $N = 10000$ runs, we can convert the rank counts R_{is} into MSR_i and SI_i by using Equations (12) and (13). R_{is}, MSR_i and SI_i of the alternatives are reported in Table 8. According to the value SI_i in Table 8, we obtain the ultimate ranking of the alternatives: $A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$.

The proposed simulation based approach was performed by varying the number of simulation replications from 50 to 10000. The obtained SI values of alternatives in each of the respective cases are given in Table 9. From Table 9, we can see that the ranking of all alternatives becomes consistent after the $N = 200$ case.

Table 8
Rank counts of alternatives

R_{is}	A_1	A_2	A_3	A_4	A_5
R_{i1}	2601	2402	124	1160	3713
R_{i2}	2791	2115	337	2064	2693
R_{i3}	2518	2125	762	2743	1852
R_{i4}	1696	2335	1815	2821	1333
R_{i5}	394	1023	6962	1212	409
MSR_i	4898.2	5492.4	9030.8	6172.2	4406.4
SI_i	0.3623	0.4365	0.8788	0.5215	0.3008

Table 9
Statistical regularity of different simulation runs

$N=$	50	100	200	500	1000	5000	10000
SI_1	0.395	0.310	0.378	0.366	0.362	0.358	0.362
SI_2	0.370	0.443	0.423	0.418	0.420	0.436	0.437
SI_3	0.885	0.878	0.868	0.882	0.875	0.884	0.879
SI_4	0.555	0.488	0.523	0.531	0.530	0.518	0.522
SI_5	0.295	0.383	0.310	0.305	0.313	0.304	0.301

5.2. Comparative study

In this subsection, we apply the proposed approach to an example adopted in Chen et al. [52] and compare its performance with that of the model developed in [52]. This example involves the evaluation of university faculty for tenure and promotion. In particular, the attributes used by a university are teaching (C_1), research (C_2), and service (C_2), which carry unknown weighting vectors. Suppose that 5 candidates x_i ($i = 1, 2, 3, 4, 5$) need to be evaluated by 10 experts $E = \{e_1, e_2, \ldots, e_{10}\}$ under these three attributes. Due to the uncertainty involved in evaluating the candidates, the experts may either use single linguistic terms or comparative linguistic expressions to provide their preferences. Thus, the problem can be viewed as a special case of the problems considered in this paper by letting $n_1 = n$ and $H = \{(w_1, \cdots, w_n) \mid w_j \geq 0,\ \sum_{j=1}^{n} w_j = 1\}$ in our model. According to the decision matrix constructed on the basis of the proportional comparative linguistic pairs $Pll_{ij}^{\tau} = (ll_{ij}^{\tau}, p_{ij}^{\tau})$ in [52], we can use the proposed approach to solve the MAGDM problem.

Table 7
The overall performances of alternatives in round t

	A_1	A_2	A_3	A_4	A_5
$\bar{x}_i^t$	$(s_3, 0.2952)$	$(s_5, -0.4633)$	$(s_2, -0.3860)$	$(s_3, -0.0457)$	$(s_4, -0.4328)$
β_i^t	3.2952	4.5367	1.6140	2.9543	3.5672

Table 10
Transformed random preferences

P_{ij}	s_0	s_1	s_2	s_3	s_4	s_5	s_6
P_{11}	0.000	0.000	0.000	0.100	0.400	0.400	0.100
P_{21}	0.150	0.150	0.000	0.000	0.000	0.600	0.100
P_{31}	0.040	0.040	0.207	0.432	0.232	0.025	0.025
P_{41}	0.117	0.117	0.177	0.177	0.177	0.177	0.060
P_{51}	0.000	0.000	0.000	0.100	0.333	0.333	0.233
P_{12}	0.067	0.067	0.200	0.250	0.250	0.117	0.050
P_{22}	0.060	0.060	0.260	0.118	0.218	0.158	0.125
P_{32}	0.000	0.200	0.500	0.000	0.100	0.100	0.100
P_{42}	0.000	0.000	0.100	0.100	0.300	0.300	0.200
P_{52}	0.100	0.100	0.100	0.400	0.300	0.000	0.000
P_{13}	0.000	0.000	0.200	0.350	0.350	0.050	0.050
P_{23}	0.040	0.040	0.040	0.115	0.115	0.575	0.075
P_{33}	0.000	0.000	0.167	0.392	0.192	0.125	0.125
P_{43}	0.133	0.133	0.208	0.075	0.175	0.175	0.100
P_{53}	0.000	0.000	0.000	0.325	0.125	0.275	0.275

In step 4 of our approach, for the attribute value of alternative x_i $(i = 1, 2, \cdots, m)$ with respect to attribute C_j $(j = 1, 2, \cdots, n)$, the individual comparative linguistic expressions ll_{ij}^d $(d = 1, 2, \cdots, D)$ are aggregated into a random preference P_{ij}^C of the group. Similarly, in [52], the proportional comparative linguistic pairs $Pll_{ij}^\tau = (ll_{ij}^\tau, p_{ij}^\tau)$ $(\tau = 1, 2, \cdots, \Gamma)$ are transformed into a proportional hesitant fuzzy linguistic term set (PHFLTS) $P_{H_S}^{ij}$. In fact, both P_{ij}^C and $P_{H_S}^{ij}$ can be considered as the probability distributions over the linguistic term set $S = \{s_0, s_1, \cdots, s_g\}$, indicating the collective preferences about the performance of alternative x_i with respect to attribute C_j. Assuming that each DM in the group are equally important, we can obtain the random preferences reported in Table 10 by using the method in step 4 of our approach. The corresponding PHFLTSs are listed in the matrix $R_p = (R_p^1, R_p^2, R_p^3)$ in [52]. For the majority of the entries in the two matrices, we find there are differences between the probability distributions obtained by these two different methods of transformation. In fact, a closer analysis of the transformation algorithm in [52] shows that, if the formula in step 4 of the algorithm in [52], i.e.,

$$p_\tau^{ij} = p_{ij}^\tau / (\Sigma_{\tau=1}^{\Gamma} \varphi_\tau^{ij} p_{ij}^\tau) \qquad \tau = 1, 2, \cdots, \Gamma \quad (14)$$

is replaced by the following equation:

$$p_\tau^{ij} = p_{ij}^\tau / \varphi_\tau^{ij} \qquad \tau = 1, 2, \cdots, \Gamma, \qquad (15)$$

then the probability distributions obtained will be identical to those of Table 10. In the following, we argue that Equation (15) is more reasonable than Equation (14) in terms of calculating the proportions (probability) of linguistic term s_{τ_k} in $E_{G_H}(ll_{ij}^\tau)$. Notice that $\Sigma_{\tau=1}^{\Gamma} p_{ij}^\tau = 1$. Thus, each linguistic term s_{τ_k} in $E_{G_H}(ll_{ij}^\tau)$ should share the same proportion of ll_{ij}^τ. Since the number of elements of $E_{G_H}(ll_{ij}^\tau)$ is φ_τ^{ij}, we can calculate the proportion for linguistic term s_{τ_k} in $E_{G_H}(ll_{ij}^\tau)$ by using Equation (15) if there is no additional information. In fact, if Equation (14) is replaced by Equation (15), then the transformation algorithm in [52] is equivalent to the transformation method in step 4 in our approach. Moreover, the relative importance of DMs can be considered in our method. However, in [52], each DM is assumed to be equally important due to the fact that the proportional comparative linguistic pair $Pll_{ij}^\tau = (ll_{ij}^\tau, p_{ij}^\tau)$ $(\tau = 1, 2, \cdots, \Gamma)$ is obtained by simply combining individual comparative linguistic expressions.

After the completion of information transformation, Monte Carlo simulation is conducted in order to rank the alternatives. By using the proposed approach, we obtain the information about the rank counts and the ultimate ranking of the alternatives, which is reported in Table 11 (the numbers before the slash). Table 11 also shows the simulation results based on the transformed decision matrix in [52]. From Table 11, we see that the ranking of the alternatives is $A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$ when the decision matrix is transformed using the algorithm in [52]. This ranking is the same as that in [52], which implies that there is no difference in ranking of alternatives between the simulation based approach and the operator based approach in this problem. However, we can see that the ranking is $A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$ if the proposed approach is applied. Obviously, the reverse in the ranking order between A_2 and A_5 is mainly attributed to the use of different

Table 11
Rank counts of alternatives

R_{is}	A_1	A_2	A_3	A_4	A_5
R_{i1}	1520/1493	3107/3374	815/847	1866/1939	2692/2347
R_{i2}	2258/2293	2265/2222	1000/1091	1764/1830	2713/2564
R_{i3}	2799/2765	1537/1442	1622/1715	1729/1667	2313/2411
R_{i4}	2210/2188	1302/1320	3002/3000	1977/1780	1509/1712
R_{i5}	1213/1261	1789/1642	3561/3347	2664/2784	773/966
SI_i	0.4835/0.4858	0.4100/0.3909	0.6874/0.6727	0.5452/0.5410	0.3740/0.4096
ranking	3/3	2/1	5/5	4/4	1/2

Table 12
Comparison with the related MAGDM approaches

	Performance ratings	Attribute weights	Method	Final results
Proposed approach	comparative linguistic expressions and real numbers	completely unknown or partially known	stochastic	distribution for possible rankings
Mousavi's approach [12]	single linguistic term	known(linguistic term)	stochastic	distribution for possible rankings
Bayram's approach [13]	real numbers	completely known	stochastic	distribution for possible rankings
Zhang's approach [53]	heterogeneous information	completely unknown or partially known	fuzzy	single ranking

transformation methods. Since the improvement in the method of transformation is meaningful, it can be concluded that the ranking results obtained by our approach is more reasonable than that of [52]. More importantly, the rank counts shown in Table 11 may also be important for DMs because they reveal the distribution information among the ranking alternatives, which is not available in other approaches.

In order to demonstrate the relationship and differences between the proposed approach and other MAGDM approaches under uncertain environment and emphasize the advantages and characteristics of the proposed method, in the following we further compare the proposed approach with the related latest work on MAGDM [12, 13, 53]. Based on the analysis of different approaches, the comparison results are listed in Table 12.

As mentioned in the introduction, Monte Carlo simulation has been used to deal with MAGDM under uncertain environment in literature. In particular, in [12] and [13], different stochastic MAGDM approaches are developed to handle MAGDM problem, which motivate the proposed approach in our work. As shown in Table 12, the differences between the proposed approach and the methods developed in [12] and [13] are obvious. First of all, our proposal captures DMs' hesitancy about linguistic expressions by allowing the use of comparative linguistic expressions under a simulation framework. In the framework of simulation, the proposed approach can combine comparative linguistic expressions and numerical information in MAGDM. However, the approach developed in [12] is only applicable to MAGDM problem where the decision makers describe a value for an alternative with respect to an attribute by the use of linguistic variables. On the other hand, the method in [13] is only capable of solving MAGDM problems where attribute values and weights are described in crisp numbers. Secondly, the proposed approach can effectively deal with MAGDM problems with incomplete information about attributes. In our work, attribute weights are assumed to be partially known, whereas they are completely known in [13] and known as linguistic variables in [12]. In addition, we remark that if attribute weights are completely unknown, the proposed approach is still valid because the weight vector of attributes can be regarded as a random vector uniformly distributed over the simplex $H = \{w = (w_1, w_2, \ldots, w_n) \geq 0 | \sum_{i=1}^{n} w_i = 1\}$.

In [53], a deviation model is developed to handle heterogeneous MAGDM problems with incomplete weight information in which the decision information is expressed in multiple formats of attribute values (such as real numbers, interval numbers, and linguistic variables). In fact, the fuzzy approach is adopted to model uncertainty in the considered MCGDM problem. Notice that the output of the decision model is a single ranking of alternatives in [53]. For the MAGDM problem under uncertainty, a single ranking of alternatives may have limitations for decision making because because it cannot reflect the possibilities and distribution of the ranking of alternatives. In the proposed simulation-based approach, the output is the distribution information for possible rankings, which can provide a more complete understanding of possible outcomes for MAGDM under uncertainty. This is also the main feature of the simulation-based approach, which distinguishes itself from the fuzzy approach.

6. Conclusions

In real world MAGDM, DMs often do not have complete knowledge about the attribute weights and the performance ratings of alternatives due to the complexity of the decision problems and the limitation of human cognitive ability. Thus, how to model the uncertainty and imprecision in the decision making problems is a challenging issue in MAGDM. This paper proposes a novel simulation based approach for MAGDM under uncertainty. The approach is divided into three stage: transformation, simulation, and selection, each of which consists of multiple steps.

Compared with the decision making methods based on fuzzy set theory, the advantage of proposed approach lies in its ability to provide a complete understanding of alternatives' preference structure. The main contributions of this work are as follows. First, the DMs hesitancy about linguistic expressions for performance ratings with respect to qualitative attributes is well modeled under the proposed simulation framework. In the proposed approach, the comparative linguistic expressions generated by context-free grammar are used by DMs to express their preference. Then, individual comparative linguistic expressions are aggregated into random preferences over the linguistic term set, which are used as inputs to Monte Carlo simulation. This distinguishes our work from previous studies [12] that consider qualitative attributes in simulation based MAGDM approaches. Second, by treating the attribute weight vector as a random vector uniformly distributed over the space characterized by weight constraints, the proposed approach provides a viable way to cope with the incomplete information about attributes weights. Third, the proposed approach can be used to solve MAGDM problems in which both quantitative attributes and qualitative attributes are involved. Although many models have been proposed in the literature to solve MAGDM involving both qualitative and quantitative attributes, most of these models are established from the fuzzy viewpoint. The proposed simulation based approach can provide a more complete understanding of alternatives' preference structure.

For future research, utilizing other multi-attribute decision making methods, such as TOPSIS and VIKOR, to rank the alternatives in the stage of Monte Carlo simulation is suggested to improve the overall performance of the proposed approach. In addition, developing effective algorithms for unifying non-homogenous information about attribute values will also be helpful for further validating the effectiveness of the proposed approach.

Acknowledgments

The authors are very grateful to the anonymous reviewers and the editor for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work of Shengbao Yao was supported by the Humanities and Social Sciences Foundation of Ministry of Education of China (No. 14YJA630086), the Fundamental Research Funds for the Central Universities (No. 31541410811). The work of Jiaqiao Hu was supported by the National Science Foundation under Grant CMMI-1634627.

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