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Time-consistent stopping under decreasing impatience

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Abstract Under non-exponential discounting, we develop a dynamic theory for stopping problems in continuous time. Our framework covers discount functions that induce decreasing impatience. Due to the inherent time inconsistency, we look for equilibrium stopping policies, formulated as fixed points of an operator. Under appropriate conditions, fixed-point iterations converge to equilibrium stopping policies. This iterative approach corresponds to the hierarchy of strategic reasoning in game theory and provides "agent-specific" results: it assigns one specific equilibrium stopping policy to each agent according to her initial behavior. In particular, it leads to a precise mathematical connection between the naive behavior and the sophisticated one. Our theory is illustrated in a real options model.

Keywords Time inconsistency · Optimal stopping · Hyperbolic discounting · Decreasing impatience · Subgame-perfect Nash equilibrium · Iterative approach

Mathematics Subject Classification (2010) 60G40 · 91B06

JEL Classification C61 · D81 · D90 · G02

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1 Introduction

Time inconsistency is known to exist in stopping decisions, such as casino gambling in [2] and [11], optimal stock liquidation in [37], and real options valuation in [15]. A general treatment, however, has not been proposed in continuous-time models. In this article, we develop a dynamic theory for time-inconsistent stopping problems in continuous time, under non-exponential discounting. In particular, we focus on log-subadditive discount functions (Assumption 3.12), which capture *decreasing impatience*, an acknowledged feature of empirical discounting in behavioral economics; see e.g. [36, 22, 21]. Hyperbolic and quasi-hyperbolic discount functions are special cases under our consideration.

The seminal work by Strotz [34] identifies three types of agents under time inconsistency—the *naive*, the *pre-committed*, and the *sophisticated*. Among them, only the sophisticated agent takes the possible change of future preferences seriously, and works on *consistent planning*: she aims to find a strategy that once being enforced over time, none of her future selves would want to deviate from. How to precisely formulate such a *sophisticated strategy* has been a challenge in continuous time. For stochastic control, Ekeland and Lazrak [12] resolved this issue by defining sophisticated controls as subgame-perfect Nash equilibria in a continuous-time intertemporal game of multiple selves. This has aroused vibrant research on time inconsistency in mathematical finance; see e.g. [14, 13, 16, 38, 7, 10, 6, 5]. There is, nonetheless, no equivalent development for stopping problems.

This paper contributes to the literature of time inconsistency in three ways. First, we provide a precise definition of a sophisticated stopping policy (or equilibrium stopping policy) in continuous time (Definition 3.7). Specifically, we introduce the operator Θ in (3.6), which describes the game-theoretic reasoning of a sophisticated agent. Sophisticated policies are formulated as fixed points of Θ, which connects to the concept of subgame-perfect Nash equilibrium invoked in [12].

Second, we introduce a new, iterative approach for finding equilibrium strategies. For any initial stopping policy τ, we apply the operator Θ to τ repetitively until it converges to an equilibrium stopping policy. Under appropriate conditions, this fixed-point iteration indeed converges (Theorem 3.16), which is the main result of this paper. Recall that the standard approach for finding equilibrium strategies in continuous time is solving a system of nonlinear equations, as proposed in [14] and [5]. Solving this system of equations is difficult; and even when it is solved (as in the special cases in [14] and [5]), we only obtain one particular equilibrium, and it is unclear how other equilibrium strategies can be found. Our iterative approach can be useful here: we find different equilibria simply by starting the fixed-point iteration with different initial strategies τ. In some cases, we are able to find *all* equilibria; see Proposition 4.5.

Third, when an agent starts to do game-theoretic reasoning and to look for equilibrium strategies, she is not satisfied with an arbitrary equilibrium. Instead, she works on improving her initial strategy to turn it into an equilibrium. This improving process is absent from [12, 14, 5] and subsequent research, although it is well known in game theory as the hierarchy of strategic reasoning in [32] and [33]. Our iterative approach specifically represents this improving process: for any initial strategy τ, each application of Θ to τ corresponds to an additional level of strategic reasoning. As a result, the iterative approach complements the existing literature of time inconsistency in that it not only facilitates the search for equilibrium strategies, but provides "agent-specific" equilibria: it assigns one specific equilibrium to each agent according to her initial behavior.

Upon completion of our paper, we noticed the recent work by Pedersen and Peskir [28] on mean–variance optimal stopping. They introduced "dynamic optimality" to deal with time inconsistency. As explained in detail in [28], this new concept is different from *consistent planning* in Strotz [34], and does not rely on game-theoretic modeling. Therefore, our equilibrium stopping policies are different from their dynamically optimal stopping times. That being said, a few connections between our paper and [28] do exist, as pointed out later in Remarks 2.3, 3.4 and 4.9.

The paper is organized as follows. In Sect. 2, we introduce the setup of our model, and demonstrate time inconsistency in stopping decisions through examples. In Sect. 3, we formulate the concept of equilibrium for stopping problems in continuous time, search for equilibrium strategies via fixed-point iterations, and establish the required convergence result. Section 4 illustrates our theory in detail in a real options model. Most of the proofs are relegated to appendices.

2 Preliminaries and motivation

Consider the canonical space $\Omega := \{\omega \in C([0,\infty);\mathbb{R}^d) : \omega_0 = 0\}$. Let $(W_t)_{t\ge 0}$ be the coordinate process $W_t(\omega) = \omega_t$ and $\mathbb{F}^W = (\mathcal{F}^W_s)_{s\ge 0}$ the natural filtration generated by W. Let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F}^W_\infty)$, where $\mathcal{F}^W_\infty := \bigcup_{s\ge 0}\mathcal{F}^W_s$. For each $t \ge 0$, we introduce the filtration $\mathbb{F}^{t,W} = (\mathcal{F}^{t,W}_s)_{s\ge 0}$ with

$$\mathcal{F}^{t,W}_s = \sigma(W_{u\vee t} - W_t : 0 \le u \le s),$$

and let $\mathbb{F}^t = (\mathcal{F}^t_s)_{s\ge 0}$ be the $\mathbb{P}$-augmentation of $\mathbb{F}^{t,W}$. We denote by $\mathcal{T}_t$ the collection of all $\mathbb{F}^t$-stopping times τ with $\tau \ge t$ a.s. For the case where $t = 0$, we simply write $\mathbb{F}^0 = (\mathcal{F}^0_s)_{s\ge 0}$ as $\mathbb{F} = (\mathcal{F}_s)_{s\ge 0}$, and $\mathcal{T}_0$ as $\mathcal{T}$.

Remark 2.1 For any $0 \le s \le t$, $\mathcal{F}^t_s$ is the σ-algebra generated by only the $\mathbb{P}$-negligible sets. Moreover, for any $s, t \ge 0$, $\mathcal{F}^t_s$-measurable random variables are independent of $\mathcal{F}_t$; see Bouchard and Touzi [9, Remark 2.1] for a similar setup.

Consider the space $\mathbb{X} := [0,\infty) \times \mathbb{R}^d$, equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{X})$. For any $\tau \in \mathcal{T}$ and $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-measurable ξ, let X be the solution to the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \quad \text{for } t \ge \tau, \qquad \text{with } X_\tau = \xi \text{ a.s.} \tag{2.1}$$

We assume that $b : \mathbb{X} \to \mathbb{R}$ and $\sigma : \mathbb{X} \to \mathbb{R}$ satisfy Lipschitz and linear growth conditions in $x \in \mathbb{R}^d$, uniformly in $t \in [0,\infty)$. Then for any $\tau \in \mathcal{T}$ and $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-measurable ξ with $\mathbb{E}[|\xi|^2] < \infty$, (2.1) admits a unique strong solution. Alternatively, we can consider $X_s := f(s, W_s)$, $s \ge 0$, for some measurable function $f : \mathbb{X} \to \mathbb{R}$. When f is twice differentiable, this formulation can be turned into (2.1) by Itô's formula. It is, however, not covered by (2.1) for a general measurable f; see Sect. 4 for an example.

For any $(t,x)\in\mathbb{X}$, we denote by $X^{t,x}$ the solution to (2.1) with $X_t=x$, and by $\mathbb{E}^{t,x}$ the expectation conditioned on $X_t=x$.

2.1 Classical optimal stopping

Consider a *payoff function* $g:\mathbb{R}^d\to\mathbb{R}$, assumed to be nonnegative and continuous, and a *discount function* $\delta:\mathbb{R}_+\to[0,1]$, assumed to be continuous, decreasing and satisfy $\delta(0)=1$. Moreover, we assume that

$$\mathbb{E}^{t,x}\left[\sup_{t\le s\le\infty}\delta(s-t)g(X_s)\right]<\infty\quad\forall(t,x)\in\mathbb{X},$$

where we interpret $\delta(\infty-t)g(X^{t,x}_\infty):=\limsup_{s\to\infty}\delta(s-t)g(X^{t,x}_s)$; this is in line with Karatzas and Shreve [17, Appendix D]. Given $(t,x)\in\mathbb{X}$, classical optimal stopping asks if there is a $\tau\in\mathcal{T}_t$ such that the expected discounted payoff

$$J(t,x;\tau):=\mathbb{E}^{t,x}[\delta(\tau-t)g(X_\tau)] \tag{2.2}$$

can be maximized. The associated value function

$$v(t,x):=\sup_{\tau\in\mathcal{T}_t}J(t,x;\tau) \tag{2.3}$$

has been widely studied, and the existence of an optimal stopping time is affirmative. The following is a standard result taken from [17, Appendix D] and [29, Chapter I.2].

Proposition 2.2 *For any $(t,x)\in\mathbb{X}$, let $(Z^{t,x}_s)_{s\ge t}$ be a right-continuous process with*

$$Z^{t,x}_s(\omega)=\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,X^{t,x}_s(\omega)}[\delta(\tau-t)g(X_\tau)]\qquad a.s., \forall s\ge t, \tag{2.4}$$

and define $\widetilde{\tau}_{t,x}\in\mathcal{T}_t$ by

$$\widetilde{\tau}_{t,x}:=\inf\{s\ge t:\delta(s-t)g(X^{t,x}_s)=Z^{t,x}_s\}. \tag{2.5}$$

Then $\widetilde{\tau}_{t,x}$ is an optimal stopping time for (2.3), i.e.,

$$J(t,x;\widetilde{\tau}_{t,x})=\sup_{\tau\in\mathcal{T}_t}J(t,x;\tau). \tag{2.6}$$

Moreover, $\widetilde{\tau}_{t,x}$ is the smallest, if not unique, optimal stopping time.

Remark 2.3 The classical optimal stopping problem (2.3) is *static* in the sense that it involves only the preference of the agent at time t. Following the terminology of Definition 1 in Pedersen and Peskir [28], $\widetilde{\tau}_{t,x}$ in (2.5) is "statically optimal".

2.2 Time inconsistency

Following Strotz [34], a naive agent solves the classical problem (2.3) repeatedly at every moment as time passes by. That is, given initial $(t,x)\in\mathbb{X}$, the agent solves

$$\sup_{\tau\in\mathcal{T}_s} J(s, X_s^{t,x};\tau) \qquad \text{at every moment } s\geq t.$$

In view of Proposition 2.2, the agent at time s intends to employ the stopping time $\widetilde{\tau}_{s,X_s^{t,x}}\in\mathcal{T}_s$, for all $s\geq t$. This raises the question of whether optimal stopping times obtained at different moments, $\widetilde{\tau}_{t,x}$ and $\widetilde{\tau}_{t',X_{t'}^{t,x}}$ with $t'>t$, are consistent with each other.

Definition 2.4 We say the problem (2.3) is *time-consistent* if for any $(t,x)\in\mathbb{X}$ and $s>t$, $\widetilde{\tau}_{t,x}(\omega)=\widetilde{\tau}_{s,X_s^{t,x}(\omega)}(\omega)$ for a.e. $\omega\in\{\widetilde{\tau}_{t,x}\geq s\}$. We say the problem (2.3) is *time-inconsistent* if the above does not hold.

In the classical literature of mathematical finance, the discount function usually takes the form $\delta(s)=e^{-\rho s}$ for some $\rho\geq 0$. This already guarantees time consistency of (2.3). To see this, first observe the identity

$$\delta(s)\delta(t)=\delta(s+t) \qquad \forall s,t\geq 0. \tag{2.7}$$

Fix $(t,x)\in\mathbb{X}$ and pick $t'>t$ such that $\mathbb{P}[\widetilde{\tau}_{t,x}\geq t']>0$. For a.e. $\omega\in\{\widetilde{\tau}_{t,x}\geq t'\}$, set $y:=X_{t'}^{t,x}(\omega)$. We observe from (2.5), (2.4) and $X_s^{t,x}(\omega)=X_s^{t',y}(\omega)$ that

$$\widetilde{\tau}_{t,x}(\omega)=\inf\left\{s\geq t':\delta(s-t)g\big(X_s^{t',y}(\omega)\big)\geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,X_s^{t',y}(\omega)}[\delta(\tau-t)g(X_\tau)]\right\},$$

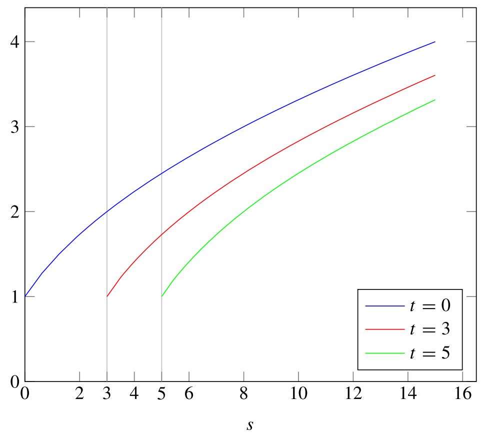
$$\widetilde{\tau}_{t',y}(\omega)=\inf\left\{s\geq t':\delta(s-t')g\big(X_s^{t',y}(\omega)\big)\geq \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s}\mathbb{E}^{s,X_s^{t',y}(\omega)}[\delta(\tau-t')g(X_\tau)]\right\}.$$

Then (2.7) guarantees $\widetilde{\tau}_{t,x}(\omega)=\widetilde{\tau}_{t',y}(\omega)$ as $\frac{\delta(\tau-t)}{\delta(s-t)}=\frac{\delta(\tau-t')}{\delta(s-t')}=\delta(\tau-s)$. For non-exponential discount functions, the identity (2.7) no longer holds, and the problem (2.3) is in general time-inconsistent.

Example 2.5 Suppose a smoker has a fixed lifetime $T>0$. Consider a deterministic cost process $X_s:=x_0e^{\frac{1}{2}s}$, $s\in[0,T]$, for some $x_0>0$. Thus we have $X_s^{t,x}=xe^{\frac{1}{2}(s-t)}$ for $s\in[t,T]$. The smoker can (i) quit smoking at some time $s<T$ (with cost X_s) and die peacefully at time T (with no cost), or (ii) never quit smoking (thus incurring no cost) but die painfully at time T (with cost X_T). With the hyperbolic discount function $\delta(s):=\frac{1}{1+s}$ for $s\geq 0$, (2.3) becomes minimizing the cost

$$\inf_{s\in[t,T]}\delta(s-t)X_s^{t,x}=\inf_{s\in[t,T]}\frac{xe^{\frac{1}{2}(s-t)}}{1+(s-t)}.$$

Fig. 1 The free boundary $s \mapsto \sqrt{1+(s-t)}$ with different initial times t



By basic calculus, the optimal stopping time $\widetilde{\tau}_{t,x}$ is given by

$$\widetilde{\tau}_{t,x} = \begin{cases} t+1, & \text{if } t < T-1, \\ T, & \text{if } t \geq T-1. \end{cases} \tag{2.8}$$

Time inconsistency can be easily observed, and it illustrates the procrastination behavior: the smoker never quits smoking.

Example 2.6 Suppose $d=1$ and $X_s := |W_s|$, $s \geq 0$. Consider the payoff function $g(x) := x$ for $x \in \mathbb{R}_+$ and the hyperbolic discount function $\delta(s) := \frac{1}{1+s}$ for $s \geq 0$. The problem (2.3) reduces to $v(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x[\frac{X_\tau}{1+\tau}]$. This can be viewed as a real options problem in which the management of a large non-profitable insurance company has the intention to liquidate or sell the company, and would like to decide when to do so; see the explanations under (4.2) for details.

By the argument in Pedersen and Peskir [27], we prove in Proposition 4.1 below that the optimal stopping time $\widetilde{\tau}_x$, defined in (2.5) with $t=0$, has the explicit formula $\widetilde{\tau}_x = \inf\{s \geq 0 : X_s^x \geq \sqrt{1+s}\}$. If one solves the same problem at any time $t > 0$ with $X_t = x \in \mathbb{R}_+$, the optimal stopping time is

$$\widetilde{\tau}_{t,x} = t + \widetilde{\tau}_x = \inf\{s \geq t : X_s^{t,x} \geq \sqrt{1+(s-t)}\}.$$

The free boundary $s \mapsto \sqrt{1+(s-t)}$ is unusual in its dependence on the initial time t. From Fig. 1, we clearly observe time inconsistency: $\widetilde{\tau}_{t,x}(\omega)$ and $\widetilde{\tau}_{t',X_{t'}^{t,x}}(\omega)$ do not agree in general, for any $t' > t$, as they correspond to different free boundaries.

As proposed in Strotz [34], to deal with time inconsistency, we need a strategy that is either *pre-committed* or *sophisticated*. A pre-committed agent finds $\widetilde{\tau}_{t,x}$ in (2.5) at time t, and forces her future selves to follow $\widetilde{\tau}_{t,x}$ through a commitment mechanism (e.g. a contract). By contrast, a sophisticated agent works on "consistent planning": she anticipates the change of future preferences, and aims to find a stopping strategy that once being enforced, none of her future selves would want to deviate from.

How to precisely formulate sophisticated stopping strategies has been a challenge in continuous time, and the next section focuses on resolving this.

3 Equilibrium stopping policies

3.1 Objective of a sophisticated agent

Since one may re-evaluate and change one's choice of stopping times over time, a stopping strategy is not a single stopping time, but a stopping policy defined below.

Definition 3.1 A Borel-measurable function $\tau : \mathbb{X} \to \{0, 1\}$ is called a *stopping policy*. We denote by $\mathcal{T}(\mathbb{X})$ the set of all stopping policies.

Given current time and state $(t, x) \in \mathbb{X}$, a policy $\tau \in \mathcal{T}(\mathbb{X})$ governs when an agent stops: the agent stops at the first time that $\tau(s, X_s^{t,x})$ yields the value 0, i.e., at the moment

$$\mathcal{L}\tau(t, x) := \inf\{s \geq t : \tau(s, X_s^{t,x}) = 0\}. \tag{3.1}$$

To show that $\mathcal{L}\tau(t, x)$ is a well-defined stopping time, we introduce the set

$$\ker(\tau) := \{(t, x) \in \mathbb{X} : \tau(t, x) = 0\}. \tag{3.2}$$

It is called the *kernel* of τ, which is the collection of (t, x) at which the policy τ suggests immediate stopping. Then $\mathcal{L}\tau(t, x)$ can be expressed as

$$\mathcal{L}\tau(t, x) = \inf\{s \geq t : (s, X_s^{t,x}) \in \ker(\tau)\}. \tag{3.3}$$

Lemma 3.2 *For any* $\tau \in \mathcal{T}(\mathbb{X})$ *and* $(t, x) \in \mathbb{X}$, *we have* $\ker(\tau) \in \mathcal{B}(\mathbb{X})$ *and* $\mathcal{L}\tau(t, x) \in \mathcal{T}_t$.

Proof The Borel-measurability of $\tau \in \mathcal{T}(\mathbb{X})$ immediately implies that $\ker(\tau) \in \mathcal{B}(\mathbb{X})$. In view of (3.3), $\mathcal{L}\tau(t, x)(\omega) = \inf\{s \geq t : (s, \omega) \in E\}$, where

$$E := \left\{(r, \omega) \in [t, \infty) \times \Omega : \left(r, X_r^{t,x}(\omega)\right) \in \ker(\tau)\right\}.$$

With $\ker(\tau) \in \mathcal{B}(\mathbb{X})$ and the process $X^{t,x}$ being progressively measurable, E is a progressively measurable set. Since the filtration $\mathbb{F}^t$ satisfies the usual conditions, [3, Theorem 2.1] asserts that $\mathcal{L}\tau(t, x)$ is an $\mathbb{F}^t$-stopping time. □

Remark 3.3 Recall the optimal stopping time $\widetilde{\tau}_{t,x}$ defined in (2.5) for all $(t, x) \in \mathbb{X}$. Define $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ by

$$\widetilde{\tau}(t, x) := \begin{cases} 0, & \text{if } \widetilde{\tau}_{t,x} = t, \\ 1, & \text{if } \widetilde{\tau}_{t,x} > t. \end{cases} \tag{3.4}$$

Note that $\widetilde{\tau}:\mathbb{X}\to\{0,1\}$ is indeed Borel-measurable because $\widetilde{\tau}_{t,x}=t$ if and only if

$$(t,x)\in\left\{(t,x)\in\mathbb{X}:g(x)=\sup_{\tau\in\mathcal{T}_t}\mathbb{E}^{t,x}[\delta(\tau-t)g(X_\tau)]\right\}\in\mathcal{B}(\mathbb{X}).$$

Following the standard terminology (see e.g. [34, 30]), we call $\widetilde{\tau}$ the naive stopping policy as it describes the behavior of a naive agent, discussed in Sect. 2.2.

Remark 3.4 Despite its name, the naive stopping policy $\widetilde{\tau}$ may readily satisfy certain optimality criteria. For example, the "dynamic optimality" recently proposed in Pedersen and Peskir [28] can be formulated in our case as follows: $\tau\in\mathcal{T}(\mathbb{X})$ is dynamically optimal if there is no other $\pi\in\mathcal{T}(\mathbb{X})$ such that

$$\mathbb{P}^{t,x}\Big[J\Big(\mathcal{L}\tau(t,x),X^{t,x}_{\mathcal{L}\tau(t,x)};\mathcal{L}\pi\big(\mathcal{L}\tau(t,x),X^{t,x}_{\mathcal{L}\tau(t,x)}\big)\Big)>g(X^{t,x}_{\mathcal{L}\tau(t,x)})\Big]>0$$

for some $(t,x)\in\mathbb{X}$. By (3.4) and Proposition 2.2, $\widetilde{\tau}$ is dynamically optimal as the above probability is always 0.

Example 3.5 Recall the setting of Example 2.6. A naive agent follows $\widetilde{\tau}\in\mathcal{T}(\mathbb{X})$, and the actual moment of stopping is

$$\mathcal{L}\widetilde{\tau}(t,x)=\inf\{s\geq t:\widetilde{\tau}(s,X^{t,x}_s)=0\}=\inf\{s\geq t:X^{t,x}_s\geq 1\},$$

which differs from the agent's original decision $\widetilde{\tau}_{t,x}$ in Example 2.6.

We now introduce equilibrium policies. Suppose that a stopping policy $\tau\in\mathcal{T}(\mathbb{X})$ is given to a sophisticated agent. At any $(t,x)\in\mathbb{X}$, the agent carries out the following game-theoretic reasoning: "assuming that all my future selves will follow $\tau\in\mathcal{T}(\mathbb{X})$, what is the best stopping strategy at the current time t in response to that?" Note that the agent at time t has only two possible actions: stopping and continuation. If she stops at time t, she gets $g(x)$ immediately. If she continues at time t, given that all her future selves will follow $\tau\in\mathcal{T}(\mathbb{X})$, she will eventually stop at the moment

$$\mathcal{L}^*\tau(t,x):=\inf\{s>t:\tau(s,X^{t,x}_s)=0\}=\inf\{s>t:(s,X^{t,x}_s)\in\ker(\tau)\},\qquad(3.5)$$

leading to the payoff

$$J\big(t,x;\mathcal{L}^*\tau(t,x)\big)=\mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\tau(t,x)-t\big)g(X_{\mathcal{L}^*\tau(t,x)})\big].$$

By the same argument as in Lemma 3.2, $\mathcal{L}^*\tau(t,x)$ is a well-defined stopping time in $\mathcal{T}_t$. Note the subtle difference between $\mathcal{L}\tau(t,x)$ and $\mathcal{L}^*\tau(t,x)$: with the latter, the agent at time t simply chooses to continue, with no regard to what $\tau\in\mathcal{T}(\mathbb{X})$ suggests at time t. This is why we have "$s>t$" in (3.5), instead of "$s\geq t$" in (3.1).

Now, we separate the space $\mathbb{X}$ into the three distinct regions

$$S_\tau:=\big\{(t,x)\in\mathbb{X}:g(x)>J\big(t,x;\mathcal{L}^*\tau(t,x)\big)\big\},$$
$$C_\tau:=\big\{(t,x)\in\mathbb{X}:g(x)<J\big(t,x;\mathcal{L}^*\tau(t,x)\big)\big\},$$
$$I_\tau:=\big\{(t,x)\in\mathbb{X}:g(x)=J\big(t,x;\mathcal{L}^*\tau(t,x)\big)\big\}.$$

Some conclusions can be drawn:

1. If $(t,x) \in S_\tau$, the agent should stop immediately at time t.
2. If $(t,x) \in C_\tau$, the agent should continue at time t.
3. If $(t,x) \in I_\tau$, the agent is indifferent between stopping and continuation at the current time; there is then no incentive for the agent to deviate from the originally assigned stopping strategy $\tau(t,x)$.

To summarize, for any $(t,x) \in \mathbb{X}$, the best stopping strategy at the current time (in response to future selves following $\tau \in \mathcal{T}(\mathbb{X})$) is

$$\Theta\tau(t,x) := \begin{cases} 0 & \text{for } (t,x) \in S_\tau, \\ 1 & \text{for } (t,x) \in C_\tau, \\ \tau(t,x) & \text{for } (t,x) \in I_\tau. \end{cases} \tag{3.6}$$

The next result shows that $\Theta\tau : \mathbb{X} \to \{0,1\}$ is again a stopping policy.

Lemma 3.6 *For any $\tau \in \mathcal{T}(\mathbb{X})$, S_τ, C_τ and I_τ belong to $\mathcal{B}(\mathbb{X})$, and $\Theta\tau \in \mathcal{T}(\mathbb{X})$.*

Proof Since $\mathcal{L}^*\tau(t,x)$ is the first hitting time to the Borel set $\ker(\tau)$, the mapping $(t,x) \mapsto J(t,x;\mathcal{L}^*\tau(t,x)) = \mathbb{E}^{t,x}[\delta(\mathcal{L}^*\tau(t,x)-t)g(X_{\mathcal{L}^*\tau(t,x)})]$ is Borel-measurable. It follows that S_τ, I_τ and C_τ all belong to $\mathcal{B}(\mathbb{X})$. Now, in view of (3.6), we have $\ker(\Theta\tau) = S_\tau \cup (I_\tau \cap \ker(\tau)) \in \mathcal{B}(\mathbb{X})$, which implies that $\Theta\tau \in \mathcal{T}(\mathbb{X})$. □

By Lemma 3.6, Θ can be viewed as an operator acting on the space $\mathcal{T}(\mathbb{X})$. For any initial $\tau \in \mathcal{T}(\mathbb{X})$, $\Theta : \mathcal{T}(\mathbb{X}) \to \mathcal{T}(\mathbb{X})$ generates a new policy $\Theta\tau \in \mathcal{T}(\mathbb{X})$. The switch from τ to $\Theta\tau$ corresponds to an additional level of strategic reasoning in game theory, as discussed later below Corollary 3.17.

Definition 3.7 $\tau \in \mathcal{T}(\mathbb{X})$ is an *equilibrium stopping policy* if $\Theta\tau(t,x) = \tau(t,x)$ for all $(t,x) \in \mathbb{X}$. We denote by $\mathcal{E}(\mathbb{X})$ the collection of all equilibrium stopping policies.

The term "equilibrium" is used as a connection to subgame-perfect Nash equilibria in an intertemporal game among current self and future selves. This equilibrium idea was invoked in stochastic control under time inconsistency; see e.g. [12, 14, 13, 6]. A contrast with the stochastic control literature needs to be pointed out.

Remark 3.8 In time-inconsistent stochastic control, using local perturbations of strategies on small time intervals $[t, t+\varepsilon]$ is the standard way to define equilibrium controls. In our case, the local perturbation is carried out instantaneously at time t. This is because an instantaneously modified stopping strategy may already change the expected discounted payoff significantly, whereas a control perturbed only at time t yields no effect.

The first question concerning Definition 3.7 is the existence of an equilibrium stopping policy. Finding at least one such a policy turns out to be easy.

Remark 3.9 Define $\tau \in \mathcal{T}(X)$ by $\tau(t,x) := 0$ for all $(t,x) \in \mathbb{X}$. Then by definition, $\mathcal{L}\tau(t,x) = \mathcal{L}^*\tau(t,x) = t$, and thus $J(t,x;\mathcal{L}^*\tau(t,x)) = g(x)$ for all $(t,x) \in \mathbb{X}$.

This implies $I_\tau = \mathbb{X}$. We then conclude from (3.6) that $\Theta\tau(t,x) = \tau(t,x)$ for all $(t,x) \in \mathbb{X}$, which shows $\tau \in \mathcal{E}(\mathbb{X})$. We call it the trivial equilibrium stopping policy.

Example 3.10 Recall the setting in Example 2.5. Observe from (2.8) and (3.4) that $\mathcal{L}^*\widetilde{\tau}(t,x) = T$ for all $(t,x) \in \mathbb{X}$. Then we obtain

$$\delta\big(\mathcal{L}^*\widetilde{\tau}(t,x) - t\big) X^{t,x}_{\mathcal{L}^*\widetilde{\tau}(t,x)} = \frac{X^{t,x}_T}{1+T-t} = \frac{x e^{\frac{1}{2}(T-t)}}{1+T-t}.$$

Since $e^{\frac{1}{2}s} = 1+s$ has two solutions $s=0$ and $s = s^* \approx 2.513$, and $e^{\frac{1}{2}s} > 1+s$ if and only if $s > s^*$, the above equation implies that we have $S_{\widetilde{\tau}} = \{(t,x) : t < T - s^*\}$, $C_{\widetilde{\tau}} = \{(t,x) : t \in (T-s^*, T)\}$ and $I_{\widetilde{\tau}} = \{(t,x) : t = T - s^* \text{ or } T\}$. We therefore get

$$\Theta\widetilde{\tau}(t,x) = \begin{cases} 0 & \text{for } t < T - s^*, \\ 1 & \text{for } t \geq T - s^*. \end{cases}$$

Whereas a naive smoker delays quitting smoking indefinitely (as in Example 2.5), the first level of strategic reasoning (i.e., applying Θ to $\widetilde{\tau}$ once) recognizes this procrastination behavior and pushes the smoker to quit immediately, unless he is already too old (i.e., $t \geq T - s^*$). It can be checked that $\Theta\widetilde{\tau}$ is already an equilibrium, i.e., $\Theta^2\widetilde{\tau}(t,x) = \Theta\widetilde{\tau}(t,x)$ for all $(t,x) \in \mathbb{X}$.

It is worth noting that in the classical case of exponential discounting characterized by (2.7), the naive stopping policy $\widetilde{\tau}$ in (3.4) is already an equilibrium.

Proposition 3.11 *Under* (2.7), $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ *defined in* (3.4) *belongs to* $\mathcal{E}(\mathbb{X})$.

Proof The proof is relegated to Appendix A.1. □

3.2 The main result

In this subsection, we look for equilibrium policies through fixed-point iterations. For any $\tau \in \mathcal{T}(\mathbb{X})$, we apply Θ to τ repetitively until we reach an equilibrium policy. In short, we define τ_0 by

$$\tau_0(t,x) := \lim_{n\to\infty} \Theta^n \tau(t,x) \qquad \forall (t,x) \in \mathbb{X}, \tag{3.7}$$

and take it as a candidate equilibrium policy. To make this argument rigorous, we need to show (i) the limit in (3.7) exists, so that τ_0 is well defined; (ii) τ_0 is indeed an equilibrium policy, i.e., $\Theta\tau_0 = \tau_0$. To this end, we impose the following condition.

Assumption 3.12 The function δ satisfies $\delta(s)\delta(t) \leq \delta(s+t)$ for all $s,t \geq 0$.

Assumption 3.12 is closely related to *decreasing impatience (DI)* in behavioral economics. It is well documented in empirical studies, e.g. [36, 22, 21], that people exhibit DI: when choosing between two rewards, they are more willing to wait for the larger reward (more patient) when these two rewards are further away in time. For instance, in the two scenarios (i) getting \$100 today or \$110 tomorrow, and (ii) getting \$100 in 100 days or \$110 in 101 days, people tend to choose \$100 in (i), but \$110 in (ii).

Following [31, Definition 1] and [23, 24], DI can be formulated in the current context as follows: the discount function δ induces DI if

$$\text{for any } s > 0, \text{ the function } t \mapsto \frac{\delta(t+s)}{\delta(t)} \text{ is strictly increasing.} \tag{3.8}$$

Many discount functions in behavioral economics satisfy (3.8). This includes *hyperbolic discounting* $\delta(t) := \frac{1}{1+\beta t}$ with $\beta > 0$ (see e.g. [1, 31]), *generalized hyperbolic discounting* $\delta(t) := \frac{1}{(1+\beta t)^k}$ with $\beta, k > 0$ (see e.g. [21, 20]), and *pseudo-exponential discounting* $\delta(t) := \lambda e^{-\rho_1 t} + (1-\lambda)e^{-\rho_2 t}$ with $\lambda \in (0,1)$ and $\rho_1, \rho_2 > 0$ (see e.g. [12, 18, 14]), among others. Observe that (3.8) readily implies Assumption 3.12, as $\delta(t+s)/\delta(t) \geq \delta(s)/\delta(0) = \delta(s)$ for all $s, t \geq 0$. That is, Assumption 3.12 is automatically true under DI. Note that Assumption 3.12 is more general than DI, as it obviously includes the classical case of exponential discounting characterized by (2.7).

The main convergence result for (3.7) is the following.

Proposition 3.13 *Let Assumption 3.12 hold. If $\tau \in \mathcal{T}(\mathbb{X})$ satisfies*

$$\ker(\tau) \subseteq \ker(\Theta\tau), \tag{3.9}$$

then

$$\ker(\Theta^n \tau) \subseteq \ker(\Theta^{n+1}\tau) \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Hence, τ_0 in (3.7) is a well-defined element in $\mathcal{T}(\mathbb{X})$, with $\ker(\tau_0) = \bigcup_{n\in\mathbb{N}} \ker(\Theta^n \tau)$.

Proof The proof is relegated to Appendix A.2. □

Condition (3.9) means that at any $(t,x) \in \mathbb{X}$ where the initial policy τ indicates immediate stopping, the new policy $\Theta\tau$ agrees with it; however, it is possible that at some $(t,x) \in \mathbb{X}$ where τ indicates continuation, $\Theta\tau$ suggests immediate stopping based on the game-theoretic reasoning in Sect. 3.1. Note that (3.9) is not very restrictive, as it already covers all hitting times to subsets of $\mathbb{X}$ that are open (or more generally, half-open in $[0,\infty)$ and open in $\mathbb{R}^d$), as explained below.

Remark 3.14 Let E be a subset of $\mathbb{X}$ that is "open" in the following sense: for any $(t,x) \in E$, there exists $\varepsilon > 0$ such that $(t,x) \in [t, t+\varepsilon) \times B_\varepsilon(x) \subseteq E$, with $B_\varepsilon(x)$ defined by $\{y \in \mathbb{R}^d : |y-x| < \varepsilon\}$. Define $\tau \in \mathcal{T}(\mathbb{X})$ by $\tau(t,x) = 0$ if and only if $(t,x) \in E$. Since $\ker(\tau) = E$ is "open", for any $(t,x) \in \ker(\tau)$, we have $\mathcal{L}^*\tau(t,x) = t$, which implies $(t,x) \in I_\tau$. Thus, $\ker(\tau) \subseteq I_\tau$. It follows that (3.9) holds, as

$$\ker(\tau) \subseteq S_\tau \cup \ker(\tau) = S_\tau \cup (I_\tau \cap \ker(\tau)) = \ker(\Theta\tau),$$

where the last equality is due to (3.6).

The stopping policy τ in this setting corresponds to the hitting times

$$T_{t,x} := \inf\{s \geq t : (s, X_s^{t,x}) \in E\}$$

for all $(t, x) \in \mathbb{X}$. In particular, if $E = [0, \infty) \times F$ where F is an open set in $\mathbb{R}^d$, the corresponding hitting times are $T'_{t,x} := \inf\{s \geq t : X_s^{t,x} \in F\}$, $(t, x) \in \mathbb{X}$.

Moreover, the naive stopping policy $\widetilde{\tau}$ also satisfies (3.9).

Proposition 3.15 $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ *defined in* (3.4) *satisfies* (3.9).

Proof The proof is relegated to Appendix A.3. □

The next theorem is the main result of our paper. It shows that the fixed-point iteration in (3.7) indeed converges to an equilibrium policy.

Theorem 3.16 *Let Assumption* 3.12 *hold. If* $\tau \in \mathcal{T}(\mathbb{X})$ *satisfies* (3.9), *then* τ_0 *defined in* (3.7) *belongs to* $\mathcal{E}(\mathbb{X})$.

Proof The proof is relegated to Appendix A.4. □

The following result for the naive stopping policy $\widetilde{\tau}$ defined in (3.4) is a direct consequence of Proposition 3.15 and Theorem 3.16.

Corollary 3.17 *Let Assumption* 3.12 *hold. The stopping policy* $\widetilde{\tau}_0 \in \mathcal{T}(\mathbb{X})$ *defined by*

$$\widetilde{\tau}_0(t, x) := \lim_{n\to\infty} \Theta^n \widetilde{\tau}(t, x) \qquad \forall (t, x) \in \mathbb{X} \tag{3.11}$$

belongs to $\mathcal{E}(\mathbb{X})$.

Our iterative approach in (3.7) contributes to the literature of time inconsistency in two ways. First, the standard approach for finding equilibrium strategies in continuous time is solving a system of nonlinear equations (the so-called extended HJB equation), as proposed in [14] and [5]. Solving this system of equations is difficult; and even when it is solved (as in the special cases in [14] and [5]), we just obtain one particular equilibrium, and it is unclear how other equilibrium strategies can be found. Our iterative approach provides a potential remedy here. We can find different equilibria simply by starting the iteration (3.7) with different initial policies $\tau \in \mathcal{T}(\mathbb{X})$. In some cases, we are able to find *all* equilibria, and obtain a complete characterization of $\mathcal{E}(\mathbb{X})$; see Proposition 4.5 below.

Second, while the continuous-time formulation of equilibrium strategies was initiated in [12], the "origin" of an equilibrium strategy has not been addressed. This question is important as people do not start with using equilibrium strategies. People have their own initial strategies, determined by a variety of factors such as classical optimal stopping theory, personal habits, and popular rules of thumb in the market. Once an agent starts to do game-theoretic reasoning and look for equilibrium strategies, she is not satisfied with an arbitrary equilibrium. Instead, she works on improving her initial strategy to turn it into an equilibrium. This improving process is absent from [12, 14, 5], but it is in fact well known in game theory as the hierarchy of strategic reasoning in [32] and [33]. Our iterative approach embodies this framework: given an initial $\tau \in \mathcal{T}(\mathbb{X})$, $\Theta^n\tau \in \mathcal{T}(\mathbb{X})$ corresponds to level-n strategic reasoning in [33], and $\tau_0 := \lim_{n\to\infty} \Theta^n\tau$ reflects full rationality of "smart$_\infty$" players in [32]. Hence our formulation complements the literature of time inconsistency in that it not only defines what an equilibrium is, but explains where an equilibrium is coming from. This in turn provides "agent-specific" results: it assigns one specific equilibrium to each agent according to her initial behavior.

In particular, Corollary 3.17 specifies the connection between the naive behavior and the sophisticated one. While these behaviors have been widely discussed in the literature, their relation has not been stated mathematically as precisely as in (3.11).

3.3 The time-homogeneous case

Suppose the state process X is time-homogeneous, i.e., $X_s = f(W_s)$ for some measurable $f : \mathbb{R}^d \to \mathbb{R}$, or the coefficients b and σ in (2.1) do not depend on t. The objective function (2.2) then reduces to $J(x;\tau) := \mathbb{E}^x[\delta(\tau)g(X_\tau)]$ for $x \in \mathbb{R}^d$ and $\tau \in \mathcal{T}$, where the superscript of $\mathbb{E}^x$ means $X_0 = x$. The decision to stop or to continue then depends on the current state x only. The formulation in Sect. 3.1 reduces to

Definition 3.18 When X is time-homogeneous, a Borel-measurable $\tau : \mathbb{R}^d \to \{0,1\}$ is called a *stopping policy*, and we denote by $\mathcal{T}(\mathbb{R}^d)$ the set of all stopping policies. Given $\tau \in \mathcal{T}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define, in analogy to (3.2), (3.1), and (3.5), $\ker(\tau) := \{x \in \mathbb{R}^d : \tau(x) = 0\}$, $\mathcal{L}\tau(x) := \inf\{t \geq 0 : \tau(X_t^x) = 0\}$, together with $\mathcal{L}^*\tau(x) := \inf\{t > 0 : \tau(X_t^x) = 0\}$. Furthermore, we say that $\tau \in \mathcal{T}(\mathbb{R}^d)$ is an *equilibrium stopping policy* if $\Theta\tau(x) = \tau(x)$ for all $x \in \mathbb{R}^d$, where

$$\Theta\tau(x) := \begin{cases} 0, & \text{if } x \in S_\tau := \{x : g(x) > \mathbb{E}^x[\delta(\mathcal{L}^*\tau(x))g(X_{\mathcal{L}^*\tau(x)})]\}, \\ 1, & \text{if } x \in C_\tau := \{x : g(x) < \mathbb{E}^x[\delta(\mathcal{L}^*\tau(x))g(X_{\mathcal{L}^*\tau(x)})]\}, \\ \tau(x), & \text{if } x \in I_\tau := \{x : g(x) = \mathbb{E}^x[\delta(\mathcal{L}^*\tau(x))g(X_{\mathcal{L}^*\tau(x)})]\}. \end{cases} \tag{3.12}$$

Remark 3.19 When X is time-homogeneous, all the results in Sect. 3.2 hold with $\mathcal{T}(\mathbb{X})$, $\mathcal{E}(\mathbb{X})$, $\ker(\tau)$ and Θ replaced by the corresponding ones in Definition 3.18. Proofs of these statements are similar to, and in fact easier than, those in Sect. 3.2, thanks to the homogeneity in time.

4 A detailed case study: stopping of BES(1)

In this section, we recall the setup of Example 2.6, with hyperbolic discount function

$$\delta(s) := \frac{1}{1+\beta s} \qquad \forall s \geq 0, \tag{4.1}$$

where $\beta > 0$ is a fixed parameter. The state process X is a one-dimensional Bessel process, i.e., $X_t = |W_t|$, $t \geq 0$, where W is a one-dimensional Brownian motion. With

X being time-homogeneous, we follow Definition 3.18 and Remark 3.19. Also, the classical optimal stopping problem (2.3) reduces to

$$v(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^x \left[\frac{X_\tau}{1+\beta\tau} \right] \qquad \text{for } x \in \mathbb{R}_+. \tag{4.2}$$

This can be viewed as a real options problem, as explained below.

By [35] and the references therein, when the surplus (or reserve) of an insurance company is much larger than the size of each individual claim, the dynamics of the surplus process can be approximated by $dR_t = \mu dt + \sigma dW_t$ with $\mu = p - \mathbb{E}[Z]$ and $\sigma = \sqrt{\mathbb{E}[Z^2]}$. Here, $p > 0$ is the premium rate, and Z is a random variable that represents the size of each claim. Suppose that an insurance company is non-profitable with $\mu = 0$, i.e., it uses all the premiums collected to cover incoming claims. Also assume that the company is large enough to be considered "systemically important", so that when its surplus hits zero, the government will provide monetary support to bring it back to positivity, as in the recent financial crisis. The dynamics of R is then a Brownian motion reflected at the origin. Thus, (4.2) describes a real options problem in which the management of a large non-profitable insurance company has the intention to liquidate or sell the company, and would like to decide when to do so.

An unusual feature of (4.2) is that the discounted process $(\delta(s)v(X_s^x))_{s\geq 0}$ may fail to be a supermartingale. This makes solving (4.2) for the optimal stopping time $\widetilde{\tau}_x$, defined in (2.5) with $t = 0$, nontrivial. As shown in Appendix B.1, we need an auxiliary value function, and we use the method of time-change in [27].

Proposition 4.1 *For any $x \in \mathbb{R}_+$, the optimal stopping time $\widetilde{\tau}_x$ of* (4.2) (*defined in* (2.5) *with* $t = 0$) *admits the explicit formula*

$$\widetilde{\tau}_x = \inf\{s \geq 0 : X_s^x \geq \sqrt{1/\beta + s}\}. \tag{4.3}$$

Hence, the naive stopping policy $\widetilde{\tau} \in \mathcal{T}(\mathbb{R}_+)$ defined in (3.4) *is given by*

$$\widetilde{\tau}(x) := \mathbf{1}_{[0,\sqrt{1/\beta})}(x) \qquad \forall x \in \mathbb{R}_+. \tag{4.4}$$

Proof The proof is relegated to Appendix B.1. □

4.1 Characterization of equilibrium policies

Lemma 4.2 *For any $\tau \in \mathcal{T}(\mathbb{R}_+)$, consider $\tau' \in \mathcal{T}(\mathbb{R}_+)$ with* $\ker(\tau') := \overline{\ker(\tau)}$. *Then $\mathcal{L}^*\tau(x) = \mathcal{L}\tau(x) = \mathcal{L}\tau'(x) = \mathcal{L}^*\tau'(x)$ for all $x \in \mathbb{R}_+$. Hence $\tau \in \mathcal{E}(\mathbb{R}_+)$ if and only if $\tau' \in \mathcal{E}(\mathbb{R}_+)$.*

Proof If $x \in \mathbb{R}_+$ is in the interior of $\ker(\tau)$, then $\mathcal{L}^*\tau(x)$, $\mathcal{L}\tau(x)$, $\mathcal{L}\tau'(x)$ and $\mathcal{L}^*\tau'(x)$ are all equal to 0. Since a one-dimensional Brownian motion W is monotone in no interval, if $x \in \ker(\tau') \setminus \ker(\tau)$, $\mathcal{L}^*\tau(x) = \mathcal{L}\tau(x) = 0 = \mathcal{L}\tau'(x) = \mathcal{L}^*\tau'(x)$; if $x \notin \ker(\tau')$, then

$$\begin{aligned}\mathcal{L}^*\tau(x) = \mathcal{L}\tau(x) &= \inf\{s \geq 0 : |W_s^x| \in \ker(\tau)\} \\ &= \inf\{s \geq 0 : |W_s^x| \in \overline{\ker(\tau)}\} = \mathcal{L}\tau'(x) = \mathcal{L}^*\tau'(x).\end{aligned}$$

Finally, we deduce from (3.12) and $\mathcal{L}^*\tau(x) = \mathcal{L}^*\tau'(x)$ for all $x \in \mathbb{R}_+$ that $\tau \in \mathcal{E}(\mathbb{R}_+)$ implies $\tau' \in \mathcal{E}(\mathbb{R}_+)$, and vice versa. □

The next result shows that every equilibrium policy corresponds to the hitting time to a certain threshold. Recall that a set $E \subseteq \mathbb{R}_+$ is called totally disconnected if the only nonempty connected subsets of E are singletons, i.e., E contains no interval.

Lemma 4.3 *For any* $\tau \in \mathcal{E}(\mathbb{R}_+)$, *define* $a := \inf \ker(\tau) \geq 0$. *Then the Borel set* $E := \{x \geq a : x \notin \ker(\tau)\}$ *is totally disconnected. Hence* $\overline{\ker(\tau)} = [a, \infty)$, *and the stopping policy* τ_a *defined by* $\tau_a(x) := \mathbf{1}_{[0,a)}(x)$ *for* $x \in \mathbb{R}_+$ *belongs to* $\mathcal{E}(\mathbb{R}_+)$.

Proof The proof is relegated to Appendix B.2. □

The converse question is for which $a \geq 0$ the policy $\tau_a \in \mathcal{T}(\mathbb{R})$ is an equilibrium. To answer this, we need to find the sets S_{τ_a}, C_{τ_a} and I_{τ_a} in (3.12). By Definition 3.18,

$$\mathcal{L}\tau_a(x) = T_a^x := \inf\{s \geq 0 : X_s^x \geq a\}, \quad \mathcal{L}^*\tau_a(x) = \inf\{s > 0 : X_s^x \geq a\}. \tag{4.5}$$

Note that $\mathcal{L}\tau_a(x) = \mathcal{L}^*\tau_a(x)$ by an argument similar to the proof of Lemma 4.2. As a result, for $x \geq a$, we have $J(x; \mathcal{L}^*\tau_a(x)) = J(x; 0) = x$, which implies

$$[a, \infty) \subseteq I_{\tau_a}. \tag{4.6}$$

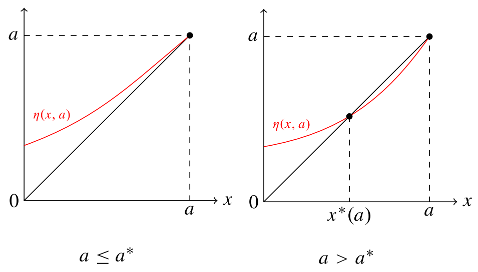
For $x \in [0, a)$, we need the lemma below, whose proof is relegated to Appendix B.3.

Lemma 4.4 *Recall* T_a^x *in* (4.5). *On the space* $\{(x, a) \in \mathbb{R}_+^2 : a \geq x\}$, *define*

$$\eta(x, a) := \mathbb{E}^x\left[\frac{a}{1 + \beta T_a^x}\right].$$

(i) *For any* $a \geq 0$, $x \mapsto \eta(x, a)$ *is strictly increasing and strictly convex on* $[0, a]$ *and satisfies* $0 < \eta(0, a) < a$ *and* $\eta(a, a) = a$.
(ii) *For any* $x \geq 0$, $\eta(x, a) \to 0$ *as* $a \to \infty$.
(iii) *There exists a unique* $a^* \in (0, 1/\sqrt{\beta})$ *such that for any* $a > a^*$, *there is a unique solution* $x^*(a) \in (0, a^*)$ *of* $\eta(x, a) = x$. *Hence* $\eta(x, a) > x$ *for* $x < x^*(a)$ *and* $\eta(x, a) < x$ *for* $x > x^*(a)$. *On the other hand,* $a \leq a^*$ *implies that* $\eta(x, a) > x$ *for all* $x \in (0, a)$.

The figure below illustrates $x \mapsto \eta(x, a)$ *under the scenarios* $a \leq a^*$ *and* $a > a^*$.



We now separate the case $x \in [0, a)$ into two sub-cases:

1. If $a \le a^*$, Lemma 4.4 (iii) shows that $J(x; \mathcal{L}^*\tau_a(x)) = \eta(x, a) > x$, and thus
$$[0, a) \subseteq C_{\tau_a}. \tag{4.7}$$

2. If $a > a^*$, then by Lemma 4.4 (iii),
$$J\big(x; \mathcal{L}^*\tau_a(x)\big) = \eta(x, a) \begin{cases} > x, & \text{if } x \in [0, x^*(a)), \\ = x, & \text{if } x = x^*(a), \\ < x, & \text{if } x \in (x^*(a), a). \end{cases} \tag{4.8}$$

By (4.6)–(4.8) and the definition of Θ in (3.12),

$$\begin{aligned} &\text{if } a \le a^*, \text{ then } \Theta\tau_a(x) = \mathbf{1}_{[0,a)}(x) + \tau_a(x)\mathbf{1}_{[a,\infty)}(x) \equiv \tau_a(x); \\ &\text{if } a > a^*, \text{ then } \Theta\tau_a(x) = \mathbf{1}_{[0,x^*(a))}(x) + \tau_a(x)\mathbf{1}_{\{x^*(a)\}\cup[a,\infty)}(x) \not\equiv \tau_a(x). \end{aligned} \tag{4.9}$$

Proposition 4.5 *τ_a defined in Lemma 4.3 belongs to $\mathcal{E}(\mathbb{R}_+)$ if and only if $a \in [0, a^*]$, where $a^* > 0$ is characterized by $a^* \int_0^\infty e^{-s}\sqrt{2\beta s}\tanh(a^*\sqrt{2\beta s})ds = 1$. Moreover,*
$$\mathcal{E}(\mathbb{R}_+) = \{\tau \in \mathcal{T}(\mathbb{R}_+) : \overline{\ker(\tau)} = [a, \infty) \text{ for some } a \in [0, a^*]\}. \tag{4.10}$$

Proof The derivation of the claimed equivalence is presented in the discussion above. By the proof of Lemma 4.4 in Appendix B.3, a^* satisfies $\eta_a(a^*, a^*) = 1$, which leads to the characterization of a^*. Now, for any $\tau \in \mathcal{T}(\mathbb{R}_+)$ with $\overline{\ker(\tau)} = [a, \infty)$ and $a \in [0, a^*]$, Lemma 4.2 implies $\tau \in \mathcal{E}(\mathbb{R}_+)$. On the other hand, for any $\tau \in \mathcal{E}(\mathbb{R}_+)$, set $a := \inf\ker(\tau)$. By Lemma 4.3, $\overline{\ker(\tau)} = [a, \infty)$ and $\tau_a \in \mathcal{E}(\mathbb{R}_+)$. The latter implies $a \in [0, a^*]$ and thus completes the proof. □

Remark 4.6 With $\beta = 1$, numerical computation gives $a^* \approx 0.946475$. It follows that for a general $\beta > 0$, we have $a^* \approx 0.946475/\sqrt{\beta}$.

For $a > a^*$, although $\tau_a \notin \mathcal{E}(\mathbb{R}_+)$ by Proposition 4.5, we may use the iteration in (3.7) to find a stopping policy in $\mathcal{E}(\mathbb{R}_+)$. Here, the repetitive application of Θ to τ_a has a simple structure: to reach an equilibrium, we need only *one* iteration.

Remark 4.7 Fix $a > a^*$, and recall $x^*(a) \in (0, a^*)$ in Lemma 4.4 (iii). By (4.9),
$$\Theta\tau_a(x) = \tau'_{x^*(a)}(x) := \mathbf{1}_{[0,x^*(a)]}(x) \qquad \text{for all } x \in \mathbb{R}_+.$$

Equivalently, $\ker(\Theta\tau_a) = \ker(\tau'_{x^*(a)}) = (x^*(a), \infty)$. Since $\overline{\ker(\tau'_{x^*(a)})} = [x^*(a), \infty)$ and $x^*(a) \in (0, a^*)$, we conclude from (4.10) that $\tau'_{x^*(a)} \in \mathcal{E}(\mathbb{R}_+)$.

Recall (3.11) which connects the naive and sophisticated behaviors. With the naive strategy $\widetilde{\tau} \in \mathcal{T}(\mathbb{R}_+)$ given explicitly in (4.4), Proposition 4.5 and Remark 4.6 imply $\widetilde{\tau} \notin \mathcal{E}(\mathbb{R}_+)$. We may find the corresponding equilibrium as in Remark 4.7.

Remark 4.8 Set $\widetilde{a} := 1/\sqrt{\beta}$. By (4.4) and Remark 4.7, $\Theta\widetilde{\tau} = \Theta\tau_{\widetilde{a}} = \tau'_{x^*(\widetilde{a})} \in \mathcal{E}(\mathbb{R}_+)$. In view of the proof of Lemma 4.4 in Appendix B.3, we can find $x^*(\widetilde{a})$ by solving $\eta(1/\sqrt{\beta}, x) = x$, i.e., $\frac{1}{\sqrt{\beta}}\int_0^\infty e^{-s}\cosh(x\sqrt{2\beta s})\,\mathrm{sech}(\sqrt{2s})ds = x$, for x. Numerical computation shows that $x^*(\widetilde{a}) \approx 0.92195/\sqrt{\beta}$, and thus $x^*(\widetilde{a}) < a^*$ by Remark 4.6. This verifies $\tau'_{x^*(\widetilde{a})} \in \mathcal{E}(\mathbb{R}_+)$, thanks to (4.10).

Remark 4.9 Recall the notions of "static optimality" and "dynamic optimality" from Remarks 2.3 and 3.4. By Proposition 4.1, $\widetilde{\tau}_x$ in (4.3) is statically optimal for fixed $x \in \mathbb{R}_+$, while $\widetilde{\tau}$ in (4.4) is dynamically optimal. This is reminiscent of the situation in Theorem 3 of [28]. Moreover, $\tau \in \mathcal{T}(\mathbb{R}_+)$ defined by $\tau(x) := \mathbf{1}_{[0,b)}(x)$, $x \in \mathbb{R}_+$, is dynamically optimal for all $b \geq \sqrt{1/\beta}$, thanks again to Proposition 4.1.

4.2 Further considerations on selecting equilibrium policies

In view of (4.10), it is natural to ask which equilibrium in $\mathcal{E}(\mathbb{R}_+)$ one should employ. According to the standard game theory literature discussed below Corollary 3.17, a sophisticated agent should employ the specific equilibrium generated by her initial stopping policy τ through the iteration (3.7). Now, imagine that an agent is "born" sophisticated: she does not have any previously determined initial stopping policy, and intends to apply an equilibrium policy straightaway. A potential way to formulate her stopping problem is to consider

$$\sup_{\tau\in\mathcal{E}(\mathbb{R}_+)} J\big(x;\mathcal{L}\tau(x)\big) = \sup_{a\in[0,a^*]} J\big(x;\mathcal{L}\tau_a(x)\big) = \sup_{a\in[x,a^*\vee x]} \mathbb{E}^x\left[\frac{a}{1+\beta T_a^x}\right], \tag{4.11}$$

where the first equality follows from Proposition 4.5 and Lemma 4.2.

Proposition 4.10 $\tau_{a^*} \in \mathcal{E}(\mathbb{R}_+)$ *solves* (4.11) *for all* $x \in \mathbb{R}_+$.

Proof Fix $a \in [0, a^*)$. For any $x \leq a$, we have $T_a^x \leq T_{a^*}^x$. Thus,

$$\begin{aligned} J(x;\mathcal{L}\tau_{a^*}(x)) &= \mathbb{E}^x\left[\frac{a^*}{1+\beta T_{a^*}^x}\right] = \mathbb{E}^x\left[\mathbb{E}^x\left[\frac{a^*}{1+\beta T_{a^*}^x}\,\middle|\, \mathcal{F}_{T_a^x}\right]\right] \\ &\geq \mathbb{E}^x\left[\frac{1}{1+\beta T_a^x}\mathbb{E}^a\left[\frac{a^*}{1+\beta T_{a^*}^a}\right]\right] > \mathbb{E}^x\left[\frac{a}{1+\beta T_a^x}\right] = J\big(x;\mathcal{L}\tau_a(x)\big), \end{aligned}$$

where the last inequality follows from Lemma 4.4 (iii). □

The conclusion is twofold. First, it is possible, at least under the current setting, to find one single equilibrium policy that solves (4.11) *for all* $x \in \mathbb{R}_+$. Second, this "optimal" equilibrium policy τ_{a^*} is different from $\tau'_{x^*(\widetilde{a})}$, the equilibrium generated by the naive policy $\widetilde{\tau}$ (see Remark 4.8). This indicates that the map

$$\Theta^* := \lim_{n\to\infty}\Theta^n : \mathcal{T}(\mathbb{X}) \to \mathcal{E}(\mathbb{X})$$

is in general nonlinear: while $\widetilde{\tau} \in \mathcal{T}(\mathbb{X})$ is constructed from optimal stopping times $\{\widetilde{\tau}_x\}_{x\in\mathbb{R}_+}$ (or "dynamically optimal" as in Remark 4.9), $\Theta^*(\widetilde{\tau}) = \tau'_{x^*(\widetilde{a})} \in \mathcal{E}(\mathbb{X})$ is not optimal under (4.11). As a matter of fact, this is not that surprising once we realize that $\widetilde{\tau}_x > \mathcal{L}\widetilde{\tau}(x) > \mathcal{L}\tau'_{x^*(\widetilde{a})}(x)$ for some $x \in \mathbb{R}_+$. The first inequality is essentially another way to describe time inconsistency, and the second follows from $\ker(\widetilde{\tau}) \subseteq \ker(\Theta\widetilde{\tau}) = \ker(\tau'_{x^*(\widetilde{a})})$. Thus the optimality of $\widetilde{\tau}_x$ for $\sup_{\tau\in\mathcal{T}} J(x;\tau)$ does not necessarily translate into the optimality of $\tau'_{x^*(\widetilde{a})}$ for $\sup_{\tau\in\mathcal{E}(\mathbb{R}_+)} J(x;\mathcal{L}\tau(x))$.

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Appendix A: Proofs for Sect. 3

Throughout this appendix, we constantly use the notation

$$\tau_n := \Theta^n \tau, \qquad n \in \mathbb{N}, \text{for any } \tau \in \mathcal{T}(\mathbb{X}). \tag{A.1}$$

A.1 Proof of Proposition 3.11

Fix $(t,x)\in\mathbb{X}$. We deal with the two cases $\widetilde{\tau}(t,x)=0$ and $\widetilde{\tau}(t,x)=1$ separately. If $\widetilde{\tau}(t,x)=0$, i.e., $\widetilde{\tau}_{t,x}=t$, then by (2.6),

$$g(x) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{t,x}[\delta(\tau-t)g(X_\tau)] \geq \mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\widetilde{\tau}(t,x)-t\big)g(X_{\mathcal{L}^*\widetilde{\tau}(t,x)})\big],$$

which implies $(t,x)\in S_{\widetilde{\tau}}\cup I_{\widetilde{\tau}}$. We then conclude from (3.6) that

$$\Theta\widetilde{\tau}(t,x) = \begin{Bmatrix} 0, & \text{if } (t,x)\in S_{\widetilde{\tau}} \\ \widetilde{\tau}(t,x), & \text{if } (t,x)\in I_{\widetilde{\tau}} \end{Bmatrix} = \widetilde{\tau}(t,x).$$

If $\widetilde{\tau}(t,x)=1$, then $\mathcal{L}^*\widetilde{\tau}(t,x)=\mathcal{L}\widetilde{\tau}(t,x)=\inf\{s\geq t:\widetilde{\tau}_{s,X_s^{t,x}}=s\}$. By (2.5) and (2.4), $\widetilde{\tau}_{s,X_s^{t,x}}=s$ means that

$$g\big(X_s^{t,x}(\omega)\big) = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}(\omega)}[\delta(\tau-s)g(X_\tau)],$$

which is equivalent to

$$\begin{aligned}\delta(s-t)g\big(X_s^{t,x}(\omega)\big) &= \delta(s-t)\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}(\omega)}[\delta(\tau-s)g(X_\tau)] \\ &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_s} \mathbb{E}^{s,X_s^{t,x}(\omega)}[\delta(\tau-t)g(X_\tau)] = Z_s^{t,x}(\omega),\end{aligned}$$

where the second equality follows from (2.7). As a result, we can conclude that $\mathcal{L}^*\widetilde{\tau}(t,x)=\inf\{s\geq t:\delta(s-t)g(X_s^{t,x})=Z_s^{t,x}\}=\widetilde{\tau}_{t,x}$. This together with (2.6) shows that

$$\mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\widetilde{\tau}(t,x)-t\big)g(X_{\mathcal{L}^*\widetilde{\tau}(t,x)})\big] = \mathbb{E}^{t,x}[\delta(\widetilde{\tau}_{t,x}-t)g(X_{\widetilde{\tau}_{t,x}})] \geq g(x),$$

which implies $(t,x) \in I_{\widetilde{\tau}} \cup C_{\widetilde{\tau}}$. By (3.6), we have

$$\Theta\widetilde{\tau}(t,x) = \begin{cases} \widetilde{\tau}(t,x), & \text{if } (t,x) \in I_{\widetilde{\tau}} \\ 1, & \text{if } (t,x) \in C_{\widetilde{\tau}} \end{cases} = \widetilde{\tau}(t,x).$$

We therefore have $\Theta\widetilde{\tau}(t,x) = \widetilde{\tau}(t,x)$ for all $(t,x) \in \mathbb{X}$, i.e., $\widetilde{\tau} \in \mathcal{E}(\mathbb{X})$.

A.2 Derivation of Proposition 3.13

To prove the technical result in Lemma A.1 below, we need to introduce shifted random variables as in Nutz [25]. Recall from Sect. 2 that Ω is the canonical path space. For any $t \geq 0$ and $\omega \in \Omega$, we define the concatenation of ω and $\tilde{\omega} \in \Omega$ at time t by

$$(\omega \otimes_t \tilde{\omega})_s := \omega_s \mathbf{1}_{[0,t)}(s) + \big(\tilde{\omega}_s - (\tilde{\omega}_t - \omega_t)\big)\mathbf{1}_{[t,\infty)}(s), \quad s \geq 0.$$

For any $\mathcal{F}_\infty$-measurable random variable $\xi : \Omega \to \mathbb{R}$, we define the shifted random variable $[\xi]_{t,\omega} : \Omega \to \mathbb{R}$, which is $\mathcal{F}^t_\infty$-measurable, by

$$[\xi]_{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}) \quad \forall \tilde{\omega} \in \Omega.$$

Given $\tau \in \mathcal{T}$, we write $\omega \otimes_{\tau(\omega)} \tilde{\omega}$ as $\omega \otimes_\tau \tilde{\omega}$, and $[\xi]_{\tau(\omega),\omega}(\tilde{\omega})$ as $[\xi]_{\tau,\omega}(\tilde{\omega})$. A detailed analysis of shifted random variables can be found in [4, Appendix A]; Proposition A.1 there implies that for fixed $(t,x) \in \mathbb{X}$, any $\theta \in \mathcal{T}_t$ and $\mathcal{F}^t_\infty$-measurable ξ with $\mathbb{E}^{t,x}[|\xi|] < \infty$ satisfy

$$\mathbb{E}^{t,x}[\xi \mid \mathcal{F}^t_\theta](\omega) = \mathbb{E}^{t,x}\big[[\xi]_{\theta,\omega}\big] \quad \text{for a.e. } \omega \in \Omega. \tag{A.2}$$

Lemma A.1 *For any $\tau \in \mathcal{T}(\mathbb{X})$ and $(t,x) \in \mathbb{X}$, define $t_0 := \mathcal{L}^*\tau_1(t,x) \in \mathcal{T}_t$ and $s_0 := \mathcal{L}^*\tau(t,x) \in \mathcal{T}_t$, with τ_1 as in (A.1). If $t_0 \leq s_0$, then for a.e. $\omega \in \{t < t_0\}$,*

$$g\big(X^{t,x}_{t_0}(\omega)\big) \leq \mathbb{E}^{t,x}[\delta(s_0 - t_0)g(X_{s_0}) \mid \mathcal{F}^t_{t_0}](\omega).$$

Proof For a.e. $\omega \in \{t < t_0\} \in \mathcal{F}_t$, we deduce from $t_0(\omega) = \mathcal{L}^*\tau_1(t,x)(\omega) > t$ that $\tau_1(s, X^{t,x}_s(\omega)) = 1$ for all $s \in (t, t_0(\omega))$. In view of (A.1) and (3.6), this implies $(s, X^{t,x}_s(\omega)) \notin S_\tau$ for all $s \in (t, t_0(\omega))$. Thus,

$$g\big(X^{t,x}_s(\omega)\big) \leq \mathbb{E}^{s,X^{t,x}_s(\omega)}\big[\delta\big(\mathcal{L}^*\tau(s,X_s) - s\big)g(X_{\mathcal{L}^*\tau(s,X_s)})\big] \quad \forall s \in \big(t, t_0(\omega)\big). \tag{A.3}$$

For any $s \in (t, t_0(\omega))$, note that

$$[t_0]_{s,\omega}(\tilde{\omega}) = t_0(\omega \otimes_s \tilde{\omega}) = \mathcal{L}^*\tau_1(t,x)(\omega \otimes_s \tilde{\omega}) = \mathcal{L}^*\tau_1\big(s, X^{t,x}_s(\omega)\big)(\tilde{\omega})$$

for all $\tilde{\omega} \in \Omega$. As $t_0 \leq s_0$, a similar calculation gives

$$[s_0]_{s,\omega}(\tilde{\omega}) = \mathcal{L}^*\tau\big(s, X^{t,x}_s(\omega)\big)(\tilde{\omega}).$$

We thus conclude from (A.3) that

$$g\big(X_s^{t,x}(\omega)\big) \le \mathbb{E}^{s,X_s^{t,x}(\omega)}\big[\delta([s_0]_{s,\omega}-s)g([X_{s_0}]_{s,\omega})\big]$$
$$\le \mathbb{E}^{s,X_s^{t,x}(\omega)}\big[\delta([s_0]_{s,\omega}-[t_0]_{s,\omega})g([X_{s_0}]_{s,\omega})\big] \quad \forall s \in \big(t, t_0(\omega)\big), \tag{A.4}$$

where the second line holds because δ is decreasing and also δ and g are both nonnegative. On the other hand, by (A.2), it holds a.s. that

$$\mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_s^t](\omega)$$
$$= \mathbb{E}^{t,x}\left[\delta([s_0]_{s,\omega}-[t_0]_{s,\omega})g([X_{s_0}^{t,x}]_{s,\omega})\right] \quad \forall s \ge t, s \in \mathbb{Q}.$$

Note that we used the countability of $\mathbb{Q}$ to obtain the above almost sure statement. This together with (A.4) shows that it holds a.s. that

$$g\big(X_s^{t,x}(\omega)\big)\, \mathbf{1}_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s) \le \mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_s^t](\omega)\, \mathbf{1}_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s). \tag{A.5}$$

Since our sample space Ω is the canonical space for Brownian motion with the right-continuous Brownian filtration $\mathbb{F}$, the martingale representation theorem holds under the current setting. This implies that every martingale has a continuous version. Let $(M_s)_{s\ge t}$ be the continuous version of the martingale $(\mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_s^t])_{s\ge t}$. Then (A.5) immediately implies that it holds a.s. that

$$g\big(X_s^{t,x}(\omega)\big)\mathbf{1}_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s) \le M_s(\omega)\, \mathbf{1}_{\{(t,t_0(\omega))\cap\mathbb{Q}\}}(s). \tag{A.6}$$

Also, using the right-continuity of M and (A.2), one can show that for any $\tau \in \mathcal{T}_t$, we have $M_\tau = \mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_\tau^t]$ a.s. Now we can take some $\Omega^* \in \mathcal{F}_\infty$ with $\mathbb{P}[\Omega^*]=1$ such that (A.6) holds true and $M_{t_0}(\omega) = \mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_{t_0}^t](\omega)$ for all $\omega \in \Omega^*$. For any $\omega \in \Omega^* \cap \{t < t_0\}$, take $(k_n) \subseteq \mathbb{Q}$ such that $k_n > t$ and $k_n \uparrow t_0(\omega)$. Then (A.6) implies that $g(X_{k_n}^{t,x}(\omega)) \le M_{k_n}(\omega)$, $\forall n \in \mathbb{N}$. As $n \to \infty$, we obtain from the continuity of $s \mapsto X_s$ and $z \mapsto g(z)$ and the left-continuity of $s \mapsto M_s$ that $g(X_{t_0}^{t,x}(\omega)) \le M_{t_0}(\omega) = \mathbb{E}^{t,x}[\delta(s_0-t_0)g(X_{s_0}) \mid \mathcal{F}_{t_0}^t](\omega)$. □

Now we are ready to prove Proposition 3.13.

Proof of Proposition 3.13 We prove (3.10) by induction. We know that the result holds for $n=0$ by (3.9). Now assume that (3.10) holds for $n=k \in \mathbb{N}\cup\{0\}$, and we intend to show that (3.10) also holds for $n=k+1$. Recall the notation in (A.1). Fix $(t,x) \in \ker(\tau_{k+1})$, i.e., $\tau_{k+1}(t,x)=0$. If $\mathcal{L}^*\tau_{k+1}(t,x)=t$, then (t,x) belongs to $I_{\tau_{k+1}}$. By (3.6), we get $\tau_{k+2}(t,x) = \Theta\tau_{k+1}(t,x) = \tau_{k+1}(t,x) = 0$, i.e., $(t,x) \in \ker(\tau_{k+2})$ as desired. We therefore assume below that $\mathcal{L}^*\tau_{k+1}(t,x) > t$.

By (3.6), $\tau_{k+1}(t,x)=0$ implies

$$g(x) \ge \mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\tau_k(t,x)-t\big)g(X_{\mathcal{L}^*\tau_k(t,x)})\big]. \tag{A.7}$$

Let $t_0 := \mathcal{L}^*\tau_{k+1}(t,x)$ and $s_0 := \mathcal{L}^*\tau_k(t,x)$. Under the induction hypothesis that $\ker(\tau_k) \subseteq \ker(\tau_{k+1})$, we have $t_0 \le s_0$ as t_0 and s_0 are hitting times to $\ker(\tau_{k+1})$ and

$\ker(\tau_k)$, respectively; see (3.5). Using (A.7), $t_0 \le s_0$, Assumption 3.12 and g being nonnegative, we obtain

$$\begin{aligned} g(x) &\ge \mathbb{E}^{t,x}[\delta(s_0 - t)g(X_{s_0})] \\ &\ge \mathbb{E}^{t,x}[\delta(t_0 - t)\delta(s_0 - t_0)g(X_{s_0})] \\ &= \mathbb{E}^{t,x}\big[\delta(t_0 - t)\mathbb{E}^{t,x}[\delta(s_0 - t_0)g(X_{s_0}) \mid \mathcal{F}^t_{t_0}]\big] \\ &\ge \mathbb{E}^{t,x}[\delta(t_0 - t)g(X_{t_0})], \end{aligned}$$

where the third line follows from the tower property of conditional expectations and the fourth is due to Lemma A.1. This implies $(t, x) \notin C_{\tau_{k+1}}$ and thus

$$\tau_{k+2}(t, x) = \begin{Bmatrix} 0 & \text{for } (t, x) \in S_{\tau_1} \\ \tau_{k+1}(t, x) & \text{for } (t, x) \in I_{\tau_1} \end{Bmatrix} = 0.$$

That is, $(t, x) \in \ker(\tau_{k+2})$. Thus, we conclude that $\ker(\tau_{k+1}) \subseteq \ker(\tau_{k+2})$ as desired.

It remains to show that τ_0 defined in (3.7) is a stopping policy. Observe that for any $(t, x) \in \mathbb{X}$, $\tau_0(t, x) = 0$ if and only if $\Theta^n\tau(t, x) = 0$, i.e., $(t, x) \in \ker(\Theta^n\tau)$, for n large enough. This together with (3.10) implies that

$$\{(t, x) \in \mathbb{X} : \tau_0(t, x) = 0\} = \bigcup_{n\in\mathbb{N}} \ker(\Theta^n\tau) \in \mathcal{B}(\mathbb{X}).$$

Hence $\tau_0 : \mathbb{X} \to \{0, 1\}$ is Borel-measurable and thus an element in $\mathcal{T}(\mathbb{X})$. □

A.3 Proof of Proposition 3.15

Fix $(t, x) \in \ker(\widetilde{\tau})$. Since $\widetilde{\tau}(t, x) = 0$, i.e., $\widetilde{\tau}_{t,x} = t$, (2.5), (2.4) and (2.6) imply

$$g(x) = \sup_{\tau\in\mathcal{T}_t} \mathbb{E}^{t,x}[\delta(\tau - t)g(X_\tau)] \ge \mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\widetilde{\tau}(t, x) - t\big)g(X_{\mathcal{L}^*\widetilde{\tau}(t,x)})\big].$$

This shows that $(t, x) \in S_{\widetilde{\tau}} \cup I_{\widetilde{\tau}}$. Thus we have $\ker(\widetilde{\tau}) \subseteq S_{\widetilde{\tau}} \cup I_{\widetilde{\tau}}$. It follows that

$$\ker(\widetilde{\tau}) = \big(\ker(\widetilde{\tau}) \cap S_{\widetilde{\tau}}\big) \cup \big(\ker(\widetilde{\tau}) \cap I_{\widetilde{\tau}}\big) \subseteq S_{\widetilde{\tau}} \cup \big(\ker(\widetilde{\tau}) \cap I_{\widetilde{\tau}}\big) = \ker(\Theta\widetilde{\tau}),$$

where the last equality follows from (3.6).

A.4 Derivation of Theorem 3.16

Lemma A.2 *Suppose Assumption* 3.12 *holds and* $\tau \in \mathcal{T}(\mathbb{X})$ *satisfies* (3.9). *Then* τ_0 *defined in* (3.7) *satisfies*

$$\mathcal{L}^*\tau_0(t, x) = \lim_{n\to\infty} \mathcal{L}^*\Theta^n\tau(t, x) \quad \forall (t, x) \in \mathbb{X}.$$

Proof We use the notation in (A.1). Recall that we have $\ker(\tau_n) \subseteq \ker(\tau_{n+1})$ for all $n \in \mathbb{N}$ and $\ker(\tau_0) = \bigcup_{n\in\mathbb{N}} \ker(\tau_n)$ from Proposition 3.13. By (3.5), this implies that $(\mathcal{L}^*\tau_n(t, x))_{n\in\mathbb{N}}$ is a nonincreasing sequence of stopping times and

$$\mathcal{L}^*\tau_0(t, x) \le t_0 := \lim_{n\to\infty} \mathcal{L}^*\tau_n(t, x).$$

It remains to show that $\mathcal{L}^*\tau_0(t, x) \ge t_0$. We deal with the following two cases.

(i) On $\{\omega \in \Omega : \mathcal{L}^*\tau_0(t,x)(\omega) = t\}$: By (3.5), there exists a sequence $(t_m)_{m\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on $\omega \in \Omega$, such that $t_m \downarrow t$ and $\tau_0(t_m, X^{t,x}_{t_m}(\omega)) = 0$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, by the definition of τ_0 in (3.7), there exists $n^* \in \mathbb{N}$ large enough such that $\tau_{n^*}(t_m, X^{t,x}_{t_m}(\omega)) = 0$, which implies $\mathcal{L}^*\tau_{n^*}(t,x)(\omega) \le t_m$. Since $(\mathcal{L}^*\tau_n(t,x))_{n\in\mathbb{N}}$ is nonincreasing, we have $t_0(\omega) \le \mathcal{L}^*\tau_{n^*}(t,x)(\omega) \le t_m$. With $m \to \infty$, we obtain $t_0(\omega) \le t = \mathcal{L}^*\tau_0(t,x)(\omega)$.

(ii) On $\{\omega \in \Omega : \mathcal{L}^*\tau_0(t,x)(\omega) > t\}$: Set $s_0 := \mathcal{L}^*\tau_0(t,x)$ and focus on the value of $\tau_0(s_0(\omega), X^{t,x}_{s_0}(\omega))$. If $\tau_0(s_0(\omega), X^{t,x}_{s_0}(\omega)) = 0$, then by (3.7) there exists $n^* \in \mathbb{N}$ large enough such that $\tau_{n^*}(s_0(\omega), X^{t,x}_{s_0}(\omega)) = 0$. Since $(\mathcal{L}^*\tau_n(t,x))_{n\in\mathbb{N}}$ is nonincreasing, $t_0(\omega) \le \mathcal{L}^*\tau_{n^*}(t,x)(\omega) \le s_0(\omega)$ as desired. If $\tau_0(s_0(\omega), X^{t,x}_{s_0}(\omega)) = 1$, then by (3.5), there exists a sequence $(t_m)_{m\in\mathbb{N}}$ in $\mathbb{R}_+$, depending on $\omega \in \Omega$, such that $t_m \downarrow s_0(\omega)$ and $\tau_0(t_m, X^{t,x}_{t_m}(\omega)) = 0$ for all $m \in \mathbb{N}$. Then we can argue as in case (i) to show that $t_0(\omega) \le s_0(\omega)$ as desired. □

Now we are ready to prove Theorem 3.16.

Proof of Theorem 3.16 By Proposition 3.13, $\tau_0 \in \mathcal{T}(\mathbb{X})$ is well defined. For simplicity, we use the notation in (A.1). Fix $(t,x) \in \mathbb{X}$. If $\tau_0(t,x) = 0$, then (3.7) gives $\tau_n(t,x) = 0$ for n large enough. Since $\tau_n(t,x) = \Theta\tau_{n-1}(t,x)$, we deduce from "$\tau_n(t,x) = 0$ for n large enough" and (3.6) that $(t,x) \in S_{\tau_{n-1}} \cup I_{\tau_{n-1}}$ for n large enough. That is, $g(x) \ge \mathbb{E}^{t,x}[\delta(\mathcal{L}^*\tau_{n-1}(t,x) - t)g(X_{\mathcal{L}^*\tau_{n-1}(t,x)})]$ for n large enough. With $n \to \infty$, the dominated convergence theorem and Lemma A.2 yield

$$g(x) \ge \mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\tau_0(t,x) - t\big)g(X_{\mathcal{L}^*\tau_0(t,x)})\big],$$

which shows that $(t,x) \in S_{\tau_0} \cup I_{\tau_0}$. We then deduce from (3.6) and $\tau_0(t,x) = 0$ that $\Theta\tau_0(t,x) = \tau_0(t,x)$. On the other hand, if $\tau_0(t,x) = 1$, then (3.7) gives $\tau_n(t,x) = 1$ for n large enough. Since $\tau_n(t,x) = \Theta\tau_{n-1}(t,x)$, we deduce from "$\tau_n(t,x) = 1$ for n large enough" and (3.6) that $(t,x) \in C_{\tau_{n-1}} \cup I_{\tau_{n-1}}$ for n large enough. That is, $g(x) \le \mathbb{E}^{t,x}[\delta(\mathcal{L}^*\tau_{n-1}(t,x) - t)g(X_{\mathcal{L}^*\tau_{n-1}(t,x)})]$ for n large enough. With $n \to \infty$, the dominated convergence theorem and Lemma A.2 yield

$$g(x) \le \mathbb{E}^{t,x}\big[\delta\big(\mathcal{L}^*\tau_0(t,x) - t\big)g(X_{\mathcal{L}^*\tau_0(t,x)})\big],$$

which shows that $(t,x) \in C_{\tau_0} \cup I_{\tau_0}$. We then deduce from (3.6) and $\tau_0(t,x) = 1$ that $\Theta\tau_0(t,x) = \tau_0(t,x)$. We therefore conclude that $\tau_0 \in \mathcal{E}(\mathbb{X})$. □

Appendix B: Proofs for Sect. 4

B.1 Derivation of Proposition 4.1

In the classical case of exponential discounting, (2.7) ensures that for all $s \ge 0$,

$$\delta(s)v(X^x_s) = \sup_{\tau\in\mathcal{T}} \mathbb{E}^{X^x_s}[\delta(s+\tau)g(X_\tau)] = \sup_{\tau\in\mathcal{T}_s} \mathbb{E}^x[\delta(\tau)g(X_\tau)\,|\,\mathcal{F}_s], \tag{B.1}$$

which shows that $(\delta(s)v(X^x_s))_{s\ge 0}$ is a supermartingale. Under hyperbolic discounting (4.1), since $\delta(r_1)\delta(r_2) \le \delta(r_1 + r_2)$ for all $r_1, r_2 \ge 0$, $(\delta(s)v(X^x_s))_{s\ge t}$ need no longer be a supermartingale as the first equality in the above equation fails.

To overcome this, we introduce an auxiliary value function: for $(s,x)\in\mathbb{R}_+^2$,

$$V(s,x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^x[\delta(s+\tau)g(X_\tau)] = \sup_{\tau\in\mathcal{T}} \mathbb{E}^x\left[\frac{X_\tau}{1+\beta(s+\tau)}\right].$$

By definition, $V(0,x)=v(x)$, and $(V(s,X_s^x))_{s\ge 0}$ is a supermartingale as $V(s,X_s^x)$ is equal to the right-hand side of (B.1).

Proof of Proposition 4.1 Recall that $X_s=|W_s|$ for a one-dimensional Brownian motion W. Let $y\in\mathbb{R}$ be the initial value of W and define $\bar V(s,y):=V(s,|y|)$. The associated variational inequality for $\bar V(s,y)$ is the following: for $(s,y)\in[0,\infty)\times\mathbb{R}$,

$$\min\left\{w_s(s,y)+\frac{1}{2}w_{yy}(s,y),\ w(s,y)-\frac{|y|}{1+\beta s}\right\}=0. \tag{B.2}$$

Taking $s\mapsto b(s)$ as the free boundary to be determined, we can rewrite (B.2) as

$$\begin{cases} w_s(s,y)+\frac{1}{2}w_{yy}(s,y)=0 \text{ and } w(s,y)>\frac{|y|}{1+\beta s} & \text{for } |y|<b(s),\\ w(s,y)=\frac{|y|}{1+\beta s} & \text{for } |y|\ge b(s). \end{cases} \tag{B.3}$$

Following [27], we propose the ansatz $w(s,y)=\frac{1}{\sqrt{1+\beta s}}h(\frac{y}{\sqrt{1+\beta s}})$. Equation (B.3) then becomes a one-dimensional free boundary problem, namely

$$\begin{cases} -\beta z h'(z)+h''(z)=\beta h(z) \text{ and } h(z)>|z| & \text{for } |z|<\frac{b(s)}{\sqrt{1+\beta s}},\\ h(z)=|z| & \text{for } |z|\ge\frac{b(s)}{\sqrt{1+\beta s}}. \end{cases} \tag{B.4}$$

As the variable s does not appear in the above ODE, take $b(s)=\alpha\sqrt{1+\beta s}$ for some $\alpha\ge 0$. The general solution of the differential equation in the first line of (B.4) is

$$h(z)=e^{\frac{\beta}{2}z^2}\left(c_1+c_2\sqrt{\frac{2}{\beta}}\int_0^{\sqrt{\beta/2}\,z}e^{-u^2}du\right),\quad (c_1,c_2)\in\mathbb{R}^2\,.$$

We then have

$$w(s,y)=\begin{cases} \frac{e^{\frac{\beta y^2}{2(1+\beta s)}}}{\sqrt{1+\beta s}}\left(c_1+c_2\sqrt{\frac{2}{\beta}}\int_0^{\frac{\sqrt{\beta/2}}{\sqrt{1+\beta s}}y}e^{-u^2}du\right), & |y|<\alpha\sqrt{1+\beta s},\\ \frac{|y|}{1+\beta s}, & |y|\ge\alpha\sqrt{1+\beta s}. \end{cases}$$

To find the parameters c_1,c_2 and α, we equate the values of $w(s,y)$ and its partial derivatives on both sides of the free boundary. This yields the equations

$$\alpha=e^{\frac{\beta}{2}\alpha^2}\left(c_1\pm c_2\sqrt{\frac{2}{\beta}}\int_0^{\sqrt{\beta/2}\,\alpha}e^{-u^2}du\right)\qquad\text{and}\qquad \alpha^2\beta+c_2=1.$$

The first equation implies $c_2 = 0$. Then, these equations together yield $\alpha = 1/\sqrt{\beta}$ and $c_1 = \alpha e^{-1/2}$. Thus we obtain

$$w(s, y) = \begin{cases} \frac{1}{\sqrt{\beta}\sqrt{1+\beta s}} \exp(\frac{1}{2}(\frac{\beta y^2}{1+\beta s} - 1)), & |y| < \sqrt{1/\beta + s}, \\ \frac{|y|}{1+\beta s}, & |y| \geq \sqrt{1/\beta + s}. \end{cases} \tag{B.5}$$

Note that $w(s, y) > \frac{|y|}{1+\beta s}$ for $|y| < \sqrt{1/\beta + s}$. Indeed, by defining the function

$$h(y) := \frac{1}{\sqrt{\beta}\sqrt{1+\beta s}} \exp\left(\frac{1}{2}\Big(\frac{\beta y^2}{1+\beta s} - 1\Big)\right) - \frac{y}{1+\beta s}$$

and observing that $h(0) > 0$, $h(\sqrt{1/\beta + s}) = 0$ and $h'(y) < \frac{1}{1+\beta s} - \frac{1}{1+\beta s} = 0$ for all $y \in (0, \sqrt{1/\beta + s})$, we conclude that $h(y) > 0$ for all $y \in [0, \sqrt{1/\beta + s})$, i.e., $w(s, y) > \frac{|y|}{1+\beta s}$ for $|y| < \sqrt{1/\beta + s}$. Also note that w is $\mathcal{C}^{1,1}$ on $[0, \infty) \times \mathbb{R}$ and $\mathcal{C}^{1,2}$ on the domain $\{(s, y) \in [0, \infty) \times \mathbb{R} : |y| < \sqrt{1/\beta + s}\}$. Moreover, by (B.5), $w_s(s, y) + \frac{1}{2} w_{yy}(s, y) < 0$ for $|y| > \sqrt{1/\beta + s}$. We then conclude from a standard verification theorem (see e.g. [26, Theorem 3.2]) that $\bar{V}(s, y) = w(s, y)$ is a smooth solution of (B.3). This implies that $(\bar{V}(s, W_s^y))_{s\geq 0}$ is a supermartingale, and $(\bar{V}(s \wedge \tau_y^*, W_{s\wedge\tau_y^*}^y))_{s\geq 0}$, with $\tau_y^* := \inf\{s \geq 0 : |W_s^y| \geq \sqrt{1/\beta + s}\}$, is a true martingale.

It then follows from standard arguments that τ_y^* is the smallest optimal stopping time for $\bar{V}(0, y)$. As a consequence, $\hat{\tau}_x := \inf\{s \geq 0 : X_s^x \geq \sqrt{1/\beta + s}\}$ is the smallest optimal stopping time for (4.2). In view of Proposition 2.2, $\widetilde{\tau}_x = \hat{\tau}_x$. □

Remark B.1 With X being reflected at the origin, it is expected that the variational inequality of the value function $V(s, x)$ should admit a Neumann boundary condition at $x = 0$. This is not explicitly seen in (B.2) because of the change of variable $\bar{V}(s, y) := V(s, |y|)$ in the second line of the proof above, which shifts our analysis to a Brownian motion with no reflection at the origin. In fact, one may check directly from (B.5) that $V(s, x) = \bar{V}(s, x) = w(s, x)$ indeed satisfies the Neumann boundary condition $V_x(s, 0+) = 0$ for all $s \geq 0$.

B.2 Proof of Lemma 4.3

First, we prove that E is totally disconnected. If $\ker(\tau) = [a, \infty)$, then $E = \emptyset$ and there is nothing to prove. Assume that there exists $x^* > a$ such that $x^* \notin \ker(\tau)$. Define

$$\ell := \sup\{b \in \ker(\tau) : b < x^*\} \qquad \text{and} \qquad u := \inf\{b \in \ker(\tau) : b > x^*\}.$$

We claim that $\ell = u = x^*$. Assume to the contrary that $\ell < u$. Then $\tau(x) = 1$ for all $x \in (\ell, u)$. Thus, given $y \in (\ell, u)$, $\mathcal{L}^*\tau(y) = T^y := \inf\{s \geq 0 : X_s^y \notin (\ell, u)\} > 0$ and

$$J\big(y; \mathcal{L}^*\tau(y)\big) = \mathbb{E}^y\left[\frac{X_{T^y}}{1 + \beta T^y}\right] < \mathbb{E}^y[X_{T^y}] = \ell\mathbb{P}[X_{T^y} = \ell] + u\mathbb{P}[X_{T^y} = u]. \tag{B.6}$$

Since $X_s = |W_s|$ for a one-dimensional Brownian motion W and $0 < \ell < y < u$, by the optional sampling theorem, $\mathbb{P}[X_{T^y} = \ell] = \mathbb{P}[W_s^y$ hits ℓ before hitting $u] = \frac{u-y}{u-\ell}$ and $\mathbb{P}[X_{T^y} = u] = \mathbb{P}[W_s^y$ hits u before hitting $\ell] = \frac{y-\ell}{u-\ell}$. Alternatively, one may evaluate $\mathbb{P}[X_{T^y} = \ell]$ and $\mathbb{P}[X_{T^y} = u]$ directly by using the fact that the scale function of a one-dimensional Bessel process is the identity mapping (see e.g. [8, Part I, Chap. 6, Sect. 15]). This together with (B.6) gives $J(y; \mathcal{L}^*\tau(y)) < y$. This implies $y \in S_\tau$, and thus $\Theta\tau(y) = 0$ by (3.12). Then $\Theta\tau(y) \neq \tau(y)$, a contradiction to $\tau \in \mathcal{E}(\mathbb{R}_+)$. This already implies that E is totally disconnected, and thus $\overline{\ker(\tau)} = [a, \infty)$. The rest of the proof follows from Lemma 4.2.

B.3 Proof of Lemma 4.4

(i) Given $a \geq 0$, it is obvious from the definition that $\eta(0, a) \in (0, a)$ and $\eta(a, a) = a$. Fix $x \in (0, a)$ and let f_a^x denote the density of T_a^x. We obtain

$$\mathbb{E}^x\left[\frac{1}{1+\beta T_a^x}\right] = \int_0^\infty \frac{1}{1+\beta t} f_a^x(t)dt = \int_0^\infty \int_0^\infty e^{-(1+\beta t)s} f_a^x(t)ds\, dt$$
$$= \int_0^\infty e^{-s}\left(\int_0^\infty e^{-\beta st} f_a^x(t)dt\right) ds$$
$$= \int_0^\infty e^{-s}\mathbb{E}^x[e^{-\beta s T_a^x}]ds. \tag{B.7}$$

Since T_a^x is the first hitting time of a one-dimensional Bessel process, we compute its Laplace transform by using [19, Theorem 3.1] (or [8, Part II, Sect. 3, Formula 2.0.1]), as

$$\mathbb{E}^x[e^{-\frac{\lambda^2}{2}T_a^x}] = \frac{\sqrt{x}I_{-\frac{1}{2}}(x\lambda)}{\sqrt{a}I_{-\frac{1}{2}}(a\lambda)} = \cosh(x\lambda)\,\mathrm{sech}(a\lambda) \qquad \text{for } x \leq a.$$

Here, I_ν denotes the modified Bessel function of the first kind. Thanks to the above formula with $\lambda = \sqrt{2\beta s}$, we obtain from (B.7) that

$$\eta(x, a) = a\int_0^\infty e^{-s}\cosh(x\sqrt{2\beta s})\,\mathrm{sech}(a\sqrt{2\beta s})ds. \tag{B.8}$$

It is then obvious that $x \mapsto \eta(x, a)$ is strictly increasing. Moreover,

$$\eta_{xx}(x, a) = 2a\beta^2\int_0^\infty e^{-s}s\cosh(x\sqrt{2\beta s})\,\mathrm{sech}(a\sqrt{2\beta s})ds > 0 \qquad \text{for } x \in [0, a],$$

which shows the strict convexity.

(ii) This follows from (B.8) and the dominated convergence theorem.

(iii) We first prove the desired result with $x^*(a) \in (0, a)$, and then upgrade it to $x^*(a) \in (0, a^*)$. Fix $a \geq 0$. In view of the properties in (i), we observe that the two curves $y = \eta(x, a)$ and $y = x$ intersect at some $x^*(a) \in (0, a)$ if and only if $\eta_x(a, a) > 1$. Define $k(a) := \eta_x(a, a)$. By (B.8),

$$k(a) = a\int_0^\infty e^{-s}\sqrt{2\beta s}\tanh(a\sqrt{2\beta s})ds. \tag{B.9}$$

Thus we see that $k(0) = 0$ and $k(a)$ is strictly increasing on (0, 1) since for any $a > 0$,

$$k'(a) = \int_0^\infty e^{-s}\sqrt{2s}\left(\tanh(a\sqrt{2s}) + \frac{a\sqrt{2s}}{\cosh^2(a\sqrt{2s})}\right)ds > 0.$$

By numerical computation, $k(1/\sqrt{\beta}) = \int_0^\infty e^{-s}\sqrt{2s}\tanh(\sqrt{2s})ds \approx 1.07461 > 1$. It follows that there must exist $a^* \in (0, 1/\sqrt{\beta})$ such that $k(a^*) = \eta_x(a^*, a^*) = 1$. Monotonicity of $k(a)$ then gives the desired result.

Now, for any $a > a^*$, we intend to upgrade the previous result to $x^*(a) \in (0, a^*)$. Fix $x \geq 0$. By the definition of η and (ii), on the domain $a \in [x, \infty)$, the mapping $a \mapsto \eta(x, a)$ must either first increase and then decrease to 0, or directly decrease to 0. From (B.8), we have

$$\eta_a(x, x) = 1 - x\int_0^\infty e^{-s}\sqrt{2\beta s}\tanh(x\sqrt{2\beta s})ds = 1 - k(x),$$

with k as in (B.9). Recalling $k(a^*) = 1$, we have $\eta_a(a^*, a^*) = 0$. Notice that

$$\begin{aligned}\eta_{aa}(a^*, a^*) &= -\frac{2}{a^*}k(a^*) - 2\beta a^* + a^*\int_0^\infty 4\beta s e^{-s}\tanh^2(a^*\sqrt{2\beta s})ds \\ &\leq -\frac{2}{a^*} + 2\beta a^* < 0,\end{aligned}$$

where the second line follows from $\tanh(x) \leq 1$ for $x \geq 0$ and $a^* \in (0, 1/\sqrt{\beta})$. Since $\eta_a(a^*, a^*) = 0$ and $\eta_{aa}(a^*, a^*) < 0$, we conclude that on the domain $a \in [a^*, \infty)$, the mapping $a \mapsto \eta(a^*, a)$ decreases to 0. On the other hand, for any $a > a^*$, since $\eta(a^*, a) < \eta(a^*, a^*) = a^*$, we must have $x^*(a) < a^*$.

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