CODEGREE TURÁN DENSITY OF COMPLETE r-UNIFORM HYPERGRAPHS

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ABSTRACT. Let $r \geq 3$. Given an r-graph H, the minimum codegree $\delta_{r-1}(H)$ is the largest integer t such that every $(r-1)$-subset of $V(H)$ is contained in at least t edges of H. Given an r-graph F, the codegree Turán density $\gamma(F)$ is the smallest $\gamma > 0$ such that every r-graph on n vertices with $\delta_{r-1}(H) \geq (\gamma + o(1))n$ contains F as a subhypergraph. Using results on the independence number of hypergraphs, we show that there are constants $c_1, c_2 > 0$ depending only on r such that

$$1 - c_2 \frac{\ln t}{t^{r-1}} \leq \gamma(K_t^r) \leq 1 - c_1 \frac{\ln t}{t^{r-1}},$$

where K_t^r is the complete r-graph on t vertices. This gives the best general bounds for $\gamma(K_t^r)$.

1. INTRODUCTION

An r-uniform hypergraph (r-graph) H consists of a vertex set $V(H)$ and an edge set $E(H)$, which is a family of r-subsets of $V(H)$. A fundamental problem in extremal combinatorics is to determine the Turán number $\mathrm{ex}(n, F)$, which is the largest number of edges in an r-graph on n vertices not containing a given r-graph F as a subhypergraph (namely, F-free). When $r \geq 3$, we only know $\mathrm{ex}(n, F)$, or its asymptotics $\pi(F) := \lim_{n\to\infty} \mathrm{ex}(n, F)/\binom{n}{r}$ for very few F. Let K_t^r denote the complete r-graph on t vertices. Determining $\pi(K_t^r)$ for any $t > r \geq 3$ is a well known open problem, in particular, Turán [18] conjectured in 1941 that $\pi(K_4^3) = 5/9$. The best (general) bounds for $\pi(K_t^r)$ are due to Sidorenko [17] and de Caen [1]

$$1 - \left(\frac{r-1}{t-1}\right)^{r-1} \leq \pi(K_t^r) \leq 1 - \frac{1}{\binom{t-1}{r-1}}. \tag{1.1}$$

For more Turán-type results on hypergraphs, see surveys [7, 9].

A natural variation on the Turán problem is to ask how large the minimum ℓ-degree can be in an F-free r-graph. Given an r-graph H, the *degree* $\deg(S)$ of a set $S \subset V(H)$ is the number of the edges that contain S. Given $1 \leq \ell < r$, the minimum ℓ-degree $\delta_\ell(H)$ is the minimum $\deg(S)$ over all $S \subset V(H)$ of size ℓ. Mubayi and Zhao [14] introduced the *codegree Turán number* $\mathrm{ex}_{r-1}(n, F)$, which is the largest $\delta_{r-1}(H)$ among all F-free r-graphs on n vertices, and *codegree (Turán) density* $\pi_{r-1}(F) := \lim_{n\to\infty} \mathrm{ex}_{r-1}(n, F)/n$ (it was shown [14] that this limit exists). The corresponding ℓ-degree Turán number $\mathrm{ex}_\ell(n, F)$ and density $\pi_\ell(F)$ were defined similarly and studied by Lo and Markström [12].[1]

Most codegree Turán problems do not seem easier than the original Turán problems. We only know the codegree densities of the following r-graphs. Let Fano denote the Fano plane (a 3-graph on seven vertices and seven edges). Mubayi [13] showed that $\pi_2(\text{Fano}) = 1/2$ and Keevash [8] later showed that $\mathrm{ex}_2(n, \text{Fano}) = \lfloor n/2 \rfloor$ for sufficiently large n (DeBiasio and Jiang [2] gave another proof). Keevash and Zhao [10] studied the codegree density for other projective geometries and constructed a family of 3-graphs whose codegree densities are $1 - 1/t$ for all integers $t \geq 1$. Falgas-Ravry, Marchant, Pikhurko, and Vaughan [5] determined $\mathrm{ex}_2(F_{3,2})$ for sufficiently large n, where $F_{3,2}$ is the 3-graph on $\{1, 2, 3, 4, 5\}$ with edges $123, 124, 125, 345$.

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[1] A simple averaging argument shows that $\pi_1(F) = \pi(F)$ for every F.

Falgas-Ravry, Pikhurko, Vaughan and Volec [6] also proved that $\pi_2(K_4^{3-}) = 1/4$, where K_4^{3-} is the (unique) 3-graphs on four vertices with three edges.

In this note we obtain asymptotically matching bounds for $\pi_{r-1}(K_t^r)$ for any fixed $r \geq 3$ and sufficiently large t. Since its value is close to one, it is more convenient to write $\pi_{r-1}(K_t^r)$ in the complementary form. Given an r-graph H and $\ell < r$, let $\Delta_\ell(H)$ denote the *maximum ℓ-degree* of H and $\alpha(H)$ denote the *independence number* (the largest size of a set of vertices containing no edge) of H. Define

$$T_\ell(n,t,r) = \min\{\Delta_\ell(H) : H \text{ is an } r\text{-graph on } n \text{ vertices with } \alpha(H) < t\}$$

and $\tau_\ell(t,r) = \lim_{n\to\infty} T_\ell(n,t,r)/\binom{n-\ell}{r-\ell}$. It is clear that $T_\ell(n,t,r) = \binom{n-\ell}{r-\ell} - \mathrm{ex}_\ell(n,K_t^r)$ and $\tau_\ell(t,r) = 1 - \pi_\ell(K_t^r)$. Falgas-Ravry [4] showed that $\tau_2(t,3) \leq 1/(t-2)$ for $t \geq 4$ while Lo and Markström [12] showed that $\tau_{r-1}(t,r) \leq 1/(t-r+1)$ for $t > r \geq 3$. Recently Sidorenko [16] used zero-sum-free sequences in $\mathbb{Z}_3^d$ to get $\tau_2(t,3) \leq O(\frac{1}{t\ln t})$.

We show that $\tau_{r-1}(t,r) = \Theta(\ln t/t^{r-1})$ as $t \to \infty$.

Theorem 1.1. *For all $r \geq 3$, there exist $c_1, c_2 > 0$ such that*

$$c_1 \ln t/t^{r-1} \leq \tau_{r-1}(t,r) \leq c_2 \ln t/t^{r-1}.$$

In fact, the upper bound immediately follows from a construction of Kostochka, Mubayi and Verstraëte [11] (see Construction 2.1). The lower bound can be deduced from either the main result of [11] or a result of Duke, Lefmann, and Rödl [3]. However, since both results require $\Delta_{r-1}(H) = o(n)$, we need to extend them slightly by allowing $\Delta_{r-1}(H)$ to be a linear function of n (see Theorem 2.2).

We prove Theorem 1.1 in the next section and give concluding remarks and open questions in the last section.

2. Proof of Theorem 1.1

A *partial Steiner (n,r,ℓ)-system* is an r-graph on n vertices in which every set of ℓ vertices is contained in at most one edge. Rödl and Šinajová [15] showed that there exists $a_2 > 0$ such that for every m, there is a partial Steiner $(m,r,r-1)$-system S with $\alpha(S) \leq a_2(m\ln m)^{1/(r-1)}$. Kostochka, Mubayi and Verstraëte [11, Section 3.1] used the blowup of this Steiner system to obtain the following construction. A similar construction (but not using the result of [15]) was given in [4].

Construction 2.1. [11] *Let S be the partial Steiner $(m,r,r-1)$-system given by Rödl and Šinajová. Let V be a union of disjoint sets $V_1,\dots,V_m$ each of size d. For each edge $e = \{i_1,\dots,i_r\}$ of S, let $E_e := \{v_1v_2\dots v_r \colon v_j \in V_{i_j} \text{ for } j \in [r]\}$. Let H be the r-graph with vertex set V and edge set $\bigcup_{i\in[m]}\binom{V_i}{r} \cup \bigcup_{e\in S} E_e$. It is easy to see that*

$$\Delta_{r-1}(H) = d \quad \text{and} \quad \alpha(H) = (r-1)\alpha(S) \leq a_2(r-1)(m\ln m)^{\frac{1}{r-1}}.$$

Construction 2.1 will be used to prove the upper bound of Theorem 1.1. The lower bound of Theorem 1.1 follows from the following theorem, which will be proved at the end of the section.

Theorem 2.2. *For all $r \geq 3$, there exist $c_0, \delta_0 > 0$ such that for every $0 < \delta \leq \delta_0$, the following holds for sufficiently large n. Every r-graph on n vertices with $\Delta_{r-1}(H) \leq \delta n$ satisfies $\alpha(H) \geq c_0\left(\frac{1}{\delta}\ln\frac{1}{\delta}\right)^{1/(r-1)}$.*

Proof of Theorem 1.1. Fix $r \geq 3$. Without loss of generality, we assume that t is sufficiently large. We first prove the upper bound with $c_2 = (r-1)^r a_2^{r-1}$, where a_2 is from Construction 2.1. Our goal is to construct r-graphs H on n vertices (for infinitely many n) with $\alpha(H) < t$ and $\Delta_{r-1}(H) \leq c_2 n\ln t/t^{r-1}$. To achieve this, we apply Construction 2.1 with $m = \lceil t^{r-1}/(c_2\ln t)\rceil$ and $d = n/m \leq c_2 n\ln t/t^{r-1}$ obtaining an r-graph H on n vertices with $\Delta_{r-1}(H) = d$ and

$\alpha(H) \le a_2(r-1)(m\ln m)^{1/(r-1)}$. Since t is sufficiently large, it follows that $\ln\left\lceil \frac{t^{r-1}}{c_2 \ln t}\right\rceil < \ln t^{r-1} - 1$ and

$$m\ln m = \left\lceil \frac{t^{r-1}}{c_2\ln t}\right\rceil \ln\left\lceil \frac{t^{r-1}}{c_2\ln t}\right\rceil < \left(\frac{t^{r-1}}{c_2\ln t}+1\right)\left(\ln t^{r-1}-1\right) < \frac{(r-1)t^{r-1}}{c_2}.$$

Consequently $\alpha(H) < a_2(r-1)(\frac{(r-1)t^{r-1}}{c_2})^{1/(r-1)} = t$ by the choice of c_2.

We now prove the lower bound. Suppose c_0, δ_0 are as in Theorem 2.2. Let $c_1 = (r-1)c_0^{r-1}/2$ and $\delta = c_1 \ln t/t^{r-1}$. Since t is large, we have $\delta \le \delta_0$. Let n be sufficiently large. We need to show that every r-graph H on n vertices with $\alpha(H) < t$ satisfies $\Delta_{r-1}(H) \ge \delta n$. Indeed, by Theorem 2.2, any r-graph H on n vertices with $\Delta_{r-1}(H) = d \le \delta n$ satisfies

$$\alpha(H) \ge c_0\left(\frac{1}{\delta}\ln\frac{1}{\delta}\right)^{1/(r-1)} > c_0\left(\frac{t^{r-1}}{2c_1\ln t}\ln t^{r-1}\right)^{1/(r-1)} = t$$

because t is large and and $c_1 = (r-1)c_0^{r-1}/2$. □

The rest of the section is devoted to the proof of Theorem 2.2. We need [11, Theorem 1] of Kostochka, Mubayi, Verstraëte and [14, Lemma 2.1] of Mubayi and Zhao.[2]

Theorem 2.3. [11] *For all $r \ge 3$, there exists $b_1 > 0$ such that every r-graph with $\Delta_{r-1}(H) \le d$ for some $0 < d < n/(\ln n)^{3(r-1)^2}$ satisfies $\alpha(H) \ge b_1\left(\frac{n}{d}\ln\frac{n}{d}\right)^{1/(r-1)}$.*

Lemma 2.4. [14] *Let $r \ge 2$ and $\varepsilon > 0$. Let m be the positive integer such that $m \ge 2(r-1)/\varepsilon$ and $\binom{m}{r-1}e^{-\varepsilon^2(m-r+1)/12} \le 1/2$. Every r-graph H on $n \ge m$ vertices contains an induced sub-r-graph H' on m vertices with $\Delta_{r-1}(H')/m \le \Delta_{r-1}(H)/n + \varepsilon$.*

Proof of Theorem 2.2. Fix $r \ge 3$. Let $0 < \delta_0 < 1/4$ such that

$$24(r-1)\ln\left(\left\lceil\frac{1}{\delta^4}\right\rceil\right) \le \frac{1}{\delta^2} \quad \text{and} \quad \frac{1}{\delta^4} \le \exp\left(\left(\frac{1}{2\delta}\right)^{\frac{1}{3(r-1)^2}}\right) - 1 \tag{2.1}$$

for all $0 < \delta \le \delta_0$. Let $m = \lceil 1/\delta^4\rceil$. We claim that m satisfies the assumption of Lemma 2.4 when $\varepsilon = \delta$. Indeed, it follows from the first inequality of (2.1) that

$$m \ge \frac{24(r-1)\ln m}{\delta^2} > \frac{2(r-1)}{\delta},$$

which further implies that

$$\binom{m}{r-1}e^{-\delta^2(m-r+1)/12} \le \frac{1}{2}m^{r-1}e^{-\frac{m\delta^2}{24}} \le \frac{1}{2}.$$

Let $c_0 = 4^{-1/(r-1)}b_1$, where b_1 is defined in Theorem 2.3. Suppose H is an r-graph on $n \ge m$ vertices with $\Delta_{r-1}(H) \le \delta n$. By Lemma 2.4, there exists an induced subhypergarph H' on m vertices such that

$$\Delta_{r-1}(H') \le 2\delta m < \frac{m}{(\ln m)^{3(r-1)^2}},$$

which follows from the second inequality of (2.1) and $m = \lceil 1/\delta^4\rceil$. We now apply Theorem 2.3 to H' with $d = 2\delta m$ and obtain that

$$\alpha(H) \ge \alpha(H') \ge b_1\left(\frac{1}{2\delta}\ln\frac{1}{2\delta}\right)^{1/(r-1)} \ge b_1\left(\frac{1}{4\delta}\ln\frac{1}{\delta}\right)^{1/(r-1)} = c_0\left(\frac{1}{\delta}\ln\frac{1}{\delta}\right)^{1/(r-1)}$$

by the choice of c_0 and the assumption that $\delta \le 1/4$. □

[2]Alternatively we could apply [3, Theorem 3] of Duke, Lefmann, and Rödl – we choose [11, Theorem 1] because it provides a better constant.

3. Concluding remarks

Theorem 1.1 shows that $c_1 \ln t/t^{r-1} \le \tau_{r-1}(t,r) \le c_2 \ln t/t^{r-1}$. Our proofs of Theorems 1.1 and 2.2 together give that $c_1 = (r-1)b_1^{r-1}/8$, where b_1 comes from Theorem 2.3. A slightly more careful calculation allows us to take $c_1 = (1+o_t(1))(r-1)b_1^{r-1}$ (where $o_t(1) \to 0$ as $t \to \infty$). The equation (7) in [11] shows that $b_1^{r-1} = (1+o_r(1))(r-3)!/3$ and thus

$$c_1 = (1+o_r(1))\frac{r-1}{3}(r-3)!.$$

On the other hand, our proof of Theorem 1.1 gives $c_2 = (r-1)^r a_2^{r-1}$, where a_2 comes from Construction 2.1. Unfortunately, we do not know the smallest a_2 such that there is a partial Steiner $(m,r,r-1)$-system S with $\alpha(S) \le a_2(m\ln m)^{1/(r-1)}$ for every m. However, the random construction in [11, Section 3.2] yields a constant that asymptotically equals b_1 but requires $\ln \Delta_{r-1}(H) = o(\ln n)$. Nevertheless, we can use the blowup of this construction and add some additional edges when $r \ge 4$ to derive that[3]

$$c_2 = \begin{cases} (1+o_t(1))r\cdot r! & \text{if } r \text{ is even,} \\ (1+o_t(1))2^{r-1}r\cdot r! & \text{if } r \text{ is odd.} \end{cases}$$

When r is even, above refined values of c_1 and c_2 differ by a factor of $3r^3$ asymptotically. We tend to believe that $\tau_{r-1}(t,r) \sim r\cdot r!\ln t/t^{r-1}$ when $t \gg r \gg 1$.

Given any r-graph H on n vertices, $\Delta_\ell(H)/\binom{n-\ell}{r-\ell}$ is an increasing function of ℓ. As a result, $\tau_\ell(t,r)$ is an increasing function of ℓ. When $t \to \infty$, we have $\tau_1(t,r) = 1-\pi(K_t^r) = \Theta(1/t^{r-1})$ from (1.1) and $\tau_{r-1}(t,r) = \Theta(\ln t/t^{r-1})$ from Theorem 1.1. Putting these together, we have

$$\Theta\left(\frac{1}{t^{r-1}}\right) = \tau_1(t,r) \le \tau_2(t,r) \le \cdots \le \tau_{r-1}(t,r) = \Theta\left(\frac{\ln t}{t^{r-1}}\right).$$

It is interesting to know if $\tau_\ell(t,r) = \Theta(\ln t/t^{r-1})$ for all $\ell \ge 2$.

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