

# Updating Probability: Tracking Statistics as Criterion

Bas C. van Fraassen and Joseph Y. Halpern

---

## ABSTRACT

For changing opinion, represented by an assignment of probabilities to propositions, the criterion proposed is motivated by the requirement that the assignment should have, and maintain, the possibility of matching in some appropriate sense statistical proportions in a population. This ‘tracking’ criterion implies limitations on policies for updating in response to a wide range of types of new input. Satisfying the criterion is shown equivalent to the principle that the prior must be a convex combination of the possible posteriors. Furthermore, this is equivalent to the requirement that prior expected values must fall inside the range spanned by possible posterior expected values. The tracking criterion is liberal; it allows for, but does not require, a policy such as Bayesian conditionalization, and can be offered as a general constraint on policies for managing opinion over time. Examples are given of non-Bayesian policies, both ones that satisfy and ones that violate the criterion.

- 1 Introduction
- 2 Alternative Updating Policies
- 3 Modelling the Situation for Normal Updating
- 4 Tracking: A Criterion for Updating Policies
- 5 Tracking: Precise Formulation and Relation to Convexity
- 6 The Spanning Criterion
- 7 Non-Bayesian Policies that Satisfy the Spanning and Tracking Criteria
- 8 Policies that Violate the Spanning and Tracking Criteria

Appendix

---

## 1 Introduction

What is generally called Bayesian conditionalization is a policy for updating probability assignments. It specifies as admissible input (‘evidence’) elements of the domain of the prior probability function, and allows as possible posteriors the conditionalizations on such elements (‘propositions’). Under

certain ideal conditions, this is the only coherent policy (Teller and Fine [1975]; Teller [1976]; Diaconis and Zabell [1982]; van Fraassen [1999]). When those conditions are not met, other policies might be appropriate. Putative examples include updating by Richard Jeffrey's generalized conditionalization ('Jeffrey conditionalization') or Edwin T. Jaynes' rule to maximize relative entropy ('MAXENT'). This raises the question of what general constraints any such policy should satisfy. We will propose an answer, guided initially by some intuitive considerations that also motivate the Bayesian policy.

It is no coincidence that the probability calculus is typically introduced with examples of urns full of red and black, or marble and wooden, balls. The relations between the statistical proportions of finite sets, with their intersections and unions, introduce the basic principles that are extrapolated to present probability theory, and that extrapolation is motivated by the insistence that a probability assignment should in principle be able to track, in some appropriate sense, the relevant statistics. It is well known that the extrapolation goes far beyond the theory of finite proportions, and beyond the theory of limits of relative frequencies, but there are also far-reaching theorems to show that the relationship of these to probability theory remains appropriately close.<sup>1</sup>

That this possibility of tracking the relevant statistics should be preserved under updating also provides a motivation for the Bayesian policy. If the proportions are known, and we are given the information that a ball drawn from the urn is marble, then the probability that it is red should be updated from the proportion of red balls in the urn to the proportion of red balls among the marble ones in the urn. And if the proportions are not known, but the prior probabilities were assigned on the basis of opinion or evidence, then the updating should take the same form, in order to guarantee the following: if the prior probability assignment had the possibility that it was tracking the relevant statistics for draws of balls from the urn, then that possibility was not lost for updating to draws of balls from among the marble balls in the urn.

## 2 Alternative Updating Policies

Alternatives to the Bayesian policy have been discussed for two main reasons. The first is that the input that triggers a change in probabilities may not be of the sort this policy takes into account. Both the above examples, of Jeffrey

<sup>1</sup> For example, the strong law of large numbers implies that if  $G$  is a finitely generated field of subsets of  $S$ , and  $p$  a probability function defined on  $G$ , then there exists a countable sequence,  $s$ , of members of  $S$  such that for each element  $E$  of  $G$ ,  $p(E)$  equals the limit of the relative frequency of  $E$  in  $s$ .

conditionalization and Jaynes' MAXENT, were introduced in this way. Richard Jeffrey proposed allowing for input beyond the propositional, with the following example:

In examining a piece of cloth by candlelight one might come to attribute probabilities 0.6 and 0.4 to the propositions G that the cloth is green and B that it is blue, without there being any proposition E for which the direct effect of the observation is anything near changing the observer's degree of belief in E to I. (Jeffrey [1968], p. 172)

The proper transformation of probability he offered, now known as Jeffrey conditionalization, redistributes the probability over a partition whose cells are the affected alternatives:

$$P^0_A P \frac{1}{4} \sum_i q_i P_A C_i P;$$

with  $\{C_i : i \in \{1, 2, \dots, n\}\}$  the relevant partition, such as 'green', 'blue', . . . , 'red', and  $\{q_i : i \in \{1, 2, \dots, n\}\}$ , the weights of the new posterior probabilities of those alternatives.

Edwin T. Jaynes introduced his maximum entropy updating rule; a typical motivating example takes the following sort of input:

We have a table which we cover with black cloth, and some dice, but [...] they are black dice with white spots. A die is tossed onto the black table. Above there is a camera [that] will record only the white spots. Now we don't change the film in between, so we end up with a multiple exposure; uniform blackening of the film after we have done this a few thousand times. From the known density of the film and the number of tosses, we can infer the average number of spots which were on top, but not the frequencies with which various faces came up. Suppose that the average number of spots turned out to be 4.5 instead of the 3.5 that we might expect from an honest die. Given only this information [...] what estimates should [we] make of the frequencies with which n spots came up? (Jaynes [2003], p. 343)

Transferring this problem into one of probability updating, the prior probability assignment—assuming a fair die—included an expectation value of 3.5 for the number of dots, and the input is a new expectation value of 4.5. There is no proposition in the domain of the prior on which to conditionalize; this input is of a different sort. What should be the posterior probability assignment? Jaynes' rule gives a precise answer that implies, for example, that the outcome with one spot has posterior probability 0.05 and the outcome with six spots has posterior probability 0.35.

The second reason that has entered the discussion of alternative policies is that there may be conditions seen as triggering a more radical change than can be accommodated by conditionalization. It may be a case, as it is often put, to 'throw away the prior', but even so the change would not be a choice at

random; the agent could be guided by prior experience and theoretical pre-suppositions that appear phenomenologically only as intuition and instinct, limiting the acceptable choices.

What we will investigate is a way to place requirements on any policy for updating that remains well motivated by the intuitive considerations offered above in terms of the possibility of tracking relevant statistics and of preserving that possibility. As Jaynes' example illustrates, such motivation remains salient when departures from the Bayesian policy are envisaged.

### 3 Modelling the Situation for Normal Updating

A policy for updating probabilities needs to start with a description of the sort of situation to which it applies. Such a situation will be one where the subject or agent has at each time a state of opinion represented by a probability function on a space (representing a range of possibilities or possible circumstances). Second, it must be specified how, in this sort of situation, there can be input of various sorts. Third, the policy must offer a prescription of what that prior probability function is allowed to be changed into, in response to a given input, to form a posterior probability function.

The Bayesian policy, applicable here, starts with a prior probability, takes the inputs to be elements of a finite partition of the possibilities, and, given an element of the partition, updates the prior to a posterior by conditionalization. We will not assume that in general a policy must prescribe a unique posterior for each possible input, nor that that the posteriors correspond to cells in a partition.

To accommodate not just the Bayesian policy but also the cases presented by, for example, Richard Jeffrey and Edwin Jaynes, we must not concentrate on the special case in which the input is an 'evidential' proposition in the domain of the prior probability assignment. Indeed, Jeffrey's example introduces an agent who has no way of expressing what triggers the change in probability assignment, hence no input to which a conscious policy could apply. Nevertheless, it can be specified that this change, and the agent's response when managing his overall opinion, takes a very special form. We note that in this case too, given a prior, the input—whether explicit or unformulated—places a constraint on the posterior. An input, whatever it may be, acts as a function that takes any given prior into a set of possible (admissible, acceptable) posteriors. As the most general form of input, we must therefore count any constraint that the agent may accept as limiting the candidates for the posterior probability function. Whatever types of input there can be can thus be represented by functions that take any element of the space (a prior) into a subset of that space (its set of possible posteriors).

A little more formally: a policy for updating (whether or not in response to inputs representable in terms of the elements of the space of possibilities)

specifies for each probability function  $p$  (the prior), a set  $R$  of possible posteriors (of cardinality greater than 1), or equivalently, a constraint on the functions into which  $p$  can change by updating. We place only one condition on how these responses are formed. Since we view the process of going from  $p$  to  $R$  as normal updating, not revolutionary change, we assume absolute continuity: updating does not raise any probability from 0. For succinctness, a model (of a doxastic situation, in the present context) is a triple  $M \triangleleft p, S, R \triangleright$ , where  $p$  is a probability measure on  $S$  and  $R$  is a set of probability measures on  $S$  that assign 0 wherever  $p$  assigns 0. As stipulated above, the number of possible posteriors, the members of  $R$ , is greater than<sup>2</sup>1.

#### 4 Tracking: A Criterion for Updating Policies

Following upon the intuitive motivation presented above, we propose a formal criterion to be met by updating policies. The motivating considerations included two main points: opinion represented in terms of a probability assignment should at least possibly track the relevant statistics, and updating the probability assignment on new input should preserve that possibility. We added that under certain ideal conditions, Bayesian conditionalization is precisely the policy that satisfies these requirements.

What the ideal conditions are, and how more practical conditions could be related to the ideal case, is illustrated in the following (partly fictional) example.

In 1994 the distribution, by party and gender, in the United States Senate was as follows:

	Men	Women	Row total
Republican	42	2	44
Democrat	51	5	56
Column total	93	7	100

Suppose a fully informed ideal agent,  $A^*$ , makes all the relevant distinctions and actually has the information about each individual Senator: about their gender, age, party affiliation, and more, like their state of health, and so on. Assume that his probabilities match these statistics, for example,

$$P(x \text{ is a woman} \wedge x \text{ is a Republican Senator}) \triangleleft 2/44.$$

A certain agent,  $A$ , who is not a fully informed ideal agent, let us assume, has no information about state of health, age, or even gender; only about party affiliation in the Senate at that time. Within this very limited space, his

<sup>2</sup> If there is only one possible posterior, we take it that this would correspond to something like a test for which the outcome is certain beforehand. In that case the prior should not change, and the posterior must be the same as the prior.

probabilities match the statistics too, but lack the male–female distinction; he just has, for example,

$$P^0(x \text{ is a Republican} \mid x \text{ is a Senator}) \approx 44/100.$$

Thus the probabilities that A assigns are the marginal probabilities assigned by  $A^*$ , matching the marginal distribution in the above table with the division by gender removed.

Now the fully informed ideal agent  $A^*$  gets the new information that there are no more women Senators, they were all removed from office for such reasons as fraud, health, or other personal circumstances. Being an ideally well-informed agent, one that not only has all the relevant statistical information for his subject matter, but is also making all the relevant distinctions, the Bayesian policy is not only applicable but the uniquely right policy to follow. Thus  $A^*$  conditionalizes on this information, reclassifying those women as no longer in the Senate. His posterior probabilities match:

	Men	Women	Row total
Republican	42	0	42
Democrat	51	0	51
Column total	93	0	93

Suppose now that the non-fully informed ideal agent A, who does not make all the same distinctions as  $A^*$  and lacks either information or opinion about at least some of the aspects to which  $A^*$  is privy, nevertheless has a prior and posterior opinion that remain correctly represented by marginals of the prior and posterior of the fully informed ideal agent. This means that we see him updating (through whatever impulse or choice) to the corresponding marginal distribution: Republican 42, Democrat 51. So now both  $A^*$  and A have, for example, the probability 42/93 for a Senator being a Republican.

Watching this subject A, who is not ideally well-informed, we see that his change of opinion was not one done by conditionalization. All we see is a redistribution over the same alternatives. From the outside, it was just an abrupt change in the probabilities for party affiliation in the Senate. However, since his posterior probability is the marginal of an updating by conditionalization by the fully informed ideal subject, the virtue of matching the actual statistics was preserved.

So the criterion for updating policies that we propose will be that however the agent assigns his probabilities and updates them, there exists in principle an ideal take on his situation, and his own opinion is in a relevant sense ‘part of’ the evolving probability assignment of an ideally well-informed (hence Bayesian) agent. Echoing the intuitive discussion, we shall call this criterion ‘Tracking’. Under what conditions is this criterion satisfied?

The question is reminiscent of discussions of hidden variable interpretations of quantum mechanics, in which unfamiliar sorts of stories, involving strange or unpredictable changes, are shown to be compatible with larger stories that follow a familiar pattern. Specifically, conditions are investigated in which an assignment of probabilities of outcomes can have an underlying classical probability model. Taking our cue from the above example, we take it that for a model to satisfy the Tracking criterion, its probability functions must be the marginals of probability functions in a larger associated model on which much stricter constraints may be imposed.<sup>3</sup>

In what follows, we shall first show that Tracking is satisfied if and only if the prior probability assignment is a convex combination of the possible posterior assignments. Second, we will formulate a distinct criterion, Spanning, to govern evolving expectation values, and show that it is equivalent to Tracking. With those results in hand, we will then show the diversity of updating policies that can satisfy those criteria, but also that there are policies that violate them.

## 5 Tracking: Precise Formulation and Relation to Convexity

In our example, we were looking at a single possible updating. For the fully informed ideal subject, there were many possible posteriors; each of them a conditionalization on some proposition that changes one of the four numbers for his four 'cells'. And the intuitive Tracking criterion requires that the non-fully informed ideal subject's prior and possible posteriors should be the marginals of some such (imaginary) fully informed ideal subject's prior and possible posteriors.

We can concentrate on a representative simple form for such a case. Throughout we will take the set of possible posteriors to be finite. But as will be clear in retrospect, the updating policy could allow for all the convex combinations of a given finite set as the set of possible posteriors.

Definition: A model  $M^* \subseteq \{p^*, S^*, R^*\}$  is an associate of model  $M \subseteq \{p, S, R\}$  if and only if there is an integer,  $n$ , such that:

- i.  $R \subseteq \{p_1, \dots, p_n\}$ ;
- ii.  $S^* \subseteq S \subseteq \{1, \dots, n\}$ ;
- iii.  $p$  is the marginal of  $p^*$  on  $S$ , that is,  $p(E) \equiv p^*(E \cap \{1, \dots, n\})$  for each measurable subset  $E$  of  $S$ ;
- iv.  $R^* \subseteq \{p_1^*, \dots, p_n^*\}$  defined by:
  - a.  $p_j^*(S \setminus \{j\}) \equiv 1$  (for  $j \in \{1, \dots, n\}$ );

<sup>3</sup> For prior results concerning the conditions under which a single prior to posterior shift involves a hidden conditionalization, see (Diaconis and Zabell [1982]; Skyrms [1980]).

b. for each  $p^0$  in  $R$  there is a single member  $p^0$  of  $R^*$  such that  $p^0$  is the marginal of  $p^0$ , that is,  $p^0(E) \leq p^*(E \setminus \{j\})$ .

At this point, we have assumed nothing about the relation between  $p^*$  and the family  $\{p_1^*, \dots, p_n^*\}$  beyond the fact that the latter are absolutely continuous with respect to the former. This implies that  $p^*(S \setminus \{j\}) > 0$  for  $j \in \{1, \dots, n\}$ .

The set  $\{1, \dots, n\}$  could represent a partition of a larger space (the hidden parameters), but what these integers represent is not relevant to the argument; in our proofs, they function simply as an index set. We can think of the members of  $R$  as also indexed by these integers in a corresponding fashion. Let  $p_j$  be the member of  $R$  such that  $p_j^*$  is the probability function in  $R^*$  indicated in clause (iii.b). Then  $p_j(E) \leq p^*(E \setminus \{j\})$  for each subset  $E$  of  $S$ .

The only constraint on the measure  $p^*$  in the definition above is given by clause (iii), so it is clear that each model,  $M$ , will have many associate models. This allows the introduction of the guiding constraint 'Tracking' in these terms:

Model  $M \in \langle p, S, R \rangle$  satisfies Tracking if and only if  $M$  has an associated model  $M^* \in \langle p^*, S^*, R^* \rangle$  in which  $p^*$  is a probability function on  $S^*$  such that the members of  $R^*$  are conditionalizations of  $p^*$ , specifically,  $p_j^*(E \setminus \{j\}) \leq p^*(E \setminus \{j\})$  for  $j \in \{1, \dots, n\}$ .

Let us spell out precisely what this involves.<sup>4</sup>

Suppose that  $M$ , as described, satisfies Tracking, with  $M^*$  the associate model, and that  $R$  is countable. Then  $p^*$  is a convex combination of the members of  $R^*$ , the weights being the probabilities  $p^*(S \setminus \{j\})$  for  $j \in \{1, \dots, n\}$ , each of these being positive.

Since  $p^*$  gives positive probability to each set  $\{n\}$ , and  $n$  has at least two members, it follows that these are strict convex combinations—that is, the values of  $p$  for elements of  $S^*$  are neither infimum nor supremum of the corresponding values assigned by members of  $R^*$ , unless those values are all the same.

The marginal  $p$  of  $p^*$  on  $S$  is:

$$\begin{aligned}
 p(E) &\leq p(E \setminus \{1\}) + p(E \setminus \{2\}) + \dots + p(E \setminus \{n\}) \\
 &\leq \frac{1}{n} p(\delta E \setminus \{1\}) + \frac{1}{n} p(\delta E \setminus \{2\}) + \dots + \frac{1}{n} p(\delta E \setminus \{n\}) \\
 &\leq \frac{1}{n} p_j(\delta E \setminus \{j\}) + \frac{1}{n} p(\delta E \setminus \{j\}) \\
 &\leq \frac{1}{n} p_j(\delta E \setminus \{j\}) + p^*(S \setminus \{j\})
 \end{aligned}$$

<sup>4</sup> The policy of conditionalization satisfies this criterion. But our focus here is on the more interesting question of how the criterion applies to updating policies allowing for a larger variety of inputs, and a less restrictive specification of allowable posteriors in response.

<sup>5</sup> We include here infinite (countable) combinations as convex, when the coefficients are real, positive, and sum to 1.

So  $p$  is that same convex combination of the members of  $R$ . Therefore:

**Theorem 1**

If  $M \triangleleft \langle p, S, R \rangle$ , with  $R$  countable, satisfies Tracking, then  $p$  is a strict convex combination of the members of  $R$ .

The converse to Theorem 1 also holds. Suppose that  $p \triangleleft \sum_j c_j p_j$ , with the weights  $c_j$  non-negative and summing to 1. We can then construct an associate model  $M^*$  as above, filling in the single 'blank' left in the definition by setting  $p^* \triangleleft \sum_j c_j p_j^*$ , taking  $p^*(S \setminus \{j\}) \triangleleft c_j$ , for  $j \in \{1, \dots, n\}$ . It is easy to see then that  $p$  is the marginal of  $p^*$  on  $S$  since that is how the probabilities  $p^*(S \setminus \{j\})$  were chosen. Thus:

**Theorem 2**

If in  $M \triangleleft \langle p, S, R \rangle$ , with  $R$  countable, the prior  $p$  is a strict convex combination of the members of  $R$ , then  $M$  satisfies Tracking.

## 6 The Spanning Criterion

It would be preferable to have a formulation of the criterion that can be applied directly to the relation between the prior and the possible posteriors (allowed by the policy in question, in the situation in question), without recourse to a study of other models.

For this we offer the following criterion, which has some history in the literature:

Model  $M \triangleleft \langle p, S, R \rangle$  satisfies Spanning if and only, for each random variable  $g$ ,  $E_p[g]$ , the expected value of  $g$  with respect to  $p$ , lies strictly inside the interval spanned by  $E_p^0[g]$  for  $p^0 \in R$ .

Again, 'strictly inside' an interval means that it is neither infimum nor supremum unless the interval is a single point.<sup>6</sup>

We note that Spanning is the same as the General Reflection Principle (van Fraassen [1995],[1999]), except that in the present formulation there is no reference to prior probabilities about what the future probabilities will be like. With suitable assumptions, the requirement to satisfy Spanning implies the original Reflection Principle (van Fraassen [1995], pp. 18–9) and under special conditions coincides with the Bayesian updating policy (van Fraassen [1999], p. 96).<sup>7</sup>

<sup>6</sup> This is deliberately formulated in such a way that it would apply also to interval-valued probability, but here we focus solely on the sharp probability case.

<sup>7</sup> For the special case in which the possible posteriors each assign 1 to distinct cells of a single partition, the result on conglomerability in Theorem 1 of (Dubins [1975]) is closely related to the following theorems. (Thanks to Teddy Seidenfeld for this reference.)

**Theorem 3**

If  $M \triangleleft \langle p, S, R \rangle$  satisfies Tracking and both  $S$  and  $R$  are countable, then  $M$  satisfies Spanning.

If  $M$  satisfies Tracking then, by Theorem 1,  $p$  is a convex combination  $\sum_{j \in N} c_j p_j$  of the members of  $R$ , with all coefficients positive. Suppose that  $g$  is a random variable; since  $S$  is countable, the expected value of  $g$  with respect to  $p$  is

$$\begin{aligned} E_p[g] &\triangleq \frac{1}{|S|} \sum_{x \in S} p(x) g(x) \\ &= \frac{1}{|S|} \sum_{j \in N} c_j p_j(x) g(x) \\ &= \frac{1}{|S|} \sum_{j \in N} c_j E_{p_j}[g] \\ &= \sum_{j \in N} c_j E_{p_j}[g]; \end{aligned}$$

So  $E_p[g]$  is a convex combination of the expected values of  $g$  with respect to the members of  $R$  and, since all the coefficients are positive, it lies strictly inside the interval spanned by the latter.

**Theorem 4**

If  $M \triangleleft \langle p, S, R \rangle$ , where  $S$  and  $R$  are finite and  $|R| > 1$ , satisfies Spanning then  $M$  satisfies Tracking.

In view of Theorem 2, it suffices here to prove the following lemma<sup>8</sup>.

**Lemma**

If  $M$  satisfies Spanning, where  $S$  and  $R$  are finite and  $|R| > 1$ , then  $p$  is a strict convex combination of the members of  $R$ <sup>9</sup>.

The random variables defined on  $S$  are the functions that map  $S$  into the real numbers. These form a vector space (isomorphic to the familiar  $\mathbb{R}$ , the vector space of  $n$ -tuples of real numbers), with addition and scalar multiplication defined point-wise:

$$cg(x) \triangleq \sum_{x \in S} g(x) c_x; \quad \delta g(x) \triangleq \sum_{x \in S} g(x) \delta_x;$$

The probability measures form a convex subset of this vector space, defined by

$$q(x) = 0; \quad \sum_{x \in S} q(x) = 1;$$

The fact that the set is convex follows from the observation that if  $p$  and  $q$  are probability measures on  $S$ , then so is  $cp + (1 - c)q$ , for  $c$  in  $[0, 1]$ .

<sup>8</sup> We would have a much simpler proof, using the Separating Hyperplane Theorem (Gruber [2007], p. 59; Luenberger [1969], pp. 133–4) if we did not require the probability  $p$  to lie strictly inside the interval spanned by  $E_p[g]$  for  $p \in R$ . Our proof is for the case where  $S$  and  $R$  are finite; we do not know if the result continues to hold if  $R$  is infinite (but see the Appendix).

<sup>9</sup> For the special case in which the posteriors have disjoint support, that was proved in (van Fraassen [1999]). The argument there does not apply to the general case.

The expected value of  $g$  with respect to a probability measure  $q$  is, in vector space terms, just the scalar product:

$$\langle g, p \rangle \triangleq \frac{1}{4} \sum_{x \in S} q(x)p(x):$$

If  $R$  is finite, the convex hull  $[R]$  of  $R$  (that is, the set of all probability measures that are convex combinations of the elements of  $R$ ) is called a polytope. The extreme points are those that are not convex combinations of other members; a polytope has extreme points. Since  $R$  has more than one member, it has at least two extreme points and also non-extreme points. (The simplest case is the one where  $R$  has just two members which are the extreme points of  $[R]$ .)

We need to show that unless  $p$  is in  $[R]$ , but not an extreme point of  $[R]$ , then Spanning is violated. The latter means that there is some random variable,  $g$ , such that the scalar product  $\langle g, p \rangle$  is not strictly in the interval spanned by the set  $\{\langle g, p^0 \rangle : p^0 \in [R]\}$ .

Two vectors,  $g$  and  $h$ , are orthogonal if and only if  $\langle g, h \rangle \leq 0$ . In the familiar three-dimensional vector space, the two-dimensional sub-spaces are the planes, and a plane is the set of vectors orthogonal to a given vector. So a plane is defined by an equation of the form  $\langle g, q \rangle \leq 0$  for a fixed vector  $g$ ; that is,  $q$  is in the plane if and only if  $\sum_{x \in S} g(x)q(x) \leq 0$ . In our context, where  $q$  is a probability function, this means that the expected value of  $g$  with respect to  $q$  is 0.

In general, the maximal proper sub-spaces of a vector space are called the hyperplanes and, again, any such sub-space is the ortho-complement of a single vector, defined in the same way. If  $H$  is the hyperplane  $\{h : \langle g, h \rangle \leq 0\}$ , then it divides the space into two half-spaces, overlapping only in  $H$  itself, namely,  $\{h : \langle g, h \rangle < 0\}$  and  $\{h : \langle g, h \rangle \geq 0\}$ . A polytope is the intersection of a finite set of half-spaces and, equivalently, the convex hull of a finite set of vectors. Its extreme points are those that are not convex combinations of other members. A polytope has a finite set of extreme points, all of which are in it, and it is the convex hull of its set of extreme points.

Since  $R$  is finite,  $[R]$  is a polytope.<sup>10</sup> Let  $[R]$  thus be the intersection of a finite set of half-spaces,  $H_1, \dots, H_m$  defined by the inequalities

$$\langle h_i, x \rangle \leq 0; \dots; \langle h_m, x \rangle \leq 0;$$

The corresponding supporting hyperplanes,  $T_1, \dots, T_m$ , which contain the faces of  $[R]$ , are defined by the equalities

$$\langle h_i, x \rangle \leq 0; \dots; \langle h_m, x \rangle \leq 0;$$

<sup>10</sup> In fact, since  $[R]$  is part of the convex set of all probability functions on this space, it is bounded (all members have a norm less than or equal to 1).

Now suppose that  $p$  is not a strict convex combination of members of  $R$ . Then  $p$  is either outside  $[R]$  or else is an extreme point of  $[R]$ . If  $p$  is outside  $[R]$ , then there is a half-space,  $H$  to which  $p$  does not belong. Hence  $(hp)$  is positive, and thus greater than  $(h, x)$  for any member  $x$  of  $[R]$ .

If  $p$  is an extreme point of  $[R]$ , then it is the unique intersection of a subfamily of the hyperplanes above:

$$p \not\in T_{i1} \cup \dots \cup T_{ik}$$

(cf. Gruber [2007], pp. 246–7). So the random variable  $g \not\in h_{i1} \cup \dots \cup h_{ik}$  takes value 0 on  $p$ . But if  $x$  is any other point in  $[R]$ , it will lie in all the corresponding half-spaces, but not on all of these hyperplanes, hence  $g(x) < 0$ . Hence the expected value of  $g$  with respect to  $p$  is not strictly in the interval spanned by the expected values of  $g$  with respect to members of  $R$ . Thus, in that case, Spanning is violated.

## 7 Non-Bayesian Policies that Satisfy the Spanning and Tracking Criteria

The close connection between Tracking and conditionalization may give the impression that the results proved here impose Bayesian conditionalization as the sole admissible policy. That is not so, for a number of reasons.

The orthodox Bayesian policy is this:

- . accept as admissible input only propositions;
- . as response to such an input, the only admissible change is conditioning the prior on the proposition in question.

So an updating policy can depart from the Bayesian in one or more of three ways:

1. Accept as admissible a wider variety of inputs (for example, expected values).
2. An admissible response to such an input can be a change in the prior that is not the result of conditioning.
3. An admissible response to such an input may be non-unique, that is, the posterior may not be uniquely determined by the prior plus input.

For example, then, a policy could be non-Bayesian by differing in the third way, even if the posterior is formed from the prior by conditionalizing. This could be so if the updating involved a choice or, instead of a free choice, involved some factor in addition to prior and explicit evidential input that helps to determine the posterior—a ‘hidden variable’, of which the agent might or might not be aware.

Second, we have included no initial assumption here about how the possible posterior opinions relate to the possible inputs. It is not assumed that for each

input there is only one admissible posterior opinion, nor that they are conditionalizations of the prior.

Third, the embedding in a larger probability space (associate model) is in no way unique. The 'hidden variables' are underdetermined. Nor is there any reason to think that the same larger probability space would be pertinent for transitions over subsequent time intervals  $(t, t+a)$ ,  $(t+a, t+a+b)$ ,  $(t, t+a+b)$ .

Fourth, the possible posteriors in a model that satisfies Tracking are in general not conditionalizations of the prior. That was quite clear in our 'Senate' example above. For the marginals of conditionalizations of a probability function are only in special cases conditionalizations of a marginal. A special case is the one in which the 'hidden variables' over which an average is taken are actually independent of the 'surface' variables.

Finally, the criteria can with minor modifications in phrasing be applied to policies governing interval-valued rather than sharp probabilities, where it is still even far from clear what the proper analogue to Bayesian conditionalization must be.

It is helpful to look at some simple examples of how Spanning and Tracking can be satisfied by non-Bayesian updating policies. Our first illustration already provided a good example: a policy that consists in adopting as probabilities certain marginals of the probabilities assigned by Bayesian conditionalization in a larger space. That sort of example can illustrate the various differences. But we add here two more examples of updating policies that involve some leeway, and are clearly not the Bayesian conditionalizing policy, but satisfy Spanning and Tracking.

To begin, consider Jeffrey conditionalization. Recall Jeffrey's example, cited above, of the agent who examines a piece of cloth by candlelight and acts on an input that does not take the form of a proposition on which he could conditionalize his probability function. In response to this input, the agent redistributes the probability over a partition whose cells are the affected alternatives:

$$p \in \bigcup_{i=1}^n q_i P(A_i)$$

with  $\{C_i : i \in \{1, 2, \dots, n\}\}$  the relevant partition, such as 'green', 'blue', ..., 'red', and the weights  $\{q_i : i \in \{1, 2, \dots, n\}\}$  the new posterior probabilities of those alternatives. If the agent's policy in this case dictates no constraints on those new weights then Spanning is satisfied, since the prior,  $p$ , is itself among the possible posteriors. Thus Jeffrey's proposal, by itself, is for a policy that satisfies our criteria.

We can imagine that certain other factors in the policy do place constraints on the new weights, in which case Spanning could be violated.

When observing cloth by candlelight this would happen, for example, if the agent could only raise his probability that the cloth is green, and not lower it.

We can imagine a more complex situation in which the transformation requires not only a selection of new weights on a given partition, but also a choice of relevant partition. This would increase the set of possible posteriors accordingly. Jeffrey's example would become something like this: In examining a piece of cloth by candlelight, one might come to attribute probabilities 0.6 and 0.4 to the propositions *G*, that the cloth is green, and *B*, that it is blue, or alternatively come to attribute probabilities 0.7 and 0.3 to the propositions *C*, that the cloth is cotton, and *L*, that it is linen, without there being any proposition *E* for which the direct effect of the observation is anything near changing the observer's degree of belief in *E* to 1.

The criterion that Spanning be satisfied applies to the policy that allows these changes under those epistemic circumstances, rather than to the specific or actual change of posterior weights in either partition. Thus, Spanning is satisfied here.

As above, violation is possible if that policy has some further features that prevent a change in probability in one direction, either upward or downward, while allowing a change in the other. While on the face of it, it is hard to see how a policy could appear rational while doing so, we shall see below that a well-known updating policy does exactly that.

For the next two examples, staying rather close to the Bayesian format, we consider the case where the agent is going to make an observation and knows that the event to be observed is a member of the partition  $\{\mathbb{E} \in J\}$ . The first policy dictates conditionalizing the prior,  $p$ , on event  $\mathbb{E}$  if the agent witnesses that event. But if the agent is not sure whether the event witnessed was, say,  $E$  or  $E_m$ , then the policy dictates that he Jeffrey conditionalizes—that is, adopt as posterior a convex combination of  $p_{\mathbb{E}}$  and  $p_{E_m}$ . But the policy does not dictate the weights in that convex combination, which can be chosen spontaneously from some given finite set. Hence  $R$  consists of certain convex combinations of the result of conditioning  $p$  on members of the partition, so Spanning is satisfied. This example already involves all three of the departures from the orthodox Bayesian mold: the input is not a proposition but something involving two propositions (with the agent's attitude not belief but ambivalence) and a choice of weights, so that the posterior is not uniquely determined by the input and prior.

A still different policy links posteriors to both the event witnessed and the agent's frame of mind, which is not determined by either the prior or the witnessed events. Suppose that  $p_j$  is the probability that the agent uses when in frame of mind  $j$ , for  $j$  in some index set  $J$ . That is, the probability he assigns to events depends on his mood or on the context, as characterized by his frame

of mind. As long as he stays in frame of mind  $j$ , he updates by conditionalizing, but if his frame of mind changes from frame  $j$  to frame  $k$ , he updates by conditioning on  $p_k$ . The agent might not even be aware that he assigns different probabilities in different frames of mind.

Does this last policy satisfy Spanning? It does. Suppose that the prior is  $p$  and the agent is about to begin an experiment with outcomes that yield the mutually exclusive propositions  $E_m$  for  $m \in M$ . It is possible that the agent stays in frame of mind 1, but it is also possible that he will shift into some frame  $j$  in  $J$  before conditioning. The set of possible posteriors is  $\{p_j(j|E_m) : j \in J, m \in M\}$ , of which  $p_1$  is a convex combination.

## 8 Policies that Violate the Spanning and Tracking Criteria

We now turn to the limits set by the Tracking criterion.

A simple example will show that a violation of our criteria may be quite salient in an ordinary, easily imaginable situation. Imagine a doctor who announces to a patient that the probability that he has a certain virus is  $x$ . He prescribes a blood test that has no adverse effects of any sort. Then he announces that if the test outcome is positive he will conclude that probability is at least twice as high, but if the outcome is negative they won't know any more than before, his current opinion will not change, and they will need more tests. (Various acquaintances of ours readily imagined that their doctors might announce something of this form.)

Only a little reflection shows that the doctor's probabilities, as they can evolve over this interval of time, cannot possibly be in accordance with the actual statistics throughout. If his prior probability, before the test is given, matches the statistics, so that  $x\%$  of the population has the virus, how could the proportions in the three sub-populations (test positive, test negative, not tested) be  $2x\%$ ,  $x\%$ , and  $x\%$ ? It is clear that to satisfy Spanning, the prior value needs to fall strictly inside the interval spanned by the posterior values.

Orthodox Bayesians would have the same complaint against the doctor, for this test situation, with its 'definite' propositional outcomes, is their paradigm example, and he is supposed to conditionalize (and to know beforehand that his possible posteriors will be conditionalizations on the outcomes). Thus the orthodox Bayesian policy of conditionalizing on new 'evidence' propositions satisfies Spanning. Currently, objective Bayesians recognize a wider array of possible input forms, and follow the policy of maximizing relative entropy (MAXENT), originally proposed by Jaynes (cf. Williamson [2011]). The debate between orthodox and objective Bayesians goes back to the 1970s. Specifically, Myron Tribus and Hector Motroni ([1972]) and Kenneth Friedman ([1973]) debated an example in statistical physics where MAXENT violates Spanning.

Friedman insists on the criterion, though without naming it as a specific criterion or elaboration, viewing it as corollary to the Bayesian policy:

According to a Bayesian account that a prior probability cannot be revised upward [...] but will with probability  $p [ > 0 ]$  be revised downward, implies that [...] the prior probability must be too high.  
(Friedman [1973], p. 266)

The currently more familiar, and simpler, ‘Judy Benjamin’ problem illustrates the same difficulty (van Fraassen [1981]; van Fraassen et al. [1986]; Grove and Halpern [1997]; Hartmann and Rad [unpublished]).

In such examples, where the input is a new value for a conditional probability of a given event A on assumption B, there will be an event C disjoint from A [ $B$  whose probability MAXENT will definitely raise or keep equal, regardless of what that new input value is (cf. van Fraassen [1981]; the point is generalized in (Seidenfeld [1987], Corollary 1, p. 283)). If those probabilities are the relevant possible posteriors then Spanning is violated; therefore, by Theorem 3, Tracking is violated as well.

Many discussions have shown how this difficulty can disappear if the situation is described differently, either by adding information to the input, or constraining the range of possible inputs not ruled out by the prior, or constraining the range of possible posteriors allowed by the policy. Each of these alters the situation; Jaynes’ prescription was specifically for the case in which there is nothing else to go on. Even today MAXENT remains controversial (see, for example, Giñowald and Halpern [2003]). On the one hand, there is an explicit defense of MAXENT’s violation of Spanning by Jon Williamson ([2011], pp. 68, 72, 80–1); on the other hand, Brian Skyrms concluded on the basis of his results about the relation between MAXENT and conditiona-  
lization that ‘MAXENT escapes dynamic incoherence by a hair’s breadth’ (Skyrms [2013], p. 82; see also his [1985], [1987]).

## Appendix: Generalizing to the Countably Infinite Case

In the proof of Theorem 4, the underlying space,  $S$ , was assumed finite, so that the set of random variables formed a finite-dimensional vector space. In the case where  $S$  is countable, it becomes less clear how the criterion of possibly tracking real statistics is to be understood. Presumably, ‘tracking the real statistics’ would need to be cashed out in terms of matching limits of relative frequencies in countable sequences of samplings or events.

However, the proof of Theorem 4 can be generalized to include the case of  $R$  finite but  $S$  countable. In that case, the vector space whose elements are the random variables defined on  $S$ , including the probability functions, must be chosen such that the scalar products (hence, the expected values) are well defined.

Let  $S \subseteq \{x_j : j \in \mathbb{N}\}$ . We restrict to well-behaved random variables, that is, the function  $g$  such that  $\sum_j g(x_j)^2$  is finite ('square integrable functions'). Then the random variables form a separable Hilbert space familiar from quantum mechanics, standardly called  $\mathcal{H}$ . This includes the probability functions defined on  $S$ , in the same way as before, and the scalar product is well defined.

The definitions we gave above for the finite case carry over naturally. A hyperplane is a set of vectors orthogonal to a given vector and, equivalently, the set of vectors  $x$  such that for a certain vector,  $y$ ,  $(x, y) \neq 0$ . A hyperplane  $H$  divides the space into two half-spaces,  $H^+ \subseteq \{x : (x, y) \geq 0\}$  and  $H^- \subseteq \{x : (x, y) \leq 0\}$ .

Although  $S$  is now allowed to be countably infinite, we continue to require that  $R$  be finite, so its convex hull is a polytope with the members of  $R$  its vertices. Since a subspace of  $\mathcal{H}$  is convex, all of this hull, together with the prior  $p$ , is a subset of finite-dimensional sub-space. Nothing in the argument in Theorem 4 required reference to the properties of the ambient space, so it applies without change to this setting as well.

### Acknowledgements

We thank the referees for their detailed reading of the article and many useful comments, which resulted in significant improvements. Van Fraassen thanks Simon Kochen for a helpful discussion of issues about convexity in infinite-dimensional spaces. Halpern's work was supported in part by the National Science Foundation (NSF) under grants IIS-0911036 and CCF-1214844, the Air Force Office of Sponsored Research (AFOSR) under grant FA9550-08-1-0438, the Army Research Office (ARO) under grant W911NF-14-1-0017, and by the Department of Defence Multidisciplinary University Research Initiative (MURI) programme administered by AFOSR under grant FA9550-12-1-0040.

Bas C. van Fraassen  
 Department of Philosophy  
 San Francisco State University  
 1600 Holloway Avenue  
 San Francisco, CA 94132, USA  
 fraassen@princeton.edu

Joseph Y. Halpern  
 Department of Computer Science  
 Cornell University, 414 Gates Hall  
 Ithaca, NY 14850, USA  
 halpern@cs.cornell.edu

## References

Diaconis, P. and Zabell, S. L. [1982]: 'Updating Subjective Probability', *Journal of the American Statistical Association*, 77, pp. 822–30.

Dubins, L. E. [1975]: 'Finitely Additive Conditional Probabilities, Conglomerability, and Disintegrations', *The Annals of Probability*, 3, pp. 89–99.

Friedman, K. [1973]: 'Replies to Tribus and Motroni and to Gage and Hestenes', *Journal of Statistical Physics*, 9, pp. 265–9.

Grove, A. and Halpern, J. Y. [1997]: 'Probability Update: Conditioning versus Cross-Entropy', in D. Geiger and P. Shenoy (eds) *Proceedings of the 13th Conference on Uncertainty in AI*, San Francisco, CA: Morgan Kaufmann, pp. 208–14.

Gruber, P. M. [2007]: *Convex and Discrete Geometry*, Berlin: Springer.

Grünwald, P. D. and Halpern, J. Y. [2003]: 'Updating Probabilities', *Journal of Artificial Intelligence Research*, 19, pp. 243–78.

Hartmann, S. and Rad, S. R. [unpublished]: 'Updating on Conditionals 1/4 Kullback–Leibler + Causal Structure'.

Jaynes, E. T. [1957]: 'Information Theory and Statistical Mechanics', *Physical Review*, 106, pp. 620–30.

Jaynes, E. T. [2003]: *Probability Theory: The Logic of Science*, Cambridge: Cambridge University Press.

Jeffrey, R. [1968]: 'Probable Knowledge', in I. Lakatos (ed.), *The Problem of Inductive Logic*, Amsterdam: North Holland, pp. 166–80, 189–90.

Luenberger, D. G. [1969]: *Optimization by Vector Space Methods*, New York: Wiley.

Seidenfeld, T. [1987]: 'Entropy and Uncertainty', in I. B. MacNeill and G. J. Umphrey (eds), *Foundations of Statistical Inference*, Dordrecht: Reidel, pp. 259–87.

Skyrms, B. [1980]: 'Higher Order Degrees of Belief', in M. Hugh (ed.), *Prospects for Pragmatism*, Cambridge: Cambridge University Press, pp. 109–38.

Skyrms, B. [1985]: 'Maximum Entropy Inference as a Special Case of Conditionalization', *Synthese*, 63, pp. 55–74.

Skyrms, B. [1987]: 'Updating, Supposing, and MAXENT', *Theory and Decision*, 22, pp. 225–46.

Skyrms, B. [2013]: *From Zeno to Arbitrage: Essays on Quantity, Coherence, and Induction*, New York: Oxford University Press.

Teller, P. [1976]: 'Conditionalization, Observation, and Change of Preference', in W. L. Harper and C. A. Hooker (eds), *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, Volume 1, Dordrecht: Reidel, pp. 205–53.

Teller, P. and Fine, A. [1975]: 'A Characterization of Conditional Probability', *Mathematics Magazine*, 48, pp. 267–70.

Tribus, M. and Motroni, H. [1972]: 'Comments on the Paper "Jaynes' Maximum Entropy Prescription and Probability Theory"', *Journal of Statistical Physics*, 4, pp. 227–8.

van Fraassen, B. C. [1981]: 'A Problem for Relative Information Minimizers in Probability Kinematics', *British Journal for the Philosophy of Science*, 32, pp. 375–79.

van Fraassen, B. C. [1995]: 'Belief and the Problem of Ulysses and the Sirens', *Philosophical Studies*, 77, pp. 7–37.

van Fraassen, B. C. [1999]: 'Conditionalization, a New Argument For', *Topoi*, 18, pp. 93–6.

van Fraassen, B. C., Hughes, R. I. G. and Harman, G. [1986]: 'A Problem for Relative Information Minimizers in Probability Kinematics, Continued', *British Journal for the Philosophy of Science*, 37, pp. 453–75.

Williamson, J. [2011]: 'Objective Bayesianism, Bayesian Conditionalization, and Voluntarism', *Synthese*, 178, pp. 67–85.