



PAPER

Green function of the Poisson equation: $D = 2, 3, 4$

OPEN ACCESS

RECEIVED

13 November 2017

ACCEPTED FOR PUBLICATION

22 December 2017

PUBLISHED

29 January 2018

Original content from this work may be used under the terms of the [Creative Commons Attribution 3.0 licence](#).

Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

UD Jentschura¹ and J Sapirstein²¹ Department of Physics, Missouri University of Science and Technology, Rolla, MO 65409, United States of America² Department of Physics, University of Notre Dame, Notre Dame, IN 46556, United States of America**Keywords:** Green functions, Poisson equation, cusp condition, special functions, associated Gegenbauer polynomials

Abstract

We study the Green function of the Poisson equation in two, three and four dimensions. The solution g of the equation $\vec{\nabla}^2 g(\vec{x} - \vec{x}') = \delta^{(D)}(\vec{x} - \vec{x}')$, where \vec{x} and \vec{x}' are D -dimensional position vectors, is customarily expanded into radial and angular coordinates. For the two-dimensional case ($D = 2$), we find a subtle interplay of the necessarily introduced scale L with the radial component of zero magnetic quantum number. For $D = 3$, the well-known expressions are briefly recalled; this is done in order to highlight the analogy with the four-dimensional case, where we uncover analogies of the four-dimensional spherical harmonics with the familiar three-dimensional case. Remarks on the $SO(4)$ symmetry of the hydrogen atom complete the investigations.

1. Introduction

Solutions of the equation

$$\vec{\nabla}^2 g(\vec{x} - \vec{x}') = \delta^{(D)}(\vec{x} - \vec{x}') \quad (1)$$

enter a myriad of physical problems, from the elementary Coulomb problem in electrostatics ($D = 3$), to the attraction among vortices in two-dimensional systems ($D = 2$), and on to the four-dimensional formulation of the hydrogen Green function ($D = 4$, see [1]). Here, we shall attempt to provide a unified treatment of the radial and angular decompositions of the two-, three- and four-dimensional Green functions, which are solutions to equation (1).

In $D = 2$, a scale has to be introduced, which corresponds to a physically irrelevant overall constant term, while in $D = 3$, the formulas are very familiar (see [2, 3]). In $D = 4$, we attempt to reveal a structure of (re-) defined associated ultraspherical polynomials (Gegenbauer polynomials), which highlights analogies to the associated Legendre functions that enter the case $D = 3$.

2. Two-dimensional case

Spherical coordinates in two dimensions have a cylindrical symmetry; hence, for definiteness, we denote the two-dimensional position vectors as $\vec{\rho}$ and $\vec{\rho}'$, and their moduli as $\rho = |\vec{\rho}|$ and $\rho' = |\vec{\rho}'|$. The Green function solution of the Poisson equation,

$$\vec{\nabla}^2 g(\vec{\rho} - \vec{\rho}') = \delta^{(2)}(\vec{\rho} - \vec{\rho}'), \quad g(\vec{\rho} - \vec{\rho}') = \frac{1}{2\pi} \ln \left(\frac{|\vec{\rho} - \vec{\rho}'|}{L} \right), \quad (2)$$

introduces a scale L , which ensures that the argument of the natural logarithm is dimensionless. In terms of the Green function, the scale L adds nothing but an overall constant term,

$$g \rightarrow g - \frac{1}{2\pi} \ln L. \quad (3)$$

In order to show that g fulfills the Poisson equation, one specializes the divergence theorem to an infinitesimal area A . For example, A might be chosen as the inner area of a circle of infinitesimal radius ϵ , about the centre $\vec{\rho}'$. With ∂A denoting the boundary of A , i.e., the circle of radius ϵ about $\vec{\rho}'$, one must have

$$\int_{\partial A} \vec{\nabla} g(\vec{\rho} - \vec{\rho}') \cdot d\vec{S} = \int_A \vec{\nabla}^2 g(\vec{\rho} - \vec{\rho}') d^2\rho = \int_A \delta^{(2)}(\vec{\rho} - \vec{\rho}') d^2\rho = 1. \quad (4)$$

Using the formula (2) and the radial component of the gradient operator in two-dimensional coordinates, one verifies that indeed,

$$\int_{\partial A} \vec{\nabla} g(\vec{\rho} - \vec{\rho}') \cdot d\vec{S} = \int_0^{2\pi} \left. \frac{\partial}{\partial \rho} g(\vec{\rho} - \vec{\rho}') \right|_{|\vec{\rho} - \vec{\rho}'| = \epsilon} \epsilon d\varphi = \int \frac{1}{2\pi\epsilon} \epsilon d\varphi = 1,$$

independent of ϵ .

Let us now turn to the angular-momentum decomposition of equation (2). The spherical representation of the two-dimensional Dirac- δ is

$$\delta^{(2)}(\vec{\rho} - \vec{\rho}') = \frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi'). \quad (5)$$

An appropriate *ansatz* for the Green function is

$$g(\vec{\rho} - \vec{\rho}') = \sum_{m=-\infty}^{\infty} f_m(\rho, \rho', \varphi') e^{im\varphi}. \quad (6)$$

The two-dimensional representation of the Laplacian is

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}. \quad (7)$$

It acts on the Green function as follows,

$$\vec{\nabla}^2 g(\vec{\rho} - \vec{\rho}') = \sum_{m=-\infty}^{\infty} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} \right) f_m(\rho, \rho', \varphi') e^{im\varphi} = \frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi'). \quad (8)$$

Now, one multiplies both sides with the factor

$$\frac{1}{2\pi} e^{-im'\varphi}, \quad (9)$$

and integrates over $d\varphi$, resulting in the equation

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{m'^2}{\rho^2} \right) f_{m'}(\rho, \rho', \varphi') = \frac{1}{2\pi\rho} \delta(\rho - \rho') e^{-im'\varphi'}. \quad (10)$$

Setting

$$f_{m'}(\rho, \rho', \varphi') = g_{m'}(\rho, \rho') e^{-im'\varphi'}, \quad (11)$$

and renaming $m' \rightarrow m$ after this operation, one obtains the radial equation

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} \right) g_m(\rho, \rho') = \frac{1}{2\pi\rho} \delta(\rho - \rho'). \quad (12)$$

Inspired by textbook treatments [2, 3] of the three-dimensional Green function, one uses the following *ansatz* for nonzero m ,

$$g_m(\rho, \rho') = C \left(\frac{\rho_{<}}{\rho_{>}} \right)^{|m|}, \quad m \neq 0, \quad (13)$$

where $\rho_{<} = \min(\rho, \rho')$ and $\rho_{>} = \max(\rho, \rho')$, and integrates equation (12) from $\rho = \rho' - \epsilon$ to $\rho = \rho' + \epsilon$,

$$\int_{\rho=\rho'-\epsilon}^{\rho=\rho'+\epsilon} \left(\frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{m^2}{\rho^2} \right) g_m(\rho, \rho') d\rho = \int_{\rho=\rho'-\epsilon}^{\rho=\rho'+\epsilon} \frac{1}{2\pi} \delta(\rho - \rho') d\rho. \quad (14)$$

This results in the relation

$$\left[\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho') \right] \Big|_{\rho=\rho'-\epsilon}^{\rho=\rho'+\epsilon} = \frac{1}{2\pi}, \quad (15)$$

and amounts to the condition

$$C \left(\left[\rho \frac{\partial}{\partial \rho} \left(\frac{\rho}{\rho'} \right)^{|m|} \right] \Big|_{\rho=\rho'} - \left[\rho \frac{\partial}{\partial \rho} \left(\frac{\rho}{\rho'} \right)^{|m|} \right] \Big|_{\rho=\rho'} \right) = C (-|m| - |m|) \rho \frac{\rho^{|m|}}{\rho^{|m|+1}} = \frac{1}{2\pi}, \quad (16)$$

with the result

$$C = -\frac{1}{2\pi} \frac{1}{2|m|}. \quad (17)$$

The case $m = 0$ requires special treatment. One sets

$$g_m(\rho, \rho') = D \ln\left(\frac{\rho_{>}}{L}\right), \quad (18)$$

because this term matches the asymptotic limit of equation (2) for $\rho \rightarrow \infty$, $\rho' \rightarrow 0$. In this case, equation (15) translates into the condition

$$D \left(\left[\rho \frac{\partial}{\partial \rho} \ln\left(\frac{\rho}{L}\right) \right] \Big|_{\rho=\rho'} - \left[\rho \frac{\partial}{\partial \rho} \ln\left(\frac{\rho'}{L}\right) \right] \Big|_{\rho=\rho'} \right) = D - 0 = \frac{1}{2\pi}, \quad (19)$$

with the result $D = 1/(2\pi)$. Adding the terms for $m = 0$ and $m \neq 0$, one has

$$\begin{aligned} g(\vec{\rho} - \vec{\rho}') &= \frac{1}{2\pi} \ln\left(\frac{\rho_{>}}{L}\right) - \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{4\pi|m|} \left(\frac{\rho_{<}}{\rho_{>}}\right)^{|m|} e^{i m (\varphi - \varphi')} \\ &= \frac{1}{2\pi} \left[\ln\left(\frac{\rho_{>}}{L}\right) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_{<}}{\rho_{>}}\right)^m \cos(m(\varphi - \varphi')) \right]. \end{aligned} \quad (20)$$

A numerical check of this relations is successful. For $\vec{\rho} = 0.2 \hat{e}_x + 0.1 \hat{e}_y$ and $\vec{\rho}' = 1.1 \hat{e}_x + 1.5 \hat{e}_y$, the expression in equation (2) evaluates to

$$T_1 = g(\vec{\rho} - \vec{\rho}') = \frac{1}{2\pi} \ln\left(\frac{|\vec{\rho} - \vec{\rho}'|}{L}\right) = -0.296\,159 \quad (21)$$

while the $m = 0$ term from equation (20) is

$$T_2 = \frac{1}{2\pi} \ln\left(\frac{\rho_{>}}{L}\right) = -0.278\,459. \quad (22)$$

Adding the sum over the nonzero m , one obtains

$$T_3 = - \sum_{m=1}^{\infty} \frac{1}{2\pi m} \left(\frac{\rho_{<}}{\rho_{>}}\right)^m \cos[m(\varphi - \varphi')] = -0.017\,700. \quad (23)$$

We have checked the equality $T_1 = T_2 + T_3$ for a number of example cases. It is interesting to note that equation (20) does not seem to have appeared in the literature before.

3. Three-dimensional case

Let \vec{r} and \vec{r}' denote coordinate vectors in three-dimensional space. It is well known that

$$g(\vec{r} - \vec{r}') = -\frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|}, \quad g(\vec{k}) = - \int d^3r \, e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')} \frac{1}{4\pi} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{1}{\vec{k}^2}, \quad (24)$$

fulfills the Poisson equation

$$\vec{\nabla}^2 g(\vec{r} - \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}'). \quad (25)$$

The well-known expansion into (three-dimensional) spherical harmonics reads as follows,

$$g(\vec{r} - \vec{r}') = - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\theta', \varphi'), \quad (26)$$

where $r_{<} = \min(r, r')$, $r_{>} = \max(r, r')$. The Laplacian in three dimensions reads as

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\vec{L}^2}{r^2}, \quad (27)$$

where $\vec{L} = -i\vec{r} \times \vec{\nabla}$. The radial part of the Green function (26) is assembled from homogeneous solutions of the radial equation, in much the same way as in the derivation extending from equations (13) to (17). The transformation from Cartesian to spherical coordinates is, with $\vec{r} = \sum_{i=1}^3 x_i \hat{e}_i$,

$$x_1 = r \sin \theta \cos \varphi, \quad (28a)$$

$$x_2 = r \sin \theta \sin \varphi, \quad (28b)$$

$$x_3 = r \cos \theta. \quad (28c)$$

The infinitesimal solid angle element is

$$d^2\Omega = \sin \theta \, d\theta \, d\varphi. \quad (29)$$

The well-known spherical harmonics are given as

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell}^m(\cos(\theta)) e^{im\varphi}, \quad (30)$$

with the orthonormality and completeness properties

$$\int d^2\Omega Y_{\ell' m'}^*(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \delta_{\ell' \ell} \delta_{m m'}, \quad (31a)$$

$$\sum_{\ell m} Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}^*(\theta', \varphi') = \frac{1}{\sin \theta} \delta(\theta - \theta') \delta(\varphi - \varphi'). \quad (31b)$$

The summation limits are $\ell = 0, \dots, \infty$ and $m = -\ell, \dots, \ell$. The generating function for the Legendre polynomials [4] is

$$P_{\ell}^0(x) = P_{\ell}(x), \quad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} P_{\ell}(x) t^{\ell}, \quad (32)$$

while the associated Legendre polynomials are given by

$$P_{\ell}^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_{\ell}(x)}{dx^m}. \quad (33)$$

They have the property

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{m^2}{\sin^2 \theta} \right) P_{\ell}^m(\cos \theta) = -\ell(\ell+1) P_{\ell}^m(\cos \theta). \quad (34)$$

These formulas are recalled with the notion of clarifying the analogies with the four-dimensional case, as will be done in the following.

4. Four-dimensional case

Let us denote four-dimensional vectors like $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ in bold face. Just for clarity, we should stress that we are assuming a Euclidean metric. The formula analogous to equation (24) is

$$g(\boldsymbol{\xi} - \boldsymbol{\xi}') = -\frac{1}{4\pi^2} \frac{1}{(\boldsymbol{\xi} - \boldsymbol{\xi}')^2}, \quad g(\mathbf{k}) = -\int d^4\xi e^{-ik(\boldsymbol{\xi}-\boldsymbol{\xi}')} \frac{1}{4\pi^2} \frac{1}{(\boldsymbol{\xi} - \boldsymbol{\xi}')^2} = -\frac{1}{k^2}, \quad (35)$$

where $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^4$. The Green function $g(\boldsymbol{\xi} - \boldsymbol{\xi}')$ fulfills the equation

$$\nabla^2 g(\boldsymbol{\xi} - \boldsymbol{\xi}') = \delta^{(4)}(\boldsymbol{\xi} - \boldsymbol{\xi}'). \quad (36)$$

The expansion into (four-dimensional) spherical harmonics [1] introduces an additional quantum number, which we denote as n , and the analogue of equation (26) is

$$g(\boldsymbol{\xi} - \boldsymbol{\xi}') = -\sum_{n\ell m} \frac{1}{2(n+1)} \frac{\xi_{<}^n}{\xi_{>}^{n+2}} Y_{n\ell m}(\chi, \theta, \varphi) Y_{n\ell m}^*(\chi', \theta', \varphi'), \quad (37)$$

where the $Y_{n\ell m}(\chi, \theta, \varphi)$ are four-dimensional spherical harmonics, and $\xi_{<} = \min(|\boldsymbol{\xi}|, |\boldsymbol{\xi}'|)$, and $\xi_{>} = \max(|\boldsymbol{\xi}|, |\boldsymbol{\xi}'|)$. The summation limits are $n = 0, \dots, \infty$, $\ell = 0, \dots, n$, and $m = -\ell, \dots, \ell$. The four-dimensional Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{3}{\rho} \frac{\partial}{\partial \xi} + \frac{1}{\xi^2} \left(\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} - \frac{\bar{L}^2}{\sin^2 \chi} \right), \quad (38)$$

and the radial part of the decomposition (37) is assembled from homogeneous solutions of the radial component of the four-dimensional Laplacian. The transformation from Cartesian to spherical coordinates is, with $\boldsymbol{\xi} = \sum_{i=1}^4 x_i \hat{\mathbf{e}}_i$,

$$x_1 = r \cos \varphi \sin \theta \sin \chi, \quad (39a)$$

$$x_2 = r \sin \varphi \sin \theta \sin \chi, \quad (39b)$$

$$x_3 = r \cos \theta \sin \chi, \quad (39c)$$

$$x_4 = r \cos \chi. \quad (39d)$$

The infinitesimal solid angle element is

$$d^3\Omega = \sin^2 \chi \sin \theta d\theta d\varphi. \quad (40)$$

The four-dimensional spherical harmonics can be given in terms of the analogue of equation (30) as

$$Y_{n\ell m}(\chi, \theta, \varphi) = \sqrt{\frac{2}{\pi}} \sqrt{\frac{(n+1)(n-\ell)!}{(n+\ell+1)!}} Q_n^\ell(\cos \chi) Y_{\ell m}(\theta, \varphi), \quad (41)$$

with the orthonormality and completeness properties

$$\int d^3\Omega Y_{n'\ell'm'}^*(\chi, \theta, \varphi) Y_{n\ell m}(\chi, \theta, \varphi) = \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'}, \quad (42a)$$

$$\sum_{n=0}^{\infty} \sum_{\ell=0}^n \sum_{m=-\ell}^{\ell} Y_{n\ell m}(\chi, \theta, \varphi) Y_{n'\ell' m'}^*(\chi', \theta', \varphi') = \frac{1}{\sin^2 \chi \sin \theta} \delta(\chi - \chi') \delta(\theta - \theta') \delta(\varphi - \varphi'). \quad (42b)$$

The generating function for the Gegenbauer-type Q polynomials is an analogue of equation (32),

$$Q_n^0(x) = Q_n(x), \quad \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} Q_n(x) t^n. \quad (43)$$

The associated Gegenbauer-type polynomials can be defined in complete analogy with equation (33),

$$Q_n^\ell(x) = (-1)^\ell (1 - x^2)^{\ell/2} \frac{d^\ell Q_n(x)}{dx^\ell}. \quad (44)$$

They have a property analogous to equation (34),

$$\left(\frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \sin^2 \chi \frac{\partial}{\partial \chi} - \frac{\ell(\ell+1)}{\sin^2 \chi} \right) Q_n^\ell(\cos \chi) = -n(n+2) Q_n^\ell(\cos \chi). \quad (45)$$

Steps toward a unified treatment of the four-dimensional spherical harmonics were made in [5], but it appears that the normalization prefactor in equation (41) was not given in explicit form. The connection to the usual associated Gegenbauer polynomials $C_n^\ell(x)$ (in the canonical form, see [4]) is found as

$$Q_n(x) = C_n^1(x), \quad (46a)$$

$$Q_n^\ell(x) = (-1)^n 2^\ell \ell! (1 - x^2)^{\ell/2} C_{n-\ell}^{\ell+1}(-x). \quad (46b)$$

Finally, we should mention the addition theorem

$$\sum_{n\ell m} Y_{n\ell m}(\chi, \theta, \varphi) Y_{n'\ell' m'}^*(\chi', \theta', \varphi') = \frac{n+1}{2\pi^2} Q_n(\mathbf{x} \cdot \mathbf{x}'). \quad (47)$$

Connections of these formulas to the hydrogen wave functions are discussed in the [appendix](#).

5. Conclusions

The most important formulas of this brief paper can be found in equations (20), (26) and (37): we derive (and in the case of equation (26), just recall) the decomposition of the two-, three- and four-dimensional Green functions of the Poisson equation into radial and angular parts. For $D = 2$, only one ‘quantum number’ is introduced, namely, the ‘magnetic’ (azimuthal) quantum number m ; for $D = 3$, one has the orbital angular momentum ℓ and its magnetic projection m , while in $D = 4$, a third additional quantum number has to be introduced which can be associated with a ‘principal’ quantum number n ; it is associated with the additional angular coordinate χ in four dimensions (see equation (39)). The latter interpretation is ramified by the fact that indeed, the momentum-space wave functions of the nonrelativistic hydrogen atom (for nuclear charge $z = 1$) can be written as (see p 39 of [6])

$$\psi_{n\ell m}(\vec{p}) = (2\pi)^{3/2} \frac{4[\hbar/(a_0 n)]^{5/2}}{([\hbar/(a_0 n)]^2 + \vec{p}^2)^2} Y_{(n-1)\ell m}(\chi, \theta, \varphi), \quad (48)$$

where

$$\cos \chi = \frac{[\hbar/(a_0 n)]^2 - \vec{p}^2}{[\hbar/(a_0 n)]^2 + \vec{p}^2}, \quad (49)$$

and θ and φ are the polar and azimuth angles of the unit vector in the momentum direction, i.e., in the direction of the unit vector $\hat{p} = \vec{p}/|\vec{p}|$. The Bohr radius is $a_0 = \hbar/(\alpha m_e c)$, where α is the fine-structure constant, m_e is the electron mass, and c is the speed of light. These wave functions are normalized as $(2\pi)^{-3} \int d^3 p |\psi_{n\ell m}(\vec{p})|^2 = 1$.

In our treatment of the four-dimensional Green function, we find it useful (see equations (43) and (44)) to define polynomials $Q_n(x)$, and associated function $Q_n^\ell(x)$, which are related to, but not equal to, the Gegenbauer, and associated Gegenbauer, polynomials [4]. Hence, we refer to them as ‘Gegenbauer-type’ functions. Analogies to the three-dimensional case (Legendre and associated Legendre functions) are highlighted. The most intriguing problem in the calculation of the two-dimensional Green function lies in the matching of the $m = 0$ term from equation (18) with the $m \neq 0$ term from equation (13); the consideration of the asymptotic limit $\rho_\gamma \rightarrow \infty$ helps in finding the matching coefficients (see equation (23)).

The angular-momentum decomposition (20) for $D = 2$ reveals that the dominant logarithmic term in the interaction of vortices in the two-dimensional sine-Gordon model is exclusively due to S-wave interactions. The result might become useful as one tries to augment previous studies on high- T_c Josephson-coupled, and magnetically coupled superconductors [7, 8] by the inclusion of higher-order derivative terms [9].

Acknowledgments

This research has been supported by the National Science Foundation (Grants PHY-1403973 and PHY-1710856) and by the Missouri Research Board.

Appendix. Remarks of Schwinger’s derivation

In this appendix, we provide the clarification of three points which we found to be in need of some further explanation, in regard to Schwinger’s derivation [1, 10, 11] of the Schrödinger–Coulomb Green function, which is based on the $SO(4)$ symmetry. First, one may observe that a certain prefactor in the definition of the Schrödinger–Coulomb Green function may be in need of a reconsideration. Namely, if we assume that the defining equation of the Schrödinger–Coulomb Green function in coordinate space is

$$\left(\frac{\vec{p}^2}{2m} + V - E \right) G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}'), \quad (\text{A1})$$

where $V = -\frac{Z\alpha}{r}$ is the Coulomb potential, then the defining equation of the momentum-space Green function incurs a prefactor $(2\pi)^3$, in comparison to [1]. The following conventions for the Fourier transforms

$$f(\vec{p}) = \int d^3r e^{-i\vec{k}\cdot\vec{r}} f(\vec{r}), \quad f(\vec{r}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{r}} f(\vec{p}), \quad (\text{A2})$$

with an ‘asymmetric’ distribution of the factors 2π , are almost universally adopted in the physical literature. With $\hbar = c = \epsilon_0 = 1$, the Coulomb potential, in momentum space, is $V(\vec{p} - \vec{p}') = -\frac{4\pi Z\alpha}{(\vec{p} - \vec{p}')^2}$. The defining equation for the Green function thus becomes, in momentum space,

$$\left(\frac{\vec{p}^2}{2m} - E \right) G(\vec{p}, \vec{p}') - \int \frac{d^3p''}{(2\pi)^3} \frac{4\pi Z\alpha}{(\vec{p} - \vec{p}'')^2} G(\vec{p}'', \vec{p}') = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}'). \quad (\text{A3})$$

The factor $(2\pi)^3$ is not present in the first (unnumbered) equation of [1].

The fifth (unnumbered) equation of [1] contains two nontrivial identities. It is useful to derive the equation $d^3\Omega = \frac{d^3\xi}{|\xi_0|}$ for the area element on the three-dimensional unit sphere, embedded in four-dimensional space.

Here, one should remember that the three-dimensional components of the four-dimensional vector $(\xi_0, \vec{\xi})$ may have varying magnitude, but one considers them, according to [1], on the four-dimensional unit sphere $\xi_0^2 + \vec{\xi}^2 = 1$. One needs to remember that the appropriate generalization to the three-dimensional ‘surface’ of a manifold embedded into four-dimensional space is, with $x = x(t_1, t_2, t_3)$, $y = y(t_1, t_2, t_3)$, $z = z(t_1, t_2, t_3)$, $a = a(t_1, t_2, t_3)$ being the fourth coordinate, where a is the fourth coordinate,

$$d^3\Omega = \left| \det \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z & \hat{e}_a \\ \frac{\partial x}{\partial t_1} & \frac{\partial y}{\partial t_1} & \frac{\partial z}{\partial t_1} & \frac{\partial a}{\partial t_1} \\ \frac{\partial x}{\partial t_2} & \frac{\partial y}{\partial t_2} & \frac{\partial z}{\partial t_2} & \frac{\partial a}{\partial t_2} \\ \frac{\partial x}{\partial t_3} & \frac{\partial y}{\partial t_3} & \frac{\partial z}{\partial t_3} & \frac{\partial a}{\partial t_3} \end{pmatrix} \right| dt_1 dt_2 dt_3. \quad (\text{A4})$$

One calculates first the four-dimensional vector described by the determinant, and then calculates its vector modulus. The three-dimensional unit sphere, embedded in four-dimensional space, can be interpreted as a

three-dimensional manifold, parameterized by the coordinates $x = \xi_x = t_1, y = \xi_y = t_2$, and $z = \xi_z = t_3$, while $a = \xi_a = \sqrt{1 - \xi_x^2 - \xi_y^2 - \xi_z^2}$. One finds that

$$d^3\Omega = \frac{d\xi_x d\xi_y d\xi_z}{\sqrt{1 - \xi_x^2 - \xi_y^2 - \xi_z^2}} = \frac{d^3\xi}{|\xi_0|}. \quad (\text{A5})$$

This finally shows the first identity in the fifth equation of [1]. In order to show the second identity, one has to calculate a further Jacobian, transforming $d^3\xi$ into d^3p (in the conventions of [1], keeping the zeroth (or fourth) component $X = p_0$ of the four-dimensional (Euclidean) momentum constant.

There is a second nontrivial point which we found to be not very well explained in [1], and it concerns the fourth unnumbered equation (from the bottom) on the second page of [1]. The ‘version of the expansion’ referred to in [1] necessitates the use of the following trick, which is to enter the angular-momentum expansion formula given for $D(\xi - \xi') = -g(\xi - \xi')$ with the following values for $\xi = \mathbf{r}_1$, and $\xi' = \mathbf{r}_2$, as follows,

$$\begin{aligned} \mathbf{r}_1 &= \rho \xi, & |\xi| &= 1, & |\mathbf{r}_1| &= \rho, & \hat{\mathbf{r}}_1 &= \hat{\xi}, \\ \mathbf{r}_2 &= \xi', & |\xi'| &= 1, & |\mathbf{r}_2| &= 1, & \hat{\mathbf{r}}_2 &= \hat{\xi}', \end{aligned} \quad (\text{A6})$$

with $0 < \rho < 1$, so that $r_1 = \rho = r_<$, and $r_2 = 1 = r_>$. The identity

$$\begin{aligned} |\mathbf{r}_1 - \mathbf{r}_2|^2 &= |\mathbf{r}_1|^2 + |\mathbf{r}_2|^2 - 2\rho (\xi \cdot \xi') \\ &= (1 - \rho)^2 + \rho(\xi - \xi')^2 \end{aligned} \quad (\text{A7})$$

then follows, leading to

$$\frac{1}{4\pi^2} \frac{1}{(1 - \rho)^2 + \rho(\xi - \xi')^2} = \sum_{n\ell m} \frac{\rho^n}{2(n+1)} Y_{n\ell m}(\chi, \theta, \varphi) Y_{n\ell m}^*(\chi', \theta', \varphi'), \quad (\text{A8})$$

which is the desired identity used in [1]. We note that a representation of the $Y_{n\ell m}(\chi, \theta, \varphi)$ in terms of elementary functions is not given in [1].

The calculations of [1] culminate in the integral representation (see also equation (B2) of [12])

$$G(\vec{p}, \vec{p}') = 4\pi m X^3 \left(\frac{ie^{i\pi\nu}}{2\sin(\pi\nu)} \right) \int_1^{0^+} d\rho \rho^{-\nu} \frac{\partial}{\partial \rho} \frac{(1 - \rho^2)/\rho}{\left[X^2 (\vec{p} - \vec{p}')^2 + \frac{(1 - \rho)^2}{4\rho} (X^2 + \vec{p}^2) (X^2 + \vec{p}'^2) \right]^2}, \quad (\text{A9})$$

where $E = -X^2/(2m)$ is the energy argument of the Green function ($X = p_0$), and $\nu = Z\alpha m/\sqrt{-2mE}$. In comparison to [1], the result for G adds the prefactor $(2\pi)^3$; in the latter form, it has been useful in Lamb shift calculations [12, 13].

The hydrogen wave functions in momentum space can be expressed as (see equation (48))

$$\psi_{n\ell m}(\vec{p}) = \frac{16\pi a_0^3 n^2}{Z^3} \sqrt{\frac{(n-1-\ell)!}{(n+\ell)!}} \left(1 + \frac{n^2 a_0^2 \vec{p}^2}{Z^2} \right)^{-2} Q_{n-1}^\ell \left(\frac{1 - \frac{n^2 a_0^2 \vec{p}^2}{Z^2}}{1 + \frac{n^2 a_0^2 \vec{p}^2}{Z^2}} \right) Y_{\ell m}(\theta, \varphi), \quad (\text{A10})$$

where a_0 is the Bohr radius, and Z is the nuclear charge number. In comparison to p 39 of [6], we absorb the overall prefactor $(-1)^{n-1}$ into the global phase of the wave function.

ORCID iDs

UD Jentschura  <https://orcid.org/0000-0003-4967-5458>

References

- [1] Schwinger J 1964 Coulomb Green's function *J. Math. Phys.* **5** 1606–8
- [2] Jackson J D 1998 *Classical Electrodynamics* 3rd edn (New York, NY: Wiley)
- [3] Jentschura UD 2017 *Advanced Classical Electrodynamics: Green Functions, Regularizations, Multipole Decompositions* (Singapore: World Scientific)
- [4] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* 10th edn (Washington, DC: National Bureau of Standards)
- [5] Torres del Castillo G F and Calvario Acócal J L 2007 The hydrogen atom via the four-dimensional spherical harmonics *Rev. Mex. Fis.* **53** 407–14
- [6] Bethe H A and Salpeter E E 1957 *Quantum Mechanics of One- and Two-Electron Atoms* (Berlin: Springer)

- [7] Nándori I, Jentschura U D, Nagy S, Sailer K, Vad K and Mészáros S 2007 On the applicability of the layered sine-Gordon model for Josephson-coupled high- T_c -layered superconductors *J. Phys.: Condens. Matter* **19** 236226
- [8] Nándori I, Vad K, Mészáros S, Jentschura U D, Nagy S and Sailer K 2007 Applicability of layered sine-Gordon models to layered superconductors: II. The case of magnetic coupling *J. Phys.: Condens. Matter* **19** 496211
- [9] Jentschura U D, Nagy S and Nandori I 2017 in preparation
- [10] Lieber M 1968 $SO(4)$ symmetry of the hydrogen atom and the Lamb shift *Phys. Rev.* **174** 2037
- [11] Lieber M 1989 The Coulomb Green's function *AIP Conf. Proc.* **189** 445–59
- [12] Pachucki K 1993 Higher-order binding corrections to the Lamb shift *Ann. Phys., NY* **226** 1–87
- [13] Jentschura U and Pachucki K 1996 Higher-order binding corrections to the Lamb shift of $2P$ states *Phys. Rev. A* **54** 1853–61