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Multichannel conformal blocks for scattering amplitudes

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ABSTRACT

By performing resummation of small fermion–antifermion pairs within the pentagon form factor program to scattering amplitudes in planar $\mathcal{N}=4$ superYang–Mills theory, we construct multichannel conformal blocks within the flux-tube picture for N-sided NMHV polygons. This procedure is equivalent to summation of descendants of conformal primaries in the OPE framework. The resulting conformal partial waves are determined by multivariable hypergeometric series of Lauricella–Saran type.

1. Introduction

Symmetries of a system allow one to significantly reduce the number of degrees of freedom that require dynamical considerations. Conformal block decomposition of correlation functions $\langle \prod_j \mathcal{O}_j \rangle$ of local operators $\mathcal{O}_j \equiv \mathcal{O}_j(z_j)$ is a way of implementing them in a scale-invariant field theory (or CFT) via the operator product expansion (OPE). Under the assumption of convergence, a correlator can be expanded in a complete set of primary operators Φ_{Δ_ℓ} of increasing scaling dimension and spin (cumulatively called Δ_ℓ) and their conformal descendants built with the action of derivatives $\partial^n \Phi_\ell$. It is the latter infinite tower which is conveniently packed together in the conformal block, also known as the partial wave $\mathcal{F}_{\boldsymbol{\Delta}}(\boldsymbol{w})$, which is a function of $\boldsymbol{\Delta} = \{\Delta_\ell\}$ and cross ratios $\boldsymbol{w} = \{w_\ell\}$, schematically,

$$\langle \prod_j \mathcal{O}_j \rangle = \left(\prod_{j<k} z_{jk}^{\Delta_{jk}} \right) \sum_{\boldsymbol{\Delta}} a_{\boldsymbol{\Delta}} \mathcal{F}_{\boldsymbol{\Delta}}(\boldsymbol{w}) \,, \tag{1}$$

with an overall multiplicative function of the coordinate differences with powers Δ_{jk} being functions of the operator $\mathcal{O}_j$ dimensions/spins conveniently chosen to carry the scaling dimension of the left-hand side. The conformal blocks $\mathcal{F}_{\boldsymbol{\Delta}}$ are eigenfunctions of conformal Casimir operators for successive channels in the operator product expansion and are subject to appropriate boundary conditions. While the low-point correlators are well studied, there is little to no knowledge of multichannel conformal blocks.

Conformal blocks are ubiquitous in physics so they make their natural appearance in the analysis of scattering amplitudes within the pentagon operator product expansion [1,2]. In the latter, one relies on a dual description of amplitudes in terms of excitations propagating on a color flux-tube sourced by the contour of the Wilson loop living in the four-dimensional momentum space [3–8]. The vacuum represented by the flux is in fact SL(2) invariant to lowest order in 't Hooft coupling [9,10]. This property was used in the construction of conformal blocks for (N)MHV hexagons and heptagons [11–14].

The tree-level N-particle ratio function of the NMHV to MHV tree amplitudes

$$\mathbb{R}_N = \sum_{1<j<k<N-2} [1, j, j+1, k, k+2] \tag{2}$$

is determined by the R-invariants [15,16]

$$[i, j, k, l, m] = \frac{\delta^{0|4}(\chi_i (jklm) + \text{cyclic})}{(ijkl)(jklm)(klmi)(lmij)(mijk)} \,, \tag{3}$$

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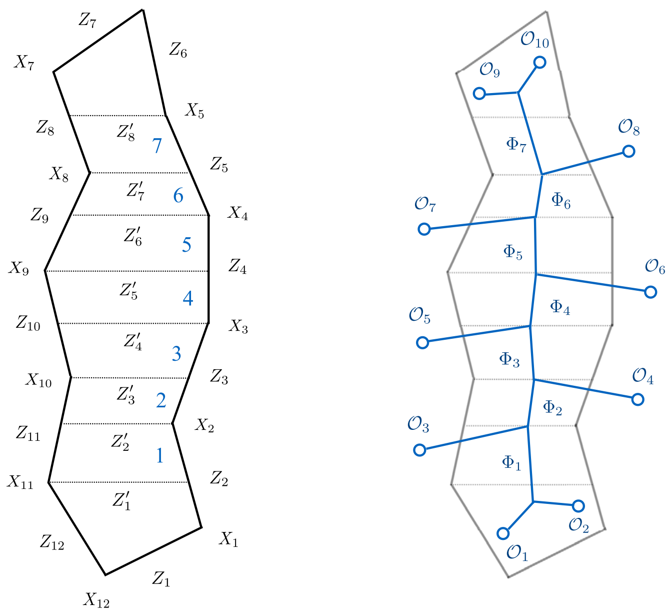

Fig. 1. A tessellation of a polygon (dodecagon on the left) and its OPE dual graph (on the right).

with the four-bracket defined by the determinant $(jklm) \equiv \varepsilon_{JKLM} Z_j^J Z_k^K Z_l^L Z_m^M$ built from the momentum twistors Z_j^J and χ_j^A being their fermionic partners. Within the pentagon form factor program, each individual Grassmann component $R^{[r_1,r_2,\dots,r_{N-5}]}$ of $\mathbb{R}_N$, with R-weights $r_1,\dots,r_{N-5}$ of all parent excitations, corresponding to the SU(4) dimensions $\mathbf{R}$ of flux-tube excitations, $r = 0,1,2,3,4$ for $\mathbf{R} = \bar{\mathbf{1}}, \bar{\mathbf{4}}, \mathbf{6}, \mathbf{4}, \mathbf{1}$, can be represented in terms of flux-tube integrals

$$
R^{[r_1,r_2,\dots,r_{N-5}]} = \sum_{\alpha_1,\dots,\alpha_{N-5}} \mathrm{e}^{-t_{\alpha_1}\tau_1 - \dots - t_{N-5}\tau_{\alpha_{N-5}} + ih_{\alpha_1}\varphi_1 + \dots + ih_{\alpha_{N-5}}\varphi_{N-5}} \times \int \prod_{j=1}^{N-5} \frac{du_j}{2\pi} \mathrm{e}^{2i\sigma_1 u_1 + \dots 2i\sigma_{N-5} u_{N-5}} I^{\mathbf{R}_1|\dots|\mathbf{R}_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) \,, \tag{4}
$$

where the $3(N-5)$ conformal invariants of Eq. (2) were traded for $N-5$ sets of triplets $(\tau_j, \sigma_j, \varphi_j)$ with their reciprocal variables interpreted as the energy (or twist), momentum and helicity, respectively, of the particles propagating on the flux and their SU(4) representation $\mathbf{R}_j$.

There is an infinite number of (parent) flux-tube excitations $\Phi_\alpha^{\mathbf{R}}$ [12,14] of different spin/R-change and increasing energy (i.e., conformal primary states, in the language of CFT) which determine the integrand $I^{\mathbf{R}_1|\dots|\mathbf{R}_{N-5}}$. Their descendants arise by gluing small fermion–antifermion pairs to $\Phi_\alpha^{\mathbf{R}}$'s. A small fermion–antifermion pair $\psi_s\bar{\psi}_s$ is equivalent to the derivative since ψ_s at zero momentum becomes the generator of Poincaré supersymmetry Q and since $\{Q, \bar{Q}\} \sim P$, according to their algebra, $(\psi_s\bar{\psi}_s)^n \Phi_\alpha^{\mathbf{R}} \sim \partial^n \Phi_\alpha^{\mathbf{R}}$ by analogy with conformal OPE alluded to above. In this note, we will construct multichannel conformal blocks for N-leg NMHV amplitudes by explicit resummation of the entire tower of small fermion–antifermions pairs accompanying parent particles, this will yield the substitution in the integrand

$$
\begin{aligned}
& I^{\mathbf{R}_1|\dots|\mathbf{R}_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) \\
& \to I^{\mathbf{R}_1|\dots|\mathbf{R}_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) \mathcal{F}^{[r_1,\dots,r_{N-5}]}_{h_{\alpha_1},t_{\alpha_1}|\dots|h_{\alpha_{N-5}},t_{\alpha_{N-5}}}(u_1,\tau_1|\dots|u_{N-5},\tau_{N-5})
\end{aligned} \tag{5}
$$

where $\mathcal{F}$ are the conformal blocks in question. This formalism is equivalent to the projection technique for computation of conventional conformal blocks in a CFT, which we briefly review by applying it to a four-point correlator in Appendix A to draw a parallel with the flux-tube physics.

2. Kinematics

Before turning to dynamics, let us introduce some kinematics first. The starting point is a tessellation of a polygon determined by the reference momentum twistors Z_j in terms of a sequence of squares formed by the polygon edges and internal light-like lines encoded in the momentum twistors Z'_k (see the left panel in Fig. 1 for the case of the dodecagon). A choice of a square automatically defines a conformal frame and thus a channel for propagation of parent flux excitations and their descendants. This is equivalent to a choice of an OPE channel for correlation functions (see the right panel in Fig. 1). To make the discussion more explicit, let us provide a choice of reference twistors for the dodecagon as a case of study (shown in Fig. 1)

$$
\begin{aligned}
&Z_1 = (6,4,12,5)\,, && Z_2 = (1,2,4,1)\,, && Z_3 = (0,1,1,0)\,, && Z_4 = (0,1,0,0)\,, \\
&Z_5 = (0,2,-1,1)\,, && Z_6 = (-1,6,-4,6)\,, && Z_7 = (-4,6,-5,12)\,, && Z_8 = (-2,1,-1,4)\,, \\
&Z_9 = (-1,0,0,1)\,, && Z_{10} = (1,0,0,0)\,, && Z_{11} = (2,0,1,1)\,, && Z_{12} = (6,1,6,4)\,,
\end{aligned} \tag{6}
$$

while the twistors connecting the cusps X_j with opposite sites of the polygon are

$$\begin{array}{llll} Z'_1=(4,1,5,3)\,, & Z'_2=(1,1,3,1)\,, & Z'_3=(1,0,1,1)\,, & Z'_4=(0,0,1,0)\,, \\ Z'_5=(0,0,0,1)\,, & Z'_6=(0,1,-1,1)\,, & Z'_7=(-1,1,-1,3)\,, & Z'_8=(-1,4,-3,5)\,. \end{array} \tag{7}$$

Every intermediate square enjoys a residual three-parameter conformal symmetry which leaves it invariant. These three parameters corresponds to the triplet $(\tau_j,\sigma_j,\varphi_j)$ introduced above. The invariance matrices for the squares can be determined successively starting with the middle one, i.e., fourth square in Fig. 1, which reads

$$M_4(\tau,\sigma,\phi)=\mathrm{diag}\left(\mathrm{e}^{\sigma-i\phi/2},\mathrm{e}^{-\sigma-i\phi/2},\mathrm{e}^{\tau+i\phi/2},\mathrm{e}^{-\tau+i\phi/2}\right)\,, \tag{8}$$

and its matrix elements fixed in a particular conformal frame as recalled in Appendix B. The symmetry transformations for the rest can be obtained by finding rotation matrices of the corresponding twistors defining adjacent squares[1] and then using them for construction of the M-matrices, namely,

$$\begin{aligned} M_7(\tau_7,\sigma_7,\phi_7)&=R_6^{-1}M_6(\tau_7,\sigma_7,\phi_7)R_6\,,\\ M_6(\tau_6,\sigma_6,\phi_6)&=R_5^{-1}M_5(\tau_6,\sigma_6,\phi_6)R_5\,,\\ M_5(\tau_5,\sigma_5,\phi_5)&=R_4^{-1}M_4(\tau_5,\sigma_5,\phi_5)R_4\,,\\ M_3(\tau_3,\sigma_3,\phi_3)&=R_3^{-1}M_4(\tau_3,\sigma_3,\phi_3)R_3\,,\\ M_2(\tau_2,\sigma_2,\phi_2)&=R_2^{-1}M_3(\tau_2,\sigma_2,\phi_2)R_2\,,\\ M_1(\tau_1,\sigma_1,\phi_1)&=R_1^{-1}M_2(\tau_1,\sigma_1,\phi_1)R_1\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &R_1=\begin{pmatrix}10&2&11&7\\-4&4&4&-2\\5&-2&0&3\\-23&-2&-21&-16\end{pmatrix},\quad R_2=\begin{pmatrix}0&1&1&0\\1&0&0&0\\1&0&1&1\\0&0&1&0\end{pmatrix},\quad R_3=\begin{pmatrix}0&1&1&0\\1&0&0&0\\1&0&1&1\\0&0&1&0\end{pmatrix},\\ &R_4=\begin{pmatrix}0&1&0&0\\1&0&0&-1\\0&0&0&1\\0&-1&1&-1\end{pmatrix},\quad R_5=\begin{pmatrix}0&1&-1&2\\0&-2&1&-1\\-1&-2&1&1\\-1&1&-1&3\end{pmatrix},\quad R_6=\begin{pmatrix}16&-16&14&-40\\2&0&2&-6\\14&-21&19&-44\\10&-13&11&-28\end{pmatrix}. \end{aligned} \tag{10}$$

In order to generate all inequivalent polygons, we act with these transformations on the twistors located either above or below it. For the case at hand, we have

$$\begin{aligned} &Z_1\to Z_1M_1M_2M_3\,,\quad Z_2\to Z_2M_2M_3\,,\quad Z_3\to Z_3\,,\quad Z_4\to Z_4\,, \\ &Z_5\to Z_5M_5^{-1}M_4^{-1}\,,\quad Z_6\to Z_6M_7^{-1}M_6^{-1}M_5^{-1}M_4^{-1}\,,\quad Z_7\to Z_7M_7^{-1}M_6^{-1}M_5^{-1}M_4^{-1}\,,\\ &Z_8\to Z_8M_6^{-1}M_5^{-1}M_4^{-1}\,,\quad Z_9\to Z_9M_4^{-1}\,,\quad Z_{10}\to Z_{10}\,,\quad Z_{11}\to Z_{11}M_3\,,\quad Z_{12}\to Z_{12}M_1M_2M_3\,. \end{aligned} \tag{11}$$

3. Dynamics: an example

Now we are in a position to turn to the flux-tube dynamics. Let us exemplify the inner workings of the formalism on the $\chi_1^2\chi_7^2$ component of the NMHV dodecagon, corresponding at lowest twist to the creation of the scalar ϕ at the bottom, its propagation through all intermediate squares and eventual absorption at the top. The integrand of Eq. (4) reads in this case

$$I^{\mathbf{6}|\mathbf{6}|\mathbf{6}|\mathbf{6}|\mathbf{6}|\mathbf{6}|\mathbf{6}}(-1,u_1|\dots|-1,u_7)=\mu_\phi(u_1)P_{\phi|\phi}(-u_1|u_2)\mu_\phi(u_2)P_{\phi|\phi}(-u_2|u_3)\dots P_{\phi|\phi}(-u_6|u_7)\mu_\phi(u_7)\,, \tag{12}$$

where $\alpha_j=-1$ in the nomenclature scheme of Ref. [14] according to which $\Phi^{\mathbf{6}}_{-1}=\phi$ with zero helicity and unit twist as reminded below. Here μ_ϕ and $P_{\phi|\phi}$ are the scalar measure and its pentagon transition at lowest order in 't Hooft coupling [17]. Fourier transform with respect to the rapidities provides the leading OPE contribution to the amplitude

$$\begin{aligned} R^{[2,2,2,2,2,2,2]}=\mathrm{e}^{-\tau_1-\dots-\tau_7}\Big[&\mathrm{e}^{\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7}+\mathrm{e}^{-\sigma_1+\sigma_2-\sigma_3+\sigma_4-\sigma_5+\sigma_6-\sigma_7}\\ &+\sum_{j=1}^{7}\mathrm{e}^{\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7-2s_j}+\sum_{j=1}^{7}\sum_{k=j+2}^{7}\mathrm{e}^{\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7-2s_j-2s_k}\\ &+\sum_{j=1}^{7}\sum_{k=j+2}^{7}\sum_{l=k+2}^{7}\mathrm{e}^{\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7-2s_j-2s_k-2s_l}\Big]^{-1}, \end{aligned} \tag{13}$$

[1] For instance, the twistors for the 5-th square $\mathbb{Z}_5=(Z'_6,Z_4,Z'_5,Z_9)$ can be determined from the 4-th one $\mathbb{Z}_4=(Z'_4,Z_4,Z'_5,Z_{10})$ via the transformation $\mathbb{Z}_5=R_5\mathbb{Z}_4$.

which agrees with the corresponding component of the ratio function (2) after the use of the twistors (10) as can immediately be verified with the package accompanying Ref. [18].

The next step is the inclusion of descendants, i.e., adding an arbitrary number of $\psi_s\bar{\psi}_s$ pairs to the parent scalar. This has to be done in every intermediate square. Let us begin with just one extra pair at the very bottom, i.e., the process $0 \to \phi\psi_s\bar{\psi}_s \to \phi \to \dots \to \phi \to 0$. Then, the integrand (12) has to be multiplied by the factor

$$
\mathrm{e}^{-2\tau_1} \int_{C_s} \frac{dv_1 dv_2}{(2\pi)^2} \frac{\mu_{\psi_s}(v_1)\mu_{\psi_s}(v_2) P_{\psi_s|\phi}(-v_1|u_2) P_{\psi_s|\phi}(-v_2|u_2)}{|P_{\phi|\psi_s}(u_1|v_1)|^2 |P_{\phi|\psi_s}(u_1|v_2)|^2 |P_{\psi_s|\bar{\psi}_s}(v_1|v_2)|^2} \left[\frac{x[v_1]}{x[v_2]}\right]^{1/2}
$$
$$
\times \frac{1}{6} [\Pi^{\mathbf{6}}_{0|\phi^{i_1 i_2}\psi^{j_1}\bar{\psi}_{j_2}}]_{k_1 k_2}(0|u_1, v_1, v_2) [\Pi^{\mathbf{1}}_{\psi^{j_2}\bar{\psi}_{j_1}\phi_{i_1 i_2}|\phi^{k_1 k_2}}](-v_2, -v_1, -u_1|u_2) ,
$$

where the integrations run over the small fermion contours C_s. The first factor in the above integrand is the factorized form of multiparticle pentagons along with the small fermion measures in conventions adopted from Ref. [19]. The second factor is the NMHV helicity form factor (on the small sheet) expressed via the Zhukowski variable $x[v] \simeq v + O(g^2)$. Last but not least, is the SU(4) tensor part. The latter is quite lengthy but their explicit form can be found in appendices to Refs. [14] and [20] in the order they appear. Substituting the lowest order expressions in 't Hooft coupling in the first line (where we already used the fact that the small fermion momentum is of order g^2) and evaluating the contour integrals via the Cauchy theorem with the poles arising from the matrix part, one finds a very simple result for the factor in question

$$
-\mathrm{e}^{-2\tau_1}(\tfrac{3}{2} + iu_1)(1 + iu_1 + iu_2) .
$$

This rapidity polynomial can be recast as a differential operator acting on the Fourier exponent in the integrand of Eq. (4) and making use of the preceding OPE result (13) successfully compared with subleading term in the near collinear expansion of (2). We have repeated similar analyses with a $\psi_s\bar{\psi}_s$ pair in other intermediate squares, i.e., $0 \to \phi \to \dots \to \phi\psi_s\bar{\psi}_s \to \dots \to \phi \to 0$ and every time found that the integrand acquires a factor

$$
\begin{aligned}
&- \mathrm{e}^{-2\tau_j}(1 + iu_{j-1} + iu_j)(1 + iu_j + iu_{j+1}) , \quad \text{for} \quad j = 2, 3, 4, 5, 6 , \\
&- \mathrm{e}^{-2\tau_7}(1 + iu_6 + iu_7)(\tfrac{3}{2} + iu_7) .
\end{aligned}
$$

The procedure was then extended further to up to three pairs either in the same or different squares. We found a recursive pattern which was summarized in the following proposal for seven-channel conformal block of the flux-tube scalar:

$$
\begin{aligned}
&\mathcal{F}^{[2,2,2,2,2,2,2]}_{1,1|\dots|1,1}(u_1, \tau_1|\dots|u_7, \tau_7) \\
&\quad = F_K\left(\begin{matrix} \frac{3}{2}+iu_1, 1+iu_1+iu_2, \dots, 1+iu_6+iu_7, \frac{3}{2}+iu_7 \\ 1,1,1,1,1,1,1 \end{matrix} \middle| -\mathrm{e}^{-2\tau_1}, \dots, -\mathrm{e}^{-2\tau_7}\right) .
\end{aligned}
\tag{14}
$$

It is given by the generalization of the Lauricella F_K series, discussed by Saran in Ref. [21] for the case of three variables, to L variables

$$
F_K\left(\begin{matrix} \alpha_1, \beta_1, \dots, \beta_{L-1}, \alpha_2 \\ \gamma_1, \dots, \gamma_L \end{matrix} \middle| z_1, \dots, z_L\right) = \sum_{n_1,\dots,n_L=0}^{\infty} \frac{(\alpha_1)_{n_1}(\beta_1)_{n_1+n_2}\dots(\beta_{L-1})_{n_{L-1}+n_L}(\alpha_2)_{n_L}}{(\gamma_1)_{n_1}\dots(\gamma_L)_{n_L}} \frac{z_1^{n_1}\dots z_L^{n_L}}{n_1!\dots n_L!} . \tag{15}
$$

This conjecture was tested numerically to very high orders in the near-collinear expansion against data produced with the help of Ref. [18] confirming its correctness.

4. *N*-sided NMHV polygons

Let us now present a generic form for the flux-tube integrands providing an exact representation for the tree level transitions

$$
0 \to \Phi^{\mathbf{R}}_{s_1,\alpha_1} \to \Phi^{\mathbf{R}}_{s_2,\alpha_2} \to \dots \to \Phi^{\mathbf{R}}_{s_L,\alpha_L} \to 0 , \tag{16}
$$

with the signature $s_j = \pm 1$. Depending on the channel, the parent excitations are

$$
\begin{aligned}
&\Phi^{\mathbf{6}}_{+,0} = \psi\psi_s , \quad \Phi^{\mathbf{6}}_{+,a>0} = g_a\psi_s\psi_s , \\
&\Phi^{\mathbf{6}}_{-,-1} = \phi , \quad \Phi^{\mathbf{6}}_{-,0} = \bar{\psi}\bar{\psi}_s , \quad \Phi^{\mathbf{6}}_{-,a>0} = \bar{g}_a\bar{\psi}_s\bar{\psi}_s ,
\end{aligned}
\tag{17}
$$

for the sextet of respective helicities

$$
h_{+,0} = 2 , \quad h_{+,a} = 2 + a , \quad h_{-,-1} = 0 , \quad h_{-,0} = -1 , \quad h_{-,-1} = -1 - a , \tag{18}
$$

they are

$$
\begin{aligned}
&\Phi^{\mathbf{4}}_{+,0} = \psi , \quad \Phi^{\mathbf{4}}_{+,a>0} = g_a\psi_s , \\
&\Phi^{\mathbf{4}}_{-,-1} = \phi\bar{\psi}_s , \quad \Phi^{\mathbf{4}}_{-,0} = \bar{\psi}\bar{\psi}_s\bar{\psi}_s , \quad \Phi^{\mathbf{4}}_{-,a>0} = \bar{g}_a\bar{\psi}_s\bar{\psi}_s\bar{\psi}_s ,
\end{aligned}
\tag{19}
$$

for the quartet of fermionic particles with helicities

$$h_{+,0} = \tfrac{1}{2}\,, \quad h_{+,a} = \tfrac{1}{2} + a\,, \quad h_{-,-1} = -\tfrac{1}{2}\,, \quad h_{-,0} = -\tfrac{3}{2}\,, \quad h_{-,-1} = -\tfrac{3}{2} - a\,, \tag{20}$$

and finally

$$\begin{aligned} &\Phi^{\mathbf{1}}_{+,a>0} = g_a\,, \\ &\Phi^{\mathbf{1}}_{-,-2} = \psi\bar{\psi}_s\,, \quad \Phi^{\mathbf{1}}_{-,-1} = \phi\bar{\psi}_s\bar{\psi}_s\,, \quad \Phi^{\mathbf{1}}_{-,0} = \bar{\psi}\bar{\psi}_s\bar{\psi}_s\bar{\psi}_s\,, \quad \Phi^{\mathbf{1}}_{-,a>0} = \bar{g}_a\bar{\psi}_s\bar{\psi}_s\bar{\psi}_s\bar{\psi}_s\,, \end{aligned} \tag{21}$$

for singlets with

$$h_{+,a} = a\,, \quad h_{-,-2} = 0\,, \quad h_{-,-1} = -1\,, \quad h_{-,0} = -2\,, \quad h_{-,-1} = -2 - a\,. \tag{22}$$

The integrands admits the same structure

$$\begin{aligned} I^{\mathbf{R}|\dots|\mathbf{R}}_{s_1|\dots|s_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) &= h^{\mathbf{R}|\dots|\mathbf{R}}_{s_1|\dots|s_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) \\ &\times \mu^{\mathbf{R}}_{s_1,\alpha_1}(u_1) P^{\mathbf{R}|\mathbf{R}}_{s_1,\alpha_1|s_2,\alpha_2}(-u_1|u_2)\mu^{\mathbf{R}}_{s_2,\alpha_2}(u_2)\dots \\ &\qquad \times \mu^{\mathbf{R}}_{s_{L-1},\alpha_{N-6}}(u_{N-6}) P^{\mathbf{R}|\mathbf{R}}_{s_{N-6},\alpha_{N-6}|s_{N-5},\alpha_{N-5}}(-u_{N-6}|u_{N-5})\mu^{\mathbf{R}}_{s_{N-5},\alpha_{N-5}}(u_{N-5})\,. \end{aligned} \tag{23}$$

The helicity NMHV form factors are

$$h^{\mathbf{6}|\dots|\mathbf{6}}_{s_1|\dots|s_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) = (-1)^{1+\alpha_1}\left(\frac{iu_1^{[-\alpha_1/2-1]}}{u_1^{[+\alpha_1/2]}}\right)^{(1-s_1)/2}\left(\frac{iu_{N-5}^{[-\alpha_L/2-1]}}{u_{N-5}^{[+\alpha_{N-5}/2]}}\right)^{(1+s_{N-5})/2}, \tag{24}$$

$$h^{\mathbf{4}|\dots|\mathbf{4}}_{s_1|\dots|s_L}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) = (-1)^{1+\alpha_1}\left(\frac{u_1^{[-\alpha_1/2-1]}u_1^{[-\alpha_1/2-2]}}{iu_1^{[+\alpha_1/2]}}\right)^{(1-s_1)/2}\left(\frac{1}{iu_{N-5}^{[+\alpha_{N-5}/2]}}\right)^{(1+s_{N-5})/2}, \tag{25}$$

$$\begin{aligned} &h^{\mathbf{1}|\dots|\mathbf{1}}_{s_1|\dots|s_{N-5}}(\alpha_1, u_1|\dots|\alpha_{N-5}, u_{N-5}) \\ &\quad = (-1)^{1+\alpha_1}\left(\frac{iu_1^{[-\alpha_1/2-1]}u_1^{[-\alpha_1/2-2]}u_1^{[-\alpha_1/2-3]}}{u_1^{[+\alpha_1/2]}}\right)^{(1-s_1)/2}\left(\frac{i}{u_{N-5}^{[+\alpha_{N-5}/2]}u_{N-5}^{[-\alpha_{N-5}/2]}}\right)^{(1+s_{N-5})/2}, \end{aligned} \tag{26}$$

where we used the notation $u^{[\alpha]} \equiv u + i\alpha$, while the measure reads

$$\mu^{\mathbf{R}}_{s,\alpha}(u) = \frac{\Gamma(1+\frac{\alpha}{2}+iu)\Gamma(1+\frac{\alpha}{2}-iu)}{\Gamma\big(2+(2-r)s+\alpha\big)}\,, \tag{27}$$

and the effective particle pentagon transitions are

$$P^{\mathbf{R}|\mathbf{R}}_{s,\alpha_1|s,\alpha_2}(u_1|u_2) = \frac{\Gamma(\frac{\alpha_1-\alpha_2}{2}+iu_1-iu_2)\Gamma\big(2+(2-r)s+\frac{\alpha_1+\alpha_2}{2}-iu_1+iu_2\big)}{\Gamma(1+\frac{\alpha_1}{2}+iu_1)\Gamma(1+\frac{\alpha_1}{2}+iu_2)\Gamma(1+\frac{\alpha_1-\alpha_2}{2}-iu_1+iu_2)}\,, \tag{28}$$

$$P^{\mathbf{R}|\mathbf{R}}_{s,\alpha_1|-s,\alpha_2}(u_1|u_2) = \frac{(-1)^{\alpha_2}\Gamma(1+\frac{\alpha_1+\alpha_2}{2}+iu_1-iu_2)}{\Gamma(1+\frac{\alpha_1}{2}+iu_1)\Gamma(1+\frac{\alpha_1}{2}+iu_2)}\,. \tag{29}$$

The resummation of the infinite number of small fermion–antifermion pairs in all intermediate transitions

$$0 \to \Phi^{\mathbf{R}}_{s_1,\alpha_1}(\psi_s\bar{\psi}_s)^\infty \to \Phi^{\mathbf{R}}_{s_2,\alpha_2}(\psi_s\bar{\psi}_s)^\infty \to \dots \to \Phi^{\mathbf{R}}_{s_L,\alpha_L}(\psi_s\bar{\psi}_s)^\infty \to 0\,, \tag{30}$$

provides the conformal blocks which we sought for

$$\begin{aligned} &\mathcal{F}^{[r_1,r_2,\dots,r_{N-5}]}_{h_1,t_1|h_2,t_2|\dots|h_{N-5},t_{N-5}}(u_1,\tau_1|u_2,\tau_2|\dots|u_{N-5},\tau_{N-5}) \\ &= F_K\left(\begin{matrix} \frac{|h_1|}{2}+\frac{2r_1+\widehat{r}_1}{4}+iu_1,\ \frac{|h_1|+|h_2|}{2}+\frac{\widehat{r}_1+\widehat{r}_2}{4}+iu_1+iu_2,\dots,\frac{|h_{N-5}|}{2}+\frac{2r_{N-5}+\widehat{r}_{N-5}-8}{4}+iu_{N-5} \\ t_1, t_2, \dots, t_{N-5} \end{matrix}\,\middle|\, -e^{-2\tau_1},\dots,-e^{-2\tau_{N-5}}\right), \end{aligned} \tag{31}$$

where

$$\widehat{r}_j = (4-r_j)\theta(h_j > 0) + r_j\theta(h_j \le 0)\,. \tag{32}$$

This is the main result of this note.

5. Conclusion

Building up on our previous work dedicated to the heptagon [14], we found the multichannel conformal blocks (31) for a polygon with any number of sides. The construction was based on resummation over descendants of parent flux-tube excitation propagating in a given NMHV component of the polygon (see Refs. [23–28] for detailed discussion). The blocks are determined by the generalizationof Lauricella hypergeometric series that was previously considered by Saran in the particular case of three variables. Multifold integral representation for the latter is available and its extension to the generic case should also be looked for since it would be of use for analytical resummation of infinite towers of flux-tube excitations of increasing helicities.

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Appendix A. 4-point correlator

For reader's convenience, let us recall a method for computation of conformal blocks in CFT based on explicit resummation of descendants, which is adopted in the main body of the paper to the case of scattering amplitudes. Here, it will suffice to discuss the holomorphic sector only (or, which is the same, a single light ray) and consider the global sl(2) subalgebra of the Virasoro algebra. Invariance under the sl(2) generators

$$\mathbb{L}^+ = z^2\partial + 2dz\,, \qquad \mathbb{L}^- = \partial\,, \qquad \mathbb{L}^0 = z\partial + d\,, \tag{33}$$

of the four-point correlator of operators $\mathcal{O}$ of the same conformal dimension d,

$$\sum_{j=1}^{4} \mathbb{L}_j^{\pm,0} \langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle = 0\,, \tag{34}$$

fixes its form

$$\langle \mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle = \frac{\mathcal{F}_4(w)}{z_{13}^{2d} z_{24}^{2d}}\,, \qquad w = \frac{z_{12}z_{34}}{z_{13}z_{24}}\,, \tag{35}$$

up to a function of the conformal cross ratio w. Let us choose a conformal frame, by setting

$$z_4 = 0\,, \qquad z_2 = 1\,, \qquad z_1 = \infty. \tag{36}$$

The operator–state correspondence (in the radial quantization) allows us to write

$$\lim_{z_1\to\infty} z_1^{2d} \langle \mathcal{O}(z_1)\mathcal{O}(1)\mathcal{O}(z_3)\mathcal{O}(0)\rangle = \langle d|\mathcal{O}(1)\mathcal{O}(z_3)|d\rangle\,. \tag{37}$$

To compute conformal blocks let us assume that the intermediate state is a primary state $|\Delta\rangle$, i.e., $L^+|\Delta\rangle = 0$, of dimension Δ and its descendants are

$$|\Delta, k\rangle \equiv (L^-)^k |\Delta\rangle\,, k > 0\,. \tag{38}$$

Obviously, $\langle\Delta, k| \equiv \langle\Delta|(L^+)^k$. Here $L^{\pm,0}$ are operators acting on the Hilbert space of states with the representation (33) on the primary fields Φ_Δ. We can project on these with

$$\Pi_\Delta = \sum_{k=0}^{\infty} \frac{|\Delta, k\rangle\langle\Delta, k|}{N_{\Delta,k}}\,, \tag{39}$$

obeying $\Pi_\Delta^2 = \Pi_\Delta$, with the normalization $N_{\Delta,k} = \langle\Delta, k|\Delta, k\rangle$. Such that

$$\mathcal{F}_4(z_3) = \langle d|\mathcal{O}(1)\Pi_\Delta\mathcal{O}(z_3)|d\rangle = \sum_{k=0}^{\infty} \frac{1}{N_{\Delta,k}} \langle d|\mathcal{O}(1)(L^-)^k|\Delta\rangle\langle\Delta|(L^+)^k\mathcal{O}(z_3)|d\rangle\,. \tag{40}$$

The calculation of the matrix elements involved is straightforward making use of the sl(2) algebra. The normalization prefactor reads

$$N_{\Delta,k} = \langle\Delta|[(L^+)^k, (L^-)^k]|\Delta\rangle = k!(2\Delta)_k\,, \tag{41}$$

which is a generalization of the elementary commutation relation

$$\langle\Delta|[(L^+)^2, (L^-)^2]|\Delta\rangle = 2(2\Delta+1)\langle\Delta|[L^+, L^-]|\Delta\rangle = 2!\,2\Delta(2\Delta+1)\langle\Delta|\Delta\rangle\,.$$

The matrix element in the numerator of the right-hand side of Eq. (40) reads

$$\langle\Delta|(L^+)^k\mathcal{O}(z_3)|d\rangle = \langle\Delta|[L^+, [L^+, \dots [L^+, \mathcal{O}(z_3)]\dots]]|d\rangle = (\mathbb{L}^+)^k\langle\Delta|\mathcal{O}(z_3)|d\rangle\,. \tag{42}$$

Since

$$\langle\Delta|\mathcal{O}(z_3)|d\rangle = 1/z_3^{2d-\Delta}\,, \tag{43}$$

is just the three-point function (fixed up to an overall normalization (set here to one) by conformal symmetry), we immediately find after repetitive differentiation

$$\langle\Delta|(L^+)^k\mathcal{O}(z_3)|d\rangle = (\Delta)_k/z_3^{2d-\Delta-k}\,. \tag{44}$$

Putting everything together, we find for $\mathcal{F}_4(z_3)$

$$\mathcal{F}_4(z_3) = z_3^{\Delta-2j}\sum_{k=0}^{\infty}\frac{(\Delta)_k^2}{k!(2\Delta)_k}z^k = z_3^{\Delta-2d}{}_2F_1\left(\left.\begin{matrix}\Delta,\Delta\\ 2\Delta\end{matrix}\right| z_3\right), \tag{45}$$

which is a well-known result [29].

The same result can be obtained making use of the eigenvalue equation for the quadratic Casimir of the sl(2) algebra,

$$\mathbb{C}_2 = \tfrac{1}{2}\left(\mathbb{L}^+\mathbb{L}^- + \mathbb{L}^-\mathbb{L}^+\right) - (\mathbb{L}^0)^2 \tag{46}$$

in a given OPE channel. For instance, in the (34)-channel, which is the same as the (12)-channel,

$$\mathbb{L}_{34}^{\pm,0} = \mathbb{L}_3^{\pm,0} + \mathbb{L}_4^{\pm,0}\,, \tag{47}$$

the equation

$$\mathbb{C}_{2,(34)}\langle\mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle = \Delta(1-\Delta)\langle\mathcal{O}(z_1)\mathcal{O}(z_2)\mathcal{O}(z_3)\mathcal{O}(z_4)\rangle\,, \tag{48}$$

immediately implies that $\mathcal{F}_4(w)$ obeys

$$w^2(w-1)\mathcal{F}_4''(w) + \left[4dw(w-1) - w^2\right]\mathcal{F}_4'(w) + \left[2d(1+2d(w-1)) + \Delta(\Delta-1)\right]\mathcal{F}_4(w) = 0\,. \tag{49}$$

It has two solutions

$$\mathcal{F}_4(w) = w^{\Delta-2d}{}_2F_1\left(\left.\begin{matrix}\Delta,\Delta\\ 2\Delta\end{matrix}\right| w\right) + \mathrm{c}\,w^{1-\Delta-2d}{}_2F_1\left(\left.\begin{matrix}1-\Delta,1-\Delta\\ 2-2\Delta\end{matrix}\right| w\right). \tag{50}$$

However, the second one does not possess correct asymptotic behavior and thus have to be dropped, i.e., $\mathrm{c} = 0$. This way, we recover our previous result for the conformal block.

Appendix B. Conformal frame for polygons

The choices made in the body for the elements of the symmetry matrices of middles squares in the tessellation of a generic polygon correspond to the following conformal cross ratios [22]

$$e^{\tau_{2j+1}} = \frac{(-j-1,j+1,j+2,j+3)(-j-2,-j-1,-j,j+2)}{(-j-2,-j-1,j+2,j+3)(-j-1,-j,j+1,j+2)}\,, \tag{51}$$

$$e^{\tau_{2j+1}+\sigma_{2j+1}-i\phi_{2j+1}} = \frac{(-j-2,-j-1,-j,-j+1)(-j-1,-j,j+2,j+3)}{(-j-2,-j-1,-j,j+3)(-j-1,-j,-j+1,j+2)}\,, \tag{52}$$

$$e^{\tau_{2j+1}+\sigma_{2j+1}+i\phi_{2j+1}} = \frac{(j+1,j+2,j+3,j+4)(-j-1,-j,j+2,j+3)}{(-j-1,j+2,j+3,j+4)(-j,j+1,j+2,j+3)}\,, \tag{53}$$

$$e^{\tau_{2j}} = \frac{(-j,j+1,j+2,j+3)(-j-1,-j,-j+1,j+2)}{(-j-1,-j,j+2,j+3)(-j,-j+1,j+1,j+2)}\,, \tag{54}$$

$$e^{\tau_{2j}+\sigma_{2j}-i\phi_{2j}} = \frac{(-j-1,-j,j+1,j+2)(j,j+1,j+2,j+3)}{(-j-1,j+1,j+2,j+3)(-j,j,j+1,j+2)}\,, \tag{55}$$

$$e^{\tau_{2j}+\sigma_{2j}+i\phi_{2j}} = \frac{(-j-2,-j-1,-j,-j+1)(-j-1,-j,j+1,j+2)}{(-j-2,-j-1,-j,j+2)(-j-1,-j,-j+1,j+1)}\,. \tag{56}$$

Here the odd and even ratios have different form due to opposite orientation of overlapping sequential pentagons.

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