Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications

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ABSTRACT. We prove nonlinear lower bounds and commutator estimates for the Dirichlet fractional Laplacian in bounded domains. The applications include bounds for linear drift-diffusion equations with nonlocal dissipation and global existence of weak solutions of critical surface quasi-geostrophic equations.

1. Introduction

Drift-diffusion equations with nonlocal dissipation naturally occur in hydrodynamics and in models of electroconvection. The study of these equations in bounded domains is hindered by a lack of explicit information on the kernels of the nonlocal operators appearing in them. In this paper we develop tools adapted for the Dirichlet boundary case: the Córdoba-Córdoba inequality ([**3**]) and a nonlinear lower bound in the spirit of ([**2**]), and commutator estimates. Lower bounds for the fractional Laplacian are instrumental in proofs of regularity of solutions to nonlinear nonlocal drift-diffusion equations. The presence of boundaries requires natural modifications of the bounds. The nonlinear bounds are proved using a representation based on the heat kernel and fine information regarding it ([**4**], [**7**], [**8**]). Nonlocal diffusion operators in bounded domains do not commute in general with differentiation. The commutator estimates are proved using the method of harmonic extension and results of ([**1**]). We apply these tools to linear drift-diffusion equations with nonlocal dissipation, where we obtain strong global bounds, and to global existence of weak solutions of the surface quasi-geostrophic equation (SQG) in bounded domains.

We consider a bounded open domain $\Omega \subset \mathbb{R}^d$ with smooth (at least $C^{2,\alpha}$) boundary. We denote by Δ the Laplacian operator with homogeneous Dirichlet boundary conditions. Its $L^2(\Omega)$ - normalized eigenfunctions are denoted w_j, and its eigenvalues counted with their multiplicities are denoted λ_j:

$$-\Delta w_j = \lambda_j w_j. \tag{1}$$

It is well known that $0 < \lambda_1 \leq ... \leq \lambda_j \to \infty$ and that $-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The ground state w_1 is positive and

$$c_0 d(x) \leq w_1(x) \leq C_0 d(x) \tag{2}$$

holds for all $x \in \Omega$, where

$$d(x) = dist(x, \partial\Omega) \tag{3}$$

and c_0, C_0 are positive constants depending on Ω. Functional calculus can be defined using the eigenfunction expansion. In particular

$$(-\Delta)^\alpha f = \sum_{j=1}^{\infty} \lambda_j^\alpha f_j w_j \tag{4}$$

with

$$f_j = \int_\Omega f(y) w_j(y) dy$$

for $f \in \mathcal{D}\left((-\Delta)^{\alpha}\right) = \{f \mid \ (\lambda_j^{\alpha} f_j) \in \ell^2(\mathbb{N})\}$. We will denote by

$$\Lambda_D^s = (-\Delta)^{\alpha}, \quad s = 2\alpha \tag{5}$$

the fractional powers of the Dirichlet Laplacian, with $0 \le \alpha \le 1$ and with $\|f\|_{s,D}$ the norm in $\mathcal{D}\left(\Lambda_D^s\right)$:

$$\|f\|_{s,D}^2 = \sum_{j=1}^{\infty} \lambda_j^s f_j^2. \tag{6}$$

It is well-known and easy to show that

$$\mathcal{D}\left(\Lambda_D\right) = H_0^1(\Omega).$$

Indeed, for $f \in \mathcal{D}\left(-\Delta\right)$ we have

$$\|\nabla f\|_{L^2(\Omega)}^2 = \int_{\Omega} f\left(-\Delta\right) f dx = \|\Lambda_D f\|_{L^2(\Omega)}^2 = \|f\|_{1,D}^2.$$

We recall that the Poincaré inequality implies that the Dirichlet integral on the left-hand side above is equivalent to the norm in $H_0^1(\Omega)$ and therefore the identity map from the dense subset $\mathcal{D}\left(-\Delta\right)$ of $H_0^1(\Omega)$ to $\mathcal{D}\left(\Lambda_D\right)$ is an isometry, and thus $H_0^1(\Omega) \subset \mathcal{D}\left(\Lambda_D\right)$. But $\mathcal{D}\left(-\Delta\right)$ is dense in $\mathcal{D}\left(\Lambda_D\right)$ as well, because finite linear combinations of eigenfunctions are dense in $\mathcal{D}\left(\Lambda_D\right)$. Thus the opposite inclusion is also true, by the same isometry argument.

Note that in view of the identity

$$\lambda^{\alpha} = c_{\alpha} \int_0^{\infty} (1 - e^{-t\lambda}) t^{-1-\alpha} dt, \tag{7}$$

with

$$1 = c_{\alpha} \int_0^{\infty} (1 - e^{-s}) s^{-1-\alpha} ds,$$

valid for $0 \le \alpha < 1$, we have the representation

$$\left((-\Delta)^{\alpha} f\right)(x) = c_{\alpha} \int_0^{\infty} \left[f(x) - e^{t\Delta} f(x)\right] t^{-1-\alpha} dt \tag{8}$$

for $f \in \mathcal{D}\left((-\Delta)^{\alpha}\right)$. We use precise upper and lower bounds for the kernel $H_D(t,x,y)$ of the heat operator,

$$(e^{t\Delta} f)(x) = \int_{\Omega} H_D(t,x,y) f(y) dy. \tag{9}$$

These are as follows ([**4**],[**7**],[**8**]). There exists a time $T > 0$ depending on the domain Ω and constants c, C, k, K, depending on T and Ω such that

$$\begin{array}{l} c \min\left(\frac{w_1(x)}{|x-y|}, 1\right) \min\left(\frac{w_1(y)}{|x-y|}, 1\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{kt}} \le \\ H_D(t,x,y) \le C \min\left(\frac{w_1(x)}{|x-y|}, 1\right) \min\left(\frac{w_1(y)}{|x-y|}, 1\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} \end{array} \tag{10}$$

holds for all $0 \le t \le T$. Moreover

$$\frac{|\nabla_x H_D(t,x,y)|}{H_D(t,x,y)} \le C \left\{ \begin{array}{ll} \frac{1}{d(x)}, & \text{if } \sqrt{t} \ge d(x), \\ \frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t} \le d(x) \end{array} \right. \tag{11}$$

holds for all $0 \le t \le T$. Note that, in view of

$$H_D(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} w_j(x) w_j(y), \tag{12}$$

elliptic regularity estimates and Sobolev embedding which imply uniform absolute convergence of the series (if $\partial\Omega$ is smooth enough), we have that

$$\partial_1^\beta H_D(t,y,x) = \partial_2^\beta H_D(t,x,y) = \sum_{j=1}^{\infty} e^{-t\lambda_j}\partial_y^\beta w_j(y)w_j(x) \tag{13}$$

for positive t, where we denoted by ∂_1^β and ∂_2^β derivatives with respect to the first spatial variables and the second spatial variables, respectively.

Therefore, the gradient bounds (11) result in

$$\frac{|\nabla_y H_D(t,x,y)|}{H_D(t,x,y)} \le C\begin{cases} \frac{1}{d(y)}, & \text{if } \sqrt{t}\ge d(y),\\ \frac{1}{\sqrt{t}}\left(1+\frac{|x-y|}{\sqrt{t}}\right), & \text{if } \sqrt{t}\le d(y). \end{cases} \tag{14}$$

2. The Córdoba - Córdoba inequality

PROPOSITION 1. *Let Φ be a C^2 convex function satisfying $\Phi(0)=0$. Let $f\in C_0^\infty(\Omega)$ and let $0\le s\le 2$. Then*

$$\Phi'(f)\Lambda_D^s f - \Lambda_D^s(\Phi(f)) \ge 0 \tag{15}$$

holds pointwise almost everywhere in Ω.

Proof. In view of the fact that both $f\in H_0^1(\Omega)\cap H^2(\Omega)$ and $\Phi(f)\in H_0^1(\Omega)\cap H^2(\Omega)$, the terms in the inequality (15) are well defined. We define

$$\left[(-\Delta)^\alpha f\right]_\epsilon(x) = c_\alpha\int_\epsilon^\infty \left[f(x)-e^{t\Delta}f(x)\right]t^{-1-\alpha}dt \tag{16}$$

and approximate the representation (8):

$$\left((-\Delta)^\alpha f\right)(x) = \lim_{\epsilon\to 0}\left[(-\Delta)^\alpha f\right]_\epsilon(x). \tag{17}$$

The limit is strong in $L^2(\Omega)$. We start the calculation with this approximation and then we rearrange terms:

$$\begin{array}{l}
\Phi'(f(x))\left[\Lambda_D^{2\alpha}f\right]_\epsilon(x) - \left[\Lambda_D^{2\alpha}(\Phi(f))\right]_\epsilon(x)\\
= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega\left\{\Phi'(f(x))\left[\frac{1}{|\Omega|}f(x)-H_D(t,x,y)f(y)\right]-\frac{1}{|\Omega|}\Phi(f(x))+H_D(t,x,y)\Phi(f(y))\right\}dy\\
= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)\left[\Phi(f(y))-\Phi(f(x))-\Phi'(f(x))(f(y)-f(x))\right]dy\\
+c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega\left[f(x)\Phi'(f(x))-\Phi(f(x))\right]\left(\frac{1}{|\Omega|}-H_D(t,x,y)\right)dy\\
= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)\left[\Phi(f(y))-\Phi(f(x))-\Phi'(f(x))(f(y)-f(x))\right]dy\\
+\left[f(x)\Phi'(f(x))-\Phi(f(x))\right]c_\alpha\int_\epsilon^\infty t^{-1-\alpha}(1-e^{t\Delta}1)dt
\end{array}$$

Because of the convexity of Φ we have

$$\Phi(b)-\Phi(a)-\Phi'(a)(b-a)\ge 0,\quad \forall\ a,b\in\mathbb{R},$$

and because $\Phi(0)=0$ we have

$$a\Phi'(a)\ge\Phi(a),\quad \forall\ a\in\mathbb{R}.$$

Consequently $f(x)\Phi'(f(x))-\Phi(f(x))\ge 0$ holds everywhere. The function

$$\theta = e^{t\Delta}1$$

solves the heat equation $\partial_t\theta-\Delta\theta=0$ in Ω, with homogeneous Dirichlet boundary conditions, and with initial data equal everywhere to 1. Although 1 is not in the domain of $-\Delta$, $e^{t\Delta}$ has a unique extension to $L^2(\Omega)$ where 1 does belong, and on the other hand, by the maximum principle $0\le\theta(x,t)\le 1$ holds for $t\ge 0$, $x\in\Omega$. It is only because $1\notin\mathcal{D}(-\Delta)$ that we had to use the ϵ approximation. Now we discard the nonnegative term

$$\left[f(x)\Phi'(f(x))-\Phi(f(x))\right]c_\alpha\int_\epsilon^\infty(1-\theta(x,t))t^{-1-\alpha}dt$$

in the calculation above, and deduce that

$$\Phi'(f(x))\left[\Lambda_D^{2\alpha}f\right]_\epsilon(x) - \left[\Lambda_D^{2\alpha}(\Phi(f))\right]_\epsilon(x) \geq 0 \tag{18}$$

as an element of $L^2(\Omega)$. (This simply means that its integral against any nonnegative $L^2(\Omega)$ function is nonnegative.) Passing to the limit $\epsilon \to 0$ we obtain the inequality (15). If Φ and the boundary of the domain are smooth enough then we can prove that the terms in the inequality are continuous, and therefore the inequality holds everywhere.

3. The Nonlinear Bound

We prove a bound in the spirit of ([**2**]). The nonlinear lower bound was used as an essential ingredient in proofs of global regularity for drift-diffusion equations with nonlocal dissipation.

THEOREM 1. *Let $f \in L^\infty(\Omega)\cap \mathcal{D}(\Lambda_D^{2\alpha})$, $0 \leq \alpha < 1$. Assume that $f = \partial q$ with $q \in L^\infty(\Omega)$ and ∂ a first order derivative. Then there exist constants c, C depending on Ω and α such that*

$$f\Lambda_D^{2\alpha}f - \frac{1}{2}\Lambda_D^{2\alpha}f^2 \geq c\|q\|_{L^\infty}^{-2\alpha}|f_d|^{2+2\alpha} \tag{19}$$

holds pointwise in Ω, with

$$|f_d(x)| = \begin{cases} |f(x)|, & \text{if } |f(x)| \geq C\|q\|_{L^\infty(\Omega)}\max\left(\frac{1}{\mathrm{diam}(\Omega)}, \frac{1}{d(x)}\right), \\ 0, & \text{if } |f(x)| \leq C\|q\|_{L^\infty(\Omega)}\max\left(\frac{1}{\mathrm{diam}(\Omega)}, \frac{1}{d(x)}\right). \end{cases} \tag{20}$$

Proof. We start the calculation using the inequality

$$f\Lambda_D^{2\alpha}f - \frac{1}{2}\Lambda_D^{2\alpha}f^2 \geq \frac{1}{2}c_\alpha\int_0^\infty \psi\left(\frac{t}{\tau}\right)t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)(f(x)-f(y))^2dy \tag{21}$$

where $\tau > 0$ is arbitrary and $0 \leq \psi(s) \leq 1$ is a smooth function, vanishing identically for $0 \leq s \leq 1$ and equal identically to 1 for $s \geq 2$. This follows repeating the calculation of the proof of the Córdoba-Córdoba inequality with $\Phi(f) = \frac{1}{2}f^2$:

$$\begin{aligned}
&f(x)\left[\Lambda_D^{2\alpha}f\right]_\epsilon(x) - \tfrac{1}{2}\left[\Lambda_D^{2\alpha}f^2\right]_\epsilon(x)\\
&= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}\int_\Omega\left\{\left[\tfrac{1}{|\Omega|}f(x)^2 - f(x)H_D(t,x,y)f(y)\right] - \tfrac{1}{2|\Omega|}f^2(x) + \tfrac{1}{2}H_D(t,x,y)f^2(y)\right\}dy\\
&= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega\left\{\tfrac{1}{2}\left[H_D(t,x,y)(f(x)-f(y))^2\right] + \tfrac{1}{2}f^2(x)\left[\tfrac{1}{|\Omega|} - H_D(t,x,y)\right]\right\}dy\\
&= c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega\left\{\tfrac{1}{2}\left[H_D(t,x,y)(f(x)-f(y))^2\right]dy + \tfrac{1}{2}f^2(x)\left[1 - e^{t\Delta}1\right](x)\right\}\\
&\geq c_\alpha\int_\epsilon^\infty t^{-1-\alpha}dt\int_\Omega \tfrac{1}{2}H_D(t,x,y)\left(f(x)-f(y)\right)^2dy
\end{aligned}$$

where in the last inequality we used the maximum principle again. Then, we choose $\tau > 0$ and let $\epsilon < \tau$. It follows that

$$f(x)\left[\Lambda_D^{2\alpha}f\right]_\epsilon(x) - \frac{1}{2}\left[\Lambda_D^{2\alpha}f^2\right]_\epsilon(x) \geq \frac{1}{2}c_\alpha\int_0^\infty\psi\left(\frac{t}{\tau}\right)t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)\left(f(x)-f(y)\right)^2dy.$$

We obtain (21) by letting $\epsilon \to 0$. We restrict to $t \leq T$,

$$\left[f\Lambda_D^{2\alpha}f - \frac{1}{2}\Lambda_D^{2\alpha}f^2\right](x) \geq \frac{1}{2}c_\alpha\int_0^T\psi\left(\frac{t}{\tau}\right)t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)\left(f(x)-f(y)\right)^2dy \tag{22}$$

and open brackets in (22):

$$\begin{aligned}
&\left[f\Lambda_D^{2\alpha}f - \tfrac{1}{2}\Lambda_D^{2\alpha}f^2\right](x)\\
&\geq \tfrac{1}{2}f^2(x)c_\alpha\int_0^T\psi\left(\tfrac{t}{\tau}\right)t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)dy - f(x)c_\alpha\int_0^T\psi\left(\tfrac{t}{\tau}\right)t^{-1-\alpha}dt\int_\Omega H_D(t,x,y)f(y)dy\\
&\geq |f(x)|\left[\tfrac{1}{2}|f(x)|I(x) - J(x)\right]
\end{aligned} \tag{23}$$

with

$$I(x) = c_\alpha \int_0^T \psi\left(\frac{t}{\tau}\right) t^{-1-\alpha} dt \int_\Omega H_D(t,x,y)dy, \tag{24}$$

and

$$\begin{aligned} J(x) &= c_\alpha \left| \int_0^T \psi\left(\tfrac{t}{\tau}\right) t^{-1-\alpha} dt \int_\Omega H_D(t,x,y) f(y) dy \right| \\ &= c_\alpha \left| \int_0^T \psi\left(\tfrac{t}{\tau}\right) t^{-1-\alpha} dt \int_\Omega \partial_y H_D(t,x,y) q(y) dy \right|. \end{aligned} \tag{25}$$

We proceed with a lower bound on I and an upper bound on J. For the lower bound on I we note that

$$\theta(x,t) = \int_\Omega H_D(t,x,y)dy \geq \int_{|x-y|\leq \frac{d(x)}{2}} H_D(t,x,y)dy$$

because H_D is positive. Using the lower bound in (2) we have that $|x-y| \leq \frac{d(x)}{2}$ implies

$$\frac{w_1(x)}{|x-y|} \geq 2c_0, \quad \frac{w_1(y)}{|x-y|} \geq c_0,$$

and then, using the lower bound in (10) we obtain

$$H_D(t,x,y) \geq 2cc_0^2 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{kt}}.$$

Integrating it follows that

$$\theta(x,t) \geq 2cc_0^2 \omega_{d-1} k^{\frac{d}{2}} \int_0^{\frac{d(x)}{2\sqrt{kt}}} \rho^{d-1} e^{-\rho^2} d\rho$$

If $\frac{d(x)}{2\sqrt{kt}} \geq 1$ then the integral is bounded below by $\int_0^1 \rho^{d-1} e^{-\rho^2} d\rho$. If $\frac{d(x)}{2\sqrt{kt}} \leq 1$ then $\rho \leq 1$ implies that the exponential is bounded below by e^{-1} and so

$$\theta(x,t) \geq c_1 \min\left\{1, \left(\frac{d(x)}{\sqrt{t}}\right)^d\right\} \tag{26}$$

for all $0 \leq t \leq T$ where c_1 is a positive constant, depending on Ω. Because

$$I(x) = \int_0^T \psi\left(\frac{t}{\tau}\right) t^{-1-\alpha} \theta(x,t) dt$$

we have

$$\begin{aligned} I(x) &\geq c_1 \int_0^{\min(T,d^2(x))} \psi\left(\tfrac{t}{\tau}\right) t^{-1-\alpha} dt \\ &= c_1 \tau^{-\alpha} \int_1^{\tau^{-1}(\min(T,d^2(x)))} \psi(s) s^{-1-\alpha} ds \end{aligned}$$

Therefore we have that

$$I(x) \geq c_2 \tau^{-\alpha} \tag{27}$$

with $c_2 = c_1 \int_1^2 \psi(s) s^{-1-\alpha} ds$, a positive constant depending only on Ω and α, provided τ is small enough,

$$\tau \leq \frac{1}{2} \min(T, d^2(x)). \tag{28}$$

In order to bound J from above we use the upper bound (14) which yields

$$\int_\Omega |\nabla_y H_D(t,x,y)| dy \leq C_1 t^{-\frac{1}{2}} \tag{29}$$

with C_1 depending only on Ω. Indeed,

$$\begin{aligned} &\int_{d(y)\geq\sqrt{t}} |\nabla_y H_D(t,x,y)| dy \\ &\leq C_2 t^{-\frac{1}{2}} \int_{\mathbb{R}^d} \left(1 + \tfrac{|x-y|}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{kt}} dy \\ &= C_3 t^{-\frac{1}{2}} \end{aligned}$$

and, in view of the upper bound in (2), $\frac{1}{d(y)}w_1(y) \le C_0$ and the upper bound in (10),

$$\begin{array}{l}\int_{d(y)\le\sqrt{t}} |\nabla_y H_D(t,x,y)|dy \\ \le C_4 \int_{\mathbb{R}^d} \frac{1}{|x-y|} t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}} dy = C_5 t^{-\frac{1}{2}}\end{array}$$

Now

$$J \le \|q\|_{L^\infty(\Omega)} \int_0^T \psi\left(\frac{t}{\tau}\right) t^{-1-\alpha} dt \int_\Omega |\nabla_y H_D(t,x,y)|dy$$

and therefore, in view of (29)

$$J \le C_1 \|q\|_{L^\infty(\Omega)} \int_0^T \psi\left(\frac{t}{\tau}\right) t^{-\frac{3}{2}-\alpha} dt$$

and therefore

$$J \le C_6 \|q\|_{L^\infty(\Omega)} \tau^{-\frac{1}{2}-\alpha} \tag{30}$$

with

$$C_6 = C_1 \int_1^\infty \psi(s) s^{-\frac{3}{2}-\alpha} ds$$

a constant depending only on Ω and α. Now, because of the lower bound (23), if we can choose τ so that

$$J(x) \le \frac{1}{4}|f(x)|I(x)$$

then it follows that

$$\left[f\Lambda_D^{2\alpha} f - \frac{1}{2}\Lambda_D^{2\alpha} f^2\right](x) \ge \frac{1}{4} f^2(x) I(x). \tag{31}$$

Because of the bounds (27), (30) the choice

$$\tau(x) = c_3 \frac{\|q\|_{L^\infty}^2}{|f(x)|^2} \tag{32}$$

with $c_3 = 16 C_6^2 c_2^{-2}$ achieves the desired bound. The requirement (28) limits the possibility of making this choice to the situation

$$c_3 \frac{\|q\|_{L^\infty}^2}{|f(x)|^2} \le \frac{1}{2}\min(T, d^2(x)) \tag{33}$$

which leads to the statement of the theorem. Indeed, if (32) is allowed then the lower bound in (31) becomes

$$\left[f\Lambda_D^{2\alpha} f - \frac{1}{2}\Lambda_D^{2\alpha} f^2\right](x) \ge c\|q\|_{L^\infty}^{-2\alpha} |f_d|^{2+2\alpha} \tag{34}$$

with $c = \frac{1}{4} c_2 c_3^{-\alpha}$.

4. Commutator estimates

We start by considering the commutator $[\nabla, \Lambda_D]$ in $\Omega = \mathbb{R}_+^d$. The heat kernel with Dirichlet boundary conditions is

$$H(x,y,t) = ct^{-\frac{d}{2}}\left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x-\widetilde{y}|^2}{4t}}\right)$$

where $\widetilde{y} = (y_1, \dots, y_{d-1}, -y_d)$. We claim that

$$\int_\Omega (\nabla_x + \nabla_y) H(x,y,t) dy \le Ct^{-\frac{1}{2}} e^{-\frac{x_d^2}{4t}}. \tag{35}$$

Indeed, the only nonzero component occurs when the differentiation is with respect to the normal direction, and then

$$(\partial_{x_d} + \partial_{y_d}) H(x,y,t) = ct^{-\frac{d}{2}} e^{-\frac{|x'-y'|^2}{4t}} \left(\frac{x_d + y_d}{t}\right) e^{-\frac{(x_d+y_d)^2}{4t}}$$

where we denoted $x' = (x_1, \dots, x_{d-1})$ and $y' = (y_1, \dots, y_{d-1})$. Therefore

$$\begin{array}{l}\int_\Omega (\nabla_x + \nabla_y) H(x,y,t) dy \le C t^{-\frac{1}{2}} \int_0^\infty \left(\frac{x_d+y_d}{t}\right) e^{-\frac{(x_d+y_d)^2}{4t}} dy_d \\ = C t^{-\frac{1}{2}} \int_{\frac{x_d}{\sqrt{t}}}^\infty \xi e^{-\frac{\xi^2}{4}} d\xi \\ = C t^{-\frac{1}{2}} e^{-\frac{x_d^2}{4t}}.\end{array}$$

Consequently

$$K(x,y) = \int_0^\infty t^{-\frac{3}{2}} (\nabla_x + \nabla_y) H(x,y,t) dt$$

obeys

$$\int_\Omega K(x,y) dy \le C \int_0^\infty t^{-2} e^{-\frac{x_d^2}{4t}} dt = \frac{C}{x_d^2}.$$

The commutator $[\nabla, \Lambda_D]$ is computed as follows

$$\begin{array}{l}[\nabla, \Lambda_D] f(x) = \int_0^\infty t^{-\frac{3}{2}} \int_\Omega \left[\nabla_x H_D(x,y,t) f(y) - H_D(x,y,t) \nabla_y f(y)\right] dy dt \\ = \int_0^\infty t^{-\frac{3}{2}} \int_\Omega (\nabla_x + \nabla_y) H_D(x,y,t) f(y) dy dt \\ = \int_\Omega K(x,y) f(y) dy.\end{array}$$

We have proved thus that the kernel $K(x,y)$ of the commutator obeys

$$\int_\Omega K(x,y) dy \le C d(x)^{-2} \tag{36}$$

and therefore we obtain, for instance, for any $p, q \in [1, \infty]$ with $p^{-1} + q^{-1} = 1$

$$\left| \int_\Omega g [\nabla, \Lambda_D] f dx \right| \le C \left(\int_\Omega d(x)^{-2} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega d(x)^{-2} |g(x)|^q dx \right)^{\frac{1}{q}}.$$

In general domains, the absence of explicit expressions for the heat kernel with Dirichlet boundary conditions requires a less direct approach to commutator estimates.

We take thus an open bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary and describe the square root of the Dirichlet Laplacian using the harmonic extension. We denote

$$Q = \Omega \times \mathbb{R}_+ = \{(x,z) \mid \ x \in \Omega, z > 0\}$$

and consider the traces of functions in $H^1_{0,L}(Q)$,

$$H^1_{0,L}(Q) = \{ v \in H^1(Q) \ \mid \ v(x,z) = 0, \ x \in \partial\Omega, \ z > 0 \}$$

$$V_0(\Omega) = \{ f \ \mid \ \exists v \in H^1_{0,L}(Q), \ f(x) = v(x,0), x \in \Omega \} \tag{37}$$

where we slightly abused notation by referring to the images of v under restriction operators as $v(x,z)$ for $x \in \partial\Omega$, and as $v(x,0)$ for $x \in \Omega$. We recall from ([**1**]) that, on one hand,

$$V_0(\Omega) = \{ f \in H^{\frac{1}{2}}(\Omega) \ \mid \ \int_\Omega \frac{f^2(x)}{d(x)} dx < \infty \} \tag{38}$$

with norm

$$\|f\|^2_{V_0} = \|f\|^2_{H^{\frac{1}{2}}(\Omega)} + \int_\Omega \frac{f^2(x)}{d(x)} dx,$$

and on the other hand $V_0(\Omega) = \mathcal{D}(\Lambda_D^{\frac{1}{2}})$, i.e.

$$V_0(\Omega) = \{ f \in L^2(\Omega) \ \mid \ f = \sum_j f_j w_j, \ \sum_j \lambda_j^{\frac{1}{2}} f_j^2 < \infty \} \tag{39}$$

with equivalent norm

$$\|f\|^2_{\frac{1}{2},D} = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} f_j^2 = \|\Lambda_D^{\frac{1}{2}} f\|^2_{L^2(\Omega)}.$$

The harmonic extension of f will be denoted v_f. It is given by

$$v_f(x,z) = \sum_{j=1}^{\infty} f_j e^{-z\sqrt{\lambda_j}} w_j(x) \tag{40}$$

and the operator Λ_D is then identified with

$$\Lambda_D f = -\left(\partial_z v_f\right)_{|\ z=0} \tag{41}$$

Note that if $f \in V_0(\Omega)$ then $v_f \in H^1(Q)$. Note also, that v_f decays exponentially in the sense that

$$\|v_f\|_{e^{zl}H^1(Q)} = \|e^{zl}\nabla v_f\|_{L^2(Q)} + \|e^{zl} v_f\|_{L^2(Q)} \le C\|f\|_{V_0} \tag{42}$$

holds with $\ell = \frac{\lambda_1}{4}$. We use a lemma in Q:

LEMMA 1. *Let $F \in H^{-1}(Q)$ (the dual of $H_0^1(Q)$). Then the problem*

$$\begin{cases} -\Delta u = F, & \text{in } Q, \\ u = 0, & \text{on } \partial Q \end{cases} \tag{43}$$

has a unique weak solution $u \in H_0^1(Q)$. If $F \in L^2(Q)$ and if there exists $l > 0$ so that

$$\|e^{zl}F\|^2_{L^2(Q)} = \int e^{2zl}|F(x,z)|^2 dxdz < \infty$$

then $u \in H_0^1(Q) \cap H^2(Q)$ and it satisfies

$$\|u\|_{H^2(Q)} \le C\|e^{zl}F\|_{L^2(Q)}$$

with C a constant depending only on Ω and l.

Proof. We consider the domain $U = \Omega \times \mathbb{R}$ and take the odd extension of F to U, $F(x,-z) = -F(x,z)$. The existence of a weak solution in $H_0^1(U)$ follows by variational methods, by minimizing

$$I(v) = \int_U \left(\frac{1}{2}|\nabla v|^2 + vF\right) dxdz$$

among all odd functions $v \in H_0^1(U)$. The domain U has finite width, so the Poincaré inequality

$$\|\nabla v\|^2_{L^2(U)} \ge c\|v\|^2_{L^2(U)}$$

is valid for functions in $H_0^1(U)$. This allows to show existence and uniqueness of weak solutions. If $F \in L^2(U)$ we obtain locally uniform elliptic estimates

$$\|u\|_{H^2(U_j)} \le C\|F\|_{L^2(V_j)}$$

where $U_j = \{(x,z)\,|\ x \in \Omega, z \in (j-1, j+1)\}$, $V_j = \{(x,z)\,|\ x \in \Omega, z \in (j-2, j+2)\}$, and $j = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$, i.e. $j \in \frac{1}{2}\mathbb{Z}$. The constant C does not depend on j. Because of the decay assumption on F, the estimates can be summed.

THEOREM 2. *Let $a \in B(\Omega)$ where $B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega)$, if $d \ge 3$, and $B(\Omega) = W^{2,p}(\Omega)$ with $p > 2$, if $d = 2$. There exists a constant C, depending only on Ω, such that*

$$\|[a, \Lambda_D]f\|_{\frac{1}{2},D} \le C\|a\|_{B(\Omega)}\|f\|_{\frac{1}{2},D} \tag{44}$$

holds for any $f \in V_0(\Omega)$, with

$$\|a\|_{B(\Omega)} = \|a\|_{W^{2,d}(\Omega)} + \|a\|_{W^{1,\infty}(\Omega)}$$

if $d \geq 3$ *and*

$$\|a\|_{B(\Omega)} = \|a\|_{W^{2,p}(\Omega)}$$

with $p > 2$*, if* $d = 2$*.*

Proof. In order to compute v_{af}, let us note that $av_f \in H^1_{0,L}(Q)$, and

$$\Delta(av_f) = v_f \Delta_x a + 2\nabla_x a \cdot \nabla v_f$$

and, because $v_f \in e^{zl} H^1(Q)$ and $a \in B(\Omega)$ we have that

$$\|\Delta(av_f)\|_{L^2(e^{zl}dzdx)} \leq C\|a\|_{B(\Omega)}\|v_f\|_{e^{zl}H^1(Q)}.$$

Solving

$$\begin{cases} \Delta u = \Delta(av_f) & \text{in } Q, \\ u = 0 & \text{on } \partial Q, \end{cases}$$

we obtain $u \in H^1_0(Q) \cap H^2(Q)$. This follows from Lemma 1 above. Note that $\partial_z u \in H^1_{0,L}(Q)$. Then

$$v_{af} = av_f - u$$

and

$$a\Lambda_D f - \Lambda_D(af) = -a(\partial_z v_f)_{|\, z=0} + \partial_z(av_f - u)_{|\, z=0} = -\partial_z u_{|\, z=0}.$$

The estimate follows from elliptic estimates and restriction estimates

$$\|\partial_z u_{|\, z=0}\|_{V_0} \leq C\|\partial_z u\|_{H^1(Q)} \leq C\|a\|_{B(\Omega)}\|v_f\|_{e^{zl}H^1(Q)} \leq C\|a\|_{B(\Omega)}\|f\|_{V_0}$$

THEOREM 3. *Let a vector field* a *have components in* $B(\Omega)$ *defined above,* $a \in (B(\Omega))^d$*. Assume that the normal component of the trace of* a *on the boundary vanishes,*

$$a_{|\, \partial\Omega} \cdot n = 0$$

(i.e the vector field is tangent to the boundary). There exists a constant C *such that*

$$\|[a \cdot \nabla, \Lambda_D]f\|_{\frac{1}{2},D} \leq C\|a\|_{B(\Omega)}\|f\|_{\frac{3}{2},D} \tag{45}$$

holds for any f *such that* $f \in \mathcal{D}\left(\Lambda_D^{\frac{3}{2}}\right)$*.*

Proof. In order to compute $v_{a\cdot\nabla f}$ we note that

$$\Delta(a \cdot \nabla v_f) = \Delta a \cdot \nabla v_f + \nabla a \cdot \nabla\nabla v_f,$$

and because $v_f \in e^{zl} H^2(Q)$ and $a \in B(\Omega)$ we have that

$$\|\Delta(a \cdot \nabla v_f)\|_{L^2(e^{zl}dzdx)} \leq C\|a\|_{B(\Omega)}\|v_f\|_{e^{zl}H^2(Q)}.$$

Then solving

$$\begin{cases} \Delta u = \Delta(a \cdot \nabla v_f) & \text{in } Q, \\ u = 0 & \text{on } \partial Q, \end{cases}$$

we obtain $u \in H^2(Q)$ (by Lemma 1) and therefore $\partial_z u \in H^1_{0,L}(Q)$. Consequently $-\partial_z u_{|\, z=0} \in V_0(\Omega)$. Because v_f vanishes on the boundary and $a \cdot \nabla$ is tangent to the boundary, it follows that $a \cdot \nabla v_f \in H^1_{0,L}(Q)$ (vanishes on the lateral boundary of Q and is in $H^1(Q)$) and therefore

$$v_{a\cdot\nabla f} = a \cdot \nabla v_f - u.$$

Consequently

$$[a \cdot \nabla, \Lambda_D]f = -\partial_z u_{|\, z=0}.$$

The estimate (45) follows from the elliptic estimates and restriction estimates on u, as above:

$$\|\partial_z u_{|\, z=0}\|_{V_0} \leq C\|\partial_z u\|_{H^1(Q)} \leq C\|a\|_{B(\Omega)}\|v_f\|_{e^{zl}H^2(Q)} \leq C\|a\|_{B(\Omega)}\|f\|_{\frac{3}{2},D}$$

5. Linear transport and nonlocal diffusion

We study the equation

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda_D\theta = 0 \tag{46}$$

with initial data

$$\theta(x,0) = \theta_0 \tag{47}$$

in the bounded open domain $\Omega\subset\mathbb{R}^d$ with smooth boundary. We assume that $u = u(x,t)$ is divergence-free

$$\nabla\cdot u = 0, \tag{48}$$

that u is smooth

$$u\in L^2(0,T;B(\Omega)^d), \tag{49}$$

and that u is parallel to the boundary

$$u_{|\,\partial\Omega}\cdot n = 0. \tag{50}$$

We consider zero boundary conditions for θ. Strictly speaking, because this is a first order equation, it is better to think of these as a constraint on the evolution equation. We satrt with initial data θ_0 which vanish on the boundary, and maintain this property in time. The transport evolution

$$\partial_t\theta + u\cdot\nabla\theta = 0$$

and, separately, the nonlocal diffusion

$$\partial_t\theta + \Lambda_D\theta = 0$$

keep the constraint of $\theta_{|\,\partial\Omega} = 0$. Because the operators $u\cdot\nabla$ and Λ_D have the same differential order, neither dominates the other, and the linear evolution needs to be treated carefully. We start by considering Galerkin approximations. Let

$$P_m f = \sum_{j=1}^m f_j w_j, \quad \text{for } f = \sum_{j=1}^\infty f_j w_j, \tag{51}$$

and let

$$\theta_m(x,t) = \sum_{j=1}^m \theta_j^{(m)}(t) w_j(x) \tag{52}$$

obey

$$\partial_t\theta_m + P_m\left(u\cdot\nabla\theta_m\right) + \Lambda_D\theta_m = 0 \tag{53}$$

with initial data

$$\theta_m(x,0) = (P_m\theta_0)(x). \tag{54}$$

These are ODEs for the coefficients $\theta_j^{(m)}(t)$, written conveniently. We prove bounds that are independent of m and pass to the limit. Note that by construction

$$\theta_m\in\mathcal{D}\left(\Lambda_D^r\right), \quad \forall r\ge 0.$$

We start with

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|^2_{L^2(\Omega)} + \|\theta_m\|^2_{V_0} = 0 \tag{55}$$

which implies

$$\sup_{0\le t\le T}\frac{1}{2}\|\theta_m(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^T \|\theta_m\|^2_{V_0}dt \le \frac{1}{2}\|\theta_0\|^2_{L^2(\Omega)}. \tag{56}$$

This follows because of the divergence-free condition and the fact that $u_{|\,\partial\Omega}$ is parallel to the boundary. Next, we apply Λ_D to (53). For convenience, we denote

$$[\Lambda_D, u\cdot\nabla]f = \Gamma f \tag{57}$$

because u is fixed throughout this section. Because P_m and Λ_D commute, we have thus

$$\partial_t\Lambda_D\theta_m + P_m\left(u\cdot\nabla\Lambda_D\theta_m + \Gamma\theta_m\right) + \Lambda_D^2\theta_m = 0. \tag{58}$$

Now, we multiply (58) by $\Lambda_D^3\theta_m$ and integrate. Note that

$$\int_\Omega P_m\left(u\cdot\nabla\Lambda_D\theta_m+\Gamma\theta_m\right)\Lambda_D^3\theta_m dx=\int_\Omega\left(u\cdot\nabla\Lambda_D\theta_m+\Gamma\theta_m\right)\Lambda_D^3\theta_m dx$$

because $P_m\theta_m=\theta_m$ and P_m is selfadjoint. We bound the term

$$\left|\int_\Omega\Gamma\theta_m\Lambda_D^3\theta_m dx\right|\leq\|\Gamma\theta_m\|_{V_0}\|\Lambda_D^{2.5}\theta_m\|_{L^2(\Omega)}$$

and use Theorem 3 (45) to deduce

$$\left|\int_\Omega\Gamma\theta_m\Lambda_D^3\theta_m dx\right|\leq C\|u\|_{B(\Omega)}\|\Lambda_D\theta_m\|_{V_0}\|\Lambda_D^{2.5}\theta_m\|_{L^2(\Omega)}.$$

We compute

$$\begin{aligned}
&\int_\Omega(u\cdot\nabla\Lambda_D\theta_m)\Lambda_D^3\theta_m dx=\int_\Omega\Lambda_D^2(u\cdot\nabla\Lambda_D\theta_m)\Lambda_D\theta_m\\
&=\int_\Omega\left[(-\Delta u)\cdot\nabla\Lambda_D\theta_m-2\nabla u\cdot\nabla\nabla\Lambda_D\theta_m\right]\Lambda_D\theta_m dx+\int_\Omega(u\cdot\nabla\Lambda_D^3\theta_m)\Lambda_D\theta_m dx\\
&=\int_\Omega\left[(-\Delta u)\cdot\nabla\Lambda_D\theta_m-2\nabla u\cdot\nabla\nabla\Lambda_D\theta_m\right]\Lambda_D\theta_m dx-\int_\Omega\Lambda_D^3\theta_m(u\cdot\nabla\Lambda_D\theta_m)dx\\
&=\int_\Omega\left[((-\Delta u)\cdot\nabla\Lambda_D\theta_m)\Lambda_D\theta_m+2\nabla u\nabla\Lambda_D\theta_m\nabla\Lambda_D\theta_m\right]dx-\int_\Omega(u\cdot\nabla\Lambda_D\theta_m)\Lambda_D^3\theta_m dx.
\end{aligned}$$

In the first integration by parts we used the fact that $\Lambda_D^3\theta_m$ is a finite linear combination of eigenfunctions which vanish at the boundary. Then we use the fact that $\Lambda_D^2=-\Delta$ is local. In the last equality we integrated by parts using the fact that $\Lambda_D\theta_m$ is a finite linear combination of eigenfunctions which vanish at the boundary and the fact that u is divergence-free. It follows that

$$\int_\Omega(u\cdot\nabla\Lambda_D\theta_m)\Lambda_D^3\theta_m dx=\frac{1}{2}\int_\Omega\left[((-\Delta u)\cdot\nabla\Lambda_D\theta_m)\Lambda_D\theta_m+2\nabla u\nabla\Lambda_D\theta_m\nabla\Lambda_D\theta_m\right]dx$$

and consequently

$$\left|\int_\Omega(u\cdot\nabla\Lambda_D\theta_m)\Lambda_D^3\theta_m dx\right|\leq C\|u\|_{B(\Omega)}\|\Lambda_D^2\theta_m\|_{L^2(\Omega)}^2$$

We obtain thus

$$\sup_{0\leq t\leq T}\|\Lambda_D^2\theta_m(\cdot,t)\|_{L^2(\Omega)}^2+\int_0^T\|\Lambda_D^2\theta_m\|_{V_0}^2dt\leq C\|\Lambda_D^2\theta_0\|_{L^2(\Omega)}^2e^{C\int_0^T\|u\|_{B(\Omega)}^2dt}. \tag{59}$$

Passing to the limit $m\to\infty$ is done using the Aubin-Lions Lemma ([**6**]). We obtain

THEOREM 4. *Let $u\in L^2(0,T;B(\Omega)^d)$ be a vector field parallel to the boundary. Then the equation (46) with initial data $\theta_0\in H_0^1(\Omega)\cap H^2(\Omega)$ has unique solutions belonging to*

$$\theta\in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap L^2(0,T;H^{2.5}(\Omega)).$$

If the initial data $\theta_0\in L^p(\Omega)$, $1\leq p\leq\infty$, then

$$\sup_{0\leq t\leq T}\|\theta(\cdot,t)\|_{L^p(\Omega)}\leq\|\theta_0\|_{L^p(\Omega)} \tag{60}$$

holds.

The estimate (60) holds because, by use of Proposition 1 for the diffusive part and integration by parts for the transport part, we have for solutions of (46)

$$\frac{d}{dt}\|\theta\|_{L^p(\Omega)}^p\leq 0,$$

$1\leq p<\infty$. The L^∞ bound follows by taking the limit $p\to\infty$ in (60).

6. SQG

We consider now the equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda_D \theta = 0 \tag{61}$$

with

$$u = R_D^{\perp} \theta \tag{62}$$

and

$$R_D = \nabla \Lambda_D^{-1} \tag{63}$$

in a bounded open domain in $\Omega \subset \mathbb{R}^2$ with smooth boundary. Local existence of smooth solutions is possible to prove using methods similar to those developed above for linear drift-diffusion equations. We will consider weak solutions (solutions which satisfy the equations in the sense of distributions).

THEOREM 5. *Let $\theta_0 \in L^2(\Omega)$ and let $T > 0$. There exists a weak solution of (61)*

$$\theta \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V_0(\Omega))$$

satisfying $lim_{t\to 0}\theta(t) = \theta_0$ weakly in $L^2(\Omega)$.

Proof. We consider Galerkin approximations, θ_m

$$\theta_m(x,t) = \sum_{j=1}^{m} \theta_j(t) w_j(x)$$

obeying the ODEs (written conveniently as PDEs):

$$\partial_t \theta_m + P_m \left[R_D^{\perp}(\theta_m) \cdot \nabla \theta_m \right] + \Lambda_D \theta_m = 0$$

with initial datum

$$\theta_m(0) = P_m(\theta_0).$$

We observe that, multiplying by θ_m and integrating we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_m\|^2 + \|\theta_m\|^2_{\frac{1}{2},D} = 0$$

which implies that the sequence θ_m is bounded in

$$\theta_m \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; V_0(\Omega))$$

It is known ([**1**]) that $V_0(\Omega) \subset L^4(\Omega)$ with continuous inclusion. It is also known ([**5**]) that

$$R_D : L^4(\Omega) \to L^4(\Omega)$$

are bounded linear operators. It is then easy to see that $\partial_t \theta_m$ are bounded in $L^2(0,T; H^{-1}(\Omega))$. Applying the Aubin-Lions lemma, we obtain a subsequence, renamed θ_m converging strongly in $L^2(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;V_0(\Omega))$ and in $L^2(0,T;L^4(\Omega))$. The limit solves the equation (61) weakly. Indeed, this follows after integration by parts because the product $(R_D^{\perp}\theta_m)\theta_m$ is weakly convergent in $L^2(0,T;L^2(\Omega))$ by weak-times-strong weak continuity. The weak continuity in time at $t = 0$ follows by integrating

$$(\theta_m(t), \phi) - (\theta_m(0), \phi) = \int_0^t \frac{d}{ds}\theta_m(s) ds$$

and use of the equation and uniform bounds. We omit further details.

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References

[1] X. Cabre, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math. 224 (2010), no. 5, 2052-2093.

[2] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, GAFA **22** (2012) 1289-1321.

[3] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations. Comm. Math. Phys. **249** (2004), 511–528.

[4] E.B. Davies, Explicit constants for Gaussian upper bounds on heat kernels, Am. J. Math **109** (1987) 319-333.

[5] D. Jerison, C. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Analysis **130** (1995), 161-212.

[6] J.L. Lions, Quelque methodes de résolution des problèmes aux limites non linéaires, Paris, Dunod (1969).

[7] Q. S. Zhang, The boundary behavior of heat kernels of Dirichlet Laplacians, J. Diff. Eqn **182** (2002), 416-430

[8] Q. S. Zhang, Some gradient estimates for the heat equation on domains and for an equation by Perelman, IMRN (2006), article ID92314, 1-39.

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