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Solving the Lagrangian dual problem for some traffic coordination problems through linear programming

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Abstract— **In a series of recent publications, we have addressed a class of traffic coordination problems that can be formulated as combinatorial scheduling problems, and we have developed a Lagrangian duality approach for obtaining (i) strong lower bounds for the corresponding optimal solution, and (ii) potential "seeds" for the construction of feasible, near-optimal solutions to these problems. In those past works, the corresponding "dual" problem was solved through a customized dual-ascent algorithm that took advantage of a distributed representation of the sub-differentials of the underlying "dual" function in order to compute efficiently ascending directions during the optimization process. In this paper we show that the fundamental insights that led to the aforementioned past developments, can also enable an efficient linear programming formulation for the considered "dual" problem.[1]**

I. Introduction

The presented work concerns a set of real-time traffic management problems for the traffic that is generated by a set of "agents" circulating over a connected digraph which is known as the "(supporting) guidepath network". The edges of the guidepath network define "zones" for the supported traffic, and in order to ensure the safety of the agent motion, it is stipulated that each zone can be occupied by at most one agent at a time. Access to the system zones is granted by a central controller, and the trip of any given agent from an "origin" location to a "destination" location constitutes a "resource allocation process" where the necessary zones that support the various legs of this trip, are requested and acquired one zone at a time. Additional constraints regulate the agent transitioning between zones, and the behavior of the agents upon reaching their destinations. The main objective of the considered problems is to enable all traveling agents to reach their destinations in a way that respects the imposed regulations, and maintains a certain notion of expediency for the generated traffic.[2]

From an application standpoint, the traffic problems outlined in the previous paragraph arise naturally in the real-time operations of various automated unit-load material handling (MH) systems, like the AGV, overhead monorail and the complex crane and gantry systems used in many production and distribution facilities [1], but also in the physical medium that implements the various elementary operations in the context of quantum computing [2]; the reader is referred to [3], [4], [5], [6], [7] for an elaboration on these connections. In addition, similar guidepath-based traffic models have drawn recently the attention of the robotics community (e.g., [8], [9], [10], [11]), while, in the past, they have been studied even by the broader CS community in the context of some classical games like the, so called, "15-puzzle" where 15 uniquely numbered "pebbles" located on a 4×4 grid have to be re-arranged in the row-major order by "pebble sliding" through the single unoccupied vertex of the grid [12], [13].

In an ongoing research program of ours, we seek to develop an effective methodological framework for the real-time management of the aforementioned traffic systems, especially as they materialize in the context of the real-time operations of unit-load, zone-controlled MH systems, and in quantum computing. This endeavor capitalizes upon perspectives and results borrowed from (i) combinatorial scheduling theory [14], (ii) the emergent theory on the logical control of complex resource allocation systems (RAS) [15], [16], and (iii) model predictive control (MPC) [17]. More specifically, in the aforementioned research program, we have adopted an MPC framework that decomposes the overall traffic management problem by seeking to route the traveling agents to their next *immediate* destination in a way that (i) minimizes the corresponding makespan and (ii) ensures the liveness of the overall generated traffic (i.e., the ability of all these agents to visit all their target destinations, in the desired order, and eventually retire to a "home" station that is provided by the underlying guidepath network). The resulting "core" scheduling (sub-)problem of this MPC-based decomposition, that concerns the expedient routing of the traveling agents to their immediate destinations, has been formally modeled as a mixed integer program (MIP) [18]. But while this MIP formulation provides a solid analytical characterization of the underlying scheduling problem, it is computationally intractable for most practical instantiations of this problem. Hence, our research program has also developed some heuristic algorithms that construct near-optimal traffic schedules through "local search"-based methods [19].[3] Furthermore, in a complementary line of research, we have

[1]This work was partially supported by NSF grants ECCS-1405156 and ECCS-1707695, and also by an IRAD grant from the Georgia Tech Research Institute.

[2]This high-level description of the considered traffic systems becomes much more detailed in the subsequent, more technical parts of this work.

[3]In fact, for the considered class of scheduling problems, even the construction of just a feasible solution is, in general, an NP-hard problem [20]. However, the considered research has identified a number of conditions that will render quite tractable the construction of a first initial solution. Subsequently, a pertinent specification and representation of the "neighborhood" structure to be used in the pursued local search, together with some novel dynamic programming (DP) formulations for effecting this search, enable the expedient identification of some very efficient schedules, even for some very hard problem instances.

pursued a Lagrangian relaxation approach [21], [22] for the aforementioned MIP formulation, that can provide (a) strong lower bounds to its optimal value, and (ii) potential "seeds" for the construction of near-optimal solutions to the original scheduling problem. In this last line of research, the corresponding (Lagrangian) dual problem, which is a challenging problem in itself [22], is solved through a customized dual-ascent algorithm that takes advantage of a distributed representation of the sub-differentials involved in the relevant optimization process. More specifically, by means of this representation, our algorithm is able to identify an improving direction at each iteration, and attains a monotonic, finite convergence to an optimal set of Lagrange multipliers. Detailed, technical treatments of the aforementioned results can be found in [3], [4], [5], [6], [7]. These publications also contain numerical results that demonstrate and assess more empirically the efficacy of the aforementioned developments.

In the context of all these past developments, the main objective of the current work is to establish the theoretically interesting and computationally useful additional result that the aforementioned Lagrangian dual problem can be reduced to a Linear Program (LP) [23]. Furthermore, the fundamental insights that have led to the customized dual-ascent algorithm discussed in the previous paragraph, also enable an efficient representation of the new LP in terms of the numbers of variables and constraints involved. Hence, the presented results render the Lagrangian dual problems addressed in this work solvable by the powerful LP solvers that are currently available. This possibility is effectively demonstrated through some numerical experimentation that is reported in the last part of the paper.

The rest of this paper is organized as follows: Section II overviews the MIP formulation for the basic traffic scheduling problem considered in this work, as well as the employed Lagrangian relaxation and the corresponding dual problem. Subsequently, Section III establishes the reduction of this dual problem to the aforementioned LP, and Section IV reports the numerical experimentation that demonstrates the tractability of this LP formulation. Finally, Section V concludes the paper and discusses some directions for future work.

II. THE CONSIDERED TRAFFIC SCHEDULING PROBLEM AND THE CORRESPONDING DUAL PROBLEM

In this section we overview the MIP formulation that constitutes the core (sub-)problem for the pursued MPC framework, and the Lagrangian relaxation approach that we have pursued in [3], [4] for the derivation of strong lower bounds to this MIP formulation. Given the imposed page limits for this document, our presentation is limited to the minimal amount of information that is necessary for the development of the main results of this work, that are presented in Section III; the interested reader is referred to [3], [4], [5], [6] and [7] for a more extensive discussion of the considered formulations and the underlying modeling assumptions.

A. The considered traffic scheduling problem and its MIP formulation

To begin with the formal representation of the considered traffic scheduling problem, let us denote the set of the traveling agents by $\mathcal{A}$, and the underlying guidepath network by the digraph $G = (V, E)$. As mentioned in the introductory section, the edges $e \in E$ of G define "zones" that are allocated sequentially and exclusively to the traveling agents by the system controller. From a physical standpoint, each zone can be modeled by an undirected edge $\{v_i, v_j\}$. But in our model, $\{v_i, v_j\}$ is represented by the pair of directed edges $e_1 = (v_i, v_j)$ and $e_2 = (v_j, v_i)$ in E, so that we can also encode a sense of direction of the traveling agent on this edge. Edges e_1 and e_2 are said to be "complementary", and this fact is denoted by writing $e_2 = \bar{e}_1$ (and, equivalently, $e_1 = \bar{e}_2$). Also, for any given edge e, $e^{\bullet}$ denotes the set of "output" edges for e, i.e., all these edges $e' \in E$ that can be accessed by an agent located in e upon leaving edge e. On the other hand, the set ${}^{\bullet}e_l$ contains the "input" edges of e, i.e., the edges $e' \in E$ that constitute entry points for e. Furthermore, in the basic positioning of the considered traffic scheduling problem, all edges $e \in E$ have a uniform length, and therefore, they can be traversed by any agent $a \in \mathcal{A}$ at a constant time. Taking this traversal time as the corresponding "time unit", we also obtain a discretization of the dynamics of the underlying traffic.

In the context of the notation introduced in the previous paragraph, the considered scheduling problem seeks to transport each traveling agent $a \in \mathcal{A}$ from an initial edge s_a to a destination edge d_a while minimizing the total time that is necessary for these transports. This time is considered in the discretized setting that was defined in the previous paragraph, it is characterized as the "makespan" of the corresponding schedule, and it will be denoted by w. Furthermore, as remarked in the introductory section, a feasible schedule must ensure an exclusive allocation of the various zones to the traveling agents, and this allocation must also observe an additional set of rules that seeks to ensure the safety of the agent motion by enforcing some further levels of separation among them. Finally, letting T denote an available upper bound for w,[4] and setting $\mathcal{T} = \{0, 1, \ldots, T\}$, we can express the considered traffic scheduling problem through the following MIP:

$$\min w \tag{1}$$

s.t.

$$\forall a \in \mathcal{A},\ \forall t \in \mathcal{T},\ \sum_{e \in E} x_{a,e,t} = 1 \tag{2}$$

$$\forall a \in \mathcal{A},\ \forall e \in E\ \ x_{a,e,0} = I_{\{e=s_a\}} \tag{3}$$

$$\forall a \in \mathcal{A}, \quad x_{a,d_a,T} = 1 \tag{4}$$

[4] A pertinent T value for any given problem instance is readily provided by any feasible traffic schedule for this problem instance. Some results for obtaining such feasible traffic schedules are provided in [5], [6], [7], [13].

$$\forall a \in \mathcal{A},\ \forall t \in \mathcal{T} \setminus \{T\}, \quad x_{a,d_a,t+1} \geq x_{a,d_a,t} \tag{5}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T} \setminus \{T\}, \\ x_{a,e,t} \leq x_{a,e,t+1} + \sum_{e' \in e^\bullet} x_{a,e',t+1} \tag{6}$$

$$\forall e = (v_i, v_j) \in E \text{ s.t. } i < j,\ \forall t \in \mathcal{T} \setminus \{0, T\}, \\ \sum_{a \in \mathcal{A}} (x_{a,e,t} + x_{a,\bar{e},t}) \leq 1 \tag{7}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T} \setminus \{0\}, \\ x_{a,e,t} + \sum_{a' \in \mathcal{A} : a' \neq a} (x_{a',e,t-1} + x_{a',\bar{e},t-1}) \leq 1 \tag{8}$$

$$\forall a \in \mathcal{A}, \quad w \ \geq \ \sum_{t=0}^{T} (1 - x_{a,d_a,t}) \ = \ T + 1 - \sum_{t=0}^{T} x_{a,d_a,t} \tag{9}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T}, \quad x_{a,e,t} \in \{0,1\} \tag{10}$$

The primary decision variables in the above MIP formulation are the binary variables $x_{a,e,t},\ a \in \mathcal{A},\ e \in E,\ t \in \mathcal{T}$, with $x_{a,e,t} = 1$ denoting that agent a occupies edge e at period t. Then, Constraint (2) of the above formulation imposes the requirement that all agents must occupy one and only one position at any time period, and Constraint (3) places the agents $a \in \mathcal{A}$ at their initial zones s_a at time $t = 0$. Constraint (4) requires that every agent must reach its destination edge, d_a, within the provided time horizon, while Constraint (5) further stipulates that agents cannot leave their destination edges after reaching them. Constraint (6) enforces the fact that agents can only transition to adjacent, directionally compatible edges, and Constraint (7) prevents any two agents to cohabit on an edge. Constraint (8) enforces the additional requirement that an agent can enter an edge at period t only if this edge was empty during the previous period, $t - 1$.[5] Constraint (9) together with Equation (1) define the objective of the considered formulation as the minimization of the makespan of the generated traffic schedule. Finally, Constraint (10) specifies the binary nature of all decision variables $x_{a,e,t}$, and it also implies a free nature for the remaining decision variable w.

The MIP formulation of Eqs (1)–(10) can be infeasible for some of its instantiations, and the assessment of the corresponding (in-)feasibility can be a hard problem in itself.[6]

[5] Hence, Constraint (8) prevents an agent from transitioning into an edge of the guidepath network while another agent is transitioning out from this edge at the same time period, and, thus, it ensures the necessary separation among the traveling agents in the face of the uncertainty that would be introduced by such simultaneous transitions.

[6] Generally speaking, the feasibility of the considered MIP will depend on (i) the topology of the underlying guidepath network and the relative positioning of the edges s_a and d_a, $a \in \mathcal{A}$, in this topology, (ii) the ability of the agents to reverse their motion within their currently allocated edges, and (iii) the selection of the parameter T; please, also c.f. Footnote 4 regarding the third issue.

But even in the cases where the MIP formulation of Eqs (1)–(10) is guaranteed to be feasible, it might be intractable for most practical instantiations of the considered problem. These computational challenges are especially aggravated by the "on-line" / "real-time" nature of the considered problem, and the strict time budgets that this feature implies for the involved computations. Hence, as remarked in the introductory section, our research program has pursued the development of efficient heuristic algorithms for the construction of near-optimal traffic schedules, and also a Lagrangian relaxation approach for the derivation of (i) lower bounds to the optimal makespan w^*, and potentially (ii) some additional information that can be used in the synthesis of the heuristic algorithms that were mentioned above. Next, we present this Lagrangian relaxation and the corresponding "dual" problem.

B. *The Lagrangian relaxation of the considered MIP and the corresponding dual problem*

The Lagrangian relaxation for the MIP formulation of Eqs (1)–(10) that has been considered in our past work, is obtained by relaxing the constraints of Eqs (7)–(9), which act as "coupling" constraints for the variable sets $\{x_{a,e,t} : e \in E,\ t \in \mathcal{T}\},\ a \in \mathcal{A}$. Denoting by $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ the (row) vectors that collect the corresponding sets of Lagrange multipliers, we obtain the following Lagrange relaxation for the original MIP:

$$\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \equiv \min_{\boldsymbol{x}, w} \Bigg\{ w + \\ + \sum_{\{e \in E : v_i < v_j\}} \sum_{t \in \mathcal{T} \setminus \{0,T\}} \lambda_{e,t} \left[\sum_{a \in \mathcal{A}} (x_{a,e,t} + x_{a,\bar{e},t}) - 1 \right] + \\ + \sum_{a \in \mathcal{A}} \sum_{e \in E} \sum_{t \in \mathcal{T} \setminus \{0\}} \mu_{a,e,t} \Bigg[x_{a,e,t} + \\ + \sum_{\{a' \in \mathcal{A} : a' \neq a\}} (x_{a',e,t-1} + x_{a',\bar{e},t-1}) - 1 \Bigg] + \\ + \sum_{a \in \mathcal{A}} \nu_a \left[T + 1 - w - \sum_{t \in \mathcal{T}} x_{a,d_a,t} \right] \Bigg\} \tag{11}$$

s.t. the primal constraint sets (2)–(6) and (10)

The function $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ constitutes the "dual" function for the considered relaxation, and it provides a lower bound to the optimal makespan, w^*, for any $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \geq \mathbf{0}$. Hence, the tightest lower bound for w^* provided by the above Lagrangian relaxation, can be obtained by solving the following mathematical programming (MP) formulation, which is known as the corresponding "dual" problem:

$$\max_{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}} \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \tag{12}$$

$$\text{s.t. } (\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \geq \mathbf{0}$$

In [4] it is further shown that any optimal solution of the aforestated dual problem must also satisfy $\sum_{a \in \mathcal{A}} \nu_a = 1$, and thus, the dual problem reduces to:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} \theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}) \equiv (T+1) + \Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} + \\ + \min_{\boldsymbol{x}\in X} \left\{ \sum_{a\in\mathcal{A}} \left[\sum_{e\in E} \sum_{t\in\mathcal{T}} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t} - \nu_a \sum_{t\in\mathcal{T}} x_{a,d_a,t} \right] \right\} \tag{13}$$

s.t.

$$\boldsymbol{\lambda} \geq \mathbf{0}; \quad \boldsymbol{\mu} \geq \mathbf{0}; \quad \boldsymbol{\nu} \geq \mathbf{0}; \quad \sum_{a\in\mathcal{A}} \nu_a = 1.0 \tag{14}$$

where $\Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and $C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t}$ are linear functions of the corresponding Lagrange multipliers,[7] and $X \equiv \left\{ \boldsymbol{x} \text{ satisfying the primary constraint sets (2)–(6) and (10)} \right\}$.

The reader should notice that Eq. (13) implies the separability of the minimization problem that is involved in the definition of the dual function $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ with respect to the agent set $\mathcal{A}$. Furthermore, the structure of the right-hand-side of Eq. (13), when combined with the aforestated linearity of the quantities $\Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and $C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t}$ with respect to the corresponding Lagrange multipliers, imply that the considered dual function can also be expressed in the following form, for appropriately defined (row) vectors $\boldsymbol{g}(\boldsymbol{x}),\ \boldsymbol{x} \in X$:

$$\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}) = \min_{\boldsymbol{x}\in X} \left\{ \boldsymbol{g}(\boldsymbol{x}) \cdot (\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})^T \right\} + (T+1) \tag{15}$$

Finally, Eq. (15), when combined with the finiteness of the involved set X, implies a concave, polyhedral structure for $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$. This structure was exploited in [3], [4] in order to develop a customized dual-ascent algorithm for the solution of the corresponding dual problem. However, in the next section, we shall show that Eq. (15) also enables the representation of the considered dual problem as an LP. Furthermore, the aforementioned separability of the minimization problem that is involved in the definition of the dual function $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ with respect to the set $\mathcal{A}$, enables a quite compact representation of this LP in terms of the involved sets of its constraints and decision variables. Hence, it can be tractable for quite large instantiations of the dual problem; this fact is demonstrated numerically in Section IV.

III. The Main Result

To derive the main result of this paper, we start with the observation that the representation of the dual function according to Eq. (15) allows us to express the dual problem of Eqs (13)–(14) as the following LP:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu},u} u \tag{16}$$

s.t.

$$\forall \boldsymbol{x} \in X, \quad u \leq \boldsymbol{g}(\boldsymbol{x}) \cdot (\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})^T \tag{17}$$

$$\sum_{a\in\mathcal{A}} \nu_a = 1.0 \tag{18}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}; \quad \boldsymbol{\mu} \geq \mathbf{0}; \quad \boldsymbol{\nu} \geq \mathbf{0} \tag{19}$$

Furthermore, from standard LP duality theory [23], the optimal value of the above LP can also be obtained by solving its dual LP, that has the following form:

$$\min_{\boldsymbol{\gamma},\boldsymbol{z},\psi} \psi \tag{20}$$

s.t.

$$[-I \ \mathbf{1}_{\boldsymbol{\nu}}] \cdot \begin{bmatrix} \boldsymbol{z} \\ \psi \end{bmatrix} = \sum_{\boldsymbol{x}\in X} \boldsymbol{\gamma}_{\boldsymbol{x}} \cdot \boldsymbol{g}(\boldsymbol{x})^T \tag{21}$$

$$\sum_{\boldsymbol{x}\in X} \boldsymbol{\gamma}_{\boldsymbol{x}} = 1.0 \tag{22}$$

$$\boldsymbol{\gamma} \geq \mathbf{0}; \quad \boldsymbol{z} \geq \mathbf{0} \tag{23}$$

In the above LP formulation, the nonnegative vector $\boldsymbol{\gamma}$ collects the dual variables that correspond to the constraints of Eq. (17) in the primal LP, and the free variable ψ is the dual variable corresponding to the constraint of Eq. (18). On the other hand, vector $\boldsymbol{z}$ is a set of "slack" variables that converts the constraint of Eq. (21) to an "equality" constraint. Furthermore, in Eq. (21), I denotes the identity matrix of dimensionality k equal to the total number of Lagrange multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, and $\mathbf{1}_{\boldsymbol{\nu}}$ is a k-dimensional binary (column) vector with its unit elements placed at the components that correspond to Lagrange multipliers $\boldsymbol{\nu}$.

While the formulations of Eqs (16)–(19) and Eqs (20)–(23) provide valid LP representations for the dual problem of Eqs (13)–(14), their practical usefulness is limited by the fact that they require a complete enumeration of set X. However, in the following we shall show that, in the considered context, X admits a distributed representation that enables the representation of the above LPs in a much more compact form in terms of the employed numbers of variables and constraints.

The starting point for obtaining these new formulations is the observation that the constraints defining the set X are totally separable across the agents $a \in \mathcal{A}$.[8] In more technical terms, vector $\boldsymbol{x}$ can be perceived as a concatenation of the vectors $\boldsymbol{x}_a,\ a \in \mathcal{A}$, and each vector $\boldsymbol{x}_a$ lives in a space X_a that is defined by the corresponding subset of the Constraints (2)–(6) and (10) that refer to agent a. The aforementioned separability of these constraint sets (2)–(6) and (10) across the agent set $\mathcal{A}$ implies that

$$X = \times_{a\in\mathcal{A}} X_a \tag{24}$$

From a more conceptual standpoint, each set $X_a,\ a \in \mathcal{A}$, encodes all the possible routes in G for taking agent a from initial location s_a to its destination location d_a within the provided time span T. A compact way to represent all these routes is by an acyclic digraph $\mathcal{G}_a$. The nodes of this graph are labeled by (e,t) and signify the placement of agent a at (directed) edge e at time period t. On the other hand, the edges of $\mathcal{G}_a$ connect nodes (e,t) for $t \in \{0,1,\ldots,T-1\}$ to nodes $(e',t+1)$ with $e' \in \{e\} \cup e^{\bullet}$. Furthermore, for T adequately large to ensure the feasibility of the original MIP of Section II-A, the digraph $\mathcal{G}_a$ will have node $(s_a,0)$ as its single "source" node, and node (d_a,T) as its single

[7] Due to the imposed space limitations, the reader is referred to [4] for the detailed definition of these quantities.

[8] This separability is a consequence of the logic that has defined the employed Lagrange relaxation for the original MIP of Section II-A.

"sink" node. Each feasible route in X_a corresponds to a path leading from node $(s_a, 0)$ to node (d_a, T). Finally, the construction of the digraphs $\mathcal{G}_a,\ a \in \mathcal{A}$, can be performed through elementary reachability analysis on the guidepath network G.

The LP formulation of Eqs (20)–(23) essentially employs the convex hull of the vector set $\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\}$. Following a proof similar to that of Lemma 5.1 in [4], we can show that this convex hull can be represented as $\{\boldsymbol{g}(\boldsymbol{q}) : \boldsymbol{q} \in Conv(X)\}$, where $Conv(X)$ denotes the convex hull of X. Eq. (24) further implies that

$$Conv(X) = \times_{a\in\mathcal{A}} Conv(X_a) \tag{25}$$

But each element $\boldsymbol{q}_a \in Conv(X_a),\ a \in \mathcal{A}$, can be represented by means of the aforementioned graph $\mathcal{G}_a$, as a flow transferring a unit of fluid from the "source" node of $\mathcal{G}_a$ to its "sink" node. Hence, for each $a \in \mathcal{A}$, the corresponding $Conv(X_a)$ can be represented parametrically by a set of linear equations

$$F_a \cdot \boldsymbol{q}_a = \boldsymbol{\beta}_a^1 \quad ; \quad \boldsymbol{q}_a \geq \boldsymbol{0} \tag{26}$$

Furthermore, we can combine the linear systems of equations that are defined in Eq. (26) for each agent $a \in \mathcal{A}$, in order to obtain a similar representation for the entire convex hull $Conv(X)$:

$$F \cdot \boldsymbol{q} = \boldsymbol{\beta}^1 \quad ; \quad \boldsymbol{q} \geq \boldsymbol{0} \tag{27}$$

Finally, let

$$\boldsymbol{g}(\boldsymbol{x})^T = A \cdot \boldsymbol{x} + \boldsymbol{\beta}^2 \tag{28}$$

for some appropriately specified matrix A and vector $\boldsymbol{\beta}^2$.

Then, in view of all the above discussion, the original LP formulation of Eqs (20)–(23) can be rewritten as

$$\min_{\boldsymbol{q},\boldsymbol{z},\psi} \psi \tag{29}$$

s.t.

$$\begin{bmatrix} F & 0 & 0 \\ -A & -I & \boldsymbol{1}_{\boldsymbol{\nu}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{z} \\ \psi \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}^1 \\ \boldsymbol{\beta}^2 \end{bmatrix} \tag{30}$$

$$\boldsymbol{q} \geq \boldsymbol{0}; \quad \boldsymbol{z} \geq \boldsymbol{0} \tag{31}$$

Also, the dual of the above LP has the form:

$$\max_{\boldsymbol{\eta},\boldsymbol{\rho}} \ (\boldsymbol{\beta}^1)^T \cdot \boldsymbol{\eta} + (\boldsymbol{\beta}^2)^T \cdot \boldsymbol{\rho} \tag{32}$$

s.t.

$$F^T \cdot \boldsymbol{\eta} - A^T \cdot \boldsymbol{\rho} \ \leq \ 0 \tag{33}$$

$$\boldsymbol{1}_{\boldsymbol{\nu}}^T \cdot \boldsymbol{\rho} \ = \ 1.0 \tag{34}$$

$$\boldsymbol{\rho} \geq \boldsymbol{0} \tag{35}$$

The vectors $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ that constitute the decision variables in this last formulation, collect, respectively, the dual variables for the constraints that correspond to the first and the second rows in Eq. (30). Furthermore, this last LP is the analogue of the original LP formulation of Eqs (16)–(19) in the distributed representation of the set X and its convex hull $Conv(X)$ that were introduced in the previous paragraphs.

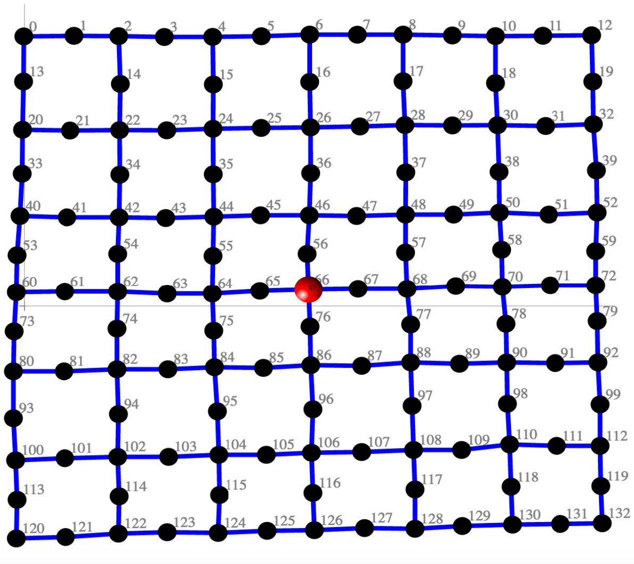

Fig. 1: The guidepath network used in our experiment.

Taking a closer look at the structure of these two LPs and their respective duals, it can be seen that the role of the variable vector $\boldsymbol{\rho}$ in the last LP is equivalent to that of the variable vector $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})^T$ in the LP of Eqs (16)–(19). Hence, we have reached to the following result:

Theorem 3.1: The LP formulation of Eqs (32)–(35) is a valid representation of the "dual" problem of Eqs (13)–(14). Furthermore, for any optimal solution of this LP, $(\boldsymbol{\eta}^*, \boldsymbol{\rho}^*)$, the vector $\boldsymbol{\rho}^*$ defines an optimal solution for the problem of Eqs (13)–(14).

IV. Some numerical results

In addition to revealing some further special structure for the "dual" problem that is defined by Eqs (13)–(14), the results of Section III have a significant practical implication for the solution of this problem, since they render it amenable to the capabilities of the various commercial solvers that are currently available for the solution of some pretty large LP formulations. To further demonstrate and assess this potential, we set up and solved the LP formulations corresponding to the "dual" problem that is defined by a number of instantiations of the traffic scheduling problem considered in Section II; all these instantiations were defined by means of the guidepath network depicted in Figure 1.

The guidepath network of Figure 1 provides 133 distinct zones for the traveling agents, organized in the depicted grid.[9] In the performed experiment, we generated randomly a number of instances of the original MIP formulation of Section II-A, while varying the number of traveling agents from 3 to 30, with a step-increase of 3. For each of these levels, we generated five replications, and for each replication, we obtained an upper bound T to the optimal makespan w^* using some of the heuristic algorithms that

[9]We should also notice that in the graph of Figure 1 the available zones are encoded by the graph nodes and not by its edges; but the translation of this structure to the corresponding model of Section II-A is pretty straightforward.

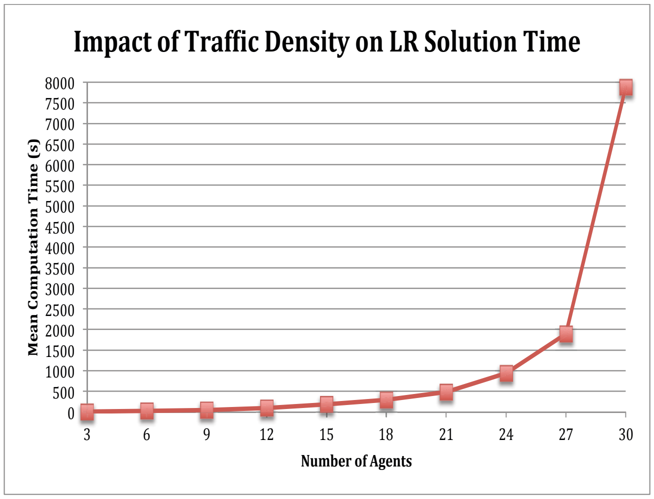

Fig. 2: A plot reporting the computational times observed in our experiments.

are reported in [6], [5]. Subsequently, we formulated and solved the corresponding LP formulation of Eqs (32)–(35). All the LP formulations were solved through CPLEX, while the preparation of the input files for CPLEX from the original problem data was performed through MATLAB. The corresponding computation was executed on a 2013 Macbook Pro with a 2.4 GHz Intel Core i5 processor and 8 GB of 1600 MHz DDR3 RAM.

Figure 2 plots the computational times required for setting up and solving these formulations, as a function of the number of traveling agents; in particular, the reported numbers are the averages of the computational times that were observed for the corresponding replications. As it can be seen from the plot of Figure 2, the computational time required by the presented approach increases with the number of agents involved and the congestion that is experienced in the underlying guidepath network,[10] but the approach remains tractable for pretty large instances of the underlying scheduling problem.[11] Finally, to provide the reader with some more concrete appreciation of the computational effort that was involved in the presented experiment, we also notice that the largest LP formulated and solved in this experiment employed 184,444 variables and 316,025 constraints.

V. Conclusions

This work has complemented our earlier developments on the problem of managing the traffic generated by a set of agents that circulate in the restricted environment of a zone-controlled guidepath network, by providing an efficient LP-based (re-)formulation of the "dual" problem that is defined by a Lagrangian relaxation of the original scheduling problem. Furthermore, the numerical experimentation reported in Section IV has demonstrated that this new LP-based representation enables the solution of the aforementioned "dual" problem through the currently available LP solvers, even for some pretty large problem instances. Our future work will seek to exploit this new computational capability in the heuristic algorithms that we have been developing for the traffic-management problems considered in this work.

[10] The impact of this congestion in the context of the considered experiment is primarily through the quality / "tightness" of the T-values that are returned by our heuristic algorithms as estimates of the optimal makespan; more specifically, the application of our heuristic algorithms to more congested problem instances tends to increase the deviation of the returned schedules from the optimal makespan, and, thus, it leads to larger estimates for parameter T.

[11] In fact, the largest part of the times reported in Figure 2 was consumed by MATLAB for setting up the corresponding formulations. We believe that the reported times can be curtailed considerably by using a more streamlined code for this task, developed in a more basic programming language like C.

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