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The congruence subgroup problem for low rank free and free metabelian groups

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A R T I C L E   I N F O

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To Efim Zelmanov, a friend and a leader

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A B S T R A C T

The congruence subgroup problem for a finitely generated group Γ asks whether $\widehat{Aut(\Gamma)} \to Aut(\hat{\Gamma})$ is injective, or more generally, what is its kernel $C(\Gamma)$? Here $\hat{X}$ denotes the profinite completion of X.
In this paper we first give two new short proofs of two known results (for $\Gamma = F_2$ and Φ_2) and a new result for $\Gamma = \Phi_3$:

(1) $C(F_2) = \{e\}$ when F_2 is the free group on two generators.
(2) $C(\Phi_2) = \hat{F}_\omega$ when Φ_n is the free metabelian group on n generators, and $\hat{F}_\omega$ is the free profinite group on $\aleph_0$ generators.
(3) $C(\Phi_3)$ contains $\hat{F}_\omega$.

Results (2) and (3) should be contrasted with an upcoming result of the first author showing that $C(\Phi_n)$ is abelian for $n \geq 4$.

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1. Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = SL_n(\mathbb{Z})$ or $G = GL_n(\mathbb{Z})$, whether every finite index subgroup of G contains a principal congruence subgroup, i.e. a subgroup of the form $G(m) = \ker(G \to GL_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. Equivalently, it asks whether the natural map $\hat{G} \to GL_n(\hat{\mathbb{Z}})$ is injective, where $\hat{G}$ and $\hat{\mathbb{Z}}$ are the profinite completions of the group G and the ring $\mathbb{Z}$, respectively. More generally, the CSP asks what is the kernel of this map. It is a classical 19th century result that the answer is negative for $n = 2$. Moreover (but not so classical, cf. [20,15]), the kernel, in this case, is $\hat{F}_\omega$ – the free profinite group on a countable number of generators. On the other hand, for $n \geq 3$, the map is injective and the kernel is therefore trivial.

The CSP can be generalized as follows: Let Γ be a group and M a finite index characteristic subgroup of it. Denote:

$$G(M) = \ker(Aut(\Gamma) \to Aut(\Gamma/M)).$$

Such a finite index normal subgroup of $G = Aut(\Gamma)$ will be called a "principal congruence subgroup" and a finite index subgroup of G which contains such a $G(M)$ for some M will be called a "congruence subgroup". Now, the CSP for Γ asks whether every finite index subgroup of G is a congruence subgroup. When Γ is finitely generated, $Aut(\hat{\Gamma})$ is profinite and the CSP is equivalent to the question (cf. [8], §1 and §3): Is the map $\hat{G} = \widehat{Aut(\Gamma)} \to Aut(\hat{\Gamma})$ injective? More generally, it asks what is the kernel $C(\Gamma)$ of this map.

As $GL_n(\mathbb{Z}) = Aut(\mathbb{Z}^n)$, the classical congruence subgroup results mentioned above can therefore be reformulated as $C(A_2) = \hat{F}_\omega$ while $C(A_n) = \{e\}$ for $n \geq 3$, when $A_n = \mathbb{Z}^n$ is the free abelian group on n generators.

Very few results are known when Γ is non-abelian. A very surprising result was proved in [2] by Asada by methods of algebraic geometry:

Theorem 1.1. $C(F_2) = \{e\}$*, i.e., the free group on two generators has the congruence subgroup property, namely* $\widehat{Aut(F_2)} \to Aut(\hat{F}_2)$ *is injective.*

A purely group theoretic proof for this theorem was given by Bux–Ershov–Rapinchuk [8]. Our first goal in this paper is to give an easier and more direct proof of Theorem 1.1, which also give a better quantitative estimate: we give an explicitly constructed congruence subgroup $G(M)$ of $Aut(F_2)$ which is contained in a given finite index subgroup H of $Aut(F_2)$ of index n. Our estimates on the index of M in F_2 as a function of n are substantially better than those of [8] – see Theorems 2.7 and 2.9.

We then turn to $\Gamma = \Phi_2$, the free metabelian group on two generators. The initial treatment of Φ_2 is similar to F_2, but quite surprisingly, the first named author showed in [4] a negative answer, i.e. $C(\Phi_2) \neq \{e\}$. We also give a shorter proof of this result, deducing that:

Theorem 1.2. $C(\Phi_2) = \hat{F}_\omega$.

We then go ahead from 2 to 3 and prove:

Theorem 1.3. *$C(\Phi_3)$ contains a copy of $\hat{F}_\omega$. In particular, the congruence subgroup property (strongly) fails for Φ_3.*

This is also surprising, especially if compared with an upcoming paper of the first author [5] showing that $C(\Phi_n)$ is abelian for $n \geq 4$. So, while the dichotomy for the abelian case $A_n = \mathbb{Z}^n$ is between $n = 2$ and $n \geq 3$, for the metabelian case, it is between $n = 2, 3$ and $n \geq 4$.

A main ingredient of the proof of Theorem 1.3 is showing that $Aut(\Phi_3)$ is large, i.e. it has a finite index subgroup which is mapped onto a non-abelian free group. For this we use the method developed by Grunewald and the second author in [13] to produce arithmetic quotients of $Aut(F_n)$. In particular, it is shown there that $Aut(F_3)$ is large. Our starting point to prove Theorem 1.3 is the observation that the same proof shows also that $Aut(\Phi_3)$ is large.

In our proof of Theorem 1.2, the largeness of $Aut(\Phi_2)$ is also playing a crucial role. But, a word of warning is needed here: largeness of $Aut(\Gamma)$ by itself is not sufficient to deduce negative answer for the CSP for Γ. For example, $Aut(F_2)$ is large but has an affirmative answer for the CSP. At the same time, as mentioned above, $Aut(F_3)$ is large and we do not know whether F_3 has the congruence subgroup property or not. To prove Theorem 1.3 we use the largeness of $Aut(\Phi_3)$ combined with the fact that every non-abelian finite simple group which is involved in $Aut(\hat{\Phi}_3)$ is already involved in $GL_3(R)$ for some finite commutative ring R, as we will show below.

The paper is organized as follows: In §2 we give a short proof for Theorem 1.1 and in §3 for Theorem 1.2. Section 4 is devoted to the proof of Theorem 1.3. We close in §5 with some remarks and open problems, about free nilpotent and solvable groups.

2. The CSP for F_2

Before we start, let us quote some general propositions which Bux–Ershov–Rapinchuk bring throughout their paper.

Proposition 2.1. (cf. [8], Lemma 2.1) *Let:*

$$1 \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 1$$

be an exact sequence of groups. Assume that G_1 is finitely generated and that the center of its profinite completion $\hat{G}_1$ is trivial. Then, the sequence of the profinite completions

$$1 \to \hat{G}_1 \xrightarrow{\hat{\alpha}} \hat{G}_2 \xrightarrow{\hat{\beta}} \hat{G}_3 \to 1$$

is also exact.

Proposition 2.2. (cf. [8], Corollaries 2.3, 2.4. and 2.7) *Let F be the free group on the set X, $|X| \geq 2$. Then:*

1. The center of $\hat{F}$, the profinite completion of F, is trivial.

2. If $x, y \in X$, $x \neq y$, then the centralizer of $[y,x]$ in $\hat{F}$ is $Z_{\hat{F}}([y,x]) = \overline{\langle [y,x] \rangle}$, the closure of the cyclic group generated by $[y,x]$.

We start now with the following lemma whose easy proof is left to the reader:

Lemma 2.3. *Let $H \leq G = Aut(\Gamma)$ be a congruence subgroup. Then:*

$$\ker(\hat{G} \to Aut(\hat{\Gamma})) = \ker(\hat{H} \to Aut(\hat{\Gamma})).$$

In particular, the map $\hat{G} \to Aut(\hat{\Gamma})$ is injective if and only if the map $\hat{H} \to Aut(\hat{\Gamma})$ is injective.

Denote now $F_2 = \langle x, y \rangle =$ the free group on x and y. It is a well known theorem of Nielsen (cf. [21], 3.5) that the kernel of the natural surjective map:

$$Aut(F_2) \to Aut(F_2/F_2') = Aut(\mathbb{Z}^2) = GL_2(\mathbb{Z})$$

is $Inn(F_2)$, the inner automorphism group of F_2. It is also well known that the group $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \cong F_2$ is free on two generators and of finite index in $GL_2(\mathbb{Z})$ which contains $\ker(GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/4\mathbb{Z}))$. Now, if we denote the preimage of $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ under the map $Aut(F_2) \to GL_2(\mathbb{Z})$ by $Aut'(F_2)$, then $Aut'(F_2)$ is of finite index in $Aut(F_2)$ and contains the principal congruence subgroup:

$$\ker\left(Aut(F_2) \to GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/4\mathbb{Z}) = Aut\left(F_2/\left(F_2^4 F_2'\right)\right)\right).$$

So, by Lemma 2.3 it is enough to prove that $\widehat{Aut'(F_2)} \to Aut(\hat{F}_2)$ is injective.

Now, by the description above, we deduce the exact sequence:

$$1 \to Inn(F_2) \to Aut'(F_2) \to \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \to 1.$$

As $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is free, this sequence splits by the map:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto \alpha = \begin{cases} x \mapsto & x \\ y \mapsto & yx^2 \end{cases}, \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mapsto \beta = \begin{cases} x \mapsto & xy^2 \\ y \mapsto & y \end{cases}$$

and thus: $Aut'(F_2) = Inn(F_2) \rtimes \langle\alpha,\beta\rangle$. By Propositions 2.1 and 2.2, the exact sequence: $1 \to Inn(F_2) \to Aut'(F_2) \to \langle\alpha,\beta\rangle \to 1$ yields the exact sequence:

$$1 \to \widehat{Inn(F_2)} \to \widehat{Aut'(F_2)} \to \widehat{\langle\alpha,\beta\rangle} \to 1$$

which gives:

$$\widehat{Aut'(F_2)} = \widehat{Inn(F_2)} \rtimes \widehat{\langle\alpha,\beta\rangle}.$$

Thus, all we need to show is that the following map is injective:

$$\widehat{Inn(F_2)} \rtimes \widehat{\langle\alpha,\beta\rangle} \to Aut(\hat{F}_2).$$

We will prove this, in three parts: The first part is that the map $\widehat{Inn(F_2)} \to Aut(\hat{F}_2)$ is injective, but this is obvious as $\widehat{Inn(F_2)} \cong \hat{F}_2$ is mapped isomorphically to $Inn(\hat{F}_2) \cong \hat{F}_2$. The second part is to show that the map $\rho : \widehat{\langle\alpha,\beta\rangle} \to Aut(\hat{F}_2)$ is injective, and the last part is to show that the intersection of the images of $\widehat{Inn(F_2)}$ and $\widehat{\langle\alpha,\beta\rangle}$ in $Aut(\hat{F}_2)$ is trivial, i.e. $Inn(\hat{F}_2) \cap \mathrm{Im}\rho = \{e\}$.

So it remains to prove the next two lemmas, Lemma 2.4 and Lemma 2.6:

Lemma 2.4. *The map $\widehat{\langle\alpha,\beta\rangle} \to Aut(\hat{F}_2)$ is injective.*

Before proving the lemma, we recall a classical result of Schreier:

Theorem 2.5. (cf. [21], 2.3 and 2.4) *Let F be the free group on the set X where $|X| = n$, and Δ a subgroup of F of index m. Let T be a right Schreier transversal of Δ (i.e. a system of representatives of right cosets containing the identity, such that the initial segment of any element of T is also in T). Then:*

1. Δ is a free group on $m \cdot (n-1) + 1$ elements.

2. The set $\left\{tx\left(\overline{tx}\right)^{-1} \neq e \,|\, t \in T,\, x \in X\right\}$ is a free generating set for Δ, where for every $g \in F$ we denote by $\bar{g}$ the unique element in T satisfying $\Delta g = \Delta\bar{g}$.

Proof of Lemma 2.4. Define $\Delta = \ker(F_2 \to (\mathbb{Z}/2\mathbb{Z})^2)$. This is a characteristic subgroup of index 4 in F_2, that by the first part of Theorem 2.5, is isomorphic to F_5. We also have: $\hat{\Delta} = \ker(\hat{F}_2 \to (\mathbb{Z}/2\mathbb{Z})^2)$, and therefore, there is a natural homomorphism: $Aut(\hat{F}_2) \to Aut(\hat{\Delta}) \cong Aut(\hat{F}_5)$ which induces the composition $\widehat{\langle\alpha,\beta\rangle} \to Aut(\hat{F}_2) \to Aut(\hat{\Delta})$. Thus, it is enough to show that the composition map $\widehat{\langle\alpha,\beta\rangle} \to Aut(\hat{\Delta})$ is injective.

Now, let $X = \{x, y\}$ and $T = \{1, x, y, xy\}$ be a right Schreier transversal of Δ. By applying the second part of Theorem 2.5 for X and T, we get the following set of free generators for Δ:

$$e_1 = x^2, \quad e_2 = yxy^{-1}x^{-1}, \quad e_3 = y^2, \quad e_4 = xyxy^{-1}, \quad e_5 = xy^2x^{-1}.$$

Hence, the automorphisms α and β act on Δ in the following way:

$$\alpha = \begin{cases} e_1 = x^2 \mapsto x^2 & = e_1 \\ e_2 = yxy^{-1}x^{-1} \mapsto yxy^{-1}x^{-1} & = e_2 \\ e_3 = y^2 \mapsto yx^2yx^2 & = e_2e_4e_3e_1 \\ e_4 = xyxy^{-1} \mapsto xyxy^{-1} & = e_4 \\ e_5 = xy^2x^{-1} \mapsto xyx^2yx & = e_4e_2e_5e_1 \end{cases}$$

$$\beta = \begin{cases} e_1 = x^2 \mapsto xy^2xy^2 & = e_5e_1e_3 \\ e_2 = yxy^{-1}x^{-1} \mapsto yxy^{-1}x^{-1} & = e_2 \\ e_3 = y^2 \mapsto y^2 & = e_3 \\ e_4 = xyxy^{-1} \mapsto xy^3xy & = e_5e_4e_3 \\ e_5 = xy^2x^{-1} \mapsto xy^2x^{-1} & = e_5 \end{cases}$$

Let us now define the map $\pi : \Delta \to \langle \alpha, \beta \rangle \cong F_2$ (yes! these are the same α and β) by the following way:

$$\pi = \begin{cases} e_1 \mapsto & \alpha \\ e_2 \mapsto & 1 \\ e_3 \mapsto & \beta \\ e_4 \mapsto & \alpha^{-1} \\ e_5 \mapsto & \beta^{-1} \end{cases}$$

It is easy to see that $N = \ker \pi$ is the normal subgroup of Δ generated as a normal subgroup by e_2, e_1e_4 and e_3e_5, and that N is invariant under the action of the automorphisms α and β, since:

$$\begin{cases} \alpha(e_2) & = e_2 \in N \\ \alpha(e_1e_4) & = e_1e_4 \in N \\ \alpha(e_3e_5) & = e_2e_4e_3e_1e_4e_2e_5e_1 \\ & = e_4\left(\left(e_4^{-1}e_2e_4\right)\left(e_3\left((e_1e_4)e_2\right)e_3^{-1}\right)(e_3e_5)(e_1e_4)\right)e_4^{-1} \in N \end{cases}$$

$$\begin{cases} \beta(e_2) & = e_2 \in N \\ \beta(e_1e_4) & = e_5e_1e_3e_5e_4e_3 = e_5\left(\left(e_1(e_3e_5)e_1^{-1}\right)(e_1e_4)(e_3e_5)\right)e_5^{-1} \in N \\ \beta(e_3e_5) & = e_3e_5 \in N \end{cases}$$

Therefore, the homomorphism $\widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{\Delta})$ induces a homomorphism: $\widehat{\langle \alpha, \beta \rangle} \to Aut(\widehat{\langle \alpha, \beta \rangle})$, and thus it is enough to show that the last map is injective. Now, under this map, α and β act on $\langle \alpha, \beta \rangle$ in the following way:

$$\alpha = \begin{cases} \alpha = e_1 N \mapsto \alpha\left(e_1 N\right) = \alpha\left(e_1\right) N = e_1 N & = \alpha \\ \beta = e_3 N \mapsto \alpha\left(e_3 N\right) = \alpha\left(e_3\right) N = e_2 e_4 e_3 e_1 N & = \alpha^{-1}\beta\alpha \end{cases}$$

$$\beta = \begin{cases} \alpha = e_1 N \mapsto \beta\left(e_1 N\right) = \beta\left(e_1\right) N = e_5 e_1 e_3 N & = \beta^{-1}\alpha\beta \\ \beta = e_3 N \mapsto \beta\left(e_3 N\right) = \beta\left(e_3\right) N = e_3 N & = \beta \end{cases}$$

Namely, α and β act via π on $\widehat{\langle \alpha, \beta \rangle}$ by the inner automorphisms α and β and hence $\widehat{\langle \alpha, \beta \rangle}$ is mapped isomorphically to $Inn(\widehat{\langle \alpha, \beta \rangle})$, yielding that the map $\widehat{\langle \alpha, \beta \rangle} \to Aut(\widehat{\langle \alpha, \beta \rangle})$ is injective and $\widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{F}_2)$ is injective as well, as required. $\square$

Lemma 2.6. *$Inn(\hat{F}_2) \cap \mathrm{Im}\rho = \{e\}$, where $\rho : \widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{F}_2)$ is the map defined above.*

Proof. First we observe that α and β fix $e_2 = [y, x]$. Thus, by the second part of Proposition 2.2, we have:

$$Inn(\hat{F}_2) \cap \mathrm{Im}\rho \subseteq Z_{Inn(\hat{F}_2)}\left(Inn\left([y, x]\right)\right) = \overline{\langle Inn\left([y, x]\right) \rangle} = \overline{\langle Inn\left(e_2\right) \rangle}.$$

Now, as $e_2 \in \ker \pi$, where π is as defined in the proof of Lemma 2.4, the image of $\overline{\langle Inn\left(e_2\right) \rangle}$ in $Inn(\widehat{\langle \alpha, \beta \rangle})$ is trivial. Thus, the image of $Inn(\hat{F}_2) \cap \mathrm{Im}\rho$ in $Inn(\widehat{\langle \alpha, \beta \rangle})$ is trivial, and isomorphic to $Inn(\hat{F}_2) \cap \mathrm{Im}\rho$ as we saw that $\mathrm{Im}\rho$ is mapped isomorphically to $Inn(\widehat{\langle \alpha, \beta \rangle})$. So $Inn(\hat{F}_2) \cap \mathrm{Im}\rho$ is trivial. $\square$

This finishes the proof of Theorem 1.1. In [8], the authors give an explicit construction of a congruence subgroup which is contained in a given finite index subgroup of $Aut(\hat{F}_2)$. They prove the following theorem:

Theorem 2.7. (cf. [8], Theorem 5.1) *Let H be a finite index normal subgroup of $G = Aut\left(F_2\right)$ such that $Inn\left(F_2\right) \leq H \leq Aut'\left(F_2\right)$ and let $n = \left[Aut'\left(F_2\right) : H\right]$. Pick two distinct odd primes $p, q \nmid n$, and set $m = n \cdot p^{n+1}$. Then, there exists an explicitly constructed normal subgroup $M \lhd F_2$ of index dividing $144 \cdot m^4 \cdot q^{36 \cdot m^4 + 1}$ such that $G\left(M\right) \leq H$, when for a general normal subgroup $M \lhd F_2$ we define:*

$$G\left(M\right) = \left\{\sigma \in G \,|\, \sigma\left(M\right) = M,\ \sigma \text{ acts trivially on } F_2/M\right\}.$$

We end this section with a much simpler explicit construction of a congruence subgroup and with a better bound for the index of M. But before, let us recall the "discrete version" of Proposition 2.2 from [8]:

Proposition 2.8. (cf. [8], Propositions 2.2 and 2.6) *Let F be the free group on the set X, $|X| \geq 2$, and let F/N be a finite quotient of F. Pick a prime p not dividing the order of F/N and set $M = N^p N'$. Then:*

1. The image of every normal abelian subgroup of F/M through the natural projection $F/M \to F/N$, is trivial.

2. If $N \subseteq F_2' F_2^6$, $x, y \in X$, $x \neq y$, then the image of the centralizer $Z_{F/M}([y,x] \cdot M)$ through the natural projection $F/M \to F/N$, is $\langle [y,x] \cdot N \rangle$.

Theorem 2.9. *Let H be a finite index normal subgroup of $G = Aut(F_2)$ such that $Inn(F_2) \leq H \leq Aut'(F_2) = Inn(F_2) \rtimes \langle \alpha, \beta \rangle$ and let $n = [Aut'(F_2) : H]$. Then for every prime $p \nmid 6n$, there exists an explicitly constructed normal subgroup $M \lhd F_2$ of index dividing $144 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$ such that $G(M) \leq H$.*

Proof. Recall the map $\pi : F_2 \supseteq \Delta \to \langle \alpha, \beta \rangle$ from the proof of Lemma 2.4, and let $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$ be the system of representatives of right cosets of Δ in F_2. Denote also $K = H \cap \langle \alpha, \beta \rangle$ and define:

$$N = F_2' F_2^6 \bigcap_{g \in F_2} g^{-1} \pi^{-1}(K) g = F_2' F_2^6 \bigcap_{i=1}^{4} t_i^{-1} \pi^{-1}(K) t_i$$

$$M = F_2' F_2^4 \cap N' N^p$$

Then $\pi^{-1}(K)$ is a subgroup of index n in Δ and $\bigcap_{i=1}^4 t_i^{-1} \pi^{-1}(K) t_i$ is a normal subgroup of F_2 of index dividing n^4 in Δ, and of index dividing $4n^4$ in F_2. So as $F_2' F_2^6$ is of index 9 in Δ, N is a normal subgroup of index dividing $36 \cdot n^4$ in F_2. Thus, by the Schreier formula, the index of $N' N^p$ in F_2 divides $36 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$ and the index of M in F_2 is dividing $4 \cdot 36 \cdot n^4 \cdot p^{36 \cdot n^4 + 1}$. So it remains to show that $G(M) \leq H$.

Let $\sigma \in G(M)$. As $M \leq F_2' F_2^4$ we have:

$$G(M) \leq \ker\left(G \to Aut\left(F_2 / \left(F_2' F_2^4\right)\right)\right) \leq Aut'(F_2) = Inn(F_2) \rtimes \langle \alpha, \beta \rangle$$

and therefore we can write $\sigma = Inn(f) \cdot \delta$ for some $f \in F_2$ and $\delta \in \langle \alpha, \beta \rangle$. By assumption, σ acts trivially on F_2/M and thus δ acts on F_2/M as $Inn\left(f^{-1}\right)$. Now, as α and β fix $[y,x]$, we deduce that so does δ. Thus, $f \cdot M \in Z_{F_2/M}([y,x] \cdot M)$ and by Proposition 2.8, $f \cdot N \in \langle [y,x] \cdot N \rangle$. Hence, δ acts on the group F_2/M as $Inn([y,x]^r \cdot n)$ for some $r \in \mathbb{Z}$ and $n \in N$. Therefore, δ acts on Δ/M as $Inn(e_2^r \cdot n)$ for some $r \in \mathbb{Z}$ and $n \in N$. So, δ acts on $\pi(\Delta)/\pi(M) = \Delta/(M \cdot \ker \pi)$ as $Inn(\pi(e_2^r \cdot n))$ for some $r \in \mathbb{Z}$ and $n \in N$. But $e_2 \in \ker \pi$, so δ acts on $\pi(\Delta)/\pi(M)$ as $Inn(\pi(n))$ for some $n \in N$. Now, by the definition of N, $\pi(N) \subseteq K$ and also $\pi(M) \subseteq K' K^p$, so δ acts on $\pi(\Delta)/K' K^p$ as $Inn(k)$ for some $k \in K$. Moreover, by the definition of π we have $\pi(\Delta) = \langle \alpha, \beta \rangle$ and by the computations we made in the proof of Lemma 2.4, δ acts on $\langle \alpha, \beta \rangle$ as $Inn(\delta)$. Thus, there exists some $k \in K$ such that $Inn(\delta) \cdot Inn(k)^{-1}$ acts trivially on $\langle \alpha, \beta \rangle / K' K^p$, i.e. $\delta \cdot k^{-1} \in Z(\langle \alpha, \beta \rangle / K' K^p)$. Now, by the first part of Proposition 2.8, as $Z(\langle \alpha, \beta \rangle / K' K^p)$ is an abelian normal subgroup of $\langle \alpha, \beta \rangle / K' K^p$ it is mapped trivially to $\langle \alpha, \beta \rangle / K$. I.e. $\delta \cdot k^{-1} \in K$, so also $\delta \in K \subseteq H$. Thus, $\sigma = Inn(f) \cdot \delta \in H$, as required. $\square$

3. The CSP for Φ_2

In this section we will prove Theorem 1.2, and will show that the congruence kernel of the free metabelian group on two generators is the free profinite group on a countable number of generators.

Before we start, let us observe that for a group Γ, one can also ask a parallel congruence subgroup problem for $G = Out\left(\Gamma\right)$. I.e. one can ask whether every finite index subgroup of G contains a principal congruence subgroup of the form:

$$G\left(M\right) = \ker\left(G \to Out\left(\Gamma/M\right)\right)$$

for some finite index characteristic subgroup $M \leq \Gamma$. When Γ is finitely generated, this is equivalent to the question whether the congruence map $\widehat{G} \to Out(\hat{\Gamma})$ is injective. Moreover, it is easy to see that Lemma 2.3 has a parallel version for G, namely, if $H \leq G$ is a congruence subgroup of G, then:

$$\ker(\widehat{G} \to Out(\hat{\Gamma})) = \ker(\widehat{H} \to Out(\hat{\Gamma})).$$

We start now with the next proposition which is slightly more general than Lemma 3.1 in [8]. Nevertheless, it is proven by the same arguments:

Proposition 3.1. (cf. [8], Lemma 3.1) *Let Γ be a finitely generated residually finite group such that $\hat{\Gamma}$ has a trivial center. Considering the congruence map $\widehat{Out\left(\Gamma\right)} \to Out(\hat{\Gamma})$, we have:*

$$C\left(\Gamma\right) = \ker(\widehat{Aut\left(\Gamma\right)} \to Aut(\hat{\Gamma})) \cong \ker(\widehat{Out\left(\Gamma\right)} \to Out(\hat{\Gamma})).$$

It is well known that Φ_2 is a residually finite group (cf. [4], Theorem 2.11). It is also proven there that $Z(\hat{\Phi}_2)$ is trivial (proposition 2.10). So by the above proposition:

$$C\left(\Phi_2\right) = \ker(\widehat{Aut\left(\Phi_2\right)} \to Aut(\hat{\Phi}_2)) \cong \ker(\widehat{Out\left(\Phi_2\right)} \to Out(\hat{\Phi}_2)).$$

In addition, it is an old result by Bachmuth [3] that the kernel of the surjective map:

$$\ker\left(Aut\left(\Phi_2\right) \to Aut\left(\Phi_2/\Phi_2'\right) = Aut\left(\mathbb{Z}^2\right) = GL_2\left(\mathbb{Z}\right)\right) = Inn\left(\Phi_2\right)$$

i.e., $Out\left(\Phi_2\right) \cong GL_2\left(\mathbb{Z}\right)$. Now, the free group $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is a congruence subgroup of $Out\left(\Phi_2\right)$ as it contains:

$$\ker\left(Out\left(\Phi_2\right) \to Out\left(\Phi_2/\Phi_2'\Phi_2^4\right)\right) = \ker\left(GL_2\left(\mathbb{Z}\right) \to GL_2\left(\mathbb{Z}/4\mathbb{Z}\right)\right).$$

So by the appropriate version of Lemma 2.3 and by Proposition 3.1, we obtain that:

$$\begin{aligned} C\left(\Phi_2\right) &= \ker(\widehat{Out\left(\Phi_2\right)} \to Out(\hat{\Phi}_2)) \\ &= \ker(\widehat{GL_2\left(\mathbb{Z}\right)} \to Out(\hat{\Phi}_2)) \\ &= \ker\left(\widehat{\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle} \to Out(\hat{\Phi}_2)\right). \end{aligned}$$

Now, as $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is a free group, we can also state that:

$$\begin{aligned} C\left(\Phi_2\right) &= \ker\left(\widehat{\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle} \to Out(\hat{\Phi}_2)\right) \\ &\cong \ker(\widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{F}_2) \to Aut(\hat{\Phi}_2) \to Out(\hat{\Phi}_2)) \end{aligned} \tag{3.1}$$

where α and β are the automorphisms of F_2 that we defined in the previous section, which are preimages of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ under the map $Aut\left(F_2\right) \to GL_2\left(\mathbb{Z}\right)$, respectively. So all we need to show is that:

Lemma 3.2. *$C\left(\Phi_2\right) = \ker(\widehat{\langle \alpha, \beta \rangle} \to Out(\hat{\Phi}_2)) = \hat{F}_\omega$.*

Proof. As the free group $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ is a congruence subgroup of the group $Aut\left(\mathbb{Z}^2\right) = Out\left(\mathbb{Z}^2\right) = GL_2\left(\mathbb{Z}\right)$, we have:

$$\begin{aligned} C\left(\mathbb{Z}^2\right) &= \ker\left(\widehat{\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle} \to Out(\hat{\mathbb{Z}}^2)\right) \\ &= \ker(\widehat{\langle \alpha, \beta \rangle} \to Out(\hat{\mathbb{Z}}^2)) \\ &= \ker(\widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{\Phi}_2) \to Out(\hat{\Phi}_2) \to Out(\hat{\mathbb{Z}}^2) = Aut(\hat{\mathbb{Z}}^2)). \end{aligned}$$

Thus, if we denote: $C = \ker(\widehat{\langle \alpha, \beta \rangle} \to Aut(\hat{\Phi}_2))$, then using equation (3.1), we have: $C \leq C\left(\Phi_2\right) \leq C\left(\mathbb{Z}^2\right)$. Now, if we consider the action of $\hat{\Phi}_2$ on $\overline{\Phi_2'} = \ker(\hat{\Phi}_2 \to \hat{\mathbb{Z}}^2)$ by conjugation, then as $\overline{\Phi_2'}$ is abelian, we actually obtain an action on $\overline{\Phi_2'}$ as a $\mathbb{Z}[\hat{\Phi}_2/\overline{\Phi_2'}] = \mathbb{Z}[\hat{\mathbb{Z}}^2]$-module, which is generated by the element $[y, x]$ as a $\mathbb{Z}[\hat{\mathbb{Z}}^2]$-module, since $\langle x, y \mid [y, x] = 1 \rangle$ is a presentation of $\mathbb{Z}^2$. Moreover, as we observed previously, α and β fix $[y, x]$. Therefore, $C\left(\mathbb{Z}^2\right)$ acts trivially not only on $\hat{\Phi}_2/\overline{\Phi_2'} = \hat{\mathbb{Z}}^2$ but also on $\overline{\Phi_2'}$.

Let us now make the following observation: if σ, τ are two automorphisms of a group Γ which act trivially on Γ/M and on M, where $M \lhd \Gamma$ is abelian, then σ and τ commute. Indeed, if $g \in \Gamma$, then $\sigma(g) = g \cdot m$ and $\tau(g) = g \cdot n$ for some $m, n \in M$, and thus:

$$\tau(\sigma(g)) = \tau(g \cdot m) = g \cdot n \cdot m = g \cdot m \cdot n = \sigma(g \cdot n) = \sigma(\tau(g)).$$

The conclusion from the above observation and from the previous discussion is that $C(\mathbb{Z}^2)/C$ is abelian, and thus, $C(\mathbb{Z}^2)/C(\Phi_2)$ is also abelian. Finally, $C(\mathbb{Z}^2)$ is known to be isomorphic to $\hat{F}_\omega$ [20,15]. Moreover, by Proposition 1.10 and Corollary 3.9 of [17] every normal closed subgroup N of $\hat{F}_\omega$ such that $\hat{F}_\omega/N$ is abelian, is also isomorphic to $\hat{F}_\omega$. Thus, $C(\Phi_2) \cong \hat{F}_\omega$ as well, as required. □

Remark 3.3. *Our proof of Theorem 1.2 is shorter than the one given in [4], but the latter gives more information. We show here that $C(\mathbb{Z}^2)/C(\Phi_2)$ is abelian, while from [4] one can deduce that, in fact, $C(\Phi_2) = C(\mathbb{Z}^2)$. See §5 for more.*

4. The CSP for Φ_3

In this section we will prove Theorem 1.3 which claims that $C(\Phi_3)$ contains a copy of $\hat{F}_\omega$. Let us start by showing that $Aut(\Phi_3)$ is large:

Proposition 4.1. *The group $Aut(\Phi_3)$ is large, i.e. it has a finite index subgroup that can be mapped onto a non-abelian free group.*

Proof. The proof will follow the method developed in [13] to produce arithmetic quotients of $Aut(F_n)$. Denote the free group on 3 generators by $F_3 = \langle x, y, z\rangle$, and the cyclic group of order 2 by $C_2 = \{1, g\}$. Define the map $\pi : F_3 \to C_2$ by: $\pi = \begin{cases} x & \mapsto g \\ y, z & \mapsto 1 \end{cases}$, and denote its kernel by $R = \ker \pi$. Then, using the right transversal $T = \{1, x\}$, we deduce by Theorem 2.5 that R is freely generated by: x^2, y, xyx^{-1}, z, xzx^{-1}. Thus, $\bar{R} = R/R' = \mathbb{Z}^5$ is generated as a free abelian group by the images:

$$v_1 = \overline{x^2},\ v_2 = \overline{y},\ v_3 = \overline{xyx^{-1}},\ v_4 = \overline{z},\ v_5 = \overline{xzx^{-1}}.$$

Now, the action of F_3 on R by conjugation induces an action of $F_3/R = C_2 = \{1, g\}$ on $\bar{R} = R/R'$, sending:

$$g \mapsto \begin{cases} v_1 = \overline{x^2} \mapsto \overline{x^2} & = v_1 \\ v_2 = \overline{y} \mapsto \overline{x^{-2}(xyx^{-1})x^2} = \overline{xyx^{-1}} & = v_3 \\ v_3 = \overline{xyx^{-1}} \mapsto \overline{y} & = v_2 \\ v_4 = \overline{z} \mapsto \overline{x^{-2}(xzx^{-1})x^2} = \overline{xzx^{-1}} & = v_5 \\ v_5 = \overline{xzx^{-1}} \mapsto \overline{z} & = v_4 \end{cases} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = B.$$

The above matrix has two eigenvalues $\lambda = \pm 1$ and the eigenspaces are:

$$\begin{aligned} V_1 &= Sp\{v_1, v_2 + v_3, v_4 + v_5\} \\ V_{-1} &= Sp\{v_2 - v_3, v_4 - v_5\}. \end{aligned}$$

Recall, $\Phi_3 = F_3/F_3''$, and as F_3/R is abelian, F_3/R' is metabelian. Thus, we have a surjective homomorphism: $\Phi_3 \twoheadrightarrow F_3/R'$. Denote now: $S = R/F_3''$, so we can identify: $F_3/R \cong \Phi_3/S$, $F_3/R' \cong \Phi_3/S'$ and $\bar{R} = R/R' \cong S/S' = \bar{S}$. So as before, $\Phi_3/S = C_2$ acts on $\bar{S}$ by the matrix B.

Denote now $G(S) = \{\sigma \in Aut(\Phi_3) \mid \sigma(S) = S\}$. It is clear that $G(S)$ is of finite index in $Aut(\Phi_3)$ with a natural map: $G(S) \to Aut(S)$ which induces a map: $\rho : G(S) \to Aut(\bar{S}) = GL_5(\mathbb{Z})$. We claim now that if $\sigma \in G(S)$ then $\rho(\sigma)$ commutes with B. First observe that there exists some $s \in S$ such that $\sigma(x) = sx$ (x now plays the role of the image of x under the map $F_3 \to \Phi_3$). Now, let $t \in S$, and remember that the action of B on $\bar{S}$ is induced by the action of x on S by conjugation. So:

$$\begin{aligned} \sigma\left(x^{-1}tx\right) &= \sigma(x)^{-1}\sigma(t)\sigma(x) = \\ &= x^{-1}s^{-1}\sigma(t)sx = \\ &= \left(x^{-1}sx\right)^{-1}\left(x^{-1}\sigma(t)x\right)\left(x^{-1}sx\right) \end{aligned}$$

and hence:

$$\begin{aligned} (\rho(\sigma)\cdot B)\left(\bar{t}\right) &= \overline{\sigma\left(x^{-1}tx\right)} = \\ &= \overline{\left(x^{-1}sx\right)^{-1}\left(x^{-1}\sigma(t)x\right)\left(x^{-1}sx\right)} = \\ &= \overline{x^{-1}\sigma(t)x} = (B\cdot\rho(\sigma))\left(\bar{t}\right). \end{aligned}$$

Therefore, $\rho(G(S))$ commutes with B. It follows that the eigenspaces of B are invariant under the action of $G(S)$. In particular, we deduce that V_{-1} is invariant under the action of $\rho(G(S))$. Thus, we obtain a homomorphism $\nu : G(S) \to Aut\left(V_{-1} \cap \bar{S}\right) = GL_2(\mathbb{Z})$.

Consider now the following automorphisms of $Aut(\Phi_3)$ (x, y, z play the role of the images of x, y, z under $F_3 \to \Phi_3$):

$$\alpha = \begin{cases} x \mapsto & x \\ y \mapsto & y \\ z \mapsto & zy \end{cases}, \quad \beta = \begin{cases} x \mapsto & x \\ y \mapsto & yz \\ z \mapsto & z \end{cases}$$

So $\alpha, \beta \in G(S)$ act on $V_{-1} = Sp\{u_1 = v_2 - v_3,\ u_2 = v_4 - v_5\}$ in the following way:

$$\begin{aligned} \alpha(u_1) &= \alpha\left(\overline{y} - \overline{xyx^{-1}}\right) = \overline{y} - \overline{xyx^{-1}} = u_1 \\ \alpha(u_2) &= \alpha\left(\overline{z} - \overline{xzx^{-1}}\right) = \overline{z} + \bar{y} - \overline{xzx^{-1}} - \overline{xyx^{-1}} = u_2 + u_1 \end{aligned}$$

$$\beta(u_1) = \beta\left(\overline{y} - \overline{xyx^{-1}}\right) = \bar{y} + \overline{z} - \overline{xyx^{-1}} - \overline{xzx^{-1}} = u_1 + u_2$$
$$\beta(u_2) = \beta\left(\overline{z} - \overline{xzx^{-1}}\right) = \overline{z} - \overline{xzx^{-1}} = u_2$$

Therefore, under the map $\nu : G(S) \to GL_2(\mathbb{Z})$ we have: $\alpha \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\beta \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, the image of $G(S)$ contains $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ which is free and of finite index in $GL_2(\mathbb{Z})$. Finally, if we denote the preimage $H = \nu^{-1}\left(\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle\right)$, then H is a finite index subgroup of $Aut(\Phi_3)$ that can be mapped onto a free group, as required. $\square$

Let us now continue with the following definition:

Definition 4.2. We say that a group P is involved in a group Q, if it isomorphic to a quotient group of some subgroup of Q.

It is not difficult to see that if a finite group P is involved in a profinite group Q, than it is involved in a finite quotient of Q. Now, we showed that $Aut(\Phi_3)$ has a finite index subgroup H which can be mapped onto F_2. Thus we have a map: $\widehat{H} \twoheadrightarrow \hat{F}_2$, but as $\hat{F}_2$ is free, the map splits, and thus $\widehat{H}$ and hence $\widehat{Aut(\Phi_3)}$, contains a copy of $\hat{F}_2$. Thus, any finite group is involved in $\widehat{Aut(\Phi_3)}$. On the other hand, we claim:

Proposition 4.3. *Let P be a non-abelian finite simple group which is involved in $Aut(\hat{\Phi}_3)$. Then, for some prime p and some $d \in \mathbb{N}$, P is involved in $SL_3(p^d)$, the special linear group over the field of order p^d.*

Proof. Let F_n be the free group on $x_1, \ldots, x_n$. Then there is a natural injective homomorphism from F_n into the matrix group:

$$\left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in F_n,\, t \in \sum_{i=1}^{n} \mathbb{Z}[F_n]\, t_i \right\}$$

defined by the map:

$$x_i \mapsto \begin{pmatrix} x_i & 0 \\ t_i & 1 \end{pmatrix}, \quad 1 \le i \le n$$

where t_i is a free basis for a right $\mathbb{Z}[F_n]$-module. This is called the Magnus embedding. Usually, its properties are studied by Fox's free differential calculus, but we will not need it here explicitly (cf. [6,25,18]).

One can prove, by induction on its length, that for a word $w \in F_n$, under the Magnus embedding, $w \mapsto \begin{pmatrix} w & 0 \\ \sum_{i=1}^{n} w_i t_i & 1 \end{pmatrix}$ where:

$$w - 1 = \sum_{i=1}^{n} (x_i - 1) w_i. \tag{4.1}$$

The identity (4.1) shows that the polynomials w_i determine the word w uniquely. Thus, we have an injective map (which is not a homomorphism) $J : End(F_n) \to M_n(\mathbb{Z}[F_n])$ defined by:

$$\alpha \overset{J}{\mapsto} \begin{pmatrix} \alpha(x_1)_1 & \cdots & \alpha(x_n)_1 \\ \vdots & & \vdots \\ \alpha(x_1)_n & \cdots & \alpha(x_n)_n \end{pmatrix}.$$

It is not difficult to check, using the identity (4.1), that the above map satisfies:

$$J(\alpha \circ \beta) = J(\alpha) \cdot \alpha(J(\beta))$$

where by $\alpha(J(\beta))$ we mean that α acts on every entry of $J(\beta)$ separately.

Now, for $m \in \mathbb{N}$, denote: $K_{n,m} = F_n^m F_n'$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Then, the natural maps $F_n \to F_n/K_{n,m} = \mathbb{Z}_m^n$ and $\mathbb{Z} \to \mathbb{Z}_m$ induce a map:

$$\begin{aligned} \pi_{n,m} : F_n &\to \left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in F_n,\, t \in \sum_{i=1}^{n} \mathbb{Z}[F_n] t_i \right\} \\ &\to \left\{ \begin{pmatrix} g & 0 \\ t & 1 \end{pmatrix} \mid g \in \mathbb{Z}_m^n,\, t \in \sum_{i=1}^{n} \mathbb{Z}_m[\mathbb{Z}_m^n] t_i \right\}. \end{aligned}$$

It is shown in [4, Proposition 2.6], that $\ker(\pi_{n,m}) = K_{n,m}^m K_{n,m}'$ and hence $\Phi_{n,m} := \mathrm{Im}(\pi_{n,m}) \cong F_n/K_{n,m}^m K_{n,m}'$. Moreover, it is proven there (Proposition 2.7) that we have the following equality:

$$\hat{\Phi}_n = \varprojlim_m \Phi_{n,m}.$$

Observe now that for every $m_2 | m_1$, $\ker(\Phi_{n,m_1} \to \Phi_{n,m_2})$ is characteristic in Φ_{n,m_1}, and for every m, $\ker(\hat{\Phi}_n \to \Phi_{n,m})$ is characteristic in $\hat{\Phi}_n$. Thus:

$$Aut(\hat{\Phi}_n) = Aut(\varprojlim_m \Phi_{n,m}) = \varprojlim_m Aut(\Phi_{n,m}).$$

Now, observe that the identity (4.1) is also valid for the entries of the elements of $\Phi_{n,m}$, and thus, every element of $\Phi_{n,m}$ is determined by its left lower coordinate. Therefore, as every automorphism of $\Phi_{n,m}$ can be lifted to an endomorphism of F_n, we have an injective map (which is not a homomorphism) $J_m : Aut\left(\Phi_{n,m}\right) \to M_n\left(\mathbb{Z}_m\left[\mathbb{Z}_m^n\right]\right)$ which satisfies the identity:

$$J_m\left(\alpha \circ \beta\right) = J_m\left(\alpha\right) \cdot \alpha\left(J_m\left(\beta\right)\right)$$

where the action of α on $\mathbb{Z}_m\left[\mathbb{Z}_m^n\right] = \mathbb{Z}_m\left[F_n/K_{n,m}\right]$ is through the natural projection $\Phi_{n,m} \cong F_n/K_{n,m}^m K_{n,m}' \to F_n/K_{n,m} \cong \mathbb{Z}_m^n$.

We denote now $KA\left(\Phi_{n,m}\right) = \ker\left(Aut\left(\Phi_{n,m}\right) \to Aut\left(\Phi_{n,m}/K_{n,m}\right)\right)$. Observe, that as $KA\left(\Phi_{n,m}\right)$ acts trivially on $\Phi_{n,m}/K_{n,m} = \mathbb{Z}_m^n$, the map J_m gives us a *homomorphism*, which is also injective, as mentioned above:

$$J_m : KA\left(\Phi_{n,m}\right) \to GL_n\left(\mathbb{Z}_m\left[\mathbb{Z}_m^n\right]\right).$$

Now, if P is a non-abelian simple group which is involved in $Aut(\hat{\Phi}_3)$, then it must be involved in $Aut\left(\Phi_{3,m}\right)$ for some m. Thus, it must be involved either in $Aut\left(\Phi_{3,m}/K_{3,m}\right) = GL_3\left(\mathbb{Z}_m\right)$ or in $KA\left(\Phi_{3,m}\right) \leq GL_3\left(\mathbb{Z}_m\left[\mathbb{Z}_m^3\right]\right)$. So it must be involved in $GL_3\left(R\right)$ for some finite commutative ring R. As every finite commutative ring is artinian, it can be decomposed as:

$$R = R_1 \times \ldots \times R_l$$

for some local finite rings $R_1, \ldots, R_l$, so:

$$GL_3\left(R\right) = GL_3\left(R_1\right) \times \ldots \times GL_3\left(R_l\right)$$

and thus P must be involved in $GL_3\left(R\right)$ for some local finite commutative ring R. Denote the unique maximal ideal of R by $M \lhd R$. As R is a finite local Noetherian ring, it is well known that $M^r = 0$ for some $r \in \mathbb{N}$.

Note now that if $S, T \lhd R$ for some commutative ring R, and

$$\begin{aligned} I + A &\in \ker\left(GL_3\left(R\right) \to GL_3\left(R/S\right)\right) \\ I + B &\in \ker\left(GL_3\left(R\right) \to GL_3\left(R/T\right)\right) \end{aligned}$$

when I denotes the identity element in $GL_3\left(R\right)$, then

$$\left[I + A, I + B\right] \in \ker\left(GL_3\left(R\right) \to GL_3\left(R/ST\right)\right).$$

Indeed, if $I + C = \left(I + A\right)^{-1}$ and $I + D = \left(I + B\right)^{-1}$ then, as $AB = CD = AD = BC = 0 \pmod{ST}$ we have:

$$\begin{aligned} \left[I + A, I + B\right] &= \left(I + A\right)\left(I + B\right)\left(I + C\right)\left(I + D\right) = \\ &= I + AC + A + BD + B + C + D \pmod{ST} \\ &= I + \left(I + A\right)\left(I + C\right) - I + \left(I + B\right)\left(I + D\right) - I = I \pmod{ST}. \end{aligned}$$

With the above observation we deduce that for every $k \geq 1$, the kernel of the map $GL_3\left(R/M^{k+1}\right) \to GL_3\left(R/M^k\right)$ is abelian. So, P must be involved in $GL_3(R/M) = GL_3\left(p^d\right)$ for some prime p and $d \in \mathbb{N}$. Finally, using the fact that $GL_3\left(p^d\right)/SL_3\left(p^d\right)$ is abelian, we obtain that P is involved in $SL_3\left(p^d\right)$, as required. $\square$

Corollary 4.4. There exists a finite simple group which is not involved in $Aut(\hat{\Phi}_3)$.

Proof. By the proposition above, it is enough to show that there is a finite simple non-abelian group which is not involved in $SL_3\left(p^d\right)$ for any prime p and $d \in \mathbb{N}$. Now, by a theorem of Jordan, there exists an integer-valued function $J(n)$ such that for every field $\mathbb{F}$, $char(\mathbb{F}) = 0$, any finite subgroup of $GL_n(\mathbb{F})$ contains a normal abelian subgroup of index at most $J(n)$. As a corollary of this theorem, Schur proved that the same holds (with the same function) for any finite subgroup $Q \leq GL_n(\mathbb{F})$ with $char(\mathbb{F}) = p > 0$, provided $p \nmid |Q|$ (cf. [26] chapter 9). Clearly, the same holds for any group which is involved in such a finite group Q.

We claim that for n large enough, $Alt(n)$ is not involved in $SL_3\left(p^d\right)$ for any p and d. Indeed, fix two different primes q_1 and q_2 larger than $J(3)$. Then, for n sufficiently large (e.g. $n > q_i^3$) the q_i-sylow subgroup S_i of $Alt(n)$ is non-abelian (since $Alt(n)$ contains the non-abelian q_i-group of order q_i^3) and every subgroup of S_i of index $\leq J(3)$ is equal to S_i, so also non-abelian. If $Alt(n)$ were involved in $SL_3\left(p^d\right)$ then for at least one of the q_i, $q_i \neq p$, a contradiction. $\square$

Corollary 4.5. The congruence kernel $C(\Phi_3)$ contains a copy of $\hat{F}_\omega$.

Proof. The immediate conclusion of Corollary 4.4 is that $Aut(\hat{\Phi}_3)$ does not contain a copy of $\hat{F}_2$. Thus, the intersection of $C(\Phi_3)$ and the copy of $\hat{F}_2$ in $\widehat{Aut(\Phi_3)}$ is not trivial. Thus, $C(\Phi_3)$ contains a non-trivial normal closed subgroup N of $\hat{F}_2$. By Theorem 3.10 in [17] it contains a copy of $\hat{F}_\omega$, as required. $\square$

5. Remarks and open problems

We end this paper with several remarks and open problems. Denote the free solvable group of derived length r on 2 generators by $\Phi_{2,r}$. By combining the results of [9, Theorem 1] and [14, Theorem 1.4] we have:

$$\ker\left(Aut(\Phi_{2,r}) \to Aut\left(\mathbb{Z}^2\right) = GL_2(\mathbb{Z})\right) = Inn(\Phi_{2,r})$$

for every r, i.e. $Out(\Phi_{2,r}) = GL_2(\mathbb{Z})$. So by the same arguments as in §3 we have:

$$C(\Phi_{2,r}) = \ker(\widehat{\langle \alpha, \beta \rangle} \to Out(\hat{\Phi}_{2,r})).$$

As $Out(\hat{\Phi}_{2,r+1})$ is mapped onto $Out(\hat{\Phi}_{2,r})$, we obtain the sequence:

$$C\left(\mathbb{Z}^2\right) = C\left(\Phi_{2,1}\right) \geq C\left(\Phi_2\right) = C\left(\Phi_{2,2}\right) \geq C\left(\Phi_{2,3}\right) \geq$$
$$\geq C\left(\Phi_{2,4}\right) \geq \ldots \geq C\left(\Phi_{2,r}\right) \geq \ldots \geq C\left(F_2\right) = \{e\}$$

and a natural question is whether the inequalities are strict or not. An equivalent reformulation of this question is the following: the cosets of the kernels

$$\ker(GL_2(\mathbb{Z}) = Out\left(\Phi_{2,r}\right) \to Out\left(\Phi_{2,r}/K\right))$$

for characteristic finite index subgroups $K \leq \Phi_{2,r}$ provide a basis for a topology $\mathscr{C}(r)$ on $GL_2(\mathbb{Z})$, called the congruence topology with respect to $\Phi_{2,r}$, which is weaker (equal) than the profinite topology $\mathscr{F}$ of $GL_2(\mathbb{Z})$, and stronger (equal) than the classical congruence topology of $GL_2(\mathbb{Z})$. The latter is equal to $\mathscr{C}(1)$. So, the question above is equivalent to the question whether these topologies are strictly weaker than $\mathscr{F}$, and whether the topology $\mathscr{C}(r)$, for a given r, is strictly weaker than $\mathscr{C}(r+1)$.

For example, Theorem 1.1 which states that $C(F_2) = \{e\}$ is equivalent to the statement that the congruence topology which $Out(\hat{F}_2)$ induces on $Out(F_2) = GL_2(\mathbb{Z})$ is equal to the profinite topology of $GL_2(\mathbb{Z})$.

Considering Theorem 1.2 we deduce that $\mathscr{C}(2) \subsetneqq \mathscr{F}$, but with the proof we gave here one can not decide whether $\mathscr{C}(1) = \mathscr{C}(2)$ or $\mathscr{C}(1) \subsetneqq \mathscr{C}(2)$. Equivalently, we can not decide whether $C\left(\mathbb{Z}^2\right) = C\left(\Phi_2\right)$ or $C\left(\mathbb{Z}^2\right) \supsetneqq C\left(\Phi_2\right)$. But, in [4] it was shown quite surprisingly, that:

Theorem 5.1. $\mathscr{C}(1) = \mathscr{C}(2)$, *or equivalently* $C\left(\mathbb{Z}^2\right) = C\left(\Phi_2\right)$.

The proof in [4] suggested to conjecture that $\mathscr{C}(1) = \mathscr{C}(2) = \mathscr{C}(r)$ for every r. But, the explicit construction of a congruence subgroup we gave in §2 gives a counterexample:

Proposition 5.2. $\mathscr{C}(1) \subsetneqq \mathscr{C}(r)$ *for every* $r \geq 3$*. Equivalently* $C\left(\mathbb{Z}^2\right) \supsetneqq C\left(\Phi_{2,r}\right)$ *for every* $r \geq 3$.

Proof. Denote $G = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \leq GL_2(\mathbb{Z})$. Then by a theorem of Reiner [24], for every $p \neq 2$, $G'G^p$ is not a congruence subgroup of $GL_2(\mathbb{Z})$ in the classical manner, i.e. $G'G^p \notin \mathscr{C}(1)$. On the other hand, applying the explicit construction given in Theorem 2.9, we obtain a finite index normal subgroup $M \lhd F_2$ such that F_2/M is of solvability length 3 such that[1]:

$$\ker\left(Out\left(F_2\right) = GL_2(\mathbb{Z}) \to Out\left(F_2/M\right)\right) \leq G'G^p.$$

[1] We remark that if one wants M to be characteristic, all we need to do, is to replace M by $\bigcap_{\sigma \in Aut(F_2)} \sigma(M)$, and this procedure does not change the solvability length of F_2/M.

This shows that $G'G^p$ is a congruence subgroup of $GL_2(\mathbb{Z})$ with respect to the congruence topology induced by $Out(\hat{\Phi}_{2,3})$. Equivalently, $\mathscr{C}(1) \subsetneq \mathscr{C}(3)$ or $C(\mathbb{Z}^2) \gneqq C(\Phi_{2,3})$, as required. $\square$

The proposition suggests the following conjecture:

Conjecture 5.3. $C(\Phi_{2,r}) \gneqq C(\Phi_{2,r+1})$ for every $r \geq 2$, or equivalently $\mathscr{C}(r) \subsetneq \mathscr{C}(r+1)$. In particular, $C(\Phi_{2,r}) \neq \{e\} = C(F_2)$ and $\mathscr{C}(r) \neq \mathscr{F}$ for every r.

We should remark that we do not even know to decide whether $C(\Phi_{2,r}) \neq \{e\}$ for $r \geq 3$, i.e. we do not know if the congruence subgroup property holds for $\Phi_{2,r}$ for $r \geq 3$ or not. Note that our proofs of Theorems 1.2 and 1.3 claiming that $\Phi = \Phi_2 = \Phi_{2,2}$ and $\Phi = \Phi_3$ do not satisfy the CSP were based on two facts:

1. $Aut(\Phi)$ is large, and hence every finite group is involved in $\widehat{Aut(\Phi)}$, and
2. not every finite group is involved in $Aut(\hat{\Phi})$.

Now, for $\Phi = \Phi_{d,r}$, the free solvable group on $d \geq 2$ generators and solvability length r, part 2 is valid for $1 \leq r \leq 2$ and every d (with the same proof as for $d = 3$ in §4). But, as $C(\Phi_{d,1}) = \{e\}$ for every $d \geq 3$, and $C(\Phi_{d,2})$ is abelian for every $d \geq 4$ (cf. [5]), part 1 is not valid in these cases. On the other hand, for $\Phi = \Phi_{2,r}$ or $\Phi = \Phi_{3,r}$, part 1 is still true for every $r \geq 2$ but not part 2. In fact, we have:

Proposition 5.4. *Let $\Phi_{d,r}$ be the free solvable group on $d \geq 2$ generators and solvability length r. Then if $r \geq 3$, then every finite group H is involved in $Aut(\hat{\Phi}_{d,r})$.*

Proof. By the same arguments of [16, 5.2], it can be deduced from Gaschutz's Lemma that for every surjective homomorphism $\pi : \hat{\Phi}_{d,r} \to \Gamma$ where Γ is finite, the homomorphism

$$Aut(\hat{\Phi}_{d,r}) \geq \left\{ \sigma \in Aut(\hat{\Phi}_{d,r}) \,|\, \sigma(\ker \pi) = \ker \pi \right\} \to Aut(\Gamma)$$

is surjective. Thus, for proving our proposition it suffices to show that $\hat{\Phi}_{d,r}$ has a finite quotient Γ such that H is involved in $Aut(\Gamma)$. Now, by Cayley's Theorem, H is a subgroup of $Sym(n-1)$ for some n and the later is a subgroup of $SL_n(p)$ for every prime p. Thus, the next lemma due to Robert Guralnick, finishes the proof of the proposition. $\square$

Lemma 5.5. *For every $n \geq 2$, there exists a prime p and a finite group Γ generated by two elements and of solvability length three, such that $SL_n(p)$ is involved in $Aut(\Gamma)$.*

Proof. Fix a prime r such that $r > n + 1$. Using Dirichlet's Theorem, pick a prime p such that r divides $p - 1$. Consider now the general affine group:

$$\Delta = AGL_1(r) = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a \in \mathbb{F}_r^*, \, b \in \mathbb{F}_r \right\} = \mathbb{F}_r \rtimes \mathbb{F}_r^*.$$

Then Δ is of order $r(r-1)$. In addition, as $r|(p-1)$, $\mathbb{F}_p$ contains the unit roots of order r, fix one of them $\xi \neq 1$, and consider the diagonal matrix:

$$D = \begin{pmatrix} \xi & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi^{r-1} \end{pmatrix} \in GL_{r-1}(p).$$

Now, we can embed Δ in $GL_{r-1}(p)$ by sending an element $b \in \{0, \ldots, r-1\} = \mathbb{F}_r$ to the diagonal matrix D^b (giving rise to a subgroup $N = \left\{ D^b \mid b \in \mathbb{F}_r \right\}$) and an element $a \in \mathbb{F}_r^*$ to the permutation matrix which normalizes N, sending D^b to D^{ba}. So Δ has a module V of dimension $r-1$ over $\mathbb{F}_p$. Now, every Δ-submodule of V is also N-submodule. The N-submodules are direct sums of different one dimensional N-modules, the eigen-spaces of D^1, on which $\mathbb{F}_r^*$ acts transitively. We deduce that V is an irreducible module.

Denote now $W = \oplus_{i=1}^{r-2} V$ and using the obvious action of Δ on W, define: $\Gamma = W \rtimes \Delta$. We claim that Γ is generated by two elements. By the description above, it is clear why Δ is generated by two element, one of them is $D \in \mathbb{F}_r$ and we denote the other one by $S \in \mathbb{F}_r^*$. Let us now define

$$D' = ((\vec{e}_1, \ldots, \vec{e}_{r-2}), D), \; S' = \left((\vec{0}, \ldots, \vec{0}), S\right) \in W \rtimes \Delta$$

where $\{\vec{e}_1, \ldots, \vec{e}_{r-1}\}$ is the standard basis of V. For a $1 \leq j \leq r-1$ denote $\eta = \xi^j$. Note, that for every $1 \leq k \leq r-2$, $1 + \eta + \ldots + \eta^k = \frac{1-\eta^{k+1}}{1-\eta} \neq 0$. It follows that $D'^k = \left((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2}\vec{e}_{r-2}), D^k\right)$ where $0 \neq \alpha_i \in \mathbb{F}_p$ for every $1 \leq k \leq r-2$. Now, there is a power S^l of S, $1 \leq l \leq r-2$, which sends $\vec{e}_{r-1}$ to $\vec{e}_1$. We have also $S^l D S^{-l} = D^{r-k}$ for some $1 \leq k \leq r-2$. Thus, for some $0 \neq \alpha_i \in \mathbb{F}_p$, we can write:

$$\begin{aligned} w &= S'^l D' S'^{-l} D'^k \\ &= ((\vec{0}, \ldots, \vec{0}), S^l)((\vec{e}_1, \ldots, \vec{e}_{r-2}), D)((\vec{0}, \ldots, \vec{0}), S^{-l})((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2}\vec{e}_{r-2}), D^k) \\ &= ((S^l(\vec{e}_1), \ldots, S^l(\vec{e}_{r-2})), S^l D S^{-l})((\alpha_1 \vec{e}_1, \ldots, \alpha_{r-2}\vec{e}_{r-2}), D^k) \\ &= (S^l(\vec{e}_1) + D^{r-k}(\alpha_1 \vec{e}_1), \ldots, S^l(\vec{e}_{r-2}) + D^{r-k}(\alpha_{r-2}\vec{e}_{r-2}), I) \in W. \end{aligned}$$

Now, as S^l sends $\vec{e}_{r-1}$ to $\vec{e}_1$, $\vec{e}_1$ does not appear in any entry of w except the first one.

Observe now, that the diagonals of $D^0, \ldots, D^{r-2}$, considered as column vectors of $V = \mathbb{F}_p^{r-1}$, form a basis for V as the matrix:

$$\begin{pmatrix} 1 & \xi & \cdots & \xi^{r-2} \\ 1 & \xi^2 & \cdots & \xi^{2(r-2)} \\ \vdots & \vdots & & \vdots \\ 1 & \xi^{r-1} & \cdots & \xi^{(r-1)(r-2)} \end{pmatrix}$$

is a Vandermonde matrix, and therefore invertible. Thus, there is a linear combination

$$C = \beta_0 D^0 + \ldots + \beta_{r-2} D^{r-2} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \beta_i \in \mathbb{F}_p.$$

Now, observe that D' acts on W by conjugation via the action of D on V. Thus, we obtain an action of C on W via its action on V, in which $C(w)$ has $\vec{0}$ in every entry except the first one. This shows, as V is irreducible, that the first copy of V in W is inside the group generated by D' and S'. In a similar way, all the $r-2$ copies of V are generated by D' and S', so Γ is generated by two elements.

Now, $\Delta \times SL_{r-2}(p)$ acts on $W = \oplus_{i=1}^{r-2} V = V \otimes \mathbb{F}_p^{r-2}$ in an obvious way. Thus $\Gamma = W \rtimes \Delta$ is normal in $W \rtimes (\Delta \times SL_{r-2}(p))$, so $SL_{r-2}(p)$ is involved in $Aut(\Gamma)$, and so as $SL_n(p)$. $\square$

Let us remark that while we do not know the answer to the congruence subgroup problem for free solvable groups on two generators and solvability rank r (unless $r = 1$ or 2), the situation with free nilpotent groups on two generators is easier:

Proposition 5.6. *For every free nilpotent group on two generators Γ, the congruence kernel contains a copy of $\hat{F}_\omega$ – the free profinite group on countable number of generators.*

Proof. It is known that if $\hat{\Gamma}$ is a pro-nilpotent group, then the kernel of the map $Aut(\hat{\Gamma}) \to Aut(\hat{\Gamma}/\overline{\hat{\Gamma}'})$ is pro-nilpotent (cf. [16], 5.3). Thus, if Γ is a free nilpotent group (of arbitrary class) then by similar arguments as we brought previously, there exists a finite group which is not involved in $Aut(\hat{\Gamma})$. On the other hand, if Γ is free nilpotent group on two generators, then $Aut(\Gamma)$ is large, as it can be mapped onto $GL_2(\mathbb{Z})$.[2] Thus, $\hat{F}_2$ is a subgroup of $\widehat{Aut(\Gamma)}$ and $C(\Gamma) \cap \hat{F}_2$ is non-trivial, hence contains a copy of $\hat{F}_\omega$ (cf. [17]). $\square$

Our last remark is about the CSP for subgroups of automorphism groups. Considering the classical congruence subgroup problem, one can take G to be a subgroup of $GL_n(R)$ where R is a commutative ring, and ask whether every finite index subgroup of G contains a subgroup of the form $\ker(G \to GL_n(R/I))$ for some finite index ideal $I \lhd R$. This direction of generalization of the classical CSP has been studied intensively during the second half of the 20th century (cf. [22,23]). One can ask for a parallel generalization for automorphism groups or outer atomorphism groups. I.e. let $G \leq Aut(\Gamma)$ (resp. $G \leq Out(\Gamma)$), does every finite index subgroup of G contain a principal congruence

[2] In general, the kernel of the map $Aut(\Gamma) \to GL_2(\mathbb{Z})$ strictly contains $Inn(\Gamma)$ (cf. [10,1]).

subgroup of the form $\ker(G \to Aut(\Gamma/M))$ (resp. $\ker(G \to Out(\Gamma/M))$) for some finite index characteristic subgroup $M \leq \Gamma$?

Now, let $\pi_{g,n}$ be the fundamental group of $S_{g,n}$, the surface of genus g with n punctures, such that $\chi(S_{g,n}) = 2-2g-n \leq 0$. Then, there is an injective map of $PMod(S_{g,n})$, the pure mapping class group, into $Out(\pi_{g,n})$ (cf. [12], chapter 8). Thus, one can ask the CSP for $PMod(S_{g,n})$ as a subgroup of $Out(\pi_{g,n})$. Considering the above problem, it is known that:

Theorem 5.7. *For $g = 0, 1, 2$ and every $n > 0$, $PMod(S_{g,n})$ has the CSP.*

The cases for $g = 0$ were proved by [11] and in [19], the cases for $g = 1$ were proved by [2], and the cases for $g = 2$ where proved by [7]. It can be shown that for every $n > 0$, $\pi_{g,n} \cong F_{2g+n-1}$ = the free group on $2g + n - 1$ generators. Thus, the above cases give an affirmative answer for various subgroups of the outer aoutomorphism group of finitely generated free groups. Though, the CSP for the full $Out(F_d)$ where $d \geq 3$ is still unsettled.

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