A MINIMAJ-PRESERVING CRYSTAL ON ORDERED MULTISET PARTITIONS

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ABSTRACT. We provide a crystal structure on the set of ordered multiset partitions, which recently arose in the pursuit of the Delta Conjecture. This conjecture was stated by Haglund, Remmel and Wilson as a generalization of the Shuffle Conjecture. Various statistics on ordered multiset partitions arise in the combinatorial analysis of the Delta Conjecture, one of them being the minimaj statistic, which is a variant of the major index statistic on words. Our crystal has the property that the minimaj statistic is constant on connected components of the crystal. In particular, this yields another proof of the Schur positivity of the graded Frobenius series of the generalization $R_{n,k}$ due to Haglund, Rhoades and Shimozono of the coinvariant algebra R_n. The crystal structure also enables us to demonstrate the equidistributivity of the minimaj statistic with the major index statistic on ordered multiset partitions.

1. INTRODUCTION

The Shuffle Conjecture [HHL+05], now a theorem due to Carlsson and Mellit [CM15], provides an explicit combinatorial description of the bigraded Frobenius characteristic of the S_n-module of diagonal harmonic polynomials. It is stated in terms of parking functions and involves two statistics, area and dinv.

Recently, Haglund, Remmel and Wilson [HRW15] introduced a generalization of the Shuffle Theorem, coined the Delta Conjecture. The Delta Conjecture involves two quasisymmetric functions $\mathsf{Rise}_{n,k}(\mathbf{x};q,t)$ and $\mathsf{Val}_{n,k}(\mathbf{x};q,t)$, which have combinatorial expressions in terms of labelled Dyck paths. In this paper, we are only concerned with the specializations $q=0$ or $t=0$, in which case [HRW15, Theorem 4.1] and [Rho16, Theorem 1.3] show

$$\mathsf{Rise}_{n,k}(\mathbf{x};0,t)=\mathsf{Rise}_{n,k}(\mathbf{x};t,0)=\mathsf{Val}_{n,k}(\mathbf{x};0,t)=\mathsf{Val}_{n,k}(\mathbf{x};t,0).$$

It was proven in [HRW15, Proposition 4.1] that

$$\mathsf{Val}_{n,k}(\mathbf{x};0,t)=\sum_{\pi\in\mathcal{OP}_{n,k+1}} t^{\mathsf{minimaj}(\pi)}\mathbf{x}^{\mathsf{wt}(\pi)}, \tag{1.1}$$

where $\mathcal{OP}_{n,k+1}$ is the set of ordered multiset partitions of the multiset $\{1^{\nu_1},2^{\nu_2},\ldots\}$ into $k+1$ nonempty blocks, where $\nu=(\nu_1,\nu_2,\ldots)$ is a weak composition of n. In addition, $\mathsf{minimaj}(\pi)$ is the minimum value of the major index of the set partition π over all possible ways to order the elements in each block of π, and $\mathsf{wt}(\pi)$ is the weight of π. The symmetric

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function $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t)$ is known [Wil16, Rho16] to be Schur positive, meaning that the coefficients are polynomials in t with nonnegative coefficients.

In this paper, we provide a crystal structure on the set of ordered multiset partitions $\mathcal{OP}_{n,k}$. Crystal bases are $q \to 0$ shadows of representations for quantum groups $U_q(\mathfrak{g})$ [Kas90, Kas91], though they can also be understood from a purely combinatorial perspective [Ste03, BS17]. In type A, the character of a connected crystal component with highest weight element of highest weight λ is the Schur function s_λ. Hence, having a type A crystal structure on a combinatorial set (in our case on $\mathcal{OP}_{n,k}$) naturally yields the Schur expansion of the associated symmetric function. Furthermore, if the statistic (in our case $\mathsf{minimaj}$) is constant on connected components, then the graded character can also be naturally computed using the crystal.

Haglund, Rhoades and Shimozono [HRS16] introduced a generalization $R_{n,k}$ for $k \leqslant n$ of the coinvariant algebra R_n, with $R_{n,n} = R_n$. Just as the combinatorics of R_n is governed by permutations in S_n, the combinatorics of $R_{n,k}$ is controlled by ordered set partitions of $\{1, 2 \ldots, n\}$ with k blocks. The graded Frobenius series of $R_{n,k}$ is (up to a minor twist) equal to $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t)$. It is still an open problem to find a bigraded S_n-module whose Frobenius image is $\mathsf{Val}_{n,k}(\mathbf{x}; q, t)$. Our crystal provides another representation-theoretic interpretation of $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t)$ as a crystal character.

Wilson [Wil16] analyzed various statistics on ordered multiset partitions, including inv, dinv, maj, and $\mathsf{minimaj}$. In particular, he gave a Carlitz type bijection, which proves equidistributivity of inv, dinv, maj on $\mathcal{OP}_{n,k}$. Rhoades [Rho16] provided a non-bijective proof that these statistics are also equidistributed with $\mathsf{minimaj}$. Using our new crystal, we can give a combinatorial proof of the equidistributivity of the $\mathsf{minimaj}$ statistic and the maj statistic on ordered multiset partitions.

The paper is organized as follows. In Section 2 we define ordered multiset partitions and the $\mathsf{minimaj}$ and maj statistics on them. In Section 3 we provide a bijection φ from ordered multiset partitions to tuples of semistandard Young tableaux that will be used in Section 4 to define a crystal structure, which preserves $\mathsf{minimaj}$. We conclude in Section 5 with a proof that the $\mathsf{minimaj}$ and maj statistics are equidistributed using the same bijection φ.

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2. Ordered multiset partitions and the minimaj and maj statistics

We consider *ordered multiset partitions* of order n with k blocks. Given a weak composition $\nu = (\nu_1, \nu_2, \ldots) \models n$ of n into nonnegative integer parts, which we denote $\nu \models n$, let $\mathcal{OP}_{\nu,k}$

be the set of partitions of the multiset $\{i^{\nu_i} \mid i \geqslant 1\}$ into k nonempty ordered blocks, such that the elements within each block are distinct. For each $i \geqslant 1$, the notation i^{ν_i} should be interpreted as saying that the integer i occurs ν_i times in such a partition. The weak composition ν is also called the *weight* $\mathsf{wt}(\pi)$ of $\pi \in \mathcal{OP}_{\nu,k}$. Let

$$\mathcal{OP}_{n,k} = \bigcup_{\nu \models n} \mathcal{OP}_{\nu,k}.$$

We now specify a particular reading order for an ordered multiset partition $\pi = (\pi_1 \mid \pi_2 \mid \ldots \mid \pi_k) \in \mathcal{OP}_{n,k}$ with blocks π_i. Start by writing π_k in increasing order. Assume π_{i+1} has been ordered, and let r_i be the largest integer in π_i that is less than or equal to the leftmost element of π_{i+1}. If no such r_i exists, arrange π_i in increasing order. When such an r_i exists, arrange the elements of π_i in increasing order, and then cycle them so that r_i is the rightmost number. Continue with $\pi_{i-1}, \ldots, \pi_2, \pi_1$ until all blocks have been ordered. This ordering of the numbers in π is defined in [HRW15] and is called the *minimaj order*.

Example 2.1. If $\pi = (157 \mid 24 \mid 56 \mid 468 \mid 13 \mid 123) \in \mathcal{OP}_{15,6}$, then the minimaj order of π is $\pi = (571 \mid 24 \mid 56 \mid 468 \mid 31 \mid 123)$.

For two sequences α, β of integers, we write $\alpha < \beta$ to mean that each element of α is less than every element of β. Suppose $\pi \in \mathcal{OP}_{n,k}$ is in minimaj order. Then each block π_i of π is nonempty and can be written in the form $\pi_i = b_i \alpha_i \beta_i$, where $b_i \in \mathbb{Z}_{>0}$, and α_i, β_i are sequences (possibly empty) of distinct increasing integers such that either $\beta_i < b_i < \alpha_i$ or $\alpha_i = \emptyset$. Inequalities with empty sets should be ignored.

Lemma 2.2. *With the above notation, $\pi \in \mathcal{OP}_{n,k}$ is in minimaj order if the following hold:*

(1) $\pi_k = b_k \alpha_k$ *with* $b_k < \alpha_k$ *and* $\beta_k = \emptyset$;
(2) *for* $1 \leqslant i < k$, *either*
   (a) $\alpha_i = \emptyset$, $\pi_i = b_i \beta_i$, *and* $b_i < \beta_i \leqslant b_{i+1}$, *or*
   (b) $\beta_i \leqslant b_{i+1} < b_i < \alpha_i$.

A sequence or word $w_1 w_2 \cdots w_n$ has a *descent* in position $1 \leqslant i < n$ if $w_i > w_{i+1}$. Let $\pi \in \mathcal{OP}_{n,k}$ be in minimaj order. Observe that a descent occurs in π_i only in Case 2 (b) of Lemma 2.2, and such a descent is either between the largest and smallest elements of π_i or between the last element of π_i and the first element of π_{i+1}.

Example 2.3. Continuing Example 2.1 with $\pi = (571 \mid 24 \mid 56 \mid 468 \mid 31 \mid 123)$, we have

$$\begin{array}{lll} b_1 = 5, \alpha_1 = 7, \beta_1 = 1 & b_2 = 2, \alpha_2 = \emptyset, \beta_2 = 4 & b_3 = 5, \alpha_3 = 6, \beta_3 = \emptyset \\ b_4 = 4, \alpha_4 = 68, \beta_4 = \emptyset & b_5 = 3, \alpha_5 = \emptyset, \beta_5 = 1 & b_6 = 1, \alpha_6 = 23, \beta_6 = \emptyset. \end{array}$$

Suppose that π in minimaj order has descents in positions

$$\mathsf{D}(\pi) = \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_\ell\}$$

for some $\ell \in [0, k-1]$ ($\ell = 0$ indicates no descents). Furthermore assume that these descents occur in the blocks $\pi_{i_1}, \pi_{i_1+i_2}, \ldots, \pi_{i_1+i_2+\cdots+i_\ell}$, where $i_j > 0$ for $1 \leqslant j \leqslant \ell$ and $i_1 + i_2 + \cdots + i_\ell < k$. Assume $d_{\ell+1}$ and $i_{\ell+1}$ are the distances to the end, that is, $d_1 + d_2 + \cdots + d_\ell + d_{\ell+1} = n$ and $i_1 + i_2 + \cdots + i_\ell + i_{\ell+1} = k$.

The *minimaj statistic* $\mathsf{minimaj}(\pi)$ of $\pi \in \mathcal{OP}_{n,k}$ as given by [HRW15] is

$$\mathsf{minimaj}(\pi) = \sum_{d \in \mathsf{D}(\pi)} d = \sum_{j=1}^{\ell} (\ell + 1 - j) d_j. \tag{2.1}$$

Example 2.4. The descents for the multiset partition $\pi = (57.1 \mid 24 \mid 56. \mid 468. \mid 3.1 \mid 123)$ occur at positions 2,7,10,11 and are designated with periods. Hence $\ell = 4$, $d_1 = 2$, $d_2 = 5$, $d_3 = 3$, $d_4 = 1$ and $d_5 = 4$, and $\mathsf{minimaj}(\pi) = 2 + 7 + 10 + 11 = 30$. The descents occur in blocks π_1, π_3, π_4, and π_5, so that $i_1 = 1$, $i_2 = 2$, $i_3 = 1$, $i_4 = 1$, and $i_5 = 1$.

To define the *major index* of $\pi \in \mathcal{OP}_{n,k}$, we consider the word w obtained by ordering each block π_i in decreasing order, called the *major index order* [Wil16]. Recursively construct a word v by setting $v_0 = 0$ and $v_j = v_{j-1} + \chi(j \text{ is the last position in its block})$ for each $1 \leqslant j \leqslant n$. Here $\chi(\text{True}) = 1$ and $\chi(\text{False}) = 0$. Then

$$\mathsf{maj}(\pi) = \sum_{j\colon w_j > w_{j+1}} v_j. \tag{2.2}$$

Example 2.5. Continuing Example 2.1, note that the major index order of $\pi = (157 \mid 24 \mid 56 \mid 468 \mid 13 \mid 123) \in \mathcal{OP}_{15,6}$ is $\pi = (751 \mid 42 \mid 65 \mid 864 \mid 31 \mid 321)$. Writing the word v underneath w (omitting $v_0 = 0$), we obtain

$$\begin{aligned} w &= 751 \mid 42 \mid 65 \mid 864 \mid 31 \mid 321 \\ v &= 001 \mid 12 \mid 23 \mid 334 \mid 45 \mid 556, \end{aligned}$$

so that $\mathsf{maj}(\pi) = 0 + 0 + 1 + 2 + 3 + 3 + 4 + 4 + 5 + 5 = 27$.

Note that throughout this section, we could have also restricted ourselves to ordered multiset partitions with letters in $\{1, 2, \ldots, r\}$ instead of $\mathbb{Z}_{>0}$. That is, let $\nu = (\nu_1, \ldots, \nu_r)$ be a weak composition of n and let $\mathcal{OP}^{(r)}_{\nu,k}$ be the set of partitions of the multiset $\{i^{\nu_i} \mid 1 \leqslant i \leqslant r\}$ into k nonempty ordered blocks, such that the elements within each block are distinct. Let

$$\mathcal{OP}^{(r)}_{n,k} = \bigcup_{\nu \models n} \mathcal{OP}^{(r)}_{\nu,k}.$$

This restriction will be important when we discuss the crystal structure on ordered multiset partitions.

3. Bijection with tuples of semistandard Young tableaux

In this section, we describe a bijection from ordered multiset partitions to tuples of semistandard Young tableaux that allows us to impose a crystal structure on the set of ordered multiset partitions in Section 4.

Recall that a *semistandard Young tableau* T is a filling of a (skew) Young diagram (also called the *shape* of T) with positive integers that weakly increase across rows and strictly increase down columns. The *weight* of T is the tuple $\mathsf{wt}(T) = (a_1, a_2, \ldots)$, where a_i records the number of letters i in T. The set of semistandard Young tableaux of shape λ, where λ is a (skew) partition, is denoted by $\mathsf{SSYT}(\lambda)$. If we want to restrict the entries in the semistandard Young tableau from $\mathbb{Z}_{>0}$ to a finite alphabet $\{1, 2, \ldots, r\}$, we denote the set by $\mathsf{SSYT}^{(r)}(\lambda)$.

The tableaux relevant for us here are of two types: a single column of boxes with entries that increase from top to bottom, or a skew ribbon tableau. If $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a skew ribbon shape with γ_j boxes in the j-th row starting from the bottom, the ribbon condition requires that row $j + 1$ starts in the last column of row j. This condition is equivalent to saying that γ is connected and contains no 2×2 block of squares. Let $\mathsf{SSYT}(1^c)$ be the set of semistandard Young tableaux obtained by filling a column of length c and $\mathsf{SSYT}(\gamma)$ be the set of semistandard Young tableaux obtained by filling the skew ribbon shape γ.

To state our bijection, we need the following notation. For fixed positive integers n and k, assume $\mathrm{D} = \{d_1, d_1+d_2, \ldots, d_1+d_2+\cdots+d_\ell\} \subseteq \{1,2,\ldots,n-1\}$ and $\mathrm{I} = \{i_1, i_1+i_2, \ldots, i_1+i_2+\cdots+i_\ell\} \subseteq \{1,2,\ldots,k-1\}$ are sets of ℓ distinct elements each. Define $d_{\ell+1} := n-(d_1+\cdots+d_\ell)$ and $i_{\ell+1} := k-(i_1+\cdots+i_\ell)$.

Proposition 3.1. *For fixed positive integers n and k and sets* D *and* I *as above, let*

$$\mathrm{M}(\mathrm{D},\mathrm{I}) = \{\pi \in \mathcal{OP}_{n,k} \mid \mathsf{D}(\pi) = \mathrm{D}, \text{ and the descents occur in } \pi_i \text{ for } i \in \mathrm{I}\}.$$

Then the following map is a weight-preserving bijection:

$$\begin{aligned} \varphi\colon \mathrm{M}(\mathrm{D},\mathrm{I}) &\to \mathsf{SSYT}(1^{c_1}) \times \cdots \times \mathsf{SSYT}(1^{c_\ell}) \times \mathsf{SSYT}(\gamma) \\ \pi &\mapsto T_1 \times \cdots \times T_\ell \times T_{\ell+1} \end{aligned} \tag{3.1}$$

where

(i) $\gamma = \{1^{d_1-i_1}, i_1, i_2, \ldots, i_{\ell+1}\}$ *and* $c_j = d_{\ell+2-j} - i_{\ell+2-j}$ *for* $1 \leqslant j \leqslant \ell$.

(ii) *The skew ribbon tableau $T_{\ell+1}$ of shape γ is constructed as follows:*
- *The entries in the first column of the skew ribbon tableau $T_{\ell+1}$ beneath the first box are the first $d_1 - i_1$ elements of π in increasing order from top to bottom, excluding any b_j in that range.*
- *The remaining rows $d_1 - i_1 + j$ of $T_{\ell+1}$ for $1 \leqslant j \leqslant \ell+1$ are filled with $b_{i_1+\cdots+i_{j-1}+1}, b_{i_1+\cdots+i_{j-1}+2}, \ldots, b_{i_1+\cdots+i_j}$.*

(iii) *The tableau T_j for $1 \leqslant j \leqslant \ell$ is the column filled with the elements of π from the positions $d_1+d_2+\cdots+d_{\ell-j+1}+1$ through and including position $d_1+d_2+\cdots+d_{\ell-j+2}$, but excluding any b_i in that range.*

Note that in item (ii), the rows of γ are assumed to be numbered from bottom to top and are filled starting with row $d_1 - i_1 + 1$ and ending with row $d_1 - i_1 + \ell + 1$ at the top.

Also observe that since the bijection stated in Proposition 3.1 preserves the weight, it can be restricted to a bijection

$$\varphi\colon \mathrm{M}(\mathrm{D},\mathrm{I})^{(r)} \to \mathsf{SSYT}^{(r)}(1^{c_1}) \times \cdots \times \mathsf{SSYT}^{(r)}(1^{c_\ell}) \times \mathsf{SSYT}^{(r)}(\gamma),$$

where $\mathrm{M}(\mathrm{D},\mathrm{I})^{(r)} = \mathrm{M}(\mathrm{D},\mathrm{I}) \cap \mathcal{OP}^{(r)}_{n,k}$.

Before giving the proof, it is helpful to consider two examples to illustrate the map φ.

Example 3.2. When the entries of $\pi \in \mathcal{OP}_{n,k}$ in minimaj order are increasing, then $\ell = 0$. In this case, $d_1 = n$ and $i_1 = k$. The mapping φ takes π to the semistandard tableau $T = T_1$ that is of ribbon-shape $\gamma = (1^{n-k}, k)$. The entries of the boxes in the first column of the tableau T are b_1, followed by the $n-k$ numbers in the sequences $\beta_1, \beta_2, \ldots, \beta_{k-1}, \alpha_k$ from top to bottom. (The fact that π has no descents means that all the $\alpha_i = \emptyset$ for $1 \leqslant i < k$ and we are in Case 2 (a) of Lemma 2.2 for $1 \leqslant i < k$ and Case 1 for $i = k$.) Columns 2 through k of T_1 are filled with the numbers $b_2, \ldots, b_k$ respectively, and $b_2 \leqslant b_3 \leqslant \cdots \leqslant b_k$. The result is a semistandard tableau T_1 of hook shape.

Now suppose that T is such a hook-shape tableau with entries $b_1, b_2, \ldots, b_k$ from left to right in its top row, and entries $b_1, t_1, \ldots, t_{n-k}$ down its first column. The inverse φ^{-1} maps T to the set partition π that has as its first block $\pi_1 = b_1\beta_1$, where $\beta_1 = t_1, \ldots, t_{m_1}$, and $t_1 < \cdots < t_{m_1} \leqslant b_2$, but $t_{m_1+1} > b_2$ so that β_1 is in the interval $(b_1, b_2]$. The second block of π is given by $\pi_2 = b_2\beta_2$, where $\beta_2 = t_{m_1+1}, \ldots, t_{m_2}$, and $t_{m_1+1} < t_{m_1+2} < \cdots < t_{m_2} \leqslant b_3$, but $t_{m_2+1} > b_3$ and $\beta_2 \subseteq (b_2, b_3]$. Continuing in this fashion, we set $\pi_k = b_k\alpha_k$, where $\alpha_k = t_{m_{k-1}+1}, \ldots, t_{n-k}$ and $\alpha_k \subseteq (b_k, +\infty)$. Then $\varphi^{-1}(T) = \pi = (\pi_1 \mid \pi_2 \mid \cdots \mid \pi_k)$, where the ordered multiset partition π has no descents.

Example 3.3. The ordered multiset partition $\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \in \mathcal{OP}_{15,7}$ has the following data:

$$\begin{array}{lll} b_1 = 1, \alpha_1 = \emptyset, \beta_1 = 24 & b_2 = 4, \alpha_2 = 5, \beta_2 = \emptyset & b_3 = 3, \alpha_3 = \emptyset, \beta_3 = \emptyset \\ b_4 = 4, \alpha_4 = 6, \beta_4 = 1 & b_5 = 2, \alpha_5 = 3, \beta_5 = 1 & b_6 = 1, \alpha_6 = \emptyset, \beta_6 = \emptyset \\ b_7 = 2, \alpha_7 = 5, \beta_7 = \emptyset & & \end{array}$$

and $\ell = 3$, $d_1 = 5, d_2 = d_3 = 3, d_4 = 4$ and $i_1 = i_2 = 2, i_3 = 1, i_4 = 2$. Then

$$\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \mapsto \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline \end{array} \times \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \times \begin{array}{|c|}\hline 6 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \begin{array}{|c|c|}\hline 2 \\ \hline \end{array} \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array} \begin{array}{|c|}\hline 2 \\ \hline \end{array} \begin{array}{|c|}\hline 4 \\ \hline \end{array} \begin{array}{|c|}\hline 5 \\ \hline \end{array}.$$

It is helpful to keep the following picture in mind during the proof of Proposition 3.1, where the map φ is taking the ordered multiset partition π to the collection of tableaux T_i as illustrated below. We adopt the shorthand notation $\eta_j := i_1 + \cdots + i_j$ for $1 \leqslant j \leqslant \ell$, where we also set $\eta_0 = 0$ and $\eta_{\ell+1} = k$:

$$\pi = (b_1\beta_1 | b_2\beta_2 | \cdots | b_{\eta_1}\alpha_{\eta_1}.\beta_{\eta_1} | b_{\eta_1+1}\beta_{\eta_1+1}| \cdots | b_{\eta_j}\alpha_{\eta_j}.\beta_{\eta_j} | b_{\eta_j+1}\beta_{\eta_j+1}| \cdots | b_k\alpha_k)$$

(3.2)

$$T_{\ell+1-j} = \begin{array}{|c|}\hline \beta_{\eta_j} \\ \hline \beta_{\eta_j+1} \\ \hline \vdots \\ \hline \beta_{\eta_{j+1}-1} \\ \hline \alpha_{\eta_{j+1}} \\ \hline \end{array} \quad \text{for } 1 \leqslant j \leqslant \ell, \qquad T_{\ell+1} = \begin{array}{|c|c|c|c|c|c|c|}\hline & & & & & b_{\eta_\ell+1} & \cdots & b_{\eta_{\ell+1}} \\ \hline & & & & & \vdots & & \\ \hline & & & b_{\eta_{j-1}+1} & \cdots & b_{\eta_j} & & \\ \hline & & & \vdots & & & & \\ \hline b_1 & \cdots & b_{\eta_1} & & & & & \\ \hline \beta_1 & & & & & & & \\ \hline \vdots & & & & & & & \\ \hline \beta_{\eta_1-1} & & & & & & & \\ \hline \alpha_{\eta_1} & & & & & & & \\ \hline \end{array}.$$

Proof of Proposition 3.1. Since the entries of π are mapped bijectively to the entries of $T_1 \times T_2 \times \cdots \times T_{\ell+1}$, the map φ preserves the total weight $\mathsf{wt}(\pi) = (p_1, p_2, \ldots) \mapsto \mathsf{wt}(T)$,

where p_i is the number of entries i in π for $i \in \mathbb{Z}_{>0}$. We need to show that φ is well defined and exhibit its inverse. For this, we can assume that $\ell \geqslant 1$, as the case $\ell = 0$ was treated in Example 3.2.

Observe first that there are d_j entries in π which are between two consecutive descents, and among these entries there are exactly i_j entries that are first elements of a block, since descents happen i_j blocks apart. This implies that the tableaux have the shapes claimed.

To see that the tableaux are semistandard, consider first $T_{\ell+1}$, and let $\eta_j = i_1 + \cdots + i_j$ as above. A row numbered $d_1 - i_1 + j$ for $1 \leqslant j \leqslant \ell+1$ is weakly increasing, because the lack of a descent in a block π_i means $b_i \leqslant b_{i+1}$, and this holds for i in the interval $\eta_{j-1}+1, \ldots, \eta_j$ between two consecutive descents. The leftmost column is strictly increasing because it consists of the elements $b_1 < \beta_1 < \beta_2 < \cdots < \beta_{\eta_1 - 1} < \alpha_{\eta_1}$ (the lack of a descent before π_{η_1} implies that $\alpha_i = \emptyset$ for $i < \eta_1$ and $b_i < \beta_i \leqslant b_{i+1} < \beta_{i+1}$ by Case 2 (a) of Lemma 2.2).

The rest of the columns of $T_{\ell+1}$ contain elements b_i, where $b_{\eta_{j-1}+1}$ is the first element in row $d_1 - i_1 + j$ and b_{η_j} is the last, and b_{η_j+1} is the first element in the row immediately above it. We have $b_{\eta_j} > b_{\eta_j+1}$, since there is a descent in block π_{i_j} which implies this inequality by the ordering condition in Case 2 (b) of Lemma 2.2.

The strict inequalities for the column tableaux $T_1, \ldots, T_\ell$ hold for the same reason that they hold for the first column in $T_{\ell+1}$. That is, the columns consist of the elements $\beta_{\eta_j} < \beta_{\eta_j+1} < \cdots < \beta_{\eta_{j+1}-1} < \alpha_{\eta_{j+1}}$, where all the α_i for $\eta_j \leqslant i < \eta_{j+1}$ are in fact $\emptyset$, since we are in Case 2 (a) of Lemma 2.2 here.

Next, to show that φ is a bijection, we describe the inverse map of φ. For $\mathrm{D} = \{d_1, d_1 + d_2, \ldots, d_1 + d_2 + \cdots + d_\ell\} \subseteq \{1, 2, \ldots, n-1\}$ and $\mathrm{I} = \{i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_\ell\} = \{\eta_1, \eta_2, \ldots, \eta_\ell\} \subseteq \{1, 2, \ldots, k-1\}$ with ℓ distinct elements each, suppose $d_{\ell+1}$ and $i_{\ell+1}$ are such that $d_1 + d_2 + \cdots + d_{\ell+1} = n$ and $\eta_{\ell+1} = i_1 + i_2 + \cdots + i_{\ell+1} = k$. Assume $T_1 \times \cdots \times T_\ell \times T_{\ell+1} \in \mathsf{SSYT}(1^{c_1}) \times \cdots \times \mathsf{SSYT}(1^{c_\ell}) \times \mathsf{SSYT}(\gamma)$, where $\gamma = (1^{d_1 - i_1}, i_2, \ldots, i_{\ell+1})$ and $c_j = d_{\ell+2-j} - i_{\ell+2-j}$ for $1 \leqslant j \leqslant \ell$. We construct π by applying the following algorithm.

Read off the bottom $d_1 - i_1$ entries of the first column of $T_{\ell+1}$. Let b_1 be the element immediately above these entries in the first column of $T_{\ell+1}$, and note that b_1 is less than all of them. Let $b_2, \ldots, b_{i_1}$ be the elements in the same row of $T_{\ell+1}$ as b_1, reading from left to right. Assign $b_{\eta_1+1}, \ldots, b_{\eta_2}$ to the elements in the next higher row, and so forth, until reaching row $d_1 - i_1 + \ell + 1$ (the top row) of $T_{\ell+1}$ and assigning $b_{\eta_\ell+1}, \ldots, b_{\eta_{\ell+1}} = b_k$ to its entries. The elements in $\beta_1, \ldots, \beta_{\eta_1 - 1}, \alpha_{\eta_1}$ are obtained by cutting the entries in the first column of $T_{\ell+1}$ above b_1, so that β_i lies in the interval $(b_i, b_{i+1}]$, and α_{η_1} lies in the interval (b_{η_1}, ∞).

Now for $1 \leqslant j \leqslant \ell$, we obtain $\beta_{\eta_j}, \beta_{\eta_j+1}, \ldots, \beta_{\eta_{j+1}-1}, \alpha_{\eta_{j+1}}$ by cutting the elements in $T_{\ell+1-j}$ into sequences as follows: $\beta_{\eta_j} = T_{\ell+1-j} \cap (-\infty, b_{\eta_j+1}]$, $\beta_{\eta_j+m} = T_{\ell+1-j} \cap (b_{\eta_j+m+1}, b_{\eta_j+m+2}]$ and $\alpha_{\eta_{j+1}} = T_{\ell+1-j} \cap (b_{\eta_{j+1}}, +\infty)$.

The inequalities are naturally forced from the inequalities in the semistandard tableaux, and the descents at the given positions are also forced, because by construction $\alpha_{\eta_j} > b_{\eta_j} > b_{\eta_j+1} \geqslant \beta_{\eta_j}$. This process constructs the b_i, α_i, and β_i for each $i = 1, \ldots, k$, where we assume that sequences that have not been defined by the process are empty. Then $\varphi^{-1}(T_1 \times T_2 \times \cdots \times T_{\ell+1}) = \pi = (\pi_1 \mid \pi_2 \mid \cdots \mid \pi_k)$, where $\pi_i = b_i \alpha_i \beta_i$. □

For a partition λ, the *Schur function* $\mathsf{s}_\lambda(\mathbf{x})$ is defined as

$$\mathsf{s}_\lambda(\mathbf{x}) = \sum_{T \in \mathsf{SSYT}(\lambda)} \mathbf{x}^{\mathsf{wt}(T)}. \tag{3.3}$$

Similarly for $m \geqslant 1$, the *m-th elementary symmetric function* $\mathsf{e}_m(\mathbf{x})$ is given by

$$\mathsf{e}_m(\mathbf{x}) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_m} x_{j_1} x_{j_2} \cdots x_{j_m}.$$

As an immediate consequence of Proposition 3.1, we have the following symmetric function identity.

Corollary 3.4. *Assume* $\mathrm{D} \subseteq \{1, 2, \ldots, n-1\}$ *and* $\mathrm{I} \subseteq \{1, 2, \ldots, k-1\}$ *are sets of* ℓ *distinct elements each and let* $\mathrm{M}(\mathrm{D}, \mathrm{I})$, γ *and* c_j *for* $1 \leqslant j \leqslant \ell$ *be as in Proposition 3.1. Then*

$$\sum_{\pi \in \mathrm{M}(\mathrm{D},\mathrm{I})} \mathbf{x}^{\mathsf{wt}(\pi)} = \mathsf{s}_\gamma(\mathbf{x}) \prod_{j=1}^{\ell} \mathsf{e}_{c_j}(\mathbf{x}).$$

4. Crystal on ordered multiset partitions

4.1. **Crystal structure.** Denote the set of words of length n over the alphabet $\{1, 2, \ldots, r\}$ by $\mathcal{W}_n^{(r)}$. The set $\mathcal{W}_n^{(r)}$ can be endowed with an $\mathfrak{sl}_r$-crystal structure as follows. The weight $\mathsf{wt}(w)$ of $w \in \mathcal{W}_n^{(r)}$ is the tuple $(a_1, \ldots, a_r)$, where a_i is the number of letters i in w. The *Kashiwara raising* and *lowering operators*

$$e_i, f_i \colon \mathcal{W}_n^{(r)} \to \mathcal{W}_n^{(r)} \cup \{0\} \qquad \text{for } 1 \leqslant i < r$$

are defined as follows. Associate to each letter i in w an open bracket ")" and to each letter $i+1$ in w a closed bracket "(". Then e_i changes the $i+1$ associated to the leftmost unmatched "(" to an i; if there is no such letter, $e_i(w) = 0$. Similarly, f_i changes the i associated to the rightmost unmatched ")" to an $i+1$; if there is no such letter, $f_i(w) = 0$.

For λ a (skew) partition, the $\mathfrak{sl}_r$-crystal action on $\mathsf{SSYT}^{(r)}(\lambda)$ is induced by the crystal on $\mathcal{W}_{|\lambda|}^{(r)}$, where $|\lambda|$ is the number of boxes in λ. Consider the row-reading word $\mathsf{row}(T)$ of $T \in \mathsf{SSYT}^{(r)}(\lambda)$, which is the word obtained from T by reading the rows from bottom to top, left to right. Then $f_i(T)$ (resp. $e_i(T)$) is the RSK insertion tableau of $f_i(\mathsf{row}(T))$ (resp. $e_i(\mathsf{row}(T))$). It is well known that $f_i(T)$ is a tableau in $\mathsf{SSYT}^{(r)}(\lambda)$ with weight equal to $\mathsf{wt}(T) - \epsilon_i + \epsilon_{i+1}$, where ϵ_i is i-th standard vector in $\mathbb{Z}^r$. Similarly, $e_i(T) \in \mathsf{SSYT}^{(r)}(\lambda)$, and $e_i(T)$ has weight $\mathsf{wt}(T) + \epsilon_i - \epsilon_{i+1}$. See for example [BS17, Chapter 3].

In the same spirit, an $\mathfrak{sl}_r$-crystal structure can be imposed on

$$\mathsf{SSYT}^{(r)}(1^{c_1}, \ldots, 1^{c_\ell}, \gamma) := \mathsf{SSYT}^{(r)}(1^{c_1}) \times \cdots \times \mathsf{SSYT}^{(r)}(1^{c_\ell}) \times \mathsf{SSYT}^{(r)}(\gamma)$$

by concatenating the reading words of the tableaux in the tuple. This yields crystal operators

$$e_i, f_i \colon \mathsf{SSYT}^{(r)}(1^{c_1}, \ldots, 1^{c_\ell}, \gamma) \to \mathsf{SSYT}^{(r)}(1^{c_1}, \ldots, 1^{c_\ell}, \gamma) \cup \{0\}.$$

Via the bijection φ of Proposition 3.1, this also imposes crystal operators on ordered multiset partitions

$$\tilde{e}_i, \tilde{f}_i \colon \mathcal{OP}_{n,k}^{(r)} \to \mathcal{OP}_{n,k}^{(r)} \cup \{0\}$$

as $\tilde{e}_i = \varphi^{-1} \circ e_i \circ \varphi$ and $\tilde{f}_i = \varphi^{-1} \circ f_i \circ \varphi$.

An example of a crystal structure on $\mathcal{OP}_{n,k}^{(r)}$ is given in Figure 1.

Theorem 4.1. *The operators* $\tilde{e}_i, \tilde{f}_i$, *and* wt *impose an* $\mathfrak{sl}_r$*-crystal structure on* $\mathcal{OP}_{n,k}^{(r)}$. *In addition,* $\tilde{e}_i$ *and* $\tilde{f}_i$ *preserve the* $\mathsf{minimaj}$ *statistic.*

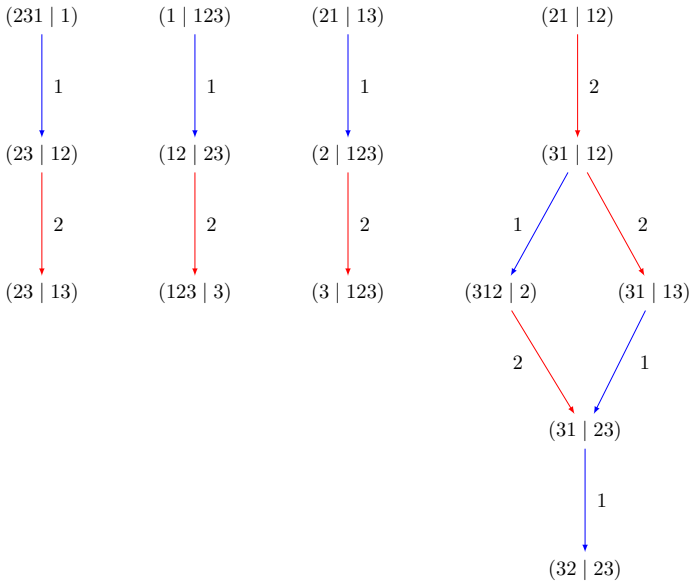

FIGURE 1. The crystal structure on $\mathcal{OP}^{(3)}_{4,2}$. The minimaj of the connected components are $2, 0, 1, 1$ from left to right.

Proof. The operators $\tilde{e}_i, \tilde{f}_i$, and wt impose an $\mathfrak{sl}_r$-crystal structure by construction since φ is a weight-preserving bijection. The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the minimaj statistic, since by Proposition 3.1, the bijection φ restricts to $\mathrm{M}(\mathrm{D}, \mathrm{I})^{(r)}$ which fixes the descents of the ordered multiset partitions in minimaj order. □

4.2. **Explicit crystal operators.** Let us now write down the crystal operator $\tilde{f}_i \colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ of Theorem 4.1 explicitly on $\pi \in \mathcal{OP}_{n,k}$ in minimaj order.

Start by creating a word w from right to left by reading the first element in each block of π from right to left, followed by the remaining elements of π from left to right. Use the crystal operator f_i on words to determine which i in w to change to an $i+1$. Circle the corresponding letter i in π. The crystal operator $\tilde{f}_i$ on π changes the circled i to $i+1$ unless we are in one of the following two cases:

$$\cdots \textcircled{i} \mid i \quad \xrightarrow{\tilde{f}_i} \quad \cdots \mid i \; \textcircled{i{+}1}\,, \tag{4.1a}$$

$$\mid \textcircled{i} \; i{+}1 \quad \xrightarrow{\tilde{f}_i} \quad i{+}1 \mid \textcircled{i{+}1}\,. \tag{4.1b}$$

Here "$\cdots$" indicates that the block is not empty in this region.

Example 4.2. In Figure 1, $\tilde{f}_2(31\textcircled{2} \mid 2) = (31 \mid 2\,\textcircled{3})$ is an example of (4.1a). Similarly, $\tilde{f}_1(31 \mid \textcircled{1}\,2) = (312 \mid \textcircled{2})$ is an example of (4.1b).

Proposition 4.3. *The above explicit description for $\tilde{f}_i$ is well defined and agrees with the definition of Theorem 4.1.*

Proof. The word w described above is precisely $\mathsf{row}(\varphi(\pi))$ on which f_i acts. Hence the circled letter i is indeed the letter changed to $i+1$. It remains to check how φ^{-1} changes the blocks. We will demonstrate this for the cases in (4.1) as the other cases are similar.

In case (4.1a) the circled letter i in block π_j does not correspond to b_j in π_j as it is not at the beginning of its block. Hence, it belongs to α_j or β_j. The circled letter is not a descent. Changing it to $i+1$ would create a descent. The map φ^{-1} distributes the letters in α_j and β_j to preserve descents, hence the circled i moves over to the next block on the right and becomes a circled $i+1$. Note also that $i+1 \notin \pi_{j+1}$, since otherwise the circled i would have been bracketed in w, contradicting the fact that f_i is acting on it.

In case (4.1b) the circled letter i in block π_j corresponds to b_j in π_j. Again, φ^{-1} now associates the $i+1 \in \pi_j$ to the previous block after applying f_i. Note that $i+1 \notin \pi_{j-1}$ since it would necessarily be b_{j-1}. But then the circled i would have been bracketed in w, contradicting the fact that f_i is acting on it. □

4.3. **Schur expansion.** The *character* of an $\mathfrak{sl}_r$-crystal B is defined as

$$\mathrm{ch}B = \sum_{b \in B} \mathbf{x}^{\mathsf{wt}(b)}.$$

Denote by $B(\lambda)$ the $\mathfrak{sl}_\infty$-crystal on $\mathsf{SSYT}(\lambda)$ defined above. This is a connected highest weight crystal with highest weight λ, and the character is the Schur function $\mathsf{s}_\lambda(\mathbf{x})$ defined in (3.3)

$$\mathrm{ch}B(\lambda) = \mathsf{s}_\lambda(\mathbf{x}).$$

Similarly, denoting by $B^{(r)}(\lambda)$ the $\mathfrak{sl}_r$-crystal on $\mathsf{SSYT}^{(r)}(\lambda)$, its character is the Schur polynomial

$$\mathrm{ch}B^{(r)}(\lambda) = \mathsf{s}_\lambda(x_1, \ldots, x_r).$$

Let us define

$$\mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t) = \sum_{\pi \in \mathcal{OP}^{(r)}_{n,k+1}} t^{\mathsf{minimaj}(\pi)} \mathbf{x}^{\mathsf{wt}(\pi)},$$

which satisfies $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t) = \mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t)$ for $r \geqslant n$, where $\mathsf{Val}_{n,k}(\mathbf{x}; 0, t)$ is as in (1.1).

As a consequence of Theorem 4.1, we now obtain the Schur expansion of $\mathsf{Val}^{(r)}_{n,k}(\mathbf{x}; 0, t)$.

Corollary 4.4. *We have*

$$\mathsf{Val}^{(r)}_{n,k-1}(\mathbf{x}; 0, t) = \sum_{\substack{\pi \in \mathcal{OP}^{(r)}_{n,k} \\ \tilde{e}_i(\pi) = 0 \ \forall\, 1 \leqslant i < r}} t^{\mathsf{minimaj}(\pi)} \mathsf{s}_{\mathsf{wt}(\pi)}.$$

When $r \geqslant n$, then by [Wil16] and [Rho16, Proposition 3.18] this is also equal to

$$\mathsf{Val}_{n,k-1}(\mathbf{x}; 0, t) = \sum_{\lambda \vdash n} \sum_{T \in \mathsf{SYT}(\lambda)} t^{\mathsf{maj}(T) + \binom{n-k}{2} - (n-k)\mathsf{des}(T)} \begin{bmatrix} \mathsf{des}(T) \\ n-k \end{bmatrix} \mathsf{s}_\lambda(\mathbf{x}),$$

where $\mathsf{SYT}(\lambda)$ is the set of standard Young tableaux of shape λ (that is, the elements in $\mathsf{SSYT}(\lambda)$ of weight $(1^{|\lambda|})$), $\mathsf{des}(T)$ is the number of descents of T, $\mathsf{maj}(T)$ is the major index of T (or the sum of descents of T), and the t-binomial coefficients in the sum are defined using the rule

$$\begin{bmatrix} m \\ p \end{bmatrix} = \frac{[m]!}{[p]!\,[m-p]!} \quad \text{where } [p]! = [p][p-1]\cdots[2][1] \text{ and } \ [p] = 1 + t + \cdots + t^{p-1}.$$

Example 4.5. The crystal $\mathcal{OP}^{(3)}_{4,2}$, displayed in Figure 1, has four highest weight elements with weights $(2,1,1)$, $(2,1,1)$, $(2,1,1)$, $(2,2)$ from left to right. Hence, we obtain the Schur expansion

$$\mathsf{Val}^{(3)}_{4,1}(\mathbf{x};0,t) = (1+t+t^2)\,\mathsf{s}_{(2,1,1)}(\mathbf{x}) + t\,\mathsf{s}_{(2,2)}(\mathbf{x}).$$

5. Equidistributivity of the minimaj and maj statistics

In this section, we describe a bijection $\psi\colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ in Theorem 5.12 with the property that $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$ for $\pi \in \mathcal{OP}_{n,k}$. This proves the link between $\mathsf{minimaj}$ and maj that was missing in [Wil16]. We can interpret ψ as a crystal isomorphism, where $\mathcal{OP}_{n,k}$ on the left is the $\mathsf{minimaj}$ crystal of Section 4 and $\mathcal{OP}_{n,k}$ on the right is viewed as a crystal of k columns with elements written in major index order.

The bijection ψ is the composition of φ of Proposition 3.1 with a certain shift operator. When applying φ to $\pi \in \mathcal{OP}_{n,k}$, we obtain the tuple $T^\bullet = T_1 \times \cdots \times T_{\ell+1}$ in (3.2). We would like to view each column in the tuple of tableaux as a block of a new ordered multiset partition. However, note that some columns could be empty, namely if $c_j = d_{\ell+2-j} - i_{\ell+2-j}$ in Proposition 3.1 is zero for some $1 \leqslant j \leqslant \ell$. For this reason, let us introduce the set of *weak ordered multiset partitions* $\mathcal{WOP}_{n,k}$, where we relax the condition that all blocks need to be nonempty sets.

Let $T^\bullet = T_1 \times \cdots \times T_{\ell+1}$ be a tuple of skew tableaux. Define $\mathsf{read}(T^\bullet)$ to be the weak ordered multiset partition whose blocks are obtained from $T^\bullet$ by reading the columns from the left to the right and from the bottom to the top; each column constitutes one of the blocks in $\mathsf{read}(T^\bullet)$. Note that given $\pi = (\pi_1|\pi_2|\cdots|\pi_k) \in \mathcal{OP}_{n,k}$ in minimaj order, $\mathsf{read}(\varphi(\pi))$ is a weak ordered multiset partition in major index order.

Example 5.1. Let $\pi = (1 \mid 56. \mid 4. \mid 37.12 \mid 2.1 \mid 1 \mid 34) \in \mathcal{OP}_{13,7}$, written in minimaj order. We have $\mathsf{minimaj}(\pi) = 22$. Then

$$T^\bullet = \varphi(\pi) = \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} \times \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \times \begin{array}{|c|}\hline 7\\\hline\end{array} \times \emptyset \times \begin{array}{|c|c|}\hline 1 & 3\\\hline \multicolumn{1}{c}{} & 2\\\cline{2-2} \multicolumn{1}{c}{} & 3\\\cline{2-2} \multicolumn{1}{c}{} & 4\\\cline{2-2} 1 & 5\\\hline 6\\\cline{1-1}\end{array}$$

and $\pi' = \mathsf{read}(T^\bullet) = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3)$.

Lemma 5.2. *Let $\mathcal{I} = \{\mathsf{read}(\varphi(\pi)) \mid \pi \in \mathcal{OP}_{n,k}\} \subseteq \mathcal{WOP}_{n,k}$, $\pi' = \mathsf{read}(\varphi(\pi)) \in \mathcal{I}$, and b_i the first elements in each block of π in minimaj order as in Lemma 2.2. Then π' has the following properties:*

1. *The last k elements of π' are $b_1, \ldots, b_k$, and b_i and b_{i+1} are in different blocks if and only if $b_i \leqslant b_{i+1}$.*
2. *If $b_1, \ldots, b_k$ are contained in precisely $k-j$ blocks, then there are at least j descents in the blocks containing the b_i's.*

Proof. Let $\pi \in \mathcal{OP}_{n,k}$, written in minimaj order. Then by (3.2), $\pi' = \mathsf{read}(\varphi(\pi))$ is of the form

$$\pi' = (\alpha^{\mathrm{rev}}_{\eta_{\ell+1}}\beta^{\mathrm{rev}}_{\eta_{\ell+1}-1}\cdots\beta^{\mathrm{rev}}_{\eta_\ell} \mid \cdots \mid \alpha^{\mathrm{rev}}_{\eta_1}\beta^{\mathrm{rev}}_{\eta_1-1}\cdots\beta^{\mathrm{rev}}_1 b_1 \cdots \mid \cdots \mid b_{\eta_1} b_{\eta_1-1}\cdots \mid \cdots \mid \cdots b_k),$$

where the superscript rev indicates that the elements are listed in decreasing order (rather than increasing order). Since the rows of a semistandard tableau are weakly increasing and

the columns are strictly increasing, the blocks of $\pi' = \mathsf{read}(\varphi(\pi))$ are empty or in strictly decreasing order. This implies that b_i and b_{i+1} are in different blocks of π' precisely when $b_i \leqslant b_{i+1}$, so a block of π' that contains a b_i cannot have a descent at its end. This proves (1).

In a weak ordered multiset partition written in major index order, any block of size $r \geqslant 2$ has $r-1$ descents. So if $b_1, \ldots, b_k$ are contained in precisely $k-j$ blocks, then at least j of these elements are contained in blocks of size at least two, so there are at least j descents in the blocks containing the b_i's. This proves (2). □

Remark 5.3. Let $\pi' \in \mathcal{WOP}_{n,k}$ be in major index order such that there are at least k elements after the rightmost occurrence of a block that is either empty or has a descent at its end. In this case, there exists a skew tableau $T^\bullet$ such that $\pi' = \mathsf{read}(T^\bullet)$. In fact, this characterizes $\mathcal{I} := \mathrm{im}(\mathsf{read} \circ \varphi)$.

Lemma 5.4. *The map* read *is invertible.*

Proof. Suppose $\pi' \in \mathcal{WOP}_{n,k}$ is in major index order such that there are at least k elements after the rightmost occurrence of a block that is either empty or has a descent at its end. Since there are no occurrences of an empty block or a descent at the end of a block amongst the last k elements of π', the blocks of π' containing the last k elements form the columns of a skew ribbon tableau $T \in \mathsf{SSYT}(\gamma)$, and the remaining blocks of π' form the column tableaux to the left of the skew ribbon tableau, so read is invertible. □

We are now ready to introduce the shift operators.

Definition 5.5. We define the *left shift operation* L on $\pi' \in \mathcal{I} = \{\mathsf{read}(\varphi(\pi)) \mid \pi \in \mathcal{OP}_{n,k}\}$ as follows. Suppose π' has $m \geqslant 0$ blocks $\pi'_{p_m}, \ldots, \pi'_{p_1}$ that are either empty or have a descent at the end, and $1 \leqslant p_m < \cdots < p_2 < p_1 < k$. Set

$$\mathsf{L}(\pi') = \mathsf{L}^{(m)}(\pi'),$$

where $\mathsf{L}^{(i)}$ for $0 \leqslant i \leqslant m$ are defined as follows:

(1) Set $\mathsf{L}^{(0)}(\pi') = \pi'$.

(2) Suppose $\mathsf{L}^{(i-1)}(\pi')$ for $1 \leqslant i \leqslant m$ is defined. By induction, the p_i-th block of $\mathsf{L}^{(i-1)}(\pi')$ is π'_{p_i}. Let S_i be the sequence of elements starting immediately to the right of block π'_{p_i} in $\mathsf{L}^{(i-1)}(\pi')$ up to and including the p_i-th descent after the block π'_{p_i}. Let $\mathsf{L}^{(i)}(\pi')$ be the weak ordered multiset partition obtained by moving each element in S_i one block to its left. Note that all blocks with index smaller than p_i in $\mathsf{L}^{(i)}(\pi')$ are the same as in π'.

Example 5.6. Continuing Example 5.1, we have $\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3)$, which is in major index order. We have $m = 2$ with $p_2 = 3 < 4 = p_1$, $S_1 = 61543$, $S_2 = 6154$ and

$$\mathsf{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3),$$
$$\mathsf{L}(\pi') = \mathsf{L}^{(2)}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3).$$

Note that $\mathsf{maj}(\pi') = 28$, $\mathsf{maj}(\mathsf{L}^{(1)}(\pi')) = 25$, and $\mathsf{maj}(\mathsf{L}(\pi')) = 22 = \mathsf{minimaj}(\pi)$.

Proposition 5.7. *The left shift operation* $\mathsf{L} \colon \mathcal{I} \to \mathcal{OP}_{n,k}$ *is well defined.*

Proof. Suppose $\pi' \in \mathcal{I}$ has $m \geqslant 0$ blocks $\pi'_{p_1}, \ldots, \pi'_{p_m}$ that are either empty or have a descent at the end, and $1 \leqslant p_m < \cdots < p_2 < p_1 < k$. If $m = 0$, then $\mathsf{L}(\pi') = \pi' \in \mathcal{OP}_{n,k}$ and we are done.

We proceed by induction on m. Note that $\mathsf{L}^{(1)}$ acts on the rightmost block π'_{p_1}. Notice that π'_{p_1} cannot contain any of the b_i's by Lemma 5.2 (1). Hence, since there are at least k elements in the $k-p_1$ blocks following π'_{p_1}, by Lemma 5.2 (2), there are at least p_1 descents after π'_{p_1}, so $\mathsf{L}^{(1)}$ can be applied to π'.

Observe that applying $\mathsf{L}^{(1)}$ to π' does not create any new empty blocks to the right of π'_{p_1}, because creating a new empty block means that the last element of S_1, which is a descent, is at the end of a block. This cannot happen, since the rightmost occurrence of an empty block or a descent at the end of its block was assumed to be in π'_{p_1}. However, note that applying $\mathsf{L}^{(1)}$ to π' does create a new block with a descent at its end, and this descent is given by the p_1-th descent after the block π'_{p_1} (which is the last element of S_1).

Now suppose $\mathsf{L}^{(i-1)}(\pi')$ is defined for $i \geqslant 2$. By induction, there are at least $p_1 > p_i$ descents following the block π'_{p_i}, so the set S_i of Definition 5.5 exists and we can move the elements in S_i left one block to construct $\mathsf{L}^{(i)}(\pi')$ from $\mathsf{L}^{(i-1)}(\pi')$. Furthermore, $\mathsf{L}^{(i)}(\pi')$ does not have any new empty blocks to the right of π'_{p_i}. To see this, note that the number of descents in S_i is p_i, so the number of descents in S_i is strictly decreasing as i increases. This implies that the $i-1$ newly created descents at the end of a block of $\mathsf{L}^{(i-1)}(\pi')$ occurs strictly to the right of S_i, and so the last element of S_i cannot be a descent at the end of a block of $\mathsf{L}^{(i-1)}(\pi')$.

Lastly, $\mathsf{L}(\pi') = \mathsf{L}^{(m)}(\pi') \in \mathcal{OP}_{n,k}$, since it does not have any empty blocks, and every block of $\mathsf{L}(\pi')$ is in decreasing order because either we moved every element of a block into an empty block or we moved elements into a block with a descent at the end. □

Definition 5.8. We define the *right shift operation* R on $\mu \in \mathcal{OP}_{n,k}$ in major index order as follows. Suppose μ has $m \geqslant 0$ blocks $\mu_{q_1}, \ldots, \mu_{q_m}$ that have a descent at the end and $q_1 < q_2 < \cdots < q_m$. Set

$$\mathsf{R}(\mu) = \mathsf{R}^{(m)}(\mu),$$

where $\mathsf{R}^{(i)}$ for $0 \leqslant i \leqslant m$ are defined as follows:

(1) Set $\mathsf{R}^{(0)}(\mu) = \mu$.
(2) Suppose $\mathsf{R}^{(i-1)}(\mu)$ for $1 \leqslant i \leqslant m$ is defined. Let U_i be the sequence of q_i elements to the left of, and including, the last element in the q_i-th block of $\mathsf{R}^{(i-1)}(\mu)$. Let $\mathsf{R}^{(i)}(\mu)$ be the weak ordered multiset partition obtained by moving each element in U_i one block to its right. Note that all blocks to the right of the (q_i+1)-th block are the same in μ and $\mathsf{R}^{(i)}(\mu)$.

Note that R can potentially create empty blocks.

Example 5.9. Continuing Example 5.6, let $\mu = \mathsf{L}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3)$. We have $m = 2$ with $q_1 = 4 < 5 = q_2$, $U_1 = 6154$, $U_2 = 61543$ and

$$\mathsf{R}^{(1)}(\mu) = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3),$$
$$\mathsf{R}(\mu) = \mathsf{R}^{(2)}(\mu) = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3),$$

which is the same as π' in Example 5.6.

Proposition 5.10. *The right shift operation* R *is well defined and is the inverse of* L.

Proof. Suppose $\mu \in \mathcal{OP}_{n,k}$ in major index order has descents at the end of the blocks $\mu_{q_1}, \ldots, \mu_{q_m}$. If $m = 0$, then $\mathsf{R}(\mu) = \mu \in \mathcal{OP}_{n,k} \subseteq \mathcal{WOP}_{n,k}$ and there is nothing to show.

We proceed by induction on m. The ordered multiset partition μ does not have empty blocks, so there are at least q_1 elements in the first q_1 blocks of μ, and $\mathsf{R}^{(1)}$ can be applied to μ.

Now suppose $\mathsf{R}^{(i-1)}(\mu)$ is defined for $i \geqslant 2$. By induction, there are at least $q_{i-1}+1$ elements in the first $q_{i-1}+1$ blocks of $\mathsf{R}^{(i-1)}(\mu)$. Since the blocks $\mu_{q_{i-1}+2}, \ldots, \mu_{q_i}$ in μ are all nonempty, there are at least $q_{i-1}+1+(q_i-(q_{i-1}+1)) = q_i$ elements in the first q_i blocks of $\mathsf{R}^{(i-1)}(\mu)$, so the set U_i of Definition 5.8 exists and we can move the elements in U_i one block to the right to construct $\mathsf{R}^{(i)}(\mu)$ from $\mathsf{R}^{(i-1)}(\mu)$.

Furthermore, every nonempty block of $\mathsf{R}(\mu)$ is in decreasing order because the rightmost element of each U_i is a descent. So $\mathsf{R}(\mu) \in \mathcal{OP}_{n,k}$ remains in major index order. This completes the proof that R is well defined.

Next we show that R is the inverse of L. Observe that if $\pi' \in \mathcal{I}$ has m occurrences of either an empty block or a block with a descent at its end, then $\mu = \mathsf{L}(\pi')$ has m blocks with a descent at its end. Hence it suffices to show that $\mathsf{R}^{(m+1-i)}$ is the inverse operation to $\mathsf{L}^{(i)}$ for each $1 \leqslant i \leqslant m$.

The property that the last element of S_i cannot be a descent at the end of a block of $\mathsf{L}^{(i-1)}(\pi')$ in the proof of Proposition 5.7 similarly holds for every element in S_i. Therefore, if the last element of S_i is in the r_i-th block of $\mathsf{L}^{(i-1)}(\pi')$, then $|S_i| = p_i + (r_i - 1 - p_i) = r_i - 1$ because the blocks are decreasing and none of the elements in S_i can be descents at the end of a block. Since the last element of S_i becomes a descent at the end of the (r_i-1)-th block of $\mathsf{L}^{(i)}(\pi)$, this implies $r_i - 1 = q_{m-i+1}$, so $U_{m-i+1} = S_i$ for every $1 \leqslant i \leqslant m$. As the operation $\mathsf{L}^{(i)}$ is a left shift of the elements of S_i by one block and the operation $\mathsf{R}^{(m+1-i)}$ is a right shift of the same set of elements by one block, they are inverse operations of each other. □

For what follows, we need to extend the definition of the major index to the set $\mathcal{WOP}_{n,k}$ of weak ordered multiset partitions of length n and k blocks, in which some of the blocks may be empty. Given $\pi' \in \mathcal{WOP}_{n,k}$ whose nonempty blocks are in major index order, if the block $\pi'_j \neq \emptyset$, then the last element in π'_j is assigned the index j, and the remaining elements in π'_j are assigned the index $j-1$ for $j = 1, \ldots, k$. Then $\mathsf{maj}(\pi')$ is the sum of the indices where a descent occurs. This agrees with (2.2) in the case when all blocks are nonempty.

Lemma 5.11. *Let $\pi' \in \mathcal{I}$. With the same notation as in Definition 5.5, we have for $1 \leqslant i \leqslant m$*

$$\mathsf{maj}(\mathsf{L}^{(i)}(\pi')) = \begin{cases} \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i + 1, & \text{if } \pi'_{p_i} = \emptyset, \\ \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i, & \text{if } \pi'_{p_i} \text{ has a descent at the end of its block.} \end{cases}$$

Proof. Assume $\pi'_{p_i} = \emptyset$. In the transformation from $\mathsf{L}^{(i-1)}(\pi')$ to $\mathsf{L}^{(i)}(\pi')$, the index of each of the first $p_i - 1$ descents in S_i decreases by one, while the index of the last descent remains the same, since it is not at the end of a block in $\mathsf{L}^{(i-1)}(\pi')$, but it becomes the last element of a block in $\mathsf{L}^{(i)}(\pi')$. The indices of elements not in S_i remain the same, so $\mathsf{maj}(\mathsf{L}^{(i)}(\pi')) = \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i + 1$ in this case.

Next assume that π'_{p_i} has a descent at the end of the block. In the transformation from $\mathsf{L}^{(i-1)}(\pi')$ to $\mathsf{L}^{(i)}(\pi')$, the indices of the descents in S_i change in the same way as in the previous case, but in addition, the index of the last descent in π'_{p_i} decreases by one, so $\mathsf{maj}(\mathsf{L}^{(i)}(\pi')) = \mathsf{maj}(\mathsf{L}^{(i-1)}(\pi')) - p_i$ in this case. □

Theorem 5.12. *Let $\psi\colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ be the map defined by*

$$\psi(\pi) = \mathsf{L}(\mathsf{read}(\varphi(\pi))) \qquad \text{for } \pi \in \mathcal{OP}_{n,k} \text{ in minimaj order.}$$

Then ψ is a bijection that maps ordered multiset partitions in minimaj order to ordered multiset partitions in major index order. Furthermore, $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$.

Proof. By Proposition 3.1, φ is a bijection. By Lemma 5.4, the map read is invertible, and by Proposition 5.10 the shift operation L has an inverse. This implies that ψ is a bijection.

It remains to show that $\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$ for $\pi \in \mathcal{OP}_{n,k}$ in minimaj order.

First suppose that $\pi' = \mathsf{read}(\varphi(\pi))$ has no empty blocks and no descents at the end of any block. In this case $\mathsf{L}(\pi') = \pi'$, so that in fact $\pi' = \psi(\pi)$. Using the definition of major index (2.2) and the representation (3.2) (where the columns in the ribbon are viewed as separate columns due to read), we obtain

$$\mathsf{maj}(\pi') = \sum_{j=1}^{\ell}(\ell+1-j)(d_j - i_j - 1) + \ell + \sum_{j=1}^{\ell}(\ell + \eta_j - j), \tag{5.1}$$

where $d_j, i_j, \eta_j = i_1 + \cdots + i_j$ are defined in Proposition 3.1 for π. Here, the first sum in the formula arises from the contributions of the first ℓ blocks and the summand ℓ compensates for the fact that b_1 is in the ℓ-th block. The second sum in the formula comes from the contributions of the b_i's. Comparing with (2.1), we find

$$\mathsf{maj}(\pi') = \mathsf{minimaj}(\pi) - \binom{\ell+1}{2} - \sum_{j=1}^{\ell}(\ell+1-j)i_j + \binom{\ell+1}{2} + \sum_{j=1}^{\ell}\eta_j = \mathsf{minimaj}(\pi),$$

proving the claim.

Now suppose that $\pi' = \mathsf{read}(\varphi(\pi))$ has a descent at the end of block π'_p. This will contribute an extra p compared to the major index in (5.1). If $\pi'_p = \emptyset$, then $c_p = d_{\ell+2-p} - i_{\ell+2-p} = 0$ and the term $j = \ell + 2 - p$ in (5.1) should be $(\ell+1-j)(d_j - i_j)$ instead of $(\ell+1-j)(d_j - i_j - 1)$ yielding a correction term of $\ell + 1 - j = \ell + 1 - \ell - 2 + p = p - 1$. Hence, with the notation of Definition 5.5, we have

$$\mathsf{maj}(\pi') = \mathsf{minimaj}(\pi) + \sum_{i=1}^{m} p_i - e,$$

where e is the number of empty blocks in π'. Since $\psi(\pi) = \mathsf{L}(\pi')$, the claim follows by Lemma 5.11. □

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