



A class of generalized Ginzburg–Landau equations with random switching

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ARTICLE INFO

Article history:

Received 13 February 2018

Available online 21 April 2018

Keywords:

Ginzburg–Landau equation

Regime-switching diffusion

Stochastic boundedness

Permanence

Two-time-scale model

ABSTRACT

This paper focuses on a class of generalized Ginzburg–Landau equations with random switching. In our formulation, the nonlinear term is allowed to have higher polynomial growth rate than the usual cubic polynomials. The random switching is modeled by a continuous-time Markov chain with a finite state space. First, an explicit solution is obtained. Then properties such as stochastic-ultimate boundedness and permanence of the solution processes are investigated. Finally, two-time-scale models are examined leading to a reduction of complexity.

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1. Introduction

This paper focuses on a class of generalized Ginzburg–Landau equations, namely, stochastic Ginzburg–Landau equations in random environment modeled by a random switching process. In contrast to the well-known Ginzburg–Landau equations with random disturbances considered in the literature, higher order nonlinearity is allowed. More importantly, the systems are in a random environment that is modeled as a random discrete event process given by a switching process. Thus, the system under consideration may be considered as a hybrid system in which continuous dynamics and discrete events coexist. Our effort in this paper is devoted to obtaining existence and uniqueness of solutions, permanence of solutions, and reduction of complexity using two-time-scale formulation.

In 1950, Ginzburg and Landau proposed a class of deterministic differential equations to describe phase transitions for superconductivity in [1]. They observed the existence of two types of superconductors depending on the energy of the interface between the normal and superconducting states. Their paper has led to significant developments to the nowadays known Ginzburg–Landau theory. Because of its prevalence in applications, this class of equations has been attracting much attention in the past decades. For instance, Ginzburg–Landau equations have been used in many areas including the theory of bistable systems, chemical turbulence, phase transitions in non-equilibrium systems, nonlinear, optics with dissipation, thermodynamics, and hydrodynamic systems, etc.; see [2–4] and references there in.

Because random noise is often unavoidable, taking into consideration of stochastic disturbances is necessary. To account for the noise effect, stochastic Ginzburg–Landau equations have received much attention in recent years. For example, Neiman and Geier [5] studied stochastic resonance in an over-damped bistable system driven by white and harmonic noises. In [6,7], delay stochastic Ginzburg–Landau equations were considered, whose solutions describe the stochastic evolution of

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the position of a particle trapped in a double well potential in the presence of a time delayed force and Gaussian white noise. Subsequently, in [8], Kloeden and Platen provided an explicit solution to the Ginzburg–Landau equation given by

$$dx(t) = [(a + \frac{\sigma^2}{2})x(t) - bx^3(t)]dt + \sigma x(t)dW(t), \quad (1.1)$$

where $W(t)$ is a standard Brownian motion, a, σ , and $b > 0$ are constants. Dung [9] studied a number of qualitative properties of the solution to a stochastic Ginzburg–Landau equation with impulsive effects.

Because of the rapid growth in science and technology, networked systems come into being. The new challenges call for more sophisticated mathematical models. One of the important advances is the development of novel models of hybrid systems in which continuous dynamics and discrete events coexist and interact. To consider both random uncertainty due to Brownian motion type of disturbances and stochastic environment represented by jump processes taking values in a finite set, switching diffusions have gained much needed attention recently; see [10,11] among others. Such switching dynamic systems are nonlinear stochastic systems with another stochastic source depicting the random environment changes as switching processes. The presence of both continuous dynamics and discrete events enables one to describe complex systems and their inherent uncertainty and randomness in the environment effectively.

Aiming at enlarging the applicability and suitability for a wider range of problems, this paper focuses on a class of real-valued systems known as generalized stochastic Ginzburg–Landau equations with regime switching. It is a generalized model since the nonlinear terms have faster growth rates than that of the usual Ginzburg–Landau equations. More importantly, we use a randomly switching process to model stochastic environment and other random factors that are not covered in the usual stochastic differential equations.

The rest of the paper is organized as follows. Section 2 presents the generalized Ginzburg–Landau equations with switching that we wish to study. Also derived in this section is an explicit solution. Section 3 investigates properties such as stochastically ultimate boundedness and permanence of the solution processes. These results may shed some light for the subsequent study on superconductivity and other desired properties. Section 4 examines a class of systems with two-time scales. The main idea here is to reduce the computational complexity. Finally, Section 5 gives some further remarks to conclude the paper.

2. Formulation and existence of solution

2.1. Formulation

Let $W(\cdot)$ be a real-valued Brownian motion, and $\alpha(\cdot)$ be a continuous-time Markov chain that is independent of $W(\cdot)$ with a state space $\mathcal{M} = \{1, \dots, m\}$ and generator $Q = (q_{ij})$. Recall, Q satisfies the conditions $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^m q_{ij} = 0$ for each $i \in \mathcal{M}$. Note that for the continuous-time Markov chain $\alpha(t)$,

$$P\{\alpha(t + \delta) = j | \alpha(t) = i\} = \begin{cases} q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + q_{ii}\delta + o(\delta), & \text{if } i = j. \end{cases}$$

The objective of this paper is to treat the generalized Ginzburg–Landau equations with random switching in which the coefficients of the systems depend on an additional time variable. Thus the coefficients of the systems are time varying in addition to the time-varying and jump properties due to the Markov chain. Consider the equation

$$dX(t) = [a(t, \alpha(t))X(t) - b(t, \alpha(t))X^{k+1}(t)]dt + \sigma(t, \alpha(t))X(t)dW(t), \quad (2.1)$$

where $k \geq 2$ is an integer. It then follows that the associated generator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}f(t, x, i) &= \frac{\partial f(t, x, i)}{\partial t} + (a(t, i)x - b(t, i)x^{k+1})\frac{\partial f(t, x, i)}{\partial x} \\ &\quad + \frac{1}{2}\sigma^2(t, i)x^2\frac{\partial^2 f(t, x, i)}{\partial x^2} + \sum_{j=1}^m q_{ij}f(t, x, j), \end{aligned} \quad (2.2)$$

for each $i \in \mathcal{M}$, where $f(\cdot, \cdot, \cdot) : [0, \infty) \times \mathbb{R} \times \mathcal{M} \mapsto \mathbb{R}$ such that for each $i \in \mathcal{M}$, $f(\cdot, \cdot, i) \in C^{1,2}$. That is, f has continuous partial derivative with respect to t , and continuous partial derivative with respect to x up to the second order.

2.2. Explicit solution

In this section, we demonstrate that (2.1) has a global explicit solution that is positive for $t \geq 0$.

(A1) For each $i \in \mathcal{M}$, $a(t, i)$, $b(t, i)$ and $\sigma(t, i)$ are bounded integrable functions defined on $[0, +\infty)$ and $b(t, i) \geq 0$.

Theorem 2.1. Assume (A1). Then for any initial condition $x_0 := X(0) > 0$, there is a unique positive solution of (2.1) on $t \geq 0$ explicitly given by

$$X(t) = \frac{\exp(\Gamma(t))}{\left[\frac{1}{x_0^k} + k \int_0^t b(s, \alpha(s)) \exp(k\Gamma(s)) ds \right]^{\frac{1}{k}}}, \quad (2.3)$$

where

$$\Gamma(t) = \int_0^t [a(s, \alpha(s)) - \frac{1}{2}\sigma^2(s, \alpha(s))]ds + \int_0^t \sigma(s, \alpha(s))dW(s).$$

Proof. Since the coefficients of (2.1) are locally Lipschitz, by the standard argument in stochastic differential equations, there exists a unique local solution $X(t)$ on $[0, t_1) \subset [0, \infty)$ for some $t_1 > 0$.

We proceed to find the solution. Define

$$U(x, t) = e^{-kx}F(t), \quad (2.4)$$

where

$$F(t) = \frac{1}{x_0^k} + k \int_0^t b(s, \alpha(s))e^{k\Gamma(s)}ds.$$

Let

$$G(t) = U(\Gamma(t), t).$$

Then by the generalized Itô formula [11, pp. 29–30] (or [10]),

$$\begin{aligned} dG(t) &= \left\{ kb(t, \alpha(t)) + \frac{k}{2}G(t) \left[(k+1)\sigma^2(t, \alpha(t)) - 2a(t, \alpha(t)) \right] \right\} dt \\ &\quad - k\sigma(t, \alpha(t))G(t)dW(t). \end{aligned} \quad (2.5)$$

Let

$$X(t) = \frac{1}{G^{\frac{1}{k}}(t)}.$$

Then $X(t)$ exists for all $t \in [0, +\infty)$, $X(t) > 0$, and is continuous. Using the generalized Itô formula again, it is straightforward to verify

$$\begin{aligned} dX(t) &= -\frac{1}{k}G^{-\frac{k+1}{k}}(t)dG(t) + \frac{k+1}{2k^2}G^{-\frac{2k+1}{k}}(t)[dG(t)]^2 \\ &= \left[a(t, \alpha(t))X(t) - b(t, \alpha(t))X^{k+1}(t) \right] dt + \sigma(t, \alpha(t))X(t)dW(t). \end{aligned}$$

Therefore, we conclude that $X(t)$ given by (2.3) is a continuous positive solution of (2.1) not only for $t \in [0, t_1)$ but for all $t \geq 0$. Thus, the solution exists for all $t \in [0, \infty)$ (i.e., it is a global solution). This completes the proof of the theorem. $\square$

3. Stochastic boundedness and permanence

This section is devoted to obtaining stochastic boundedness and permanence of the solutions of the generalized Ginzburg–Landau equations. The discussion in this section is under assumptions (A1) and (A2) given below. Recall that $\alpha(t)$ is irreducible, if the system of equations $\nu Q = 0$, $\sum_{i=1}^m \nu_i = 1$ has a unique solution $\nu = (\nu_1, \dots, \nu_m)$ with each $\nu_i > 0$. The ν is termed the stationary distribution, and is used throughout the rest of the paper.

(A2) The Markov chain $\alpha(t)$ given in (2.1) is irreducible.

For notational simplicity in the following discussion, we use

$$\begin{aligned} \check{h}(i) &:= \inf_{t \in [0, +\infty)} h(t, i), \\ \hat{h}(i) &:= \sup_{t \in [0, +\infty)} h(t, i), \end{aligned}$$

where $h(t, i)$ can be any of the $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, or $\sigma(\cdot, \cdot)$. To proceed, we define the stochastic boundedness and permanence in what follows.

Definition 3.1. For each positive initial condition, the solution $X(t)$ of (2.1) is called stochastically ultimately upper bounded, if for any $\varepsilon \in (0, 1)$, there exists a positive constant $H = H(\varepsilon)$ such that the solution of (2.1) satisfies

$$\limsup_{t \rightarrow +\infty} P\{X(t) > H\} < \varepsilon.$$

Definition 3.2. For each positive initial condition, the solution $X(t)$ of (2.1) is called stochastically ultimately lower bounded, if for any $\varepsilon \in (0, 1)$, there exists a positive constant $\delta = \delta(\varepsilon)$ such that the solution of (2.1) satisfies that

$$\limsup_{t \rightarrow +\infty} P\{X(t) < \delta\} < \varepsilon.$$

Definition 3.3. Eq. (2.1) or its solution is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exist positive constants $H = H(\varepsilon)$ and $\delta = \delta(\varepsilon)$ such that

$$\liminf_{t \rightarrow +\infty} P\{X(t) \leq H\} \geq 1 - \varepsilon, \quad \liminf_{t \rightarrow +\infty} P\{X(t) \geq \delta\} \geq 1 - \varepsilon,$$

where $X(t)$ is the solution of (2.1) with any positive initial value.

It is clear that a stochastic equation is stochastically permanent if and only if its solution is both stochastically ultimately upper bounded and lower bounded. To continue, we obtain a number of lemmas.

Lemma 3.4. Let $Y(t) = \log X(t)$, and $Z(t)$ be the solution of the SDE

$$dZ(t) = (a(t, \alpha(t)) - b(t, \alpha(t)) - \frac{\sigma^2(t, \alpha(t))}{2} - kb(t, \alpha(t))Z(t))dt + \sigma(t, \alpha(t))dW(t). \quad (3.1)$$

If $Y(0) = Z(0)$, then $Y(t) \leq Z(t)$ a.s.

Proof. By the generalized Itô formula [11, pp.29–30],

$$dY(t) = (a(t, \alpha(t)) - \frac{\sigma^2(t, \alpha(t))}{2} - b(t, \alpha(t))e^{kY(t)})dt + \sigma(t, \alpha(t))dW(t).$$

Since $e^{ky} \geq 1 + ky$ for any $y \in \mathbb{R}$, we get

$$dY(t) \leq [a(t, \alpha(t)) - b(t, \alpha(t)) - \frac{\sigma^2(t, \alpha(t))}{2} - kb(t, \alpha(t))Y(t)]dt + \sigma(t, \alpha(t))dW(t).$$

Thus by the differential form of Gronwall's inequality, $Y(t) \leq Z(t)$ a.s. $\square$

Consider the system of linear equations

$$Qc = \eta, \quad \text{where } c \in \mathbb{R}^m \text{ and } \eta \in \mathbb{R}^m. \quad (3.2)$$

Lemma 3.5 ([11, Lemma A.12]). The following assertions hold.

- Eq. (3.2) has a solution if and only if $v\eta = 0$.
- Suppose that $\tilde{c}$ and $\bar{c}$ are two solutions of (3.2). Then $\tilde{c} - \bar{c} = \gamma_0 \mathbf{1}$ for some $\gamma_0 \in \mathbb{R}$, where $\mathbf{1}$ is a column vector with all components being 1.
- Any solution of (3.2) can be written as

$$c = \gamma_0 \mathbf{1} + h_0,$$

where $\gamma_0 \in \mathbb{R}$ is an arbitrary constant and $h_0 \in \mathbb{R}^m$ is the unique solution of (3.2) satisfying $vh_0 = 0$.

Lemma 3.6. Assume $v\beta := \sum_{i=1}^m v_i \beta(i) := \sum_{i=1}^m v_i \check{b}(i) > 0$. There is a positive constant θ such that

$$\limsup_{t \rightarrow +\infty} E|Z(t)|^{2\theta} \leq K, \quad (3.3)$$

where $Z(t)$ is the solution of (3.1) and K is a constant defined in (3.10).

Proof. Since

$$v[-\beta + (v\beta)\mathbf{1}] = 0,$$

it follows from Lemma 3.5 that the equation

$$Qc = -\beta + (v\beta)\mathbf{1}$$

has a solution $c = (c_1, c_2, \dots, c_m)^T \in \mathbb{R}^m$. Therefore,

$$\check{b}(i) + \sum_{j=1}^m q_{ij}c_j = v\beta > 0, \quad i \in \mathcal{M}. \quad (3.4)$$

From (3.4), we have

$$\begin{aligned} & \frac{1}{\theta(1-c_i\theta)} \sum_{j=1}^m q_{ij}(1-c_j\theta) \\ &= -\left(\sum_{j=1}^m q_{ij}c_j + \frac{c_i\theta}{1-c_i\theta} \sum_{j=1}^m q_{ij}c_j\right) \\ &= -\left(\nu\beta - \check{b}(i) + \frac{c_i\theta}{1-c_i\theta} \sum_{j=1}^m q_{ij}c_j\right). \end{aligned} \quad (3.5)$$

Choose a small constant $\theta \in (0, \frac{1}{2})$ such that

$$\frac{1}{2} < 1 - c_i\theta < 2, \quad i \in \mathcal{M} \quad (3.6)$$

and

$$\nu\beta + \frac{c_i\theta}{1-c_i\theta} \sum_{j=1}^m q_{ij}c_j > 0, \quad i \in \mathcal{M}. \quad (3.7)$$

Now, choose a sufficiently small positive number μ such that

$$\nu\beta + \frac{c_i\theta}{1-c_i\theta} \sum_{j=1}^m q_{ij}c_j - \frac{\mu}{\theta} > 0, \quad i \in \mathcal{M}. \quad (3.8)$$

For $Z(t)$ given by (3.1), define the function $V(\cdot, \cdot, \cdot) : [0, +\infty) \times \mathbb{R} \times \mathcal{M}$ by

$$V(t, z, i) = (1 - c_i\theta)(1 + z^2)^\theta.$$

That is, V is independent of t . Then direct calculation reveals that

$$\begin{aligned} & \mathcal{L}V(t, z, i) \\ &= \theta(1 - c_i\theta)(1 + z^2)^{\theta-2} \left\{ (-2kb(t, i) + \frac{1}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta))z^4 \right. \\ & \quad + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z^3 \\ & \quad + \left[(2\theta - 1)\sigma^2(t, i) - 2kb(t, i) + \frac{2}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) \right]z^2 \\ & \quad \left. + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z + \sigma^2(t, i) + \frac{1}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) \right\}. \end{aligned}$$

This leads to

$$\begin{aligned} & \mathcal{L}(e^{\mu t}V(t, z, i)) \\ &= \theta(1 - c_i\theta)(1 + z^2)^{\theta-2}e^{\mu t} \left\{ (-2kb(t, i) + \frac{1}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) + \frac{\mu}{\theta})z^4 \right. \\ & \quad + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z^3 \\ & \quad + \left[(2\theta - 1)\sigma^2(t, i) - 2kb(t, i) + \frac{2}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) + \frac{2\mu}{\theta} \right]z^2 \\ & \quad + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z + \sigma^2(t, i) + \frac{1}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) + \frac{\mu}{\theta} \left\} \right. \\ &\leq \theta(1 - c_i\theta)(1 + z^2)^{\theta-2}e^{\mu t} \left\{ -\left[(2k - 1)\check{b}(i) + \nu\beta + \frac{c_i\theta}{1 - c_i\theta} \sum_{j=1}^m q_{ij}c_j - \frac{\mu}{\theta} \right]z^4 \right. \\ & \quad + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z^3 \\ & \quad + \left[(2\theta - 1)\sigma^2(t, i) - 2(k - 1)\check{b}(i) - 2\nu\beta - 2\frac{c_i\theta}{1 - c_i\theta} \sum_{j=1}^m q_{ij}c_j + \frac{2\mu}{\theta} \right]z^2 \\ & \quad + 2(a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i))z + \sigma^2(t, i) + \frac{1}{\theta(1 - c_i\theta)} \sum_{j=1}^m q_{ij}(1 - c_j\theta) + \frac{\mu}{\theta} \left\} \right. \\ &\leq \frac{1}{2}\mu Ke^{\mu t}, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned}
 K := & \frac{1}{2} \mu^{-1} \left\{ \sup_{(t, z, i) \in [0, +\infty) \times \mathbb{R} \times \mathcal{M}} \left\{ - \left[(2k-1) \check{b}(i) + v\beta + \frac{c_i \theta}{1 - c_i \theta} \sum_{j=1}^m q_{ij} c_j - \frac{\mu}{\theta} \right] z^4 \right. \right. \\
 & + 2(a(t, i) - \frac{1}{2} \sigma^2(t, i) - b(t, i)) z^3 \\
 & + \left[(2\theta - 1) \sigma^2(t, i) - 2(k-1) \check{b}(i) - 2v\beta - 2 \frac{c_i \theta}{1 - c_i \theta} \sum_{j=1}^m q_{ij} c_j + \frac{2\mu}{\theta} \right] z^2 \\
 & \left. \left. + 2(a(t, i) - \frac{1}{2} \sigma^2(t, i) - b(t, i)) z + \sigma^2(t, i) + \frac{1}{\theta(1 - c_i \theta)} \sum_{j=1}^m q_{ij} (1 - c_j \theta) + \frac{\mu}{\theta} \right\} \vee 1 \right\}.
 \end{aligned} \quad (3.10)$$

Let ℓ_0 be a sufficiently large integer such that $Z(0) \in [-\ell_0, \ell_0]$. Define the stopping time $\rho_\ell = \inf\{t \geq 0 : |Z(t)| > \ell\}$ for each $\ell \geq \ell_0$. It can be seen that ρ_ℓ is monotonically increasing. Since the linear SDE (3.1) has no finite explosion time, we have $\rho_\infty := \lim_{\ell \rightarrow +\infty} \rho_\ell = +\infty$ a.s. By the generalized Itô formula,

$$M(t) := e^{\mu t} V(t, Z(t), \alpha(t)) - V(0, Z(0), \alpha(0)) - \int_0^t \mathcal{L}(e^{\mu s} V(s, Z(s), \alpha(s))) ds$$

is a local martingale, which implies that $EM(t \wedge \rho_\ell) = 0$. So

$$E[e^{\mu(t \wedge \rho_\ell)} V(t \wedge \rho_\ell, Z(t \wedge \rho_\ell), \alpha(t \wedge \rho_\ell))] = EV(0, Z(0), \alpha(0)) + E \int_0^{t \wedge \rho_\ell} \mathcal{L}(e^{\mu s} V(s, Z(s), \alpha(s))) ds.$$

By the definition of ρ_ℓ , $e^{\mu(t \wedge \rho_\ell)} (1 + Z^2(t \wedge \rho_\ell))^\theta$ is monotonically increasing. It follows from $\rho_\infty = +\infty$ a.s. that

$$\lim_{\ell \rightarrow +\infty} Z(t \wedge \rho_\ell) = Z(t) \text{ a.s.},$$

and

$$\lim_{\ell \rightarrow +\infty} e^{\mu(t \wedge \rho_\ell)} (1 + Z^2(t \wedge \rho_\ell))^\theta = e^{\mu t} (1 + Z^2(t))^\theta.$$

From (3.6) and the definition of $V(\cdot, \cdot, \cdot)$,

$$e^{\mu(t \wedge \rho_\ell)} V(t \wedge \rho_\ell, Z(t \wedge \rho_\ell), \alpha(t \wedge \rho_\ell)) \leq 2e^{\mu t} (1 + Z^2(t))^\theta. \quad (3.11)$$

By (3.9), (3.11) and the dominated convergence theorem,

$$\lim_{\ell \rightarrow +\infty} E[e^{\mu(t \wedge \rho_\ell)} V(t \wedge \rho_\ell, Z(t \wedge \rho_\ell), \alpha(t \wedge \rho_\ell))] = E[e^{\mu t} V(t, Z(t), \alpha(t))],$$

and

$$\lim_{\ell \rightarrow +\infty} E \int_0^{t \wedge \rho_\ell} \mathcal{L}(e^{\mu s} V(s, Z(s), \alpha(s))) ds = E \int_0^t \mathcal{L}(e^{\mu s} V(s, Z(s), \alpha(s))) ds.$$

Therefore

$$E[e^{\mu t} V(t, Z(t), \alpha(t))] = EV(0, Z(0), \alpha(0)) + E \int_0^t \mathcal{L}(e^{\mu s} V(s, Z(s), \alpha(s))) ds,$$

which leads to

$$\frac{1}{2} E[e^{\mu t} (1 + Z^2(t))^\theta] \leq 2(1 + Z^2(0))^\theta + \frac{1}{2} K e^{\mu t}.$$

Then

$$\limsup_{t \rightarrow \infty} E[(1 + Z^2(t))^\theta] \leq K,$$

which implies (3.3). $\square$

Theorem 3.7. Assume $v\beta > 0$. Then the solution of (2.1) is stochastically ultimately upper bounded.

Proof. From Lemma 3.6, there is a constant θ such that (3.3) holds. For $\varepsilon \in (0, 1)$, let $H = e^{(K\varepsilon^{-1})^{\frac{1}{2\theta}}}$. By Chebyshev's inequality and Lemma 3.4,

$$P\{X(t) > H\} = P\{\log X(t) > \log H\} \leq P\{Z(t) > \log H\} \leq P\{|Z(t)| > \log H\} \leq \frac{E|Z(t)|^{2\theta}}{\log^{2\theta} H}.$$

Then the desired assertion is obtained from (3.3). $\square$

Lemma 3.8. Assume

$$v\gamma := \sum_{i=1}^m v_i\gamma(i) := \sum_{i=1}^m v_i(\check{a}(i) - \frac{1}{2}\hat{\sigma}^2(i)) > 0.$$

Then there is a positive constant θ such that

$$\limsup_{t \rightarrow +\infty} EX^{-k\theta}(t) \leq K, \quad (3.12)$$

where K is a constant defined in (3.20).

Proof. Define

$$Y(t) = X^{-k}(t) \text{ on } t \geq 0. \quad (3.13)$$

Then

$$\begin{aligned} dY(t) &= kY(t) \left[-a(t, \alpha(t)) + b(t, \alpha(t))X^k + \frac{1}{2}(k+1)\sigma^2(t, \alpha(t)) \right] dt - k\sigma(t, \alpha(t))Y(t)dW(t) \\ &= \left\{ kY(t) \left[-a(t, \alpha(t)) + \frac{1}{2}(k+1)\sigma^2(t, \alpha(t)) \right] + kb(t, \alpha(t)) \right\} dt \\ &\quad - k\sigma(t, \alpha(t))Y(t)dW(t) \end{aligned} \quad (3.14)$$

From Lemma 3.5,

$$Qd = -k\gamma + k(v\gamma)\mathbf{1}$$

has a solution $d = (d_1, d_2, \dots, d_m)^T \in \mathbb{R}^m$. Therefore,

$$k\gamma(i) + \sum_{j=1}^m q_{ij}d_j = kv\gamma > 0, \quad i \in \mathcal{M}. \quad (3.15)$$

From (3.15), we have

$$\frac{1}{\theta(1-d_i\theta)} \sum_{j=1}^m q_{ij}(1-d_j\theta) = -kv\gamma + k\gamma(i) - \frac{d_i\theta}{1-d_i\theta} \sum_{j=1}^m q_{ij}d_j. \quad (3.16)$$

Pick out a constant $\theta \in (0, \frac{1}{2})$ such that

$$\frac{1}{2} < 1 - d_i\theta < 2, \quad i \in \mathcal{M} \quad (3.17)$$

and

$$kv\gamma - \frac{1}{2}k^2\theta\hat{\sigma}^2(i) + \frac{d_i\theta}{1-d_i\theta} \sum_{j=1}^m q_{ij}d_j > 0, \quad i \in \mathcal{M}. \quad (3.18)$$

Now, choose a sufficiently small positive number μ such that

$$kv\gamma - \frac{1}{2}k^2\theta\hat{\sigma}^2(i) + \frac{d_i\theta}{1-d_i\theta} \sum_{j=1}^m q_{ij}d_j - \frac{\mu}{\theta} > 0, \quad i \in \mathcal{M}. \quad (3.19)$$

For $Y(t)$, define the function $V(\cdot, \cdot, \cdot) : [0, +\infty) \times \mathbb{R} \times \mathcal{M}$ by

$$V(t, y, i) = (1 - d_i\theta)(1 + y)^\theta.$$

That is, V is independent of t . It can be seen that

$$\begin{aligned} \mathcal{L}V(t, y, i) &= \theta(1 - d_i\theta)(1 + y)^{\theta-2} \\ &\quad \times \left\{ \left[-k(a(t, i) - \frac{1}{2}\sigma^2(t, i)) + \frac{1}{2}k^2\theta\sigma^2(t, i) \right. \right. \\ &\quad \left. \left. + \frac{1}{\theta(1 - d_i\theta)} \sum_{j=1}^m q_{ij}(1 - d_j\theta) \right] y^2 \right. \\ &\quad \left. + \left[-ka(t, i) + \frac{1}{2}k(k+1)\sigma^2(t, i) + kb(t, i) \right. \right. \\ &\quad \left. \left. + \frac{2}{\theta(1 - d_i\theta)} \sum_{j=1}^m q_{ij}(1 - d_j\theta) \right] y + kb(t, i) + \frac{1}{\theta(1 - d_i\theta)} \sum_{j=1}^m q_{ij}(1 - d_j\theta) \right\} \end{aligned}$$

and

$$\begin{aligned}
& \mathcal{L}(e^{\mu t} V(t, y, i)) \\
&= \theta(1 - d_i \theta) e^{\mu t} (1 + y)^{\theta-2} \left\{ kb(t, i) + \frac{1}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{\mu}{\theta} \right. \\
&\quad + \left[-ka(t, i) + \frac{1}{2} k(k+1) \sigma^2(t, i) + kb(t, i) + \frac{2}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{2\mu}{\theta} \right] y \\
&\quad + \left. \left[-k(a(t, i) - \frac{1}{2} \sigma^2(t, i)) + \frac{1}{2} k^2 \theta \sigma^2(t, i) + \frac{1}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{\mu}{\theta} \right] y^2 \right\} \\
&\leq \theta(1 - d_i \theta) e^{\mu t} (1 + y)^{\theta-2} \left\{ \left[-k\nu\gamma + \frac{1}{2} k^2 \theta \widehat{\sigma}^2(i) - \frac{d_i \theta}{1 - d_i \theta} \sum_{j=1}^m q_{ij} d_j + \frac{\mu}{\theta} \right] y^2 \right. \\
&\quad + \left[-ka(t, i) + \frac{1}{2} k(k+1) \sigma^2(t, i) + kb(t, i) + \frac{2}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{2\mu}{\theta} \right] y \\
&\quad + \left. kb(t, i) + \frac{1}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{\mu}{\theta} \right\} \\
&\leq \frac{1}{2} \mu K e^{\mu t},
\end{aligned}$$

where

$$\begin{aligned}
K &:= 2\mu^{-1} \left\{ \sup_{(t, y, i) \in [0, +\infty) \times \mathbb{R} \times \mathcal{M}} \theta(1 - d_i \theta) \right. \\
&\quad \times \left\{ \left[-k\nu\gamma + \frac{1}{2} k^2 \theta \widehat{\sigma}^2(i) - \frac{d_i \theta}{1 - d_i \theta} \sum_{j=1}^m q_{ij} d_j + \frac{\mu}{\theta} \right] y^2 \right. \\
&\quad + \left[-ka(t, i) + \frac{1}{2} k(k+1) \sigma^2(t, i) + kb(t, i) + \frac{2}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{2\mu}{\theta} \right] y \\
&\quad + \left. kb(t, i) + \frac{1}{\theta(1 - d_i \theta)} \sum_{j=1}^m q_{ij}(1 - d_j \theta) + \frac{\mu}{\theta} \right\} \vee 1 \Big\}.
\end{aligned} \tag{3.20}$$

Let ℓ_0 be a sufficiently large integer such that $Y(0) = X^{-k}(0) \in [\frac{1}{\ell_0}, \ell_0]$. Define the stopping time $\rho_\ell = \inf\{t \geq 0 : Y(t) \notin [\frac{1}{\ell}, \ell]\}$ for each $\ell \geq \ell_0$.

By the generalized Itô formula,

$$M(t) := e^{\mu t} V(t, Y(t), \alpha(t)) - V(0, Y(0), \alpha(0)) - \int_0^t \mathcal{L}(e^{\mu s} V(s, Y(s), \alpha(s))) ds$$

is a local martingale, which implies that $EM(t \wedge \rho_\ell) = 0$.

Using similar method as in the proof of Lemma 3.6 and the dominated convergence theorem, it follows that

$$E[e^{\mu t} V(t, Y(t), \alpha(t))] = EV(0, Y(0), \alpha(0)) + E \int_0^t \mathcal{L}(e^{\mu s} V(s, Y(s), \alpha(s))) ds,$$

which leads to

$$\frac{1}{2} E[e^{\mu t} (1 + Y(t))^\theta] \leq 2(1 + Y(0))^\theta + \frac{1}{2} K e^{\mu t}.$$

Then

$$\limsup_{t \rightarrow \infty} E[(1 + Y(t))^\theta] \leq K,$$

which implies (3.12). $\square$

Theorem 3.9. Assume $\nu\gamma > 0$. Then the solution of (2.1) is stochastically ultimately lower bounded.

Proof. From Lemma 3.8,

$$P\{X(t) < \delta\} = P\{X^{-k\theta}(t) > \delta^{-k\theta}\} \leq \frac{EX^{-k\theta}(t)}{\delta^{-k\theta}} = \delta^{k\theta} EX^{-k\theta}(t).$$

For any $\varepsilon > 0$, let $\delta = (K^{-1}\varepsilon)^{\frac{1}{\bar{\alpha}}}$. Then

$$\limsup_{t \rightarrow +\infty} P\{X(t) < \delta\} < \varepsilon.$$

The proof is therefore complete. $\square$

Theorem 3.10. Assume $\nu\gamma' := \sum_{i=1}^m \nu_i \gamma'(i) := \sum_{i=1}^m \nu_i (\widehat{a}(i) - \frac{1}{2}\check{\sigma}^2(i)) < 0$. Then the solution $X(t)$ of (2.1) is not stochastically ultimately lower bounded.

Proof. Let

$$V(t, x, i) = \log x. \quad (3.21)$$

Then for $(t, x, i) \in [0, +\infty) \times \mathbb{R}_+ \times \mathcal{M}$,

$$\mathcal{L}V(t, x, i) = a(t, i) - \frac{1}{2}\sigma^2(t, i) - b(t, i)x^k. \quad (3.22)$$

Eq. (3.22) together with the generalized Itô formula leads to

$$\begin{aligned} & \log X(t) - \log x_0 \\ &= \int_0^t [a(s, \alpha(s)) - \frac{1}{2}\sigma^2(s, \alpha(s)) - b(s, \alpha(s))X^k(s)]ds + \int_0^t \sigma(s, \alpha(s))dW(s). \end{aligned} \quad (3.23)$$

Therefore,

$$\begin{aligned} & \log X(t) - \log x_0 \\ &\leq \int_0^t [a(s, \alpha(s)) - \frac{1}{2}\sigma^2(s, \alpha(s))]ds + \int_0^t \sigma(s, \alpha(s))dW(s) \\ &\leq \int_0^t [\widehat{a}(\alpha(s)) - \frac{1}{2}\check{\sigma}^2(\alpha(s))]ds + \int_0^t \sigma(s, \alpha(s))dW(s). \end{aligned} \quad (3.24)$$

Let

$$M(t) = \int_0^t \sigma(s, \alpha(s))dW(s),$$

which is a martingale. The quadratic variation of $M(t)$ is

$$\langle M, M \rangle_t = \int_0^t \sigma^2(\alpha(s))ds \leq Kt.$$

By the strong law of numbers for local martingales [12],

$$\lim_{t \rightarrow +\infty} \frac{M(t)}{t} = 0, \text{ a.s.} \quad (3.25)$$

Thus

$$\limsup_{t \rightarrow \infty} \frac{\log X(t)}{t} \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t [\widehat{a}(\alpha(s)) - \frac{1}{2}\check{\sigma}^2(\alpha(s))]ds = \nu\gamma' < 0.$$

It follows that

$$\lim_{t \rightarrow +\infty} X(t) = 0, \text{ a.s.}$$

which implies the desired assertion. $\square$

The next Theorem 3.11 follows directly from Theorem 3.7 and Theorem 3.9.

Theorem 3.11. Assume $\nu\beta > 0$ and $\nu\gamma > 0$. Then (2.1) is stochastically permanent.

4. Two-time-scale models

In many applications, we encounter the case that the Markov chain is fast varying or it displays certain two-time-scale behavior. The two time scales include the usual running time and a fast or stretched time. The distinction is represented by introducing a small parameter $\varepsilon > 0$ in the generator of the Markov chain. The original system is Ginzburg–Landau with

random switching, whereas when the ε is sufficiently small, the system is close to a limit or “reduced” diffusion without switching. The closeness is demonstrated by the weak convergence of probability measures. The implication is that for all practical consideration, one can essentially replace the complex original Ginzburg–Landau equations with switching by a Ginzburg–Landau diffusion equation without switching. The coefficients of the reduced diffusion equation are averages of the original system with respect to the stationary distribution of the fast changing part of the Markov chain. This shows a substantial reduction of complexity.

To proceed, assume that the Markov chain $\alpha^\varepsilon(t)$ is generated by

$$Q^\varepsilon = \frac{Q}{\varepsilon} + Q_0, \quad (4.1)$$

where $0 < \varepsilon \ll 1$, Q is irreducible, and Q_0 is a generator of another continuous-time Markov chain. Then (2.1) becomes

$$dX_\varepsilon(t) = \left[a(t, \alpha_\varepsilon(t))X_\varepsilon(t) - b(t, \alpha_\varepsilon(t))X_\varepsilon^{k+1}(t) \right] dt + \sigma(t, \alpha_\varepsilon(t))X_\varepsilon(t)dW(t). \quad (4.2)$$

Our study is for reduction of complexity. The rationale is as follows. The switching process $\alpha_\varepsilon(\cdot)$ is fast varying. Thus it acts as a noise process. As ε is getting smaller and smaller, the noise is averaged out and the complex process (4.2) can be reduced to a limit Ginzburg–Landau equation without switching, which is much simpler than the process with switching. Then using the limit as a guide, we may study further properties of the process given in (4.2). To obtain the limit result, we replace (A1) by a slightly stronger condition.

(A1') For each $i \in \mathcal{M}$, $a(t, i)$, $b(t, i)$ and $\sigma(t, i)$ are continuous functions defined on $[0, +\infty)$ and $b(t, i) \geq 0$.

Before proceeding further, let us recall the definition of weak convergence first. Let Y_n and Y be $\mathbb{R}^r$ -valued random variables, with $r \geq 1$ being a positive integer. We say that Y_n converges weakly to Y if and only if for any bounded and continuous function $g(\cdot)$, $Eg(Y_n) \rightarrow Eg(Y)$ as $n \rightarrow \infty$. The sequence $\{Y_n\}$ is said to be tight if for each $\eta > 0$, there is a compact set K_η such that $P(Y_n \in K_\eta) \geq 1 - \eta$ for all n . The definitions of weak convergence and tightness extend to random variables in a metric space. The notion of weak convergence is a substantial generalization of convergence in distribution. It implies much more than just convergence in distribution since $g(\cdot)$ can be chosen in many interesting ways. On a complete separable metric space, the notion of tightness is equivalent to sequential compactness, which is known as Prohorov's Theorem. Due to this theorem, we are able to extract convergent subsequences once tightness is verified. Let $D^r[0, \infty)$ denote the space of $\mathbb{R}^r$ -valued functions that are right continuous and have left-hand limits, endowed with the Skorohod topology. For various notions and terms in weak convergence theory such as Skorohod topology, Skorohod representation etc., we refer to [13, Chapter 7] and the references therein.

Theorem 4.1. Assume that (A1') holds and that Q given by (4.1) is irreducible. Suppose $X_\varepsilon(0) = X_{\varepsilon,0}$ converges weakly to $x(0) = x_0$ with $X_\varepsilon(0) > 0$ a.s. Then as $\varepsilon \rightarrow 0$, $X_\varepsilon(\cdot)$ converges weakly to $x(\cdot)$ such that $x(\cdot)$ is a diffusion process, a solution of the Ginzburg–Landau equation

$$dx(t) = \left[\bar{a}(t)x(t) - \bar{b}(t)x^{k+1}(t) \right] dt + \bar{\sigma}(t)x(t)dW(t), \quad (4.3)$$

where

$$\bar{a}(t) = \sum_{i=1}^m a(t, i)v_i, \quad \bar{b}(t) = \sum_{i=1}^m b(t, i)v_i \quad \text{and} \quad \bar{\sigma}(t) = \sqrt{\sum_{i=1}^m \sigma^2(t, i)v_i}.$$

Before proceeding to the proof of the theorem, we first state a lemma, which concerns the existence and uniqueness of solution of (4.3). Thus we omit the details and only mention that the limit diffusion has a unique solution that can be established similar to that of Theorem 2.1 but only simpler since no switching is involved.

Lemma 4.2. Under the conditions of Theorem 4.1, for any initial condition $x(0) = x_0 > 0$, there is a unique positive solution of (4.3) on $t > 0$.

Proof of Theorem 4.1. To prove the desired result, we need to show that $\{X_\varepsilon(\cdot)\}$ is tight [14, p. 377]. Once the tightness is established, we characterize the limit process by means of a martingale problem formulation.

Because system (4.2) is highly nonlinear with polynomial growth in x , we first use a truncation device [13, p. 284]. Denote the N -sphere by $S_N = \{x : |x| \leq N\}$ for a fixed but otherwise arbitrary large number $N > 0$. Let $X_{\varepsilon,N}(\cdot)$ be the process that is

equal to $X_\varepsilon(\cdot)$ up until the first time exit from S_N . Consider

$$dX_{\varepsilon,N}(t) = \left[a(t, \alpha_\varepsilon(t))X_{\varepsilon,N}(t) - b(t, \alpha_\varepsilon(t))X_{\varepsilon,N}^{k+1}(t) \right] \zeta(X_{\varepsilon,N}(t))dt + \sigma(t, \alpha_\varepsilon(t))X_{\varepsilon,N}(t)\zeta(X_{\varepsilon,N}(t))dW(t), \quad (4.4)$$

where $\zeta(x)$ is a truncation function satisfying $\zeta(x) = 1$ for $x \in S_N$, $\zeta(x) = 0$ for $x \in \mathbb{R}_+ \setminus S_{N+1}$, and $\zeta(x)$ is sufficiently smooth (C^2 in x). We proceed as follows. We first show that $\{X_{\varepsilon,N}(\cdot)\}$ is tight. Then we characterize its limit. Finally, we show that the untruncated process is also convergent.

Step 1: Tightness of $\{X_{\varepsilon,N}(\cdot)\}$. For any $t > 0$, $\delta > 0$, and $0 < s \leq \delta$,

$$\begin{aligned} & X_{\varepsilon,N}(t+s) - X_{\varepsilon,N}(t) \\ &= \int_t^{t+s} \left[a(u, \alpha_\varepsilon(u))X_{\varepsilon,N}(u) - b(u, \alpha_\varepsilon(u))X_{\varepsilon,N}^{k+1}(u) \right] \zeta(X_{\varepsilon,N}(u))du \\ & \quad + \int_t^{t+s} \sigma(u, \alpha_\varepsilon(u))X_{\varepsilon,N}(u)\zeta(X_{\varepsilon,N}(u))dW(u). \end{aligned}$$

Use E_t^ε to denote the conditional expectation conditioned on $\mathcal{F}_t^\varepsilon = \{X_\varepsilon(0), W(s), \alpha^\varepsilon(s) : s \leq t\}$. Using the Hölder inequality to the first term on the right-hand side and Itô isometry on the second term together with the boundedness of the truncated process, we obtain that there is a random variable $\beta_\varepsilon(t)$ such that

$$E_t^\varepsilon |X_{\varepsilon,N}(t+s) - X_{\varepsilon,N}(t)|^2 \leq E_t^\varepsilon \beta_\varepsilon(\delta).$$

Taking limsup over ε and followed by limit as $\delta \rightarrow 0$, we have

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} E\beta_\varepsilon(\delta) = 0.$$

Thus $\{X_{\varepsilon,N}(\cdot)\}$ is tight by virtue of [15, Theorem 3, p.47].

Step 2: Characterization of the limit. Owing to the Prohorov theorem, in a complete separable metric space, tightness is equivalent to sequential compactness [13, p. 229]. Because $\{X_{\varepsilon,N}(\cdot)\}$ is tight, we can extract a weakly convergent subsequence. For notational simplicity, still denote the subsequence by $\{X_{\varepsilon,N}(\cdot)\}$ with limit $x_N(\cdot)$. We characterize the limit by means of martingale problem formulation [14, p. 378] and [13] with operator $\mathcal{L}_N$, where

$$\begin{aligned} \mathcal{L}_N f(x, t) &= \frac{\partial f(x, t)}{\partial t} + (\bar{a}(t)x - \bar{b}(t)x^{k+1})\zeta_N(x) \frac{\partial f(x, t)}{\partial x} \\ & \quad + \frac{1}{2}\sigma^2(t)x^2\zeta_N(x) \frac{\partial^2 f(x, t)}{\partial x^2}, \quad \text{for } f(\cdot) \in C_0^2, \end{aligned} \quad (4.5)$$

(the class of C^2 functions with compact support).

To proceed, for any $t, s > 0$, any $f(\cdot) \in C_0^2$, for each $i \in \mathcal{M}$, we shall show

$$f(x_N(t+s), t+s) - f(x_0, 0) - \int_0^t \mathcal{L}_N f(x_N(u), u)du \text{ is a martingale.} \quad (4.6)$$

To verify (4.6), it suffices to verify that for any bounded and continuous function $H(\cdot)$, any integer κ , and any $t_l \leq t$ with $l \leq \kappa$, we have

$$\begin{aligned} & EH(x_N(t_l) : l \leq \kappa) [f(x_N(t+s), t+s) - f(x_N(t), t)] \\ &= EH(x_N(t_l) : l \leq \kappa) \left(\int_t^{t+s} \mathcal{L}_N f(x_N(u), u)du \right). \end{aligned} \quad (4.7)$$

To verify (4.7), we work with the process indexed by ε . Because $X_{\varepsilon,N}(\cdot)$ converges weakly to $x_N(\cdot)$, by virtue of the Skorohod representation, we may assume with a slight abuse of notation (i.e., without changing notation), $X_{\varepsilon,N}(\cdot) \rightarrow x_N(\cdot)$ in the sense of convergence w.p.1. It follows that as $\varepsilon \rightarrow 0$,

$$\begin{aligned} & EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [f(X_{\varepsilon,N}(t+s), t+s) - f(X_{\varepsilon,N}(t), t)] \\ & \rightarrow EH(x_N(t_l) : l \leq \kappa) [f(x_N(t+s), t+s) - f(x_N(t), t)]. \end{aligned} \quad (4.8)$$

Expanding the difference, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [f(X_{\varepsilon,N}(t+s), t+s) - f(X_{\varepsilon,N}(t), t)] \\ &= \lim_{\varepsilon \rightarrow 0} EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) \left[\sum_{j=0}^4 g_{\varepsilon,j}(t+s, t) \right], \end{aligned} \quad (4.9)$$

where

$$\begin{aligned}
 g_{\varepsilon,0}(t+s, t) &= \int_t^{t+s} \frac{\partial f_N(X_{\varepsilon,N}(u), u)}{\partial u} du \\
 g_{\varepsilon,1}(t+s, t) &= \int_t^{t+s} \sum_{i=1}^m \frac{\partial f_N(X_{\varepsilon,N}(u), u)}{\partial x} (a(u, i)X_{\varepsilon,N}(u) - b(u, i)X_{\varepsilon,N}^{k+1}(u)) \zeta_N(X_{\varepsilon,N}(u)) v_i du \\
 g_{\varepsilon,2}(t+s, t) &= \frac{1}{2} \int_t^{t+s} \sum_{i=1}^m \frac{\partial^2 f(X_{\varepsilon,N}(u), u)}{\partial x^2} \sigma^2(u, i) X_{\varepsilon,N}^2(u) \zeta_N(X_{\varepsilon,N}(u)) v_i du \\
 g_{\varepsilon,3}(t+s, t) &= \int_t^{t+s} \sum_{i=1}^m \frac{\partial f_N(X_{\varepsilon,N}(u), u)}{\partial x} (a(u, i)X_{\varepsilon,N}(u) - b(u, i)X_{\varepsilon,N}^{k+1}(u)) \zeta_N(X_{\varepsilon,N}(u)) \\
 &\quad \times [I_{\{\alpha_\varepsilon(u)\}} - v_i] du \\
 g_{\varepsilon,4}(t+s, t) &= \frac{1}{2} \int_t^{t+s} \sum_{i=1}^m \frac{\partial^2 f(X_{\varepsilon,N}(u), u)}{\partial x^2} \sigma^2(u, i) X_{\varepsilon,N}^2(u) \zeta_N(X_{\varepsilon,N}(u)) [I_{\{\alpha_\varepsilon(u)\}} - v_i] du.
 \end{aligned} \tag{4.10}$$

By virtue of the weak convergence and the Skorohod representation, we have that as $\varepsilon \rightarrow 0$,

$$\begin{aligned}
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) g_{\varepsilon,0}(t+s, t) \\
 &\rightarrow EH(x_N(t_l) : l \leq \kappa) \left[\int_t^{t+s} \frac{\partial f(x_N(u), u)}{\partial u} du \right], \\
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) g_{\varepsilon,1}(t+s, t) \\
 &\rightarrow EH(x_N(t_l) : l \leq \kappa) \left[\int_t^{t+s} \frac{\partial f(x_N(u), u)}{\partial x} (\bar{a}(u)x_N(u) - \bar{b}(u)x_N^{k+1}(u)) \zeta_N(x_N(u)) du \right], \\
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) g_{\varepsilon,2}(t+s, t) \\
 &\rightarrow \frac{1}{2} \left[\int_t^{t+s} \frac{\partial^2 f(x_N(u), u)}{\partial x^2} \bar{\sigma}^2(u) x_N^2(u) \zeta_N(x_N(u)) du \right].
 \end{aligned} \tag{4.11}$$

Next we show that $g_{\varepsilon,j}(t+s, t)$ for $j = 3, 4$ do not contribute anything to the limit process. To this end, let $0 < \Delta < 1$, take a partition of the interval $[t, t+s]$ by using $N_\Delta = \lfloor s/\varepsilon^{(1/2)-\Delta} \rfloor$, and denote the partition as $t_0^\Delta = t < t_1^\Delta < \dots < t_{N_\Delta}^\Delta = t+s$. Note that without loss of generality, we have assumed that $t_{N_\Delta}^\Delta$ coincides with $t+s$. The modification if this does not hold is straightforward. Now define

$$\begin{aligned}
 \tilde{g}_{\varepsilon,3}(t+s, t) &= \sum_{j=0}^{N_\Delta} \int_{t_j^\Delta}^{t_{j+1}^\Delta} \sum_{i=1}^m \frac{\partial f_N(X_{\varepsilon,N}(t_j^\Delta), t_j^\Delta)}{\partial x} \\
 &\quad \times (a(t_j^\Delta, i)X_{\varepsilon,N}(t_j^\Delta) - b(t_j^\Delta, i)X_{\varepsilon,N}^{k+1}(t_j^\Delta)) \zeta_N(X_{\varepsilon,N}(t_j^\Delta)) [I_{\{\alpha_\varepsilon(u)\}} - v_i] du \\
 \tilde{g}_{\varepsilon,4}(t+s, t) &= \frac{1}{2} \sum_{j=0}^{N_\Delta} \int_{t_j^\Delta}^{t_{j+1}^\Delta} \sum_{i=1}^m \frac{\partial^2 f(X_{\varepsilon,N}(t_j^\Delta), t_j^\Delta)}{\partial x^2} \sigma^2(t_j^\Delta, i) X_{\varepsilon,N}^2(t_j^\Delta) \zeta_N(X_{\varepsilon,N}(t_j^\Delta)) \\
 &\quad \times [I_{\{\alpha_\varepsilon(u)\}} - v_i] du.
 \end{aligned} \tag{4.12}$$

Then by using the continuity of $a(\cdot, i)$, $b(\cdot, i)$, and $\sigma(\cdot, i)$, it can be shown that as $\varepsilon \rightarrow 0$,

$$\begin{aligned}
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [g_{\varepsilon,3}(t+s, t) - \tilde{g}_{\varepsilon,3}(t+s, t)] \rightarrow 0 \\
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [g_{\varepsilon,4}(t+s, t) - \tilde{g}_{\varepsilon,4}(t+s, t)] \rightarrow 0.
 \end{aligned} \tag{4.13}$$

To show that $\tilde{g}_{\varepsilon,j}(t+s, t)$ for $j = 3, 4$ lead to the zero limit, we note for example,

$$\begin{aligned}
 &EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [\tilde{g}_{\varepsilon,4}(t+s, t)] \\
 &= EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) \left[\frac{1}{2} \sum_{j=0}^{N_\Delta} \sum_{i=1}^m \frac{\partial^2 f(X_{\varepsilon,N}(t_j^\Delta), t_j^\Delta)}{\partial x^2} \sigma^2(t_j^\Delta, i) X_{\varepsilon,N}^2(t_j^\Delta) \zeta_N(X_{\varepsilon,N}(t_j^\Delta)) \right. \\
 &\quad \left. \times E_{t_j^\Delta}^\varepsilon \int_{t_j^\Delta}^{t_{j+1}^\Delta} [I_{\{\alpha_\varepsilon(u)\}} - v_i] du \right]
 \end{aligned}$$

In the above, we have used the measurability of $X_{\varepsilon,N}(t_j^\Delta)$ w.r.t. $\mathcal{F}_{t_j^\Delta}^\varepsilon$. Using an argument as in [14, Theorem 5.25], it can be shown that

$$E_{t_j^\Delta}^\varepsilon \left[\int_{t_j^\Delta}^{t_{j+1}^\Delta} [I_{\{\alpha_\varepsilon(u)\}} - v_i] du \right]^2 = O(\varepsilon).$$

Thus,

$$EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [\widetilde{g}_{\varepsilon,4}(t+s, t)] \rightarrow 0.$$

Likewise,

$$EH(X_{\varepsilon,N}(t_l) : l \leq \kappa) [\widetilde{g}_{\varepsilon,3}(t+s, t)] \rightarrow 0.$$

Combining the estimates we obtained thus far, we have $X_{\varepsilon,N}(\cdot)$ converges weakly to $x_N(\cdot)$ that is the solution of the martingale problem with operator $\mathcal{L}_N$.

Finally, we prove the untruncated process $\{X_{\varepsilon}(\cdot)\}$ also converges to the desire limit $x(\cdot)$. This process is similar to that of [15, p. 46]. The details are thus omitted. $\square$

5. Further remarks

This paper developed a class of generalized Ginzburg–Landau equations with random switching. In addition to obtaining explicit solutions, certain boundedness properties as well permanence are obtained. We also examined a two-time-scale formulation and established a weak convergence result leading reduction of computational complexity.

Because we focus on Ginzburg–Landau equations in this paper, we chose to use $k \geq 2$ in (2.1). It should be pointed out that if $k > 0$, our results still hold. It can also be seen that when $k = 1$, (2.1) is a generalized logistic equation with random switching, which is an important ecological mathematical model.

In subsequent study, further asymptotic properties such as moment bounds, stability in distribution, and recurrence deserve careful consideration. Moreover, systems with delay may also be considered in the subsequent study.

Acknowledgments

The research of first author was supported in part by the National Natural Science Foundation of China (11471015, 11401002, 11771001), Anhui Province Workshop of Prestigious Teacher, China (2016msgzs006), Anhui Provincial Nature Science Foundation, China (1508085QA01), Provincial Natural Science Research Project of Anhui Colleges, China (KJ2014A010) and Doctoral Research Start-up Funds Projects of Anhui University, China. The research of second author was supported in part by the National Science Foundation, United States under grant DMS-1710827. The research of third author was supported in part by Hubei Key Laboratory of Engineering Modeling and Scientific Computing, China.

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